MASTER

Stress-Strain Characteristics of Materials at High Strain Rates

Part VI

THE PROPAGATION OF PLASTIC WAVES IN FINITE CYLINDERS
OF STRAIN-RATE-DEPENDENT MATERIAL

by

Byron D. Tapley

Prepared for

THE SANDIA CORPORATION
CONTRACT AT(29-2)-621

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THE UNIVERSITY OF TEXAS
STRUCTURAL MECHANICS RESEARCH LABORATORY
AUSTIN, TEXAS
AUGUST 1960

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is cooperatively operated by
The Engineering Mechanics and Civil Engineering Departments
at
The Balcones Research Center
University of Texas
Austin, Texas

PREFACE

The research reported here, although supported initially by The University of Texas Research Institute, has been in its later stages supported solely by Division 5130 of the Sandia Corporation under Research Contract No. AT(29-2)-621. It represents an extension of the work reported in Stress-Strain Characteristics of Materials at High Strain Rates, Part IV, in that it seeks to determine indirectly, by studying the propagation of plastic waves, the basic dynamic-stress strain characteristics of materials.

This work represents the first successful effort, insofar as we are aware, to incorporate in the mathematical analysis of plastic wave propagation in bars, a correction for the effects of lateral inertia and shear. Full credit for the basic mathematical analysis involved should go to Dr. H. J. Plass, Jr., of the Department of Engineering Mechanics of The University of Texas. He conceived the idea for the approach used, wrote the basic equations, and supervised the investigation.

Stress-Strain Characteristics of Materials at High Strain Rates, Part VI, The Propagation of Plastic Waves in Finite Cylinders of a Strain-Rate-Dependent Material, the title of this report is also the title of a doctoral dissertation prepared by Dr. Byron D. Tapley. The results presented here are essentially a reproduction of that dissertation.

On behalf of the author and myself, I would like to acknowledge the contributions of others, in addition to those made by Dr. Plass. Particular acknowledgement should be given to Dr. E. A. Ripperger for the many informative discussions of the problem. Gratitude is expressed to the personnel of Department 5130 for making completion of this research possible.

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ABSTRACT

The elementary theory of plastic wave propagation as given by von Karman does not agree with some of the results obtained by experiments. In order to correctly predict the propagation of plastic waves, the strain rate effect must be included. The elementary strain-rate theory as expressed by Malvern gives a general agreement with the results obtained from experiments, but there is still a difference in magnitude between the experimental measurements and theoretical predictions.

An investigation is made of the effects of the lateral inertia of the bar on the propagation of a plastic disturbance in a finite copper specimen. The actual stress-strain curve of the material is used. The linear strain-rate law is assumed, and results are obtained from both the elementary Malvern theory and a theory including the effects of radial motion. Results are also obtained from each theory by applying both a "step" in velocity and a "finite" rise in velocity to one end of the bar. These results are compared with each other and with experimental measurements.

The results obtained show that:

- (1) The theory including the effects of the radial motion predicts a phenomenon with a more oscillatory nature than the elementary theory.

- (2) The theory including the effects of the radial motion agrees better with experimental results.
- (3) The results obtained with an applied velocity having a "finite" rise time agrees better with experimental results.

Recommendations for further study of the problem are outlined.

CHAPTER I

INTRODUCTION

When two bodies traveling at different velocities come into contact, the particles on the surfaces of the two bodies must acquire a common velocity in a very short interval of time. These rapid changes in the velocities of the particles, and the associated forces and deformations set up a disturbance in each of the two mediums which is propagated according to certain natural laws. Many factors affect the propagation of this disturbance and an exact mathematical description has not been given. In order to obtain an approximate answer to the question of what effect a disturbance, initiated at the surface, has on some other point in the interior, certain assumptions must be made. One such set of assumptions, and the results obtained with these assumptions, are presented here. The problem considered will be that of the propagation of a particular class of disturbance, a plastic wave, in a cylindrical, homogeneous, isotropic medium which will be called a cylindrical bar.

The theory of the propagation of plastic waves has much in common with the theory of the propagation of elastic waves. In fact, all plastic waves have an elastic front. In view of this fact, the significant contributions to the theory of elastic wave propagation are equally as important to studies in plastic wave propagation. Some of the more important of these contributions are briefly discussed here.

Propagation of Elastic Waves

Based on the assumptions that the medium in which a disturbance travels is homogeneous and isotropic, that the stress and strain are always related in a linear manner, and that all displacements are of an infinitesimal magnitude, the so-called exact equations governing the propagation of an elastic wave in a bar have been given by Pochhammer^{1*} and Chree.² The restrictions imposed by the assumptions on these equations are satisfied reasonably well by many engineering materials. However, due to their mathematical complexity, obtaining useful information from them usually involves considerable effort. Consequently, there have been many attempts at formulating an approximate theory that gives adequate agreement with the exact theory, but with which answers can be obtained with a minimum effort. These attempts are bounded by the exact theory on one extreme and the elementary theory on the other.

The elementary theory assumes that cross sections of the bar, which are initially plane and perpendicular to the longitudinal axis, remain so for all values of time. Only axial stresses are considered, and these are assumed to be uniformly distributed over the plane cross section. If a stress is applied to one end, this theory predicts that the time variation of the stress at some point farther along the bar is just that stress variation applied to the end. The stress history at the second point is delayed by the interval of time required for the distur-

*Numbers indicate references as listed in the Bibliography.

bance to travel the distance between the points.

For disturbances that do not change rapidly with time, the elementary theory agrees adequately with experimental results. However, where the disturbances do vary rapidly with time, predictions based on this theory are no longer adequate. One reason for the discrepancies is that the longitudinal extensions or compressions will cause an accompanying lateral contraction or expansion. The ratio of the lateral strain to the longitudinal strain is given by Poisson's ratio, ν . This lateral motion, and the inertia associated with it, will induce shear stresses and normal stresses in the radial direction which will eventually cause the plane cross sections to become distorted.

The first recorded attempt to correct for these discrepancies was made by Rayleigh.³ Rayleigh gave an expression for the frequencies of longitudinal vibration which included the motion of the particles in the radial direction. Love⁴ used the same correction to obtain the differential equation governing the propagation of elastic waves in a thin rod. Both Rayleigh and Love assumed that the strain in the radial direction varied linearly with the distance away from the center of the bar. The radial strain at the surface of the bar was assumed to be just the negative of the axial strain multiplied by Poisson's ratio. The original assumptions of the plane cross section and uniform distribution of the axial stress were retained.

The approximate theory given by Love gives reasonably accurate results as long as the wave lengths are longer than five bar diameters.

Although the qualitative results obtained by the "Love Theory" are good only in a very limited range, it is significant to note that the theory does substantiate the importance of including the corrections for radial motion in an approximate theory.

It has been pointed out that radial shear stresses must be associated with radial motion. In the vicinity of the wave front, where large changes occur in small intervals of time, these stresses have an important effect on the resulting wave motion.

Mindlin and Herrmann⁵ proposed a theory accounting for the effects of both the radial inertia and the shear based on the assumptions that plane cross sections remain plane, and that the radial displacements are proportional to the distance from the bar axis. However, in distinction to the assumption of Rayleigh, the radial and the axial strains are not required to be directly proportional. By including two arbitrary constants, Mindlin and Herrmann obtained a very good agreement with some of the results given by the exact theory.

Volterra,⁶ by assuming that the displacements within a bar are "constrained" in a certain prescribed manner, has obtained equations of the motion of waves in rods which include the Mindlin-Herrmann equations as a special case. Volterra's equations are more general, however, and by assuming that the displacements can be expressed in the form of a power series, any degree of freedom for distortion of the cross section may be allowed.

Bishop⁷ and Plass⁸ have also given theories which include the effects of shear and radial inertia. The results given by Plass,

although obtained in a different manner, agree with those given by Volterra.

A more detailed account of these theories, their results and the experimental work associated with them is given in surveys by Abramson, Plass, and Ripperger,⁹ and by Davies.¹⁰

It is quite evident from the previous discussion that in certain cases the propagation of waves in the elastic range is affected considerably by the inertia and shear stresses associated with radial motion. It is the purpose of this investigation to determine what effects these factors have on the propagation of plastic waves in a short bar of strain-rate-dependent material.

Plastic Wave Propagation

The propagation of transient disturbances of magnitude large enough to cause permanent distortion to the medium in which the disturbance is traveling is called plastic wave propagation. The studies of this phenomena are of comparatively recent origin. Results of the elementary theory of propagation of elastic waves in a material with a linear elastic stress-strain relation states that a disturbance introduced at some point of a bar will travel to another point along the bar with a velocity $c_0 = \sqrt{E_0/\rho}$ where ρ is the initial density, E_0 is Young's Modulus and c_0 is the "bar velocity." However, one of the fundamental assumptions from which this result is derived is that

the stress and the strain stay in the linear elastic range. Obviously, this theory is inadequate for describing the propagation of a plastic disturbance.

In a paper published in 1930, Donnell¹¹ proposed that for a non-linear stress-strain curve, the stress pulse could be assumed to be composed of a superposition of stress increments, each increment traveling at a velocity defined by $c = \sqrt{\frac{1}{\rho} \frac{d\sigma}{d\varepsilon}}$. Here ρ is the initial density and $\frac{d\sigma}{d\varepsilon}$ is the slope of the stress-strain curve obtained from "static" tensile tests for a given strain.

Extending the assumptions of Donnell and White and Griffis,¹² von Karman¹³ and Taylor,¹⁴ working independently, derived the same elementary equations for describing the propagation of a plastic wave. Due to security restrictions, neither of these reports was published until the identical results were later derived by Rakhmatulin¹⁵ and published in 1945.

The elementary equations are based on the assumption that the relation between stress and strain under dynamic loading is the same as the relation obtained in a "quasi-static" or very slow loading. That is, the rate at which a specimen is strained is assumed to have no effect on the stress-strain curve, and consequently, no effect on the propagation of the disturbance. According to the results of this theory, there will be an initial wave front consisting of the elastic portion of the disturbance which travels with the bar velocity, c_0 . The slower-moving plastic increments follow at velocities $c = \sqrt{\frac{1}{\rho} \frac{d\sigma}{d\varepsilon}}$. However, there will be some critical impact velocity which will induce a stress equal to the maximum stress on the static stress-strain curve. Since for this value of stress, $\left. \frac{d\sigma}{d\varepsilon} \right|_{\sigma_{\max}} = 0$, the disturbance will not propagate.

Failure will occur at the impacted end. One other surprising prediction of this theory is that the strain distribution in the bar for any time after the impact will show a plateau of constant strain being propagated at a velocity of $c = \sqrt{\frac{1}{\rho} \left. \frac{d\sigma}{d\varepsilon} \right|_{\sigma_{\max}}}$.

Experimental results do not show a very good agreement with the predictions of the elementary theory. Duwez and Clark¹⁶ have shown, by tensile tests on annealed copper wires, that for some "critical" velocity the specimen broke very close to the end, but that plastic strains had propagated a considerable distance down the bar. This disagrees with the prediction that for such a "critical" velocity there is no plastic disturbance propagating. The experiments of Sternglass and Stuart¹⁷ have shown that a stress increment introduced into an already plastic region will travel with the elastic bar velocity, $c_0 = \sqrt{E_0/\rho}$. Further, they were unable to find a plateau of constant strain.

One reason for the differences between the experimental results and the theoretical predictions can be attributed to the assumption of a correspondence between the dynamic and static stress-strain curves. It can be shown, experimentally, that for some materials such as copper, mild steel, or lead, the stress associated with a given value of strain will depend on the interval of time in which the straining process occurs. That is, for high rates of strain, the stress associated with each value of strain is greater than the stress for the same strain in a very slow test. However, the exact mechanism governing high

rates of straining is not known. For constant rates of straining, Prandtl,¹⁸ based on a theoretical investigation of the problem, proposed the existence of a logarithmic dependence of plastic strain rate on the stress excess above the stress present for the same strain in a quasi-static test. Deutler¹⁹ substantiated this theory with data taken from tensile tests on mild steel and copper. More recent tests on copper by Habib²⁰ gave a similar agreement.

