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**SOME PROPOSED RESEARCH ACTIVITIES
PERTAINING TO RELIABILITY**

by

R. O. Frantik

and

**SOME NOTES ON
THE ESTIMATION OF RELIABILITY**

by

R. L. Calvert

FEBRUARY 1958

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March 1960

SOME PROPOSED RESEARCH

ACTIVITIES PERTAINING TO RELIABILITY

Contributed by the Staff of the
Reliability Evaluation Division
Sandia Corporation
Albuquerque, New Mexico

Presented by R. O. Frantik

Introduction

The purpose of this paper is to outline the major problems that have appeared in reliability research at Sandia Corporation, to note the progress which has been made to date, and to generalize from our experience. Research that is not recorded in the professional journals too frequently remains in obscurity, even when the subject matter is not of a classified nature. Therefore, it is our purpose to advertise, to some extent, the progress that Sandia Corporation has made in this particular respect. We also seek information which may contribute to the solution of the problems which are discussed herein.

Before proceeding with this topic, a few comments concerning the Reliability Evaluation Division of Sandia Corporation will be of interest.

Two years ago, at the symposium at Corona, I outlined the weapon development process followed at Sandia and indicated the role of the reliability group in this process.¹ Since that time, the name of the division has been changed from the Component Analysis Division to the Reliability Evaluation Division. The division is now under the Director of Systems Development rather than under the Director of Research. There has been no essential change in personnel in the division.

As the change in division name indicates, our efforts now are directed more or less exclusively toward reliability considerations. The administrative change puts the division more closely in contact with both the producer and the consumer of reliability data within Sandia Corporation.

One of the major projects in the division is directed toward setting up machine methods for collecting, recording, and sorting data to feed into statistical studies in the evaluation of reliability. As these data become available, more effort must be expended in deriving the necessary statistical methods for their conversion into reliability estimates. Therefore, it is expected

that the time and manpower directed toward reliability research at Sandia will increase substantially in the near future.

Problems peculiar to the human engineering aspects of reliability are not considered in this paper. Such studies make a substantial and important contribution to the overall weapon or missile system reliability. However, because of the time restrictions here and because human engineering problems are considered at other symposia, this topic is not pursued in this paper.

General Areas For Reliability Research

The objective of reliability research is to derive the necessary mathematical, statistical, and laboratory methods required in the following four general activities:

- I. The evolution of a component or system design which incorporates optimum reliability.*
- II. The production of a component or a system which preserves (and possibly improves) the reliability characteristics of the design.
- III. The maintenance and evaluation of reliability characteristics in stockpile and in use.
- IV. The provision of bases for decisions by management by the military where matters of reliability are involved.

Obviously, the result of a particular reliability research project frequently will apply to more than one of these four categories. In some situations the problems discussed in the remainder of this paper are assigned somewhat arbitrarily to one of the four categories.

The research activities of the Reliability Evaluation Division generally are directed toward the derivation of mathematical and statistical tools. Research into laboratory methods usually is confined to the statistical design of experiments aspect of the problem.

Reliability Research Problems In Development

First, we consider the major problems that arise in connection with the evolution of a component or a system design. Among these problems are:

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* The term optimum reliability means maximum reliability consistent with other system requirements.

1. Analysis of the design in an attempt to predict the most probable modes of failure.
2. Determination of the test procedures which yield the maximum amount of reliability information and re-evaluation of the procedures as testing progresses.
3. Investigation of means of decreasing test costs in time and money.
4. Investigation of methods for estimating the use environment and for extrapolating reliability estimates into the use environment.
5. Application of the methods of Boolean matrix theory to the estimation of reliability in the analysis of circuit designs.
6. Investigation of methods for stating reliability specifications.

Work on the first two of these six problems has proceeded on somewhat of a routine, non-research basis at Sandia. Proposed circuit designs are usually examined by the Reliability Evaluation Division for the purpose of making a priori reliability estimates. In these examinations unsuspected modes of failure frequently are uncovered. General guide lines for applying the "physics of failure" approach to the design of a test program have been suggested by Leslie E. Ball. However, we know of no established procedure for the prediction of individual modes of failure particularly where a combination of environmental factors is involved.

Procedures for specific types of tests and for tests at particular stages in the design to stockpile process have received considerable attention. The design to stockpile sequence of events is usually long and sometimes quite complex. We believe that at least a sound philosophy or preferably a statistically sound procedure should be developed for apportioning time, money, and equipment to the series of tests required in this sequence of events. Such a procedure should yield more accurate reliability estimates and decrease the long range costs of the overall test program.

The fact that reliability varies with environmental factors is well known. However, this knowledge rarely embraces more than the qualitative aspects of such variation. Rules-of-thumb are offered occasionally as rough methods for approximating environmental effects. However, a reliability estimate with no indication of its accuracy is of little value except for purposes

of comparison. It is the reliability in use that is significant to our country's defense. Therefore, the use environment must be estimated with sufficient accuracy to allow incorporation of reasonable factors of safety into the weapon design. Furthermore, statistical and laboratory procedures need to be developed for estimating the functional relationships between reliability and the various environmental variables. Some theoretical consideration of this problem appears in a Sandia document² which considers the special case in which the environment is a random variable. Another approach may lie in the application of extreme value theory to the probability of occurrence of severe environments and environmental combinations.

The analysis of circuit designs in the early stages of development is one function of the Reliability Evaluation Division. The methods currently used have been explained in a Sandia Corporation document³. A second document⁴ illustrates some shortcut aids in the use of these methods. We hope to be able to apply Boolean matrix theory in deriving a method for more rapid circuit reliability evaluations. So far we have done little more than note an example or two which holds some promise for such an approach. Aside from establishing the necessary rules of operation, the main problem appears to be the conversion of a square matrix which describes circuit operation into a matrix with component reliability estimates which may be evaluated by machine. We believe this problem holds sufficient promise to warrant an early attack.

The valuable contributions of Robert Lusser have gone far in the development of methods for making reliability specifications. However, we believe that a complete theory must develop the relationships between interval estimates of the system reliability and the interval estimates of component reliability. Such relationships are fairly readily found for point estimates of reliability but point estimates rarely, if ever, carry any quantitative measure of their accuracy. Two Sandia studies^{5,6} make a start on this problem, but much remains to be done. The first of these studies considers the error density functions of systems (with operating time as the independent variable) in terms of the error density functions of the components. The second of these studies considers the general theory of confidence intervals and uses simple two-element systems for examples.

The problem of writing specifications also applies to test equipment. Insofar as the normal density function applies to test set errors and to errors in product performance, this problem has been considered by the Reliability Evaluation Division⁷.

Reliability Research In Production

Now, I should like to consider a few problems that appear to arise more frequently in the production stage of the system than in any of the other three general categories mentioned near the beginning of this paper. Some of these problems are:

1. The derivation of methods for using quality control and acceptance inspection data in the prediction of reliability. The suggestion of changes that will result in the data taken being more useful in deriving reliability estimates.
2. The derivation of procedures for measuring the effect of testing at various stages of production, storage, and use upon the predicted reliability in use.
3. The investigation of the theory of life curves and its application to the reliability aspects of weapon and missile components.
4. The derivation of improved sampling plans.
5. Further inquiry into the theory of testing for intermittent failures.
6. Further study of the use of confidence intervals. The study and application of the theory of confidence regions.
7. The development of criteria for making a choice between competing programs and between suppliers.

In the course of extending our data collection system, we expect to learn the extent to which quality control and acceptance inspection data can be used in estimating reliability. We are using some of these data presently. By comparing these data with data from later tests and inspections, some indication of the effects of certain environments may be obtained.

The effects of testing on reliability are closely akin to the types of life curves which may characterize many components. Therefore, in general, these two problems may be considered together. The theory of the initial failure and chance failure life curve combination has been extensively treated. However, as Marcus Acheson effectively points out, there are many other types of life curves. Where the theory of life curves applies, the effects of testing require that we learn more about the life curves of particular components as an aid in determining whether testing has an adverse effect on reliability.