In an attempt to remedy some of the discrepancies of the von Karman-Taylor theory, Malvern²¹ studied the propagation of plastic waves in a semi-infinite bar of strain-rate-dependent material. Malvern assumed that even when the strain rate is not constant, the plastic strain rate is some function of the excess stress over the static stress value which would exist for a given strain. That is,

$$\dot{\epsilon}'' = g(\sigma - f(\epsilon)) \quad - - - - - (1.1)$$

where $\dot{\epsilon}''$ is the plastic strain rate, g is some unspecified function of the stress σ and the strain ϵ , and $f(\epsilon)$ is the stress for a given value of ϵ in a quasi-static test. In order to simplify calculations, Malvern further assumed that a linear dependence of plastic strain rate on the stress and $f(\epsilon)$ existed, and that

$$\dot{\epsilon}'' = k(\sigma - f(\epsilon)) \quad - - - - - (1.2)$$

where k is some experimentally determined constant, called the strain-rate constant, which is a characteristic property of the material being studied. It is further assumed that the total strain rate is the

sum of the elastic strain rate $\dot{\epsilon}'$, and the plastic strain rate $\dot{\epsilon}''$.

This may be expressed as

$$\dot{\epsilon} = \dot{\epsilon}' + \dot{\epsilon}'' \quad - - - - - \quad (1.3)$$

Malvern made use of the method of characteristics to reformulate the basic partial differential equations as ordinary differential equations, and solved them numerically by using finite differences.

Sokolovsky²² and Rubin²³ have given solutions to Malvern's equations in closed form.

Plass and Wang²⁴ used Malvern's equations to compare the influence of the assumptions of linear and exponential strain-rate dependence on the propagation of plastic waves in elastic-perfectly plastic copper and linear-strain hardening steel bars.

The investigations mentioned above all neglected the effects of radial contraction and expansion. In doing so, the shear forces and radial inertia associated with such motion have been neglected. In the plastic impact problem, just as in the elastic case, it has been recognized that near the impact end of a semi-infinite bar or in a finite bar of short length, the radial motion may have an important effect on the phenomena occurring. At least one experimental investigator, Kolsky²⁵ has attempted to correct experimentally determined results for this motion. Experimental results obtained by Karnes²⁶ indicate that the elementary theory of von Karman and the elementary strain-rate theory of Malvern are both inadequate for describing the

propagation of plastic waves.

Plass,²⁷ in a manner similar to the procedure used by Mindlin and Herrmann in studying the propagation of elastic waves in bars, proposed a theory for predicting the propagation of plastic waves in a semi-infinite rod of strain-rate dependent material that corrects for some of the discrepancies found in the elementary theory. The linear strain-rate dependence proposed by Malvern is assumed and the effects of radial inertia and shear are included.

In this investigation, the propagation of plastic waves in a finite bar of strain-rate-dependent material will be studied, using first the elementary strain-rate theory and then the more advanced theory. Results from the two theories will be compared.

Strain-Rate Theory

In the analysis of the plastic-wave propagation as given by von Karman, Taylor, or Rakhmatulin, it is assumed that under dynamic conditions of stressing into the plastic range, the stress is exclusively determined by the existing strain. That is, for a given increment of stress, the bar undergoes an instantaneous plastic strain. However, according to the experimental work of Riparbelli²⁸ and others, there is a definite time lag between the application of a stress pulse and the appearance of plastic strain. Riparbelli concluded that when a body receives a stress increment, its first reaction is elastic, followed

by a time-dependent plastic flow. This phenomenon is explained by Simmons, Hauser, and Dorn²⁹ by assuming that plastic flow in a polycrystalline material takes place by moving dislocations. There is a certain inertia associated with each dislocation, and, due to this inertia, no instantaneous plastic deformation can occur. There is no inertia associated with elastic strain increments and consequently no time lag associated with its formation.

The above investigators have also disclosed that the plastic strain rate reaches its maximum value during an "accelerating" period which is thought to be about 5×10^{-4} μ sec. After this initial "accelerating" period, the plastic strain rate is found to be some function of the existing stress and strain, i. e.

$$\dot{\epsilon}'' = f(\sigma, \epsilon) \quad - - - - - (1.4)$$

Early investigators,^{18, 19} found from experimental evidence, and from theoretical considerations, that the plastic strain rate appeared to be some exponential function of the excess stress above that present in a quasi-static test for a given value of strain.

Malvern, on the assumption that the material behaved in a quasi-viscous manner for strains in excess of the elastic limit, gave the following equation for the strain rate.

$$E_0 \dot{\epsilon} = \dot{\sigma} + g(\sigma, \epsilon) \quad - - - - - (1.5)$$

where $\dot{\sigma}/E_0$ is the elastic strain rate and $\frac{g(\sigma, \epsilon)}{E_0}$ is the plastic strain rate.

The most realistic form for the plastic strain rate in a dynamic test appears to be

$$g(\sigma, \epsilon) = a \left(e^{\frac{\sigma - \bar{\sigma}}{b}} - 1 \right) \quad \text{--- (1.6)}$$

where a and b are experimentally determined constants.

Computations based on the exponential law are very tedious, and for this reason the majority of the investigators to date have used the linear approximation proposed by Malvern, i. e.,

$$\begin{aligned} g(\sigma, \epsilon) &= k(\sigma - \bar{\sigma}), \quad \sigma > \bar{\sigma} \\ g(\sigma, \epsilon) &= 0, \quad \sigma \leq \bar{\sigma} \end{aligned} \quad \text{--- (1.7)}$$

This linear approximation may be obtained directly from Eq. (1.6) by expanding the exponential function in a power series and neglecting all except the first terms in the power series. Obviously, this places some limit on the applicability of Eq. (1.7) and while computations are more easily performed with Eq. (1.7), they are generally done so at the expense of some accuracy. In general, where the calculated results are to be compared directly with experimental measurements, the exponential law should be used.

In this investigation, since the primary purpose will be to determine the necessity of including the effects of the radial motion in the equations describing the propagation of a plastic disturbance, the

linear approximation given in Eq. (1.7) will be used.

In attempting to compare the calculated results from either theory with experimental measurements, it should be remembered that:

1. The calculated results will be in error somewhat because of the use of the linear strain-rate law.
2. The results obtained are quite dependent on the strain-rate constant k which must be determined from experimental observations. A single value of k will not completely describe an experimental record and it is necessary to select the value of k which gives the best approximation of the experimental results.

Physical Model

The specimen studied in this investigation is a short cylindrical bar of some strain-rate dependent material such as copper, steel, lead, or aluminum. Specimens of two different lengths, 1/2-in. and 1 in., are considered. The short specimens are supported on one end by a steel back-up bar and impacted on the other end by a mass moving at a high velocity, Fig. 1. The steel back-up^{*} bar is assumed to remain elastic through the history of the disturbance and the effects of such an elastic boundary are included in the investigation. The impacting cylinder is assumed to be rigid and affects the specimen only by introducing a disturbance at the impacted end. In this investigation, the effects of the impacting cylinder are taken into account by assuming that the axial particle velocity on the impacted end varies in some

* In the experimental work, this bar is used to measure the stress at the bar end of the specimen, and as a consequence, it has been referred to in previous work as the "pressure bar."

Fig. 1. PHYSICAL MODEL FOR INVESTIGATION

prescribed manner as a function of time.

Static Stress-Strain Curve for the Material

The material composing the specimen studied here is commercially pure copper. The "quasi-static" stress-strain curve of the material as measured by Karnes²⁶ is shown in Fig. 2. One of the basic assumptions in the plastic flow of a strain-rate dependent material is that the plastic flow rate depends on the excess stress value above the stress obtained for the same strain in a static test. For purposes of computation, it is desirable to have an analytical expression of the static stress as a function of the strain. The following expressions approximate the curve in Fig. 1 quite closely:

$$\begin{aligned}
 \sigma &= E_0 \epsilon, & 0 < \epsilon < 15 \mu \text{ in./in.} &) \\
 \sigma &= E_0 (.38538389 \times 10^8 \epsilon^5 - .175057016 \times 10^7 \epsilon^4 &) \\
 &\quad + .30544835 \times 10^5 \epsilon^3 - .28254031 \times 10^3 \epsilon^2 &) \\
 &\quad + .14011972 \times 10^1 \epsilon - .17017618 \times 10^{-3}) &) \quad \text{--- (1.8)} \\
 & & , 15 \leq \epsilon \leq 124 \mu \text{ in./in.} &) \\
 \sigma &= 44,561 \text{ psi}, & 124 \leq \epsilon \mu \text{ in./in.} &)
 \end{aligned}$$

where E_0 is Young's Modulus for copper. From the measured stress-strain curve, E_0 is determined to be 15×10^6 psi. The approximation to the curve consists, then, of two straight-line portions connected by a fifth-order polynomial. The polynomial portion of the curve was

165 030

Fig. 2. STATIC STRESS-STRAIN CURVE FOR COPPER

obtained by selecting 20 points at uniform intervals throughout the section of the curve that is to be approximated by the polynomial. A weighted least-square curve-fitting program written for the IBM 650 by von Holdt and Brousseau³⁰ was then used to obtain the correct polynomial. An effort was made to fit a polynomial to the entire stress-strain curve, but it was found that the polynomial diverged from the actual curve by a considerable margin for very small and very large strains. Consequently, the above three-segment curve was finally selected.

CHAPTER II

ELEMENTARY STRAIN-RATE THEORY

It has been pointed out in Chapter I that the elementary strain-rate-dependent theory as given by Malvern is based on a rather restrictive set of assumptions. Malvern assumed that the displacement of all particles in the axial direction is a function only of their axial position and time. This assumption assures that plane cross-sections originally perpendicular to the axis remain plane and perpendicular for all values of time. There is assumed to be no radial displacement of the particles during the history of the disturbance. Under these assumptions, only the axial strain and the axial stress are nonzero.

If $\bar{x}$ is the distance measured along the axis of the bar from the impacted end, Fig. 3, the displacement will be given as a function of position and time by

$$\bar{u} = \bar{u}(\bar{x}, \bar{t}) \quad - - - - - (2.1)$$

then the axial particle velocity and the axial strain are respectively

$$\begin{aligned} \bar{v} &= \frac{\partial \bar{u}}{\partial \bar{t}} &) \\ & &) \\ & &) \\ & &) \\ \bar{\epsilon} &= \frac{\partial \bar{u}}{\partial \bar{x}} &) \\ & &) \\ & &) \end{aligned} \quad - - - - - (2.2)$$

where $\bar{t}$ denotes the time, $\bar{x}$ denotes the axial position, $\bar{v}$, the axial

Fig. 3. CO-ORDINATE SYSTEM FOR ELEMENTARY STRAIN-RATE THEORY

particle velocity and $\bar{\epsilon}$, the axial strain. The "bar" above the variables indicates that the variable is expressed in physical units. The resulting expressions will be nondimensionalized eventually and the bars will be removed.

Basic Equations

With these assumptions, the set of three partial differential equations derived by Malvern is

$$\begin{aligned}
 \frac{\partial \bar{\sigma}}{\partial x} - \rho \frac{\partial \bar{V}}{\partial t} &= 0 \\
 \frac{\partial \bar{\epsilon}}{\partial t} - \frac{\partial \bar{V}}{\partial x} &= 0 \\
 E_0 \frac{\partial \bar{\epsilon}}{\partial t} - \frac{\partial \bar{\sigma}}{\partial t} &= \bar{g}(\bar{\sigma}, \bar{\epsilon})
 \end{aligned}
 \tag{2.3}$$

where E_0 is Young's Modulus for copper, $\bar{\sigma}$ is the axial stress, ρ the density of the material, and $\bar{g}(\bar{\sigma}, \bar{\epsilon})$ is the plastic strain rate. The other quantities have been previously defined. The first of these equations is obtained by assuming that Newton's second law holds for a particle of the material. The second equation, the compatibility relationship, is obtained by differentiating the second of Eqs. (2.2) with respect to time. The third equation is the material relationship

for the total strain rate in a strain-rate-dependent material. It has been previously discussed in the section on strain-rate theory in Chapter I and is the same as Eq. (1.5).