Presently known sampling plans require inordinately large sample sizes for reasonable assurance of quality of highly reliable devices. Samples of 50 per cent or more of a lot are required in some cases. Such large samples are patently costly. Millions of dollars could be saved by successful research into new sampling techniques. However, new sampling plans must be examined for their effect on the amount and quality of information generated for a reliability evaluation program. The general theory of sequential sampling is well known. New sampling plans might be based on sequential procedures.

Certain types of failures on some types of equipment appear intermittently. The theory of the detection of intermittent failures has been developed in the Reliability Evaluation Division as it applies to acceptance sampling.^{8,9} Further work remains to be done on the effects of intermittent failures as they pertain to final reliability estimates.

Many of our reliability estimates are stated in terms of confidence intervals. The theory of confidence intervals has been studied extensively at Sandia.⁶ However, as mentioned earlier in this paper, we have yet to determine the relationships between confidence intervals for components and those for the system. Graphs and nomographs have been constructed for the determination of confidence intervals where the binomial distribution applies and where the probability of failure is expected to be small¹⁰. These graphs have also been of value in making preliminary estimates of sample sizes required to demonstrate the attainment of specified reliability values.

In many situations we are interested in greater detail than a simple success-or-failure model provides. For example, a bomb may detonate prematurely, it may detonate properly, or it may fail to detonate. In such a trichotomy of events confidence intervals can be derived for the probability of each event if only the marginal density functions are considered. However, we believe that such a situation could be described more accurately by probabilistic methods which avoid such a piecemeal approach. Perhaps the theory of confidence regions could be developed and applied in such situations.

Numerous tests exist for making choices between products or systems where special conditions apply. We know of no test that can be applied in situations where the effectiveness of a product (for example) depends on several parameters, some of which favor one design (or supplier), while other parameters favor another design. We feel that this problem is amenable to the techniques developed for use by the systems analysts. We believe it is desirable that such tests should be comprehensible by those people who make such decisions.

Reliability Problems In Stockpiling and Use of Weapons

In the maintenance and evaluation of reliability in stockpile and in use three major problems arise:

1. The derivation of methods for using surveillance data in confirming or amending earlier estimates of reliability and in extrapolating reliability estimates into the use environment.
2. The investigation of the role of reliability in determining stockpile requirements.
3. The derivation of methods for using reliability estimates in decisions to or not to retrofit.

The first of these problems is a continuation of the more general problem of keeping an up-to-date estimate of reliability beginning with the original study of the system design. As the data collection program of the Reliability Evaluation Division matures, it is expected that greater effort will be expended on the analysis of this particular type of data.

The determination of stockpile requirements is a complex problem which generally requires input data available only to high military circles. In general such problems can be treated only in terms of unknown parameters. However, some aspects of this problem appear in connection with the first problem of the next section.

The problem of when to retrofit a weapon type has been considered at Sandia but no solution has been found as yet. Insofar as retrofitting may be justified because it improves reliability, this problem may be considered in connection with the first problem in the next section.

The Provision of Bases for Decisions

The general problem of providing bases for decisions by management or by the military where matters of reliability are involved overlaps some of the problems discussed in the preceding category. In addition to these, the following problems have appeared:

1. The derivation of methods for estimating the value of improvements in reliability in terms of money, time, or other measures of value.
2. The study of relationships between reliability and performance capability, accuracy, vulnerability, operability, etc.

The problem of evaluating improvements in reliability is discussed in a Sandia document¹¹. In this document elementary methods are suggested for determining lower limits for the dollar value of reliability improvements. When these lower limits are sufficiently high, they are of some value in indicating a decision. When a decision is not indicated by these elementary means, a more detailed evaluation must be made and one immediately is confronted with the lack of input data.

In Sandia Corporation the problem of studying the relationships between reliability, performance, capability, etc. is properly the business of the Systems Evaluation Department. Under the present organization, the Reliability Evaluation Division would contribute only to the reliability phases of such a problem. In the past this division, under a different organization, has contributed to one such study with respect to a particular weapon. The solution to such a problem usually lies in the derivation of an overall "worth equation". This worth equation supposedly serves as a decision function where a choice between two or more weapons or versions of the same weapon is involved. Such decisions depend upon many factors for which quantitative measures do not yet exist. Hence, the value of any general worth equation derived to date is a matter of considerable doubt.

Some General Problems

In addition to the problems already discussed in this paper there are others of a more general nature. It is possible that game theory, search theory, and information theory can be applied to problems of interest in reliability research. In the past the Reliability Evaluation Division rarely has had more than two people at work on problems of the type outlined earlier in this paper at any one time. Therefore, our efforts have been directed to problems and methods which have shown early promise for a useful solution. In the immediate future we hope to add to the research potential of the division. As such people are added we hope to do some research of a more exploratory nature.

In retrospect it appears that we have solved few of our problems. However, we believe that real progress has been made when our major problems have been recognized. We hope that additions to our reliability research staff will allow us to attack an increasing number of these problems. Although we of the Reliability Evaluation Division of Sandia Corporation do not expect to solve all of these problems unassisted, we hope that our major contributions to future reliability symposia will lie in the domain presented here.

An Appeal

This paper has presented some proposed research activities pertaining to reliability in the context of the work being done by Sandia Corporation. However, we believe that these problems are not peculiar to our company. We have attempted in this paper to bring focus on the nature and extent of the research needed to accomplish the major tasks which face reliability people, i.e., we have attempted to define the problem. We strongly urge that the infancy of reliability efforts everywhere be recognized and that these infant efforts be bolstered by support of the research required. To this end it is recommended that the various military services and manufacturers make these important problems known to universities and other research agencies and if necessary support this research by contract or endowment.

LIST OF REFERENCES

1. Frantik, R. O., "The Reliability Program at Sandia Corporation", Proceedings of the Symposium on Guided Missile Reliability, 26, 27, 28, October 1954, NOL-CORONA Report No. 233 (Part II), U. S. Naval Ordnance Laboratory, Corona, California.
2. Clark, C. R., "Reliability Studies of Systems in Which Component Reliabilities Vary with Environment", Sandia Corporation Technical Memorandum 127-55-51, June 30, 1955, Unclassified.
3. Frantik, R. O., "The Determination, Application, and Limitations of Circuit Reliability Equations", Sandia Corporation Report No. SC-3288 (TR), April 26, 1954, Unclassified.
4. Fine, A. M., "A Guide to Reliability Analysis for System Design Engineers", Sandia Corporation Technical Memorandum 45-56-12, March 20, 1956, Secret Restricted Data.
5. Buehler, R. J., "A Study of the Accuracy and Reliability of Compound Circuits", Sandia Corporation Report No. SC-2324 (TR), May 13, 1952, Unclassified.
6. Buehler, R. J., "Theoretical Considerations In Determining Confidence Intervals for Reliability", Unclassified.
7. Clark, C. R., and Wiesen, J. M., "Determination of Allowable Test Set Errors", Sandia Corporation Technical Memorandum 201-54-51, September 1, 1954, Unclassified.

8. Clark, C. R., "Acceptance Testing of Units Subject to Intermittent Defects", Sandia Corporation Technical Memorandum 143-55-51, July 25, 1955, Unclassified.
9. Clark, C. R., "Acceptance Testing for Intermittent Defects on a Sampling Basis", Sandia Corporation Technical Memorandum 210-55-51, October 18, 1955, Unclassified.
10. Calvert, R. L., "The Determination of Confidence Intervals for Probabilities of Proper, Dud, and Premature Operation", Sandia Corporation Technical Memorandum 213-55-51, October 17, 1955, Unclassified.
11. Calvert, R. L., "Some Comments on the Value of Improvements in Reliability", Sandia Corporation Technical Memorandum 86-56-12, May 2, 1956, Secret Restricted Data.

DISCUSSION

F. A. JENNINGS, Rocketdyne (North American Aviation): You mentioned under the first group of problems, the one of tolerance stacking which was one we had some difficulty with too; that is to predict the overall reliability if you know something about the parts. We have had rather good luck using Monte Carlo methods in this, and I was wondering if you would like to comment on that? The IBM machine and its ability to pick solutions in sudden sequences has been very useful in this respect.

R. O. FRANTIK: I don't off-hand see why you went to the Monte Carlo method. Was it because you didn't know what the distribution of the population of the things that you were testing was, or you didn't know what the distribution of the test equipment was? Or were you trying to empirically determine whether or not a particular theory was valid? First, I fail to see why you went to this particular method. Just knowing you went to a Monte Carlo method doesn't mean too much to me. You had to do something else before you did this.