Now if the following dimensionless quantities are introduced:

$$\begin{aligned} \sigma &= \frac{\bar{\sigma}}{E_0} & t &= \frac{c_0 \bar{t}}{\ell} & g &= \frac{\ell}{c_0} \bar{g}(\bar{\sigma}, \bar{\epsilon}) \\ \epsilon &= \bar{\epsilon} & x &= \frac{\bar{x}}{\ell} & c_0 &= \sqrt{E_0/\rho} \\ V &= \frac{\bar{V}}{c_0} & k &= \frac{\bar{k} \ell}{c_0} \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \quad (2.4)$$

where E_0 is Young's Modulus for copper, ℓ is the length of the specimen, and c_0 is the so-called bar velocity obtained from the elementary theory of elastic wave propagation. $\bar{k}$ is the strain-rate constant discussed in Chapter I, and k is the dimensionless strain-rate constant. By substituting the dimensionless quantities for the corresponding physical quantities in Eq. (2.3), the following dimensionless equations are obtained.

$$\begin{aligned} \frac{\partial \sigma}{\partial x} - \frac{\partial V}{\partial t} &= 0 \\ \frac{\partial \epsilon}{\partial t} - \frac{\partial V}{\partial x} &= 0 \\ \frac{\partial \epsilon}{\partial t} - \frac{\partial \sigma}{\partial t} &= g(\sigma, \epsilon) \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \quad (2.5)$$

To obtain numerical results with Eqs. (2.5), the method of characteristics has been employed. For this set of equations, the characteristic curves are given in Ref. 21 or 29 as

$$\begin{array}{l} x = C_2 \\ x = t + C_1 \\ x = -t + C_1 \end{array} \quad \left. \vphantom{\begin{array}{l} x = C_2 \\ x = t + C_1 \\ x = -t + C_1 \end{array}} \right\} \quad - - - - (2.6)$$

where C_1 and C_2 are two constants. The characteristic curves for this problem are seen to be two families of straight lines intersecting at right angles and making an angle of 45° with the x -axis and one family of straight lines parallel to the t -axis. The characteristic plane for the elementary problem is shown in Fig. 4. In order to solve the system of partial differential equations given by Eqs. (2.5), it is convenient to determine the manner in which changes in the variables σ , ϵ , and V are related along each of the families of characteristics. The object is to reduce the system of partial differential equations to total differential equations holding along each of the characteristic curves. The resulting total differential equations or characteristic equations are given as²⁹

$$\begin{array}{l} d\sigma - dV = -g(\sigma, \epsilon) dt \text{ along } dx = dt \\ d\sigma + dV = -g(\sigma, \epsilon) dt \text{ along } dx = -dt \\ -d\sigma + d\epsilon = g(\sigma, \epsilon) dt \text{ along } dx = 0 \end{array} \quad \left. \vphantom{\begin{array}{l} d\sigma - dV = -g(\sigma, \epsilon) dt \text{ along } dx = dt \\ d\sigma + dV = -g(\sigma, \epsilon) dt \text{ along } dx = -dt \\ -d\sigma + d\epsilon = g(\sigma, \epsilon) dt \text{ along } dx = 0 \end{array}} \right\} \quad - - - - (2.7)$$

Fig. 4. CHARACTERISTIC PLANE FOR ELEMENTARY STRAIN-RATE THEORY

Solution of the problem has been reduced from that of integrating the original set of partial differential equations to that of integrating the set of three total differential equations. The total differential equations can be expressed as finite-difference equations and integrated numerically.

Propagation of Discontinuities

As it has been stated in Chapter I, a sudden disturbance causes very rapid changes in the stress, strain, and particle velocities at any point influenced by the disturbance. The nature of these changes usually induces discontinuities in the values of the variables. For the case where a "step" or instantaneous velocity is applied to the end of the bar, a discontinuity in all three variables is initiated at the impacted end and travels down the bar. It is of interest, then, to determine the laws governing the propagation of this discontinuity. By obtaining the relationship between the discontinuities in the different variables and how at least one of the discontinuities varies along a characteristic, the propagation of the complete discontinuity can be predicted.

Expressed in dimensionless form, the relationships between the variables across a forward-traveling discontinuity are²¹

$$\begin{array}{rcl} \Delta \sigma & = & - \Delta V \\ \Delta \sigma & = & \Delta \epsilon \\ \Delta V & = & \Delta \epsilon \end{array} \quad \left. \begin{array}{l}) \\) \\) \end{array} \right\} \quad \text{--- (2.8)}$$

while across a backward-traveling discontinuity

$$\begin{aligned} \Delta \sigma &= \Delta V \\ \Delta \sigma &= \Delta \epsilon \\ \Delta V &= \Delta \epsilon \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \end{array} \right\} \quad - - - - - \quad (2.9)$$

Now a discontinuity in the stress traveling along one of the characteristic curves is expressed as a function of time²¹ as

$$2d\sigma = -g(\sigma, \epsilon) dt \quad - - - - - \quad (2.10)$$

Across the initial wave front $d\sigma = d\epsilon$ and since initially σ and ϵ are zero, the final condition will be $\sigma = \epsilon$ and so Eq. (2.10) can be integrated, giving

$$\int_{\sigma_0}^{\sigma} \frac{d\sigma}{g(\sigma)} = -\frac{t}{2} \quad - - - - - \quad (2.11)$$

Eq. (2.11) then specifies σ and consequently $\Delta\sigma$ along the initial wave front. Eqs. (2.8) can then be used to obtain ϵ and V across the "jump" or discontinuity at all points along the wave front.

In general, for a reflected wave in a plastic material ϵ will not be equal to σ after the discontinuity passes, and so Eq. (2.10) must be integrated numerically.

Eq. (2.10) holds along a backward-traveling wave front as well as a forward-traveling wave.

Reflection of a Discontinuity

When a discontinuity comes into contact with a boundary, such as a change in cross section or a change in material properties, the instantaneous change in the discontinuity is governed by laws

different from those previously given. The boundary encountered here is a change in material properties. That is, the copper specimen is supported by a steel back-up bar. When the discontinuity reaches the end of the copper specimen, part of it is transmitted into the steel bar and part of it is reflected into the copper bar. The transmitted part induces a forward-traveling disturbance in the steel bar and the reflected portion causes a backward-traveling disturbance in the copper bar.

Now for the dynamic contact of the ends of the copper specimen and steel back-up bar, at all times

$$V = V' \quad - - - - - (2.12)$$

where the prime indicates the particle velocity in the steel bar. Further, from Newton's Third law, the stress on the supported end of the copper bar must always be equal to that on the end of the steel bar

$$\sigma = \sigma' \quad - - - - - (2.13)$$

Now if the impulse-momentum relationship is applied to an element of the bar at the supported end, the following relationships are obtained for the reflected discontinuity in the copper bar and the transmitted discontinuity in the steel bar

$$\begin{array}{lcl} \sigma_2 - \sigma_1 = V_2 - V_1 &) & \\ &) & \\ \sigma'_2 - \sigma'_1 = -\lambda(V'_2 - V'_1) &) & - - - - - (2.14) \end{array}$$

where $\lambda = \rho' c'_0 / \rho c_0$.

Now if conditions shown in Eqs. (2.12) and (2.13) are substituted into Eqs. (2.14), the following expressions for the stress, strain, and particle velocity after the reflection has occurred may be obtained:

$$\begin{aligned} \sigma_2 &= \frac{\sigma_1' + \lambda(\sigma_1 - V_1 + V_1')}{1 + \lambda} \\ V_2 &= V_1 + \sigma_2 - \sigma_1 \\ \epsilon_2 &= \epsilon_1 + \sigma_2 - \sigma_1 \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \end{array} \right\} \quad \text{--- (2.15)}$$

The third of Eqs. (2.15) follows because the reflection is assumed to occur so quickly that no plastic flow can occur, i. e., the material behaves elastically.

On the impacted end, the particle velocity is assumed to remain constant during the reflection of a disturbance. Consequently, the particle velocity after the discontinuity passes is

$$V_2 = \hat{V} \quad \text{--- (2.16)}$$

where $\hat{V}$ is the particle velocity on the impacted end.

The resulting stress and strain after the discontinuity passes are given as

$$\begin{aligned} \sigma_2 &= \sigma_1 - (\hat{V} - V_1) \\ \epsilon_2 &= \epsilon_1 - (\hat{V} - V_1) \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \end{array} \right\} \quad \text{--- (2.17)}$$

Eqs. (2.15), (2.16), and (2.17) then govern the reflection of discontinuities on either end of the copper specimen.

Where a velocity is prescribed for which there is a finite rise time to the final value on the impacted end, there will be no discontinuity traveling along the initial characteristic curve, but since the steel back-up bar behaves in an elastic manner, Eqs. (2.15) are used on the end surface.

Fig. 5 shows an element of the characteristic plane along the supported end of the bar. Since the characteristic plane does not extend past the end of the bar, the point s does not exist. The conditions at r and q are known from previous calculations. The particle velocity along the end between r and p is assumed to remain constant, i. e., to have the same value as at point r . With these assumptions, values of stress, strain, and particle velocity at point p may be calculated in a manner to be described in the next section. These values at point p are the values that would be present if there were no elastic boundary present. They differ, however, from those at point r which have been assumed to hold along $p - r$. Consequently, a small "discontinuity" in the variables will exist. This "discontinuity" is reflected to obtain the values which exist on the end surface after the disturbance coming from point q has reached point p . Eqs. (2.15) are used to calculate the reflected values.

The alternate procedure used here is to assume that the material

Fig. 5. ELEMENT OF CHARACTERISTIC PLANE ALONG
SUPPORTED END OF COPPER SPECIMEN

behaves in a "semielastic" manner. That is, the discontinuity introduced by the initial impact, on reaching the supported end, is reflected by using Eqs. (2.15) in the manner described above. However, after the initial discontinuity leaves the supported end, the particle velocity at the end is assumed to remain constant until the reflected discontinuity returns again to the supported end. Each time the "initial" discontinuity arrives at the supported end, it is reflected elastically; while during the time interval between two successive arrivals, the particle velocity at the end is maintained constant.

Numerical Computations

In order to perform the actual computations, some specified function for the plastic strain rate must be chosen. Here, it is assumed that the plastic strain rate is linearly dependent on the excess stress above the static stress for a given strain, i.e.,

$$\begin{aligned} \dot{\epsilon}'' &= k(\sigma - \bar{\sigma}), \quad \sigma > \bar{\sigma} \\ \dot{\epsilon}'' &= 0, \quad \sigma \leq \bar{\sigma} \end{aligned} \quad (2.18)$$

where $\dot{\epsilon}$, k , σ , and $\bar{\sigma}$ are expressed in dimensionless terms.

Consequently, Eqs. (2.7) expressed as finite differences become

$$\begin{aligned} (\sigma_p^i - \sigma_q) - (V_p^i - V_q) &= -k \left(\frac{\sigma_p^{i-1} + \sigma_q}{2} - \frac{\bar{\sigma}_p^{i-1} + \sigma_q}{2} \right) \Delta t \\ (\sigma_p^i - \sigma_s) + (V_p^i - V_s) &= -k \left(\frac{\sigma_p^{i-1} + \sigma_s}{2} - \frac{\bar{\sigma}_p^{i-1} + \sigma_s}{2} \right) \Delta t \\ -(\sigma_p^i - \sigma_r) + (\epsilon_p^i - \epsilon_r) &= k \left(\frac{\sigma_p^{i-1} + \sigma_r}{2} - \frac{\bar{\sigma}_p^{i-1} + \sigma_r}{2} \right) 2 \Delta t \end{aligned} \quad (2.19)$$

where the quantities on the right are averages multiplied by the time interval Δt to obtain the change across the grid element. It is assumed that values of all of the variables are known at points q , r , and s (Fig. 4) and it is desired to obtain the values of the variables at point p . Eqs. (2.19) represent three equations in three unknowns at point p .

The superscript i stands for the i^{th} approximation. The technique used to calculate the values at point p is to assume values for the first approximation of σ_p and $\bar{\sigma}_p$, i. e., ϵ_p . The quantities on the right are then determined and the variables on the left may be calculated. Eqs. (2.19) are solved for σ_p^i , ϵ_p^i , and V_p^i . These values are substituted for σ_p^{i-1} , ϵ_p^{i-1} , and V_p^{i-1} , and new values of σ_p^i , ϵ_p^i , and V_p^i are obtained.

This iterative process is repeated until two successive approximations of the strain agree to seven significant figures.