MR. JENNINGS: If you know the distributions of a bunch of components, say you know the probabilities of failures of a group of components, and you wish to know the probability failure of the overall system. It is not necessarily just multiplied probabilities of the parts, and an attempt to derive this theoretically, it becomes very difficult to determine what tolerances to apply to the probability of the overall system. Whereas, if you build a model of it and then make random choices for the component failures or successes according to the component distributions, you can come out with a tolerance or a knowledge of the distribution of the overall reliability.

MR. FRANTIK: I'm sorry, when you said tolerance I jumped to a conclusion; here I figured you were talking about test equipment.

MR. JENNINGS: I'm sorry, I meant confidence intervals.

MR. FRANTIK: I find that you really mean confidence intervals. We have considered how Monte Carlo techniques can solve a problem of this nature. This, of course, is empiricism and it would be really nice to attack this problem, and this is what we have attempted, to attack this problem on a sound

mathematical basis. I have heard statements and I have heard them repeated that, look at those people who are in the mechanical engineering field, what a fine job they are doing. But I think you should also look back and see the fine mathematical job that was done in support of mechanical engineering. Take some very simple concepts, the concept of the section modulus of a beam, do we have anything that even vaguely resembles such a concept in reliability? We need this.

K. E. PORTZ, Bendix Radio: Along the same line, if you are going to start combining distributions, we would be interested in knowing what function you intend to use for the product or for the division of two random variables?

MR. FRANTIK: With regard to what particular problem do you have reference?

MR. PORTZ: Well, if you are going to take a network and describe the tolerance of the network as a function of the tolerance of the component, then you will have to combine the tolerances, and case where you have products and divisions, you are going to have a function for doing this. The sum and difference can be handled by a convolution integral, but what do you intend to do when you get a product?

MR. FRANTIK: I am not sure that I really understand what you are trying to say, if you mean that, granted that I have the distribution which applies to each one of the components; how do I take these 2 distributions or the same distributions if they happen to be similar devices and combine them?

MR. PORTZ: That is correct.

MR. FRANTIK: Well, this is hardly the place for me to tell you about this, first of all, I can't do it myself. We have made an attempt, assuming first a success failure model and using what has been referred to as Boolean algebra. As the beginning, saying if you have two devices, the distributions should combine in such a way as described by the probability report of successful or failure or whatever the particular circuit might be, and then perhaps over simplifying the problem, but at least making a beginning to it, saying that each one of these particular devices has a distribution which is defined by some mean and variance and further by the statement that it's normally distributed.

MR. PORTZ: In other words, you are combining by moments?

MR. FRANTIK: Yes, this is correct.

BRIG GEN C. H. MITCHELL, Air Materiel Command - (Wright-Patterson):
Being a layman and not a scientist, I always have to interject something of the production side in each of these. Mr. Frantik opened his remarks by speaking of the lack of knowledge or the ignorance that exists in many places as to the realities of this business; reminds me slightly of the farmer whose friend suggested that he go to some of the extension courses and learn a little more about farming and he said, "Hell! I am not farming half as well as I know how now." I don't think that there is any subject that imposes itself on our responsibilities of producing materiel for the armed forces that has as much knowledge available, but receives as much lip service in actual application throughout management as reliability.

SOME NOTES ON THE ESTI-
MATION OF RELIABILITY

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INTRODUCTION

The purpose of this paper is to show the value of complete success-failure history of past accepted lots of an equipment or component in deriving estimates of f , the fraction successful in a lot presented for use. When sufficient past lot information has accumulated, it is possible to determine a density function $g(f)$. The use of this density function with or without a sample from a future lot L allows certain estimates of the success probability to be made for units drawn from that lot. Obviously, it must be assumed that lot L will see the same spectrum of environments that has been applied to past lots.

The number of lots for which complete success-failure history is required depends on the test which is applied to determine the "goodness-of-fit" of the $g(f)$ which is chosen to fit the data. This last subject is not discussed in this paper.