For a boundary element, Fig. 4, there is no point q . Consequently, the first of the previous finite-difference equations would have no meaning. Since there are three quantities to be determined at point p , the value of one of the variables must be specified along the boundary at $x = 0$, i. e., on the impacted end. The remaining finite difference equations in Eqs. (2.19) can then be used to determine the other two quantities at point p . The same iterative

technique as that described above can be used.

The unknown quantity to be specified on the impacted surface in this investigation is chosen to be the particle velocity. Two different cases will be investigated. The first will be that of the instantaneous "step" in particle velocity up to some value after which it is maintained constant, i. e.,

$$\left. \begin{array}{ll} V = 0 & t < 0 \\ V = \hat{V} & t \geq 0 \end{array} \right\} \text{--- (2.20)}$$

The other case considered is that where the particle velocity increases as some function of time until a given time, after which it remains constant. The particular function of time chosen here is parabolic, i. e.,

$$\left. \begin{array}{ll} V = 0 & t < 0 \\ V = \hat{V} - \phi(t - T)^2 & 0 \leq t \leq T \\ V = \hat{V} & t \geq T \end{array} \right\} \text{--- (2.21)}$$

where ϕ is a constant and T is the rise time, i. e., the interval of time required for the particles to acquire their maximum value.

On the supported end, the velocity is assumed to remain constant across a characteristic grid, but at the successive points p , it will be allowed to change in accordance with the assumed elastic reflection of the incoming disturbance.

Numerical Results

The finite-difference equations have been solved by means of an IBM 650 computer for specimens of 1/2-in. and 1-in. lengths, and rise times of 0 and 3.725 μ sec. The value of the strain-rate constant $\bar{k}$ chosen here for all calculations is $\bar{k} = 60 \times 10^4 \text{ sec}^{-1}$, and the velocity of the impacting cylinder, i.e., the velocity finally attained by the particles on the impacted surface, is chosen to be 780 in./sec in all calculations. From the copper stress-strain curve, Young's Modulus is found to be $E_0 = 15 \times 10^6 \text{ psi}$. The density of the copper is 0.32 lb/cu in. Young's Modulus for the steel back-up bar is $30 \times 10^6 \text{ psi}$ and it has a density of 0.28 lb/cu in. The dimensionless interval of time for the characteristic grid is $\Delta t = 0.05$.

Figs. 6 through 25 show the results obtained using the linear strain-rate theory. The first ten graphs show results for the 1/2-in. - length bar, while the remainder are for the 1-in. -length bar.

For both specimens, the stress value finally attained by the bar is found to be independent of the time required to attain the maximum particle velocity at the impacted end. For the "step" velocity, Figs. 11 and 21, the impacted end receives instantly

Fig. 6. DIMENSIONLESS STRESS vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 7. STRAIN vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 8. DIMENSIONLESS PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 9. DIMENSIONLESS STRESS DISTRIBUTION FOR 1/2 in. COPPER CYLINDER

Fig. 10. STRAIN DISTRIBUTION FOR 1/2 in. COPPER CYLINDER

Fig. II. DIMENSIONLESS STRESS vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 12. STRAIN vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 13. DIMENSIONLESS PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 14. DIMENSIONLESS STRESS DISTRIBUTION FOR 1/2 in. COPPER CYLINDER

Fig. 15. STRAIN DISTRIBUTION FOR 1/2 in. COPPER CYLINDER

Fig. 16. DIMENSIONLESS STRESS vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 17. STRAIN vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 18. DIMENSIONLESS PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 19. DIMENSIONLESS STRESS DISTRIBUTION FOR 1in. COPPER CYLINDER

Fig. 20. STRAIN DISTRIBUTION FOR 1in. COPPER CYLINDER

Fig. 21. DIMENSIONLESS STRESS vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 22. STRAIN vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 23. DIMENSIONLESS PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 24. DIMENSIONLESS STRESS DISTRIBUTION FOR 1 in. COPPER CYLINDER

Fig. 25. STRAIN DISTRIBUTION FOR 1 in. COPPER CYLINDER

a relatively high stress which quickly decays towards its final value. This value is the same as the final value reached respectively in Figs. 6 and 16 for the finite rise time. At dimensionless times of $T = 1$ in the 1/2-in. bar, and $T = 2$ in the 1-in. bar, the impacted end receives another instantaneous "step" in the stress. This is the arrival of the initial "step" or discontinuity that has propagated down the specimen, and, after being reflected from the supported end, returns again to the impacted end; reduced in magnitude, but still discontinuous in nature. The discontinuity is reflected from the impacted end, and starts its traversal again. However, on arriving again at the impacted end after the second traversal, it has died out to where it no longer shows up in the plotted curves.

A similar phenomenon occurs in the stress-time curve for the supported end of the specimen. However, in this case, there is no initial decay in the stress. On the contrary, following the initial step, there is a small rise in the stress followed by a gradual reduction until the discontinuity arrives at the supported end again. In all of the curves, the initiation of the disturbance at the supported end will be delayed by the interval of time required for the disturbance to travel the length of the specimen.

The particle velocity on the supported end of the specimen subjected to a "step" impact velocity (Figs. 13 and 23) changes only during the interval of time required for the discontinuity to pass the end due to the semielastic boundary assumptions discussed previously.

For the specimen subjected to a "finite rise" impact velocity, (Figs. 8 and 18) there is a continual change in the particle velocity at the supported end. In this case, the assumption of the existence of an elastic boundary for all boundary points is made. The difference between the elastic and semielastic boundaries is negligible for the 1/2-in. specimen, but there is a significant difference between the two assumptions in the case of the 1-in. specimen.

The strain rates after the initial rise are found to be the same for both initial conditions, Figs. 7, 12, 17, and 22. However, the strain for the "step" impact velocity is found to be greater at a given time than for the "finite rise" impact velocity. Further, the strain rates for the 1/2-in. specimens are greater than those for the 1-in. specimens. Both ends of the specimen eventually attain a uniform strain rate, agreeing with the uniform stress distribution which the bar finally attains. However, the strain at the impacted end is always greater in magnitude than the strain at the supported end. One factor contributing to the difference in magnitude between the two strains is the assumption of the rigid boundary for the impacting cylinder. Even if the time lag is considered, the strain on the impacted end exceeds the strain on the supported end by about 0.1 per cent. A more realistic approach to the problem would have been to assume an elastic boundary at both ends of the specimen.

CHAPTER III

MODIFIED STRAIN-RATE THEORY

In an effort to overcome some of the discrepancies of the elementary strain-rate-dependent theory, a theory has been developed which includes the effects of the radial motions of the bar. The original assumption, that cross sections originally plane and perpendicular to the neutral axis remain so for all values of time, is retained. However, a displacement of the particles in the radial direction is allowed. These radial displacements are assumed to take place independently of the axial displacements and they are assumed to vary linearly in the radial direction from zero at the bar axis to a maximum at the outer surface.

The bar, Fig. 26, will be referred to a cylindrical co-ordinate system where $\bar{x}$ is measured along the bar axis from the impacted end, $\bar{r}$ is measured in the radial direction from the neutral axis, and $\bar{\theta}$ is measured circumferentially.

Basic Equations

The axial and radial displacements are given respectively by

$$\begin{aligned} u_{\bar{x}} &= \bar{U}(\bar{x}, \bar{t}) \\ u_{\bar{r}} &= \frac{\bar{r}}{a} \bar{Y}(\bar{x}, \bar{t}) \end{aligned} \quad \begin{array}{l}) \\) \\) \\) \\) \end{array} \quad \text{--- (3.1)}$$

Fig. 26. CO-ORDINATE SYSTEM FOR MODIFIED STRAIN-RATE THEORY

where $\bar{r}$ is the radial distance of a given particle from the bar axis, $\bar{a}$ is the bar radius and $\bar{U}$ and $\bar{Y}$ are two functions of $\bar{x}$ and $\bar{t}$.

Here, as before, the "barred" quantities are expressed in physical units while the nonbarred quantities represent nondimensional quantities.

The respective particle velocities can be obtained as

$$\left. \begin{aligned} v_{\bar{x}} &= \frac{\partial \bar{U}}{\partial \bar{t}} (\bar{x}, \bar{t}) \\ v_{\bar{r}} &= \frac{\bar{r}}{\bar{a}} \frac{\partial \bar{Y}}{\partial \bar{t}} (\bar{x}, \bar{t}) = \frac{\bar{r}}{\bar{a}} \bar{W} (\bar{x}, \bar{t}) \end{aligned} \right\} \text{--- (3.2)}$$

by differentiating Eqs. (3.1) with respect to time. $\bar{W} (\bar{x}, \bar{t})$ is the radial particle velocity at the bar surface.

From Eqs. (3.1), the strains are determined as

$$\left. \begin{aligned} \epsilon_{\bar{x}} &= \frac{\partial \bar{U}}{\partial \bar{x}} \\ \epsilon_{\bar{r}} &= \epsilon_{\bar{\theta}} = \frac{\bar{Y}}{\bar{a}} \\ \gamma_{\bar{x}\bar{r}} &= \frac{\bar{r}}{\bar{a}} \frac{\partial \bar{Y}}{\partial \bar{x}} \end{aligned} \right\} \text{--- (3.3)}$$

while $\gamma_{\bar{r}\bar{\theta}}$ and $\gamma_{\bar{x}\bar{\theta}}$ are both zero.

The nonvanishing stresses will be $\sigma_{\bar{x}}$, $\sigma_{\bar{r}}$, $\sigma_{\bar{\theta}}$, and $\tau_{\bar{x}\bar{r}}$. For

reasons of symmetry $\sigma_{\bar{r}}$ and $\sigma_{\bar{\theta}}$ are equal and $\tau_{\bar{r}\bar{\theta}}$ and $\tau_{\bar{x}\bar{\theta}}$ are zero.

With these stress components, the equations of motion are given as

$$\begin{aligned} \frac{\partial \sigma_{\bar{x}}}{\partial \bar{x}} + \frac{\partial \tau_{\bar{r}\bar{x}}}{\partial \bar{r}} + \frac{\tau_{\bar{r}\bar{x}}}{\bar{r}} &= \rho \frac{\partial V_{\bar{x}}}{\partial \bar{t}} \\ \frac{\partial \tau_{\bar{r}\bar{x}}}{\partial \bar{x}} + \frac{\partial \sigma_{\bar{r}}}{\partial \bar{r}} &= \rho \frac{\partial V_{\bar{r}}}{\partial \bar{t}} \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{--- (3.4)}$$

Now, if Eqs. (3.3) are differentiated with respect to time, the equations of continuity are obtained as

$$\begin{aligned} \frac{\partial \epsilon_{\bar{x}}}{\partial \bar{t}} &= \frac{\partial V_{\bar{x}}}{\partial \bar{x}} \\ \frac{\partial \epsilon_{\bar{r}}}{\partial \bar{t}} &= \frac{1}{\bar{a}} \bar{W} \\ \frac{\partial \gamma_{\bar{r}\bar{x}}}{\partial \bar{t}} &= \frac{\bar{r}}{\bar{a}} \frac{\partial \bar{W}}{\partial \bar{x}} \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{--- (3.5)}$$

The material is assumed to behave in either an elastic or a visco-plastic manner. For sufficiently small stresses, the material behavior is assumed to be governed by Hooke's Law. However, for a

certain stress state, the material is assumed to behave as a viscoplastic substance and yielding occurs. The conditions for yielding are assumed to be those defined by the von Mises yield criterion.

This is given as,³¹

$$\frac{(\sigma_x - \sigma_r)^2}{2} + \frac{(\sigma_\theta - \sigma_x)^2}{2} + 3\tau_{rx}^2 = \sigma_o^2 \quad \text{--- (3.6)}$$

where σ_o is the stress at which the bar would yield in a pure tensile test.

Now where the yield surface is specified as some function of the stress components, the plastic flow rates are proportional to the partial derivative of the yield-surface function with respect to the corresponding stress component, i.e.,

$$\dot{\epsilon}_i'' = \frac{\partial \Phi(\sigma_i)}{\partial \sigma_i} \quad i = 1 \dots n$$

where $\dot{\epsilon}_i''$ is the i^{th} plastic strain rate, σ_i is the i^{th} stress component and $\Phi(\sigma_i)$ is the yield surface, represented here by Eq. (3.6).

Hence the plastic flow rates are obtained as

$$\begin{aligned} \dot{\epsilon}_x'' &= \beta(\sigma_x - \sigma_r) &) \\ \dot{\epsilon}_r'' &= -\frac{\beta}{2}(\sigma_x - \sigma_r) &) \\ \dot{\gamma}_{xr}'' &= 6\beta \tau_{xr} &) \end{aligned} \quad \text{--- (3.7)}$$

where β is some unspecified factor. Now, on the assumption that σ_r and τ_{xr} are small compared to σ_x , the plastic flow rates can be expressed as a function of the axial stress only. That is

$$\begin{aligned} \dot{\epsilon}_x'' &= \beta \sigma_x &) \\ \dot{\epsilon}_r'' &= -\frac{1}{2}\beta \sigma_x &) \\ \dot{\gamma}_{xr}'' &= 0 &) \end{aligned} \quad \text{--- (3.8)}$$

In this investigation, the so-far unspecified factor is replaced by the functional form Malvern used in the elementary strain-rate-dependent theory. That is, the plastic flow rates are assumed to depend in a linear manner on the excess axial stress above the stress that would exist for the same axial strain in a quasi-static test. Hence, Eqs. (3.8) become

$$\begin{aligned} \dot{\epsilon}_x'' &= \bar{k} (\sigma_x - \bar{\sigma}_x) &) \\ \dot{\epsilon}_r'' &= -\frac{\bar{k}}{2} (\sigma_x - \bar{\sigma}_x) &) \end{aligned} \quad \text{--- (3.9)}$$

where $\bar{k}$ is the previously defined strain-rate constant, $\bar{\sigma}_x$ is the stress obtained from a quasi-static stress-strain curve for a given strain. As before, the strain-rate constant $\bar{k}$ is assumed to be subject to the conditions that

$$\begin{aligned} \bar{k} &= \text{const} & , \text{ if } \sigma_x > \bar{\sigma}_x &) \\ \bar{k} &= 0 & , \text{ if } \sigma_x \leq \bar{\sigma}_x &) \end{aligned} \quad \text{--- (3.10)}$$

This insures that plastic flow will occur only when the axial stress is larger than the stress present for the same strain in a quasi-static test.

The elastic strain rates are obtained by differentiating the Hooke's Law equations and by adding the elastic and plastic strain rates, the expressions for the total strain rates are determined as

$$\begin{aligned}
 \dot{\epsilon}_{\bar{x}} &= \frac{1}{E_o} \left[\dot{\sigma}_{\bar{x}} - 2\nu\sigma_{\bar{r}} \right] + \frac{\bar{k}}{E_o} \left[\sigma_{\bar{x}} - \bar{\sigma}_{\bar{x}} \right] \\
 \dot{\epsilon}_{\bar{r}} &= \frac{1}{E_o} \left[(1 - \nu) \dot{\sigma}_{\bar{r}} - \nu \dot{\sigma}_{\bar{x}} \right] - \frac{\bar{k}}{2E_o} \left[\sigma_{\bar{x}} - \bar{\sigma}_{\bar{x}} \right] \\
 \dot{\gamma}_{\bar{xr}} &= \frac{1}{G_o} \dot{\tau}_{\bar{xr}}
 \end{aligned} \tag{3.11}$$

Eqs. (3.10) and (3.11) then give a material behavior consistent with the assumptions. These equations along with Eqs. (3.5) and (3.4) are sufficient to describe the propagation of a plastic disturbance down the bar.

However, by inspecting Eqs. (3.4), it can be seen that they depend on both the axial and the radial co-ordinates. For simplicity, it would be desirable to eliminate the $\bar{r}$ dependence, and by so doing, obtain a one-dimensional theory. To this end, Eqs. (3.11), (3.5), and the first of (3.4) are multiplied by $\bar{r}d\bar{r}d\bar{\theta}$ and the second of Eqs. (3.4) is multiplied by $\frac{\bar{r}^2}{a} d\bar{r}d\bar{\theta}$. Integration is then performed over the bar cross section. The resultant stresses and strains formed by

this process are listed below:

Integrated Stresses

$$\begin{aligned} P_{\bar{x}} &= \int_0^{\bar{a}} \sigma_{\bar{x}} \bar{r} d\bar{r} \\ P_{\bar{r}} &= \int_0^{\bar{a}} \sigma_{\bar{r}} \bar{r} d\bar{r} \\ Q_{\bar{rx}} &= \int_0^{\bar{a}} \frac{\bar{r}^2}{\bar{a}} \tau_{\bar{rx}} d\bar{r} \end{aligned}$$

----- (3.12)

Integrated Strains

$$\begin{aligned} E_{\bar{x}} &= \int_0^{\bar{a}} \epsilon_{\bar{x}} \bar{r} d\bar{r} \\ E_{\bar{r}} &= \int_0^{\bar{a}} \epsilon_{\bar{r}} \bar{r} d\bar{r} \\ \Gamma_{\bar{xr}} &= \int_0^{\bar{a}} \frac{\bar{r}^2}{\bar{a}} \gamma_{\bar{rx}} d\bar{r} \end{aligned}$$

The integrated forms of Eqs. (3.4), (3.5), and (3.11) are found to be

Equations of Motion

$$\frac{\partial \bar{P}_x}{\partial \bar{x}} = \frac{\bar{\rho} \bar{a}^2}{2} \frac{\partial \bar{V}_x}{\partial \bar{t}}$$

$$\frac{\partial \bar{Q}_{xr}}{\partial \bar{x}} - \frac{2 \bar{P}_r}{\bar{a}} = \frac{\bar{\rho} \bar{a}^2}{4} \frac{\partial \bar{W}}{\partial \bar{t}}$$

Equations of Continuity

$$\frac{\partial \bar{E}_x}{\partial \bar{t}} = \frac{\bar{a}^2}{2} \frac{\partial \bar{V}_x}{\partial \bar{x}}$$

$$\frac{\partial \bar{E}_r}{\partial \bar{t}} = \frac{\bar{a}}{2} \bar{W}$$

$$\frac{\partial \bar{\Gamma}_{xr}}{\partial \bar{t}} = \frac{\bar{a}^2}{4} \frac{\partial \bar{W}}{\partial \bar{x}}$$

(3.13)

Equations of Material Properties

$$\frac{\partial \bar{E}_x}{\partial \bar{t}} = \frac{1}{\bar{E}_0} \left[\frac{\partial \bar{P}_x}{\partial \bar{t}} - 2 \nu \frac{\partial \bar{P}_r}{\partial \bar{t}} \right] + \frac{\bar{K}}{\bar{E}_0} [\bar{P}_x - \bar{P}_x]$$

$$\frac{\partial \bar{E}_r}{\partial \bar{t}} = \frac{1}{\bar{E}_0} \left[(1 - \nu) \frac{\partial \bar{P}_r}{\partial \bar{t}} - \frac{\partial \bar{P}_x}{\partial \bar{t}} \right] - \frac{\bar{K}}{2 \bar{E}_0} [\bar{P}_x - \bar{P}_x]$$

$$\frac{\partial \bar{\Gamma}_{xr}}{\partial \bar{t}} = \frac{1}{\bar{G}_0} \frac{\partial \bar{Q}_{xr}}{\partial \bar{t}}$$

Eqs. (3.13) then describe the phenomena to be studied. Before continuing with the analysis, it is found advantageous to express Eq. (3.13) in a dimensionless form. For this purpose, the following dimensionless variables are introduced:

Dimensionless Forces

$$\begin{aligned}
 P &= \frac{\overline{P_x}}{E_o \overline{a^2}} \\
 R &= \frac{\overline{P_r}}{E_o \overline{a^2}} \\
 Q &= \frac{\overline{Q_{rx}}}{G_o \overline{a^2}}
 \end{aligned}
 \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{--- -- -- (3.14)}$$

Dimensionless Strains

$$\begin{aligned}
 A &= \frac{\overline{E_x}}{\overline{a^2}} \\
 B &= \frac{\overline{E_r}}{\overline{a^2}} \\
 \Gamma &= \frac{\overline{\Gamma_{xr}}}{\overline{a^2}}
 \end{aligned}
 \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{--- -- -- (3.15)}$$

Dimensional Particle Velocities

$$\begin{aligned}
 V &= \frac{V_{\bar{x}}}{c_D} \\
 W &= \frac{\bar{W}}{c_D}
 \end{aligned}
 \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{--- (3.16)}$$

Other Dimensionless Parameters

$$\begin{aligned}
 x &= \frac{\bar{x}}{\bar{a}} \quad \text{- dimensionless distance} \\
 t &= \frac{c_D \bar{t}}{\bar{a}} \quad \text{- dimensionless time} \\
 k &= \frac{\bar{k} \bar{a}}{c_D} \quad \text{- dimensionless strain-rate constant}
 \end{aligned}
 \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{(3.17)}$$

The following wave velocities are also used

$$\begin{aligned}
 c_o &= \sqrt{\frac{E_o}{\rho}} \quad \text{- bar velocity} \\
 c_D &= \sqrt{\frac{1-\nu}{(1-2\nu)(1+\nu)}} c_o \quad \text{- dilatational velocity} \\
 c_s &= \sqrt{\frac{G_o}{\rho}} \quad \text{- shear velocity}
 \end{aligned}
 \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{(3.18)}$$

In terms of the dimensionless quantities, Eqs. (3.13) become

Equations of Motion

$$\frac{\partial P}{\partial x} - \frac{1}{2} D^2 \frac{\partial V}{\partial t} = 0$$

$$S^2 \frac{\partial Q}{\partial x} - 2R - \frac{1}{4} D^2 \frac{\partial W}{\partial t} = 0$$

Equations of Continuity

$$\frac{\partial A}{\partial t} - \frac{1}{2} \frac{\partial V}{\partial x} = 0$$

$$\frac{\partial B}{\partial t} - \frac{1}{2} W = 0$$

$$\frac{\partial \Gamma}{\partial t} - \frac{1}{4} \frac{\partial W}{\partial x} = 0$$

(3.19)

Equations of Material Properties

$$\frac{\partial A}{\partial t} - \frac{\partial P}{\partial t} + 2\nu \frac{\partial R}{\partial t} = k (P - \bar{P})$$

$$\frac{\partial B}{\partial t} - (1 - \nu) \frac{\partial R}{\partial t} + \nu \frac{\partial P}{\partial t} = -\frac{k}{2} (P - \bar{P})$$

$$\frac{\partial \Gamma}{\partial t} - \frac{\partial O}{\partial t} = 0$$

where $D^2 = \frac{c_D^2}{c_o^2}$

$$S^2 = \frac{c_s^2}{c_o^2}$$

Following the procedure used for the elementary theory,²¹ expressions for changes in the eight variables P , R , Q , A , V , W , B , and Γ along some curve in the $x - t$ plane will be given as

$$\left. \begin{aligned} dP &= \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial t} dt \\ - - - - - \\ d\Gamma &= \frac{\partial \Gamma}{\partial x} dx + \frac{\partial \Gamma}{\partial t} dt \end{aligned} \right\} \quad - - - - - \quad (3.20)$$

Eqs. (3.19) and (3.20) now constitute a system of 16 partial differential equations in terms of the 16 unknowns $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial t}$, $\frac{\partial \Gamma}{\partial t}$. The vanishing of the determinate of the coefficients of these unknowns yields the characteristic equations

$$\left. \begin{aligned} dx &= 0 \\ dx &= \pm dt \\ dx &= \pm \frac{c_s}{c_D} dt \end{aligned} \right\} \quad - - - - - \quad (3.21)$$

It has been shown previously that any discontinuity in the variables will be propagated along one of the characteristic curves defined by Eqs. (3.21). Since any change in one of the physical quantities can be thought of as a number of very small discontinuous changes, any disturbance initiated in the bar will then travel along one of the characteristic curves.

Fig. 27 shows the families of characteristic curves described by Eqs. (3.21).

It is now necessary to obtain the manner in which the variables change along the wave fronts.

By observing that once W and Q are known, B and Γ are uniquely determined; B and Γ may be eliminated and the Eqs. (3.19) are reduced to six equations in the six unknowns, P , R , Q , A , V , and W . The equations are

$$\begin{aligned}
 \frac{\partial P}{\partial x} - \frac{1}{2} D^2 \frac{\partial V}{\partial t} &= 0 \\
 S^2 \frac{\partial Q}{\partial x} - 2R - \frac{1}{4} D^2 \frac{\partial W}{\partial t} &= 0 \\
 \frac{\partial A}{\partial t} - \frac{1}{2} \frac{\partial V}{\partial x} &= 0 \\
 \frac{\partial A}{\partial t} - \frac{\partial P}{\partial t} + 2\nu \frac{\partial R}{\partial t} &= k (P - \bar{P}) \\
 \frac{\partial B}{\partial t} - (1 - \nu) \frac{\partial R}{\partial t} + \nu \frac{\partial P}{\partial t} &= -\frac{k}{2} (P - \bar{P}) \\
 \frac{\partial Q}{\partial t} - \frac{1}{4} \frac{\partial W}{\partial x} &= 0
 \end{aligned} \tag{3.22}$$

It is now necessary to study Eqs. (3.20) and (3.22) to obtain equations relating the variables along each of the characteristic curves given by Eqs. (3.21). This is done by imposing, one at a time, each

Fig. 27 CHARACTERISTIC PLANE FOR MODIFIED
STRAIN-RATE THEORY

of the conditions implied by Eq. (3.21) on the sets (3.22) and (3.20) and, after doing so, obtaining combinations of the equations that result in a total differential equation. There will be one total differential equation for each characteristic curve. An alternative to this approach is to successively replace different columns in the characteristic matrix by the column matrix to which the original set of differential equations was equated. The vanishing of the determinates formed by this procedure will yield all of the characteristic equations. This process is discussed in Ref. 21.

By either of the above processes, the following characteristic equations are obtained.

$$\begin{aligned}
 &\underline{\text{Along } dx = dt} \\
 &dP - \frac{1}{2} D^2 dV - \frac{\nu}{1-\nu} D^2 W dt = - \frac{k}{1+\nu} (P - \bar{P}) dt \\
 &\underline{\text{Along } dx = -dt} \\
 &dP + \frac{1}{2} D^2 dV - \frac{\nu}{1-\nu} D^2 W dt = - \frac{k}{1+\nu} (P - \bar{P}) dt \\
 &\underline{\text{Along } dx = \frac{S}{D} dt} \\
 &dQ - \frac{1}{4} \frac{D}{S} dW - \frac{2R}{SD} dt = 0 \\
 &\underline{\text{Along } dx = -\frac{S}{D} dt} \\
 &dQ + \frac{1}{4} \frac{D}{S} dW + \frac{2R}{SD} dt = 0 \\
 &\underline{\text{Along } dx = 0} \\
 &dP - \frac{(1-\nu)}{\nu} dR + \frac{W}{2\nu} dt = - \frac{k}{2\nu} (P - \bar{P}) dt \\
 &dA - dP + 2\nu dR = k (P - \bar{P}) dt
 \end{aligned} \tag{3.23}$$

Eqs. (3.23) give the relations between changes in the variables along the characteristics. For example, if in Fig. 27 the values of all the variables are known at Points p, q, q', r, s, and s', then Eqs. (3.22) provide six equations for determining the six unknowns at point p. The technique for performing these calculations will be discussed later.

Since one possible manner of initiating a disturbance in the bar is to apply, instantaneously, to the end of the bar, a constant particle velocity, and by so doing introduce a discontinuity at the impacted end, it is necessary to obtain the manner in which a discontinuity is propagated along the bar and the relations of the various variables across such a discontinuity.

Propagation of Discontinuities

Consider a discontinuity traveling with some unknown velocity, c , from left to right. In Region 1, Fig. 28, the particles move with a common velocity V_x^1 in the axial direction and V_r^1 in the radial direction. Behind the wave in Region 2, the particles move with a common velocity V_x^2 in the axial direction and V_r^2 in the radial direction.

The radial and axial velocities may experience a sharp change at the wave front where the values in Region 1 are changed into the values in Region 2 in a very short period of time. Such a rapid change is called a "jump" or a discontinuity. The wave front travels with a velocity c which, for now, is assumed to be unknown.

Fig. 28. ELEMENT OF BAR AS DISCONTINUITY
PASSES

If the material in the bar is to be continuous, there must be no change in the axial or radial displacements across the wave front.

Therefore,

$$\begin{aligned} U_{\bar{x}} - U_{\bar{x}}^i &= 0 &) \\ & &) \\ & &) \\ U_{\bar{r}} - U_{\bar{r}}^i &= 0 &) \end{aligned} \quad \text{--- (3.24)}$$

Eqs. (3.24) state that the displacements must be continuous across the discontinuity.

Characteristic curves are known to be curves along which discontinuities are propagated. It is also known that the discontinuities exist only on the characteristic curve, or characteristic. As the discontinuity travels along the characteristic, it is changing with respect to both time and co-ordinate position, $\bar{x}$.

The total derivative with respect to time can be given by the symbolic operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \bar{x}} \frac{d\bar{x}}{dt} \quad \text{--- (3.25)}$$

The equation of the characteristic curve can, in general, be expressed as $\bar{x} = c\bar{t} + \text{const.}$ Therefore,

$$\frac{d\bar{x}}{d\bar{t}} = c \quad \text{--- (3.26)}$$

Using Eq. (3.26), Eq. (3.25) is expressed as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \quad \text{--- (3.27)}$$

Applying the operator $\frac{D}{Dt}$ to Eq. (3.24) gives

$$\begin{aligned} \frac{\partial}{\partial t} (U_x - U_x^i) + c \frac{\partial}{\partial x} (U_x - U_x^i) &= 0 \\ \frac{\partial}{\partial t} (U_r - U_r^i) + c \frac{\partial}{\partial x} (U_r - U_r^i) &= 0 \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \end{array} \right\} \text{--- (3.28)}$$

Substituting the values of the partial derivatives of the displacements, Eq. (3.28) becomes

$$\begin{aligned} V_x - V_x^i &= -c (\epsilon_x - \epsilon_x^i) \\ V_r - V_r^i &= -c (\gamma_{xr} - \gamma_{xr}^i) \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \end{array} \right\} \text{--- (3.29)}$$

Applying the assumption that the discontinuity behaves in an entirely elastic manner, the material relationships are:

$$\begin{aligned} E_o \epsilon_x &= (\sigma_x - 2\nu \sigma_r) \\ E_o \epsilon_r &= (1 - \nu) \sigma_r - \nu \sigma_x \\ G_o \gamma_{xr} &= \tau_{xr} \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \end{array} \right\} \text{--- (3.30)}$$

Now, if the impulse-momentum relationship in the axial direction is applied to the differential element of the bar containing the wave front, (Fig. 28) the following relationship between the axial stress and axial particle velocity is obtained:

$$-(\sigma_x - \sigma_x^i) \Delta \bar{t} = (V_x - V_x^i) \Delta \bar{x} \quad (3.31)$$

Using Eq. (3.26) for a forward-moving discontinuity, Eq. (3.31) can be expressed as

$$(\sigma_x - \sigma_x^i) = -\rho c (V_x - V_x^i) \quad (3.32)$$

Eliminating ϵ_x and V_x from the first of Eqs. (3.20), Eq. (3.31), and (3.32), results in

$$(\sigma_x - \sigma_x^i) = \frac{\rho c^2}{E} \left[\sigma_x - \sigma_x^i - 2\nu(\sigma_r - \sigma_r^i) \right] \quad (3.33)$$

If the interval of time taken for the discontinuity to pass a given point in the bar is sufficiently small, there will be no change in the radial strain. Under this assumption, the second of Eqs. (3.30) leads to the following relationship between the radial and the axial stress.

$$\sigma_x - \sigma_x^i = \frac{1-\nu}{\nu} (\sigma_r - \sigma_r^i) \quad (3.34)$$

If Eq. (3.34) is substituted into Eq. (3.33), then

$$(\sigma_x - \sigma_x^i) = \frac{\rho c^2}{E} \frac{(1-2\nu)(1+\nu)}{1-\nu} (\sigma_x - \sigma_x^i)$$

This can be true only if

$$c = \pm \sqrt{\frac{1-\nu}{(1-2\nu)(1+\nu)}} c_0 = \pm c_0 \quad \text{---} \quad (3.35)$$

Therefore, all discontinuities in the axial stress, strain, and particle velocity must be propagated at the dilatational velocity c_0 . The characteristic along which the discontinuity travels is

$$\bar{x} = \pm c_0 \bar{t} + \text{const.} \quad \text{---} \quad (3.36)$$

Eqs. (3.32) and (3.34), along with the first of Eqs. (3.30), give the relations between the discontinuities in $\sigma_{\bar{x}}$, $t_{\bar{x}}$, and $V_{\bar{x}}$ as

$$\begin{aligned} (\sigma_{\bar{r}} - \sigma_{\bar{r}}^i) &= \frac{\nu}{1-\nu} (\sigma_{\bar{x}} - \sigma_{\bar{x}}^i) \\ (V_{\bar{x}} - V_{\bar{x}}^i) &= -\frac{1}{\rho c_0} (\sigma_{\bar{x}} - \sigma_{\bar{x}}^i) \\ (\epsilon_{\bar{x}} - \epsilon_{\bar{x}}^i) &= \frac{(1-2\nu)(1+\nu)}{E_0(1-\nu)} (\sigma_{\bar{x}} - \sigma_{\bar{x}}^i) \\ (\epsilon_{\bar{r}} - \epsilon_{\bar{r}}^i) &= 0 \end{aligned} \quad \text{---} \quad (3.37)$$

Now, in a similar manner, applying the impulse-momentum relationship in the radial direction, the relationship between the radial shear stress and the radial particle velocity is obtained as

$$(\tau_{\bar{x}\bar{r}} - \tau_{\bar{x}\bar{r}}^i) \Delta \bar{t} = \rho (V_{\bar{r}} - V_{\bar{r}}^i) \Delta \bar{x} \quad \text{---} \quad (3.38)$$

Assuming that this discontinuity is propagated along some general curve in the $\bar{x} - \bar{t}$ plane whose equation is $\bar{x} = c\bar{t} + \text{const.}$, then the relationship between $\Delta\bar{x} = c \Delta\bar{t}$. Consequently, Eq. (3.38) becomes

$$(\tau_{\bar{x}\bar{r}} - \tau_{\bar{x}\bar{r}}^1) = -\rho c (V_{\bar{r}} - V_{\bar{r}}^1) \quad \text{--- (3.39)}$$

Now, eliminating $V_{\bar{r}}$ and $\gamma_{\bar{x}\bar{r}}$ from the second of Eq. (3.29), the third of Eqs. (3.30) and Eq. (3.39) results in

$$(\tau_{\bar{x}\bar{r}} - \tau_{\bar{x}\bar{r}}^1) = \frac{\rho c^2}{G_0} (\tau_{\bar{x}\bar{r}} - \tau_{\bar{x}\bar{r}}^1) \quad \text{--- (3.40)}$$

Consequently

$$c = \sqrt{\frac{G_0}{\rho}} = c_s \quad \text{--- (3.41)}$$

and all discontinuities in the radial shear force, the radial particle velocity, and the radial strain will propagate at the shear wave velocity c_s along the characteristic curve

$$\bar{x} = c_s \bar{t} + \text{const.} \quad \text{--- (3.42)}$$

Since $c_s < c_0$, this discontinuity will lag the discontinuity propagating at the dilatational velocity. In the most general case, a point

in the bar will experience a jump in the axial stress, strain, particle velocity, and in the radial stress; then at some later time, it will experience a jump in the radial particle velocity and the radial shear stress and strain.