ESTIMATES BASED ON PAST EXPERIENCE

Consider the situation in which the operation of a part or component may be classified as either a success or a failure. Suppose that such parts or components are procured in large lots and that past experience is sufficient to determine a density function $g(f)$ for the fraction f of success in past accepted lots.* Suppose there is no change in the manufacturing or acceptance inspection procedures. The success probability (reliability) of an item or combination of items from a future lot may be estimated from the density function $g(f)$.

Consider a future random lot L which is derived from the process which produced $g(f)$. The probability that f is in an interval, df is:

$$P \{ f \text{ in } df \} = g(f) df \quad (1)$$

---

*In the interest of simplicity in this exposition, information from the acceptance inspection process will be ignored except for possible contribution to the determination of $g(f)$.

The probability that a unit drawn at random from lot L is successful is

$$\int_0^1 f g(f) df = \bar{f}, \quad (2)$$

the mean of $g(f)$.

If success occurrences are not correlated, the probability that two units from L both succeed (i.e., in some situations the probability that a series system of two units succeeds) is

$$\begin{aligned} \int_0^1 f^2 g(f) df &= \int_0^1 (f - \bar{f})^2 g(f) df + \int_0^1 \bar{f}^2 g(f) df \\ &= \sigma^2 + \bar{f}^2, \end{aligned} \quad (3)$$

where σ^2 is the variance of $g(f)$.

The probability that a parallel system of two units succeeds is

$$\int_0^1 [1 - (1-f)^2] g(f) df = 2\bar{f} - \bar{f}^2 - \sigma^2 \quad (4)$$

where the individual failure probabilities are not correlated.

If $g(f)$ is a common density function, then $\bar{f}$ and σ^2 are known in terms of the parameters of the function. In this situation the preceding probabilities are readily calculated. When three or more such units are involved in a system, third or higher moments of $g(f)$ are involved in expressions comparable with (3) and (4). If more than two units are involved, it might be advantageous to derive the moment generating function of $g(f)$ in order to obtain the terms in the probability expression. This topic is not pursued further in this paper.

ESTIMATES BASED ON PAST EXPERIENCE AND A SAMPLE

Now consider the situation in which a sample of size n is taken from L in order to obtain a more accurate estimate of the value of f for the lot. Suppose that there are a successful units and $b = n - a$

unsuccessful units in the sample. Furthermore, suppose that L is large enough so that a random sample of size n taken therefrom has no appreciable effect on the value of f for the remainder of the lot. In the following development, methods are derived and illustrated for combining information from the sample and from $g(f)$ to obtain estimates of success probability for units taken from L .

The probability of observing the results of the sample is

$$\binom{n}{a} f^a (1-f)^b. \quad (5)$$

The probability that f is in df , given the sample, is (by Bayes' Theorem)

$$P \left\{ f \text{ in } df / \text{ sample} \right\} = \frac{\binom{n}{a} f^a (1-f)^b g(f) df}{\int_0^1 \binom{n}{a} f^a (1-f)^b g(f) df} \quad (6)$$

The probability that f in L is in the interval (h, k) is

$$P \left\{ h \leq f \leq k \right\} = \frac{\int_h^k \binom{n}{a} f^a (1-f)^b g(f) df}{\int_0^1 \binom{n}{a} f^a (1-f)^b g(f) df} \quad (7)$$

If P is specified and h and k are determined (subject to an additional condition), a "confidence interval" is defined with confidence coefficient $\alpha = P$. The usual condition applied to (h, k) is

$$\int_k^1 = \int_0^h = \frac{1 - \alpha}{2} \quad (8)$$

In some situations (e.g., when f is expected to be near 1) confidence intervals are given in the form $(h, 1)$. When $k = 1$, a specified value of P determines h uniquely in Equation (7). It is this situation which is considered in the following development on the assumption that high reliability is the chief concern.

THE BETA FUNCTION

Before Equation (7) can be applied, the form of $g(f)$ must be known. In this development the beta function is used for purposes of illustration. Thus

$$g(f) = \frac{(r+s+1)!}{r! s!} f^r (1-f)^s \text{ for } 0 \leq f \leq 1$$

$$= 0 \quad \text{otherwise} \quad (9)$$

for which the

$$\begin{aligned} \text{mode is at } f &= r/(r+s) \\ \text{mean} &= \bar{f} = (r+1)/(r+s+2) \\ \text{variance} &= \sigma^2 = (r+1)(s+1)/(r+s+2)^2(r+s+3) \end{aligned} \quad (10)$$

Figures 1 and 2 illustrate the latitude available in the family of beta functions. If perfect lots appear with negligible frequency, then a beta function with $s \neq 0$ may apply. If perfect lots appear with the greatest frequency, then a beta function with $s = 0$ may apply. For the purpose of this paper the functions shown in Figure 3 are used for illustration.

Situations may arise in which a beta function cannot be used for $g(f)$. In such situations (discussed in a later paragraph) the manipulations and computations generally become more involved. However, as mentioned earlier in this paper, the family of beta functions affords considerable latitude in fitting data. Beta functions are used herein in order to avoid complexities of manipulation which would make little or no contribution to the content of this paper.

A COMPARISON OF CONFIDENCE INTERVALS

Substitution from Equation (9) into Equation (7) followed by successive integrations by parts gives

$$\alpha = P \{ h \leq f \leq k \} =$$

$$\left[\sum_{i=a+r+1}^{a+r+b+s+1} \binom{a+r+b+s+1}{i} f^i (1-f)^{a+r+b+s+1-i} \right]_{f=h}^{f=k} \quad (11)$$

As it is used herein Equation (11) becomes

$$\alpha = P \{ h \leq f \leq 1 \} =$$

$$1 - \sum_{i=a+r+1}^{a+r+b+s+1} \binom{a+r+b+s+1}{i} h^i (1-h)^{a+r+b+s+1-i} \quad (12)$$

When past history is not sufficient to determine a density function $g(f)$, confidence intervals may be derived from the sample information alone. In this situation confidence intervals of the form (h, k) are determined by the following equations.

$$\sum_{i=0}^a \binom{a+b}{i} k^i (1-k)^{a+b-i} = \frac{1-\alpha}{2} \quad (13)$$

$$\sum_{i=a}^{a+b} \binom{a+b}{i} h^i (1-h)^{a+b-i} = \frac{1-\alpha}{2}$$

For a discussion of this latter type of confidence interval see Ref. 1 (among others). For the purpose of this paper the following equation corresponds to Equation (12) and is used in place of Equations (13).

$$\sum_{i=a}^{a+b} \binom{a+b}{i} h^i (1-h)^{a+b-i} = 1 - \alpha \quad (14)$$

Equations (11) through (14) may be solved readily by the use of Ref. 2 when $a + r + b + s + 1 \leq 150$ unless h (and k) are required to more than two significant decimal places.

Table 1 contains confidence intervals computed for the two $g(f)$ functions of Figure 3 using Equation (12) and from sample information alone using Equation (14). Several hypothetical samples are used (values of a and b). Confidence intervals are of the form $(h, 1)$ hence only the values of h are tabulated. The first line of the table shows values of h derived from $g(f)$ alone (i.e., the situation when $a = b = 0$). In this last situation the values of h may be obtained from the graph of the corresponding cumulative density function of Figure 3.

A study of Table 1 supports the following observations. First, the use of past history (in the form of a density function $g(f)$) in addition to a sample from lot L allows one to make closer interval estimates of lot quality than is possible if only the sample information is used. Second, interval estimates of a given size can be obtained with a smaller sample if past history is included in the derivation of a confidence interval. Third, as the sample size increases $g(f)$ plays a role of decreasing importance in the derivation of a confidence interval. (This third observation has been proved in a general form³). Fourth, even without a sample it is possible to estimate the quality of L from previous experience in the form of $g(f)$. Fifth, an estimate based on $g(f)$ and a sample does not necessarily lead to a narrower interval than that obtained from $g(f)$ alone.

OTHER REMARKS

In some situations the interval estimated from $g(f)$ and the sample may be substantially wider than the interval estimated from $g(f)$ alone. In such an event some doubt may arise concerning the assumption that the manufacturing and inspection processes have remained constant. In this situation it may be desirable to perform a test of the hypothesis that lot L came from the population of lots $g(f)$. Such a test consists of determining whether the probability of observing b or more failures in a sample of size n is greater than a prestated value (usually 0.01 or 0.05).