The relations across the discontinuity traveling at the shear velocity c_s is given as

$$\begin{aligned} (\tau_{xr} - \tau_{xr}^i) &= -\rho c_s (V_r - V_r^i) \\ (\tau_{xr} - \tau_{xr}^i) &= G_0 (\gamma_{xr} - \gamma_{xr}^i) \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \\) \end{array} \right\} \quad \text{--- (3.43)}$$

The assumption which led to Eq. (3.34), that is, that the jump in radial strain vanish along the wave front, leads to the conclusion that

$$U_r(\bar{x}, \bar{t}) = 0 \quad \text{--- (3.44)}$$

on both sides of the wave front. Since

$$\begin{aligned} \gamma_{xr} &= \frac{\bar{r}}{a} \frac{\partial U_r}{\partial \bar{x}} \\ V_r &= \frac{\bar{r}}{a} \frac{\partial U_r}{\partial \bar{t}} \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \\) \end{array} \right\} \quad \text{--- (3.45)}$$

it can be seen that there will be no discontinuity in γ_{xr} & τ_{xr} , and V_r initiated and further considerations of such a discontinuity will therefore be neglected in this analysis. In view of Eqs. (3.44)

and (3.45), Eqs. (3.43) become

$$\begin{aligned}
 (\tau_{\overline{xr}} - \tau_{\overline{xr}}^i) &= 0 \\
 (V_{\overline{r}} - V_{\overline{r}}^i) &= 0 \\
 (\gamma_{\overline{xr}} - \gamma_{\overline{xr}}^i) &= 0
 \end{aligned}
 \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{----- (3.46)}$$

Hence, Eqs. (3.46) hold not only across the initial wave front, but across all other characteristics as well.

Eqs. (3.37) and (3.46) may be integrated, and then expressed in nondimensional form in the manner indicated previously. The resulting equations relating the changes in the variables across an axial stress discontinuity are

$$\begin{aligned}
 \Delta P &= -\frac{1}{2} D^2 \Delta V \\
 \Delta R &= \frac{\nu}{1-\nu} \Delta P \\
 \Delta A &= \frac{1}{D^2} \Delta P \\
 \Delta W &= 0 \\
 \Delta Q &= 0
 \end{aligned}
 \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \\) \end{array} \right\} \text{----- (3.47)}$$

Eqs. (3.47) were developed for a forward-traveling discontinuity. A similar analysis for a backward-traveling discontinuity would yield the same equations except for the sign in the first equation. That is, along a backward-traveling discontinuity

$$\Delta P = \frac{1}{2} D^2 \Delta V \quad - - - - - (3.48)$$

The other equations remain the same.

If the manner in which a discontinuity in one of the variables is propagated, i.e., the discontinuity in one of the variables is expressed as a function of time, then at any point, the values of the discontinuities in the other variables may be predicted.

Such a relationship can be obtained by noting that along the initial side of the characteristic curve $dx = dt$

$$dP_1 - \frac{D^2}{2} dV_1 - \frac{v}{1-v} D^2 W_1 dt = - \frac{k}{1+v} (P_1 - \bar{P}_1) dt$$

and on the other side of this characteristic curve, i.e., across the discontinuity

$$dP_2 - \frac{D^2}{2} dV_1 - \frac{v}{1-v} D^2 W_1 dt = - \frac{k}{1+v} (P_2 - \bar{P}_2) dt$$

Now, subtracting these equations, results in

$$\begin{aligned}
 d(P_2 - P_1) &= \frac{D^2}{2} d(V_2 - V_1) - \frac{\nu}{1-\nu} D^2 (W_2 - W_1) dt \\
 &= -\frac{k}{1+\nu} (P_2 - P_1 - \bar{P}_2 - \bar{P}_1) dt
 \end{aligned}$$

or

$$\begin{aligned}
 d(\Delta P) &= \frac{D^2}{2} d(\Delta V) - \frac{\nu}{1-\nu} D^2 \Delta W dt \\
 &= -\frac{k}{1+\nu} (\Delta P - \Delta \bar{P}) dt
 \end{aligned} \quad (3.49)$$

where $\Delta P = P_2 - P_1$, $\Delta V = V_2 - V_1$, ... etc.

Now if the relationships given in Eqs. (3.47) are substituted into Eqs. (3.49), the following expression is obtained

$$d(\Delta P) = -\frac{\nu}{2(1+\nu)} (\Delta P - \Delta \bar{P}) dt \quad (3.50)$$

where it will be remembered that $\bar{P}$ is some function of the resultant axial strain, A .

From Eq. (3.50), it is now possible to determine the manner in which a discontinuity in the resultant axial stress will vary along the characteristic curve. Once the discontinuity in stress is determined, Eqs. (3.47) allow the calculations of the discontinuities in the other variables.

For a backward-traveling discontinuity, a similar analysis will determine that Eq. (3.50) is the same. Eq. (3.48), along with the last four of Eqs. (3.47), are then used to specify the propagation of a discontinuity in a backward-traveling wave.

Reflection of a Discontinuity

To obtain the conditions governing the discontinuity on its arrival at the supported end of the specimen, the following procedure is employed. Unless the acoustic impedances of both mediums are equal, part of the disturbance arriving at the end will be transmitted and part will be reflected. Eq. (3.48) for a backward-traveling disturbance is

$$P_2 - P_1 = \frac{D^2}{2} (V_2 - V_1) \quad - - - - - \quad (3.51a)$$

where the Subscript 2, Fig. 29, represents the conditions on the end of the bar just after the reflected disturbance has passed and the Subscript 1 stands for the conditions existing just before the disturbance has reflected from the interface between the copper and steel bar, the first of Eqs. (3.47) would be represented as

$$P_2' - P_1' = - \frac{D'^2}{2} (V_2' - V_1') \quad - - - - - \quad (3.51b)$$

Fig. 29. INTERFACE OF CHARACTERISTIC PLANE BETWEEN
STEEL BACK-UP BAR AND COPPER SPECIMEN

where the prime represents conditions in the steel bar, D' is the dimensionless dilatational velocity in the steel bar and the Subscripts 1 and 2 have the same meaning as defined above.

Now conditions for dynamic contact during the passing of the disturbance and Newton's Third law yields, respectively

$$\begin{aligned} V_2 &= V_2' \\ P_2 &= P_2' \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \end{array} \right\} \quad \text{--- (3.52)}$$

Eqs. (3.51), (3.52), and (3.53) may now be used to determine P_2 as

$$P_2 = \frac{D'^2 \left(P_1 - \frac{D^2}{2} V_1 \right) + D^2 \left(P_1' + \frac{D'^2}{2} V_1' \right)}{D^2 + D'^2} \quad (3.53)$$

and since P_1 is determined from previous calculations by Eq. (3.50), then ΔP can be obtained. Once ΔP is obtained at any point, the last four of Eqs. (3.47) may be used to obtain the values of the other variables just after the discontinuity passes. Since the incoming discontinuity will be partially transmitted and partially reflected, there is still a discontinuity of smaller amplitude traveling back along the bar. It is treated as a backward-traveling discontinuity by the methods previously described.

Numerical Computations

In order to perform the numerical computations, Eqs. (3.23) can be approximated by finite differences. Then the problem to be solved is that of obtaining the values of the six unknown variables at point p, Fig. 27, if the values of these quantities are known at points q, q', r, s, and s'. That is, Eqs. (3.23) provide six equations in terms of the known quantities at the points q, q', r, s, s' and s'. That is, Eqs. (3.23) provide six equations in terms of the known quantities at the points q, q', r, s, s', and the unknown quantities at point p.

For convenience in performing the computations, the equations along the characteristics joining p to q' and p to s' are expressed in terms of the time interval Δt . From Fig. 30, it can be seen that

$$\Delta t' = 2 \Delta t - \Delta t'' \quad \text{--- --- ---} \quad (3.54)$$

but

$$\Delta t'' = \Delta x' \tan \theta = \Delta x'$$

and along the dotted characteristic curve

$$\Delta x' = - \frac{S}{D} \Delta t'$$

Consequently,

$$\Delta t'' = - \frac{S}{D} \Delta t' \quad \text{--- --- ---} \quad (3.55)$$

Now substituting Eq. (3.55) into Eq. (3.54) gives

$$\Delta t' = \frac{2D}{(S + D)} \Delta t \quad \text{--- --- ---} \quad (3.56)$$

Fig. 30. HALF-ELEMENT OF CHARACTERISTIC GRID
SHOWING RELATION BETWEEN $\Delta t'$ AND Δt

Now by linear interpolation, the quantities at Points q' and s' may be determined in terms of the quantities at r and s as

$$\left. \begin{aligned} Q_s' &= Q_r + \frac{2S}{(S+D)} (Q_s - Q_r) \\ W_s' &= W_r + \frac{2S}{(S+D)} (W_s - W_r) \\ Q_r' &= Q_r + \frac{2S}{(S+D)} (Q_q - Q_r) \end{aligned} \right\} \text{--- -- (3.57)}$$

The finite difference form of Eqs. (3.23) with Eqs. (3.57) yields the following system of equations

Along $dx = dt$

$$\left. \begin{aligned} P_p^i - \frac{D^2}{2} V_p^i - \frac{\nu}{1-\nu} \frac{D^2 \Delta t}{2} W_p^i &= P_q - \frac{D^2}{2} \\ + \left(\frac{\nu}{1-\nu} \right) \frac{D^2 \Delta t}{2} W_q + \frac{k}{1+\nu} (P_p^{i-1} - \bar{P}_p^{i-1} + P_q - \bar{P}_q) \frac{\Delta t}{2} \end{aligned} \right\} \text{--- -- (3.58)}$$

Along $dx = -dt$

$$\left. \begin{aligned} P_p^i + \frac{D^2}{2} V_p^i - \frac{\nu}{1-\nu} \frac{D^2 \Delta t}{2} W_p^i &= P_s + \frac{D^2}{2} V_s \\ + \left(\frac{\nu}{1-\nu} \right) \frac{D^2 \Delta t}{2} W_s - \frac{k}{1+\nu} (P_p^{i-1} - \bar{P}_p^{i-1} + P_s - \bar{P}_s) \frac{\Delta t}{2} \end{aligned} \right\}$$

$$\underline{\text{Along } dx = \frac{S}{D} dt}$$

$$Q_p^i - \frac{D}{4S} W_p^i - \frac{2\Delta t}{S(S+D)} R_p^i = \frac{2S}{S+D} Q_q + Q_r \left(1 - \frac{2S}{S+D}\right)$$

$$- \frac{D}{2(S+D)} W_q - \left(\frac{D}{4S} - \frac{D}{2(S+D)}\right) W_r + \frac{4\Delta t}{(S+D)^2} R_q$$

$$+ R_r \left(1 - \frac{2S}{S+D}\right) \frac{2\Delta t}{S(S+D)}$$

$$\underline{\text{Along } dx = -\frac{S}{D} dt}$$

$$Q_p^i + \frac{D}{4S} W_p^i + \frac{2\Delta t}{S(S+D)} R_p^i = \frac{2S}{S+D} Q_s + Q_r \left(1 - \frac{2S}{S+D}\right)$$

$$+ \frac{D}{2(S+D)} W_s + \frac{D}{4S} - \frac{D}{2(S+D)} W_r - \frac{4\Delta t}{(S+D)^2} R_s$$

$$- \left(1 - \frac{2S}{S+D}\right) \frac{2\Delta t}{S(S+D)} R_r$$

$$\underline{\text{Along } dx = 0}$$

$$P_p^i + A_p^i + 2\nu R_p^i = -P_r + A_r + 2\nu R_r + k (P_p^{i-1} - \bar{P}_p^{i-1})$$

$$+ (P_r - \bar{P}_r) \Delta t$$

$$P_p^i - \left(\frac{1-\nu}{\nu}\right) R_p^i + W_p^i \frac{\Delta t}{2\nu} = P_r - \left(\frac{1-\nu}{\nu}\right) R_r$$

$$- \frac{\Delta t}{2\nu} W_r - \frac{k}{2\nu} (P_p^{i-1} - \bar{P}_p^{i-1} + P_r - \bar{P}_r) \Delta t$$

(3.58)

For the general interior point, Eqs. (3.58) furnish a means of obtaining the unknown quantities at the general point p . To do this, approximate values of P_p^{i-1} and A_p^{i-1} are assumed. The first approximations used were

$$\begin{aligned} P_p^o &= \frac{P_q + P_s}{2} \\ A_p^o &= \frac{A_q + A_s}{2} \end{aligned} \quad \begin{array}{l}) \\) \\) \\) \\) \\) \\) \end{array} \quad \text{--- (3.59)}$$

These values were substituted into the right side of Eqs. (3.58). Once this has been done, the right sides of Eqs. (3.58) are given as constants. Then the first approximation of the variables at p may be obtained. These values of P_p^i and A_p^i are then substituted into the right sides of Eqs. (3.58) to obtain new approximations of the variables. This iterative process is continued until

$$|A_p^i - A_p^{i-1}| \leq .00000001 \quad \text{--- (3.60)}$$

that is, until two successive approximations of the strain agree to seven places. It is necessary to use the strain for the convergence criterion in the iterative process, since it is possible that for some value of time, the stress in the bar will become constant. If this occurs, and the stress difference is used as the criterion for stopping the iterative process, convergence would be obtained on the first approximation, and the accuracy of the numerical calculations would be insufficient.

There are a number of methods for calculation of the unknowns on the left side of Eqs. (3.58) once the right sides have been determined. The procedure used here is to express the equations in matrix form and then solve for the unknowns by multiplying both sides of Eqs. (3.58) by the inverse of the matrix of the coefficients of the unknowns on the left.

For the boundary element on the impacted end of the bar, Fig. 31, two of Eqs. (3.58) must be deleted. Since there are no characteristic curves $dx = dt$ or $dx = \frac{S}{D} dt$, then the first and the third of Eqs. (3.58) must be omitted. It is now necessary to obtain two other relations to determine the six unknown quantities at point p. Here, the values of two of the unknowns have been specified on the impacted end, thus reducing our problem to that of four equations and four unknowns. The two variables that were specified in this study are the axial particle velocity and the integrated shear stress. The integrated shear stress is assumed to be zero at all values of time, while the axial particle velocity is assumed to have some specified magnitude for all values of time. That is

$$\begin{array}{rcl} V_{x=0} & = & V(t) \\ & &) \\ & &) \\ Q_{x=0} & = & 0 \end{array} \quad \begin{array}{l}) \\) \\) \\) \end{array} \quad \text{--- (3.61)}$$

Just as in the elementary case, two different functions for the axial particle velocity were considered. One of these specifies that the particle velocity is given an instantaneous step up to some value

Fig. 31. BOUNDARY ELEMENT ON IMPACTED END OF
COPPER SPECIMEN

after which it is maintained constant for all values of time. The other case considered is that of parabolic rise in the particle velocity up to some specified value during a finite interval of time after which the particle velocity is maintained constant for all values of time.

The boundary conditions on the impacted end are then either

$$\begin{aligned} V_{x=0} &= \hat{V}, t \geq 0 \\ Q_{x=0} &= 0, t \geq 0 \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \end{array} \right\} \text{--- -- (3.62)}$$

for the "step" particle velocity or

$$\begin{aligned} V_{x=0} &= \hat{V} - \phi(t - T)^2, \quad 0 \leq t \leq T \\ V_{x=0} &= \hat{V}, \quad t \geq T \\ Q_{x=0} &= 0, \quad t \geq 0 \end{aligned} \quad \left. \begin{array}{l}) \\) \\) \\) \\) \\) \end{array} \right\} \text{--- -- (3.63)}$$

where $\hat{V}$ is the velocity of the impacting cylinder, T is the rise time, and ϕ is a constant. That is

$$\phi = -\frac{\hat{V}}{T^2} \quad \text{--- -- (3.64)}$$

The reduced form of Eqs. (3.38) can then be expressed in matrix form and solved by methods similar to those used for the general interior point. Here, however, the first approximations for the unknown quantities at point p are assumed to be the same as the values at point s . Iteration is continued until the conditions specified by Eq. (3.60) are satisfied.

On the supported end of the specimen, a similar technique is used. The second and fourth of Eqs. (3.28) must be deleted while values for two of the unknowns must be specified. Again, in this investigation, the two unknowns specified are the axial particle velocity and the resultant shear force. Two different values for the shear force have been considered. These are

$$Q = 0 \quad \text{---} \quad (3.65a)$$

or

$$Q = mP \quad \text{---} \quad (3.65b)$$

where m is the kinetic coefficient of friction between copper and steel. Here, the value of $m = 0,5$ has been arbitrarily chosen since the actual values of m depends on the surface condition of the two materials, which may vary considerably from one specimen to another.

The axial particle velocity is held constant between points p and r (Fig. 32), but it is allowed to change at the points p along

with the other variables in the manner described by Eqs. (3.53) and the second four of Eqs. (3.47). Essentially, this process consists of first calculating the values at point p just before the incoming disturbance arrives at the interface between the steel and copper bar. This is done by using the specified shear force, the fact that the particle velocity at point p is equal to that at point r , and the deleted set of Eqs. (3.58). This calculation is accomplished by using the same iterative technique as that described for the general interior point. Once the values of the incoming "disturbance" i.e., P_p^i , R_p^i , Q_p^i , A_p^i , and W_p^i , have been obtained to a sufficient degree of accuracy, the Eqs. (3.53) and (3.47) are used to determine the reflected values of these variables, thus obtaining the values of point p after the "disturbance" has passed.

It is now possible to perform the successive calculations of all the general points p . However, to perform the first calculation, it is still necessary to know the values of all the variables along the initial wave front, that is, along the first characteristic curve $dx = dt$. To obtain these values, it will be remembered that Eqs. (3.47) can be used, once the value of one of the variables is known. Eq. (3.50) may be used to obtain the values of ΔP along the initial wave front. To do this, Eq. (3.50) is expressed in finite difference form as

$$(P_2 - P_1)_p^i = (P_2 - P_1)_r - \frac{k \Delta t}{4} \left[(P_2 - P_1)_p^{i-1} \right] + (P_2 - P_1)_s - (\bar{P}_2 - \bar{P}_1)_p^{i-1} - (\bar{P}_2 - \bar{P}_1)_s \quad (3.66)$$

Along the initial wave front, all of the variables are initially zero, so the quantities with the Subscript 1 can be omitted in Eq. (3.63).

It should be remembered, at this point, that $\bar{P}$ is a function of A ; that is, $\bar{P}$ is obtained by integrating the quasi-static stress-strain curve. Also, the strain-rate constant k has a real and positive value only when the $P_p^i > \bar{P}_p^i$ and on each iteration, this inequality should be checked. If at any point $P_p^i \leq \bar{P}_p^i$, k must be made zero for the calculations at that point.

Following the above-described procedures, the values of the variables at all points of the bar for all values of time may be obtained. The accuracy of the iterative process will be dependent on the size of the characteristic grid, i.e., the value chosen for Δt . In deciding on how large the increment Δt should be, successive values of Δt should be tried until a value is found for which decreasing the size of Δt further will have no effect on the computed results. Following this procedure, the Δt selected for this investigation is a dimensionless unit of time, $\Delta t = 0.05$.

Results

With the IBM 650 computer, results have been obtained for 1/2-in. and 1-in. specimens with two different velocities of impact, the finite rise and "step" impact velocities, and with two different boundary conditions, one with zero shear on the supported end and one

with the resultant shear force equal to mP .

For all calculations, the constants used are the same as those used in the elementary case. That is, $\bar{k} = 60 \times 10^4 \text{ sec}^{-1}$, the impacting velocity of the projectile = 780 in./sec, $E_o = 15 \times 10^6 \text{ psi}$, $E_s = 30 \times 10^6 \text{ psi}$, $\rho_c = 0.32 \text{ lb/cu in.}$, and $\rho_s = 0.28 \text{ lb/cu in.}$ In addition, a Poisson's ratio of 0.3 and a bar radius of 1 in., arbitrarily selected, were used. Since the theory is one dimensional, the radius of the bar has no significance other than to determine the magnitude of the dimensionless quantities. For conversion from dimensionless to physical quantities, the 1-in. bar radius is quite convenient.

Figs. 33 through 53 show results for the 1/2-in. -length specimens while Figs. 54 through 75 show results for the 1-in. -length specimens. Figs. 33 through 39, and 54 through 60 show results for the "step" impact velocity, while the remaining results are for the finite rise impact velocity.

In general, it can be seen that the resultant axial stress-time records for a specimen of a given length are very similar. (Figs. 33, 40, 47, and Figs. 54, 61, and 69 are the smallest. The oscillations can also be seen to be smaller for the 1-in. specimens than for the 1/2-in. specimens. From the stress distribution curves, Figs. 38, 45, 52, 59, 67, and 74, it can be seen that the tendency is for the specimen to assume a uniform stress distribution.

Fig. 33. DIMENSIONLESS AXIAL FORCE vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 34. DIMENSIONLESS RADIAL FORCE vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 35. DIMENSIONLESS AXIAL STRAIN vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 36. DIMENSIONLESS AXIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 37. DIMENSIONLESS RADIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

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Fig. 38. DIMENSIONLESS AXIAL FORCE DISTRIBUTION FOR 1/2 in. COPPER CYLINDER

Fig. 39. DIMENSIONLESS AXIAL STRAIN DISTRIBUTION FOR 1/2 in. COPPER CYLINDER

Fig. 40. DIMENSIONLESS AXIAL FORCE vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 41. DIMENSIONLESS RADIAL FORCE vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 42. DIMENSIONLESS AXIAL STRAIN vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 43. DIMENSIONLESS AXIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 44. DIMENSIONLESS RADIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1/2 in COPPER CYLINDER

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Fig. 45. DIMENSIONLESS AXIAL FORCE DISTRIBUTION FOR 1/2 in. COPPER CYLINDER

Fig. 46. DIMENSIONLESS AXIAL STRAIN DISTRIBUTION FOR 1/2 in. COPPER CYLINDER

Fig. 47. DIMENSIONLESS AXIAL FORCE vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 48. DIMENSIONLESS RADIAL FORCE vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 49. DIMENSIONLESS AXIAL STRAIN vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 50. DIMENSIONLESS AXIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 51. DIMENSIONLESS RADIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1/2 in. COPPER CYLINDER

Fig. 52. DIMENSIONLESS AXIAL FORCE DISTRIBUTION FOR 1/2 in. COPPER CYLINDER

Fig. 53. DIMENSIONLESS AXIAL STRAIN DISTRIBUTION FOR 1/2 in. COPPER CYLINDER

Fig. 54. DIMENSIONLESS AXIAL FORCE vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 55. DIMENSIONLESS RADIAL FORCE vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 56. DIMENSIONLESS AXIAL STRAIN vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 57. DIMENSIONLESS AXIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 58. DIMENSIONLESS RADIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 59. DIMENSIONLESS AXIAL FORCE DISTRIBUTION FOR 1 in. COPPER CYLINDER

Fig. 60. DIMENSIONLESS AXIAL STRAIN DISTRIBUTION FOR 1 in. COPPER CYLINDER

Fig. 61. DIMENSIONLESS AXIAL FORCE vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 62. DIMENSIONLESS RADIAL FORCE vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig.63. DIMENSIONLESS RADIAL SHEAR FORCE vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig.64. DIMENSIONLESS AXIAL STRAIN vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig.65. DIMENSIONLESS AXIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig.66. DIMENSIONLESS RADIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 67. DIMENSIONLESS AXIAL FORCE DISTRIBUTION FOR 1 in. COPPER CYLINDER

Fig. 68. DIMENSIONLESS AXIAL STRAIN DISTRIBUTION FOR 1 in. COPPER CYLINDER

Fig. 69. DIMENSIONLESS AXIAL FORCE vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 70. DIMENSIONLESS RADIAL FORCE vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 71. DIMENSIONLESS AXIAL STRAIN vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig. 72. DIMENSIONLESS AXIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig.73. DIMENSIONLESS RADIAL PARTICLE VELOCITY vs DIMENSIONLESS TIME FOR 1 in. COPPER CYLINDER

Fig.74.DIMENSIONLESS AXIAL FORCE DISTRIBUTION FOR 1 in.
COPPER CYLINDER

Fig.75.DIMENSIONLESS AXIAL STRAIN DISTRIBUTION FOR 1 in.
COPPER CYLINDER

The resultant radial stress oscillates between a positive and a negative value. In the 1/2-in. specimens, the records for the "step" and finite rise impact velocities with no shear force, Fig. 34 and 41, agree very well, but there is a quite noticeable difference between these cases and the finite rise velocity with a radial shear stress on the supported end, Fig. 48. The resultant radial stress is predominantly positive in value when the shear stress is applied to the end of the specimen.

The resultant shear stress, an example of which is shown in Fig. 63, also oscillates from positive to negative values. Note that this shear stress is also shown for $x = 0.5$, the mid point of the specimen. Figs. 61 through 68 have records plotted at the mid point of the specimen, in addition to those at either end. The behavior in the center is seen to be essentially the same as that at the ends.

From the resultant axial strain-time curves, Figs. 35, 