The required probability is

$$\int_0^1 \sum_{i=b}^n \binom{n}{i} f^{n-i} (1-f)^i g(f) df$$

$$= \frac{(r+s+1)}{(r+s+n+1)} \binom{r+s}{s} \sum_{i=b}^n \binom{n}{i} \bigg/ \binom{r+s+n}{s+1} \quad (15)$$

when $g(f)$ has the form (9). Equation (15) is expressed in a form for use with tables of the binomial coefficients.

As an example of the application of Equation (15) consider the situation in which $r = 24$, $s = 1$, $a = 7$, and $b = 3$. In this situation a 0.95 confidence interval (See Table 1) for f in L is (0.76, 1). The probability of observing 3 or more failures in 10 trials is 0.056. Therefore a change in the manufacturing or the inspection process would not be suspected on the 0.01 level but probably would be on the 0.05 level.

For some applications it may be desirable to determine the probability that f in L exceeds a specified minimum value. In such situations values of h in Equation (12) (or in Equation (7) with $k = 1$) or in Equation (14) may be specified and corresponding values of P computed. An example of such a problem appears on page 129 of Ref. 4. Further discussion of such applications is not included herein.

In some situations it may be that a beta function (Equation 9) will not fit the data from past lots. The most obvious such situation is that when $g(1) \neq 0$ or is not a maximum. In this event it may be that a triangular or a polynomial function may be used for $g(f)$. If such functions are used, then, in general, the summation of Equation (11) or of Equation (12) becomes the algebraic sum of two or more summations and a trial and error solution is indicated for h and k . (Remember that in the preceding discussion lots are assumed to be large enough that f is essentially continuous).

If $g(f)$ is a discrete function, the integrals of Equations (11) and (12) are replaced by summations and the $\alpha = P$ by $\alpha \approx P$.

The preceding discussion is offered chiefly as a means of illustrating the potential advantages of keeping a history of each lot. Before the theory is complete such a development must be made for small lots. Also the conditions on $g(f)$ which assure that confidence intervals determined by Equations (7) and (8) are smaller than those determined by Equations (13) will be of interest.

TABLE I

Confidence Intervals of the Form $(h, 1)$
 with Confidence Coefficient $\alpha = 0.95$

Hypothetical Sample		Based on Sample Information Only	Based on $g(f)$ from Fig 3A & Sample	Based on $g(f)$ from Fig. 3B & Sample
Success a	Failure b			
0	0	- -	0.83	0.91
10	0	0.74	0.87	0.92
9	1	0.60	0.83	0.90
8	2	0.49	0.80	0.87
7	3	0.38	0.76	0.85
20	0	0.86	0.90	0.93
19	1	0.78	0.87	0.91
18	2	0.72	0.84	0.89
17	3	0.65	0.81	0.87
30	0	0.90	0.92	0.94
29	1	0.85	0.89	0.92
28	2	0.80	0.87	0.90
27	3	0.76	0.84	0.89
40	0	0.93	0.93	0.95
39	1	0.89	0.91	0.93
38	2	0.85	0.89	0.92
37	3	0.82	0.87	0.90
50	0	0.94	0.94	0.95
49	1	0.91	0.92	0.94
48	2	0.88	0.90	0.92
47	3	0.85	0.88	0.91
60	0	0.95	0.94	0.96
59	1	0.92	0.93	0.94
58	2	0.90	0.91	0.93
57	3	0.85	0.90	0.92

LIST OF REFERENCES

1. Mood, A. McF., "Introduction to the Theory of Statistics", McGraw-Hill Book Co., Inc., New York, 1950.
2. Ordnance Corps. Pamphlet ORDP-20-1, "Tables of the Cumulative Binomial Probabilities", September, 1952.
3. v. Mises, R., "On the Correct Use of Bayes' Formula," Ann. Math. Stat., Vol. 13 (1942), pp. 156-165.
4. Fry, T. C., "Probability and Its Engineering Uses", D. Van Nostrand Co., Inc., New York, 1928.

Fig 1

Fig 2
19 - 11

Fig 3

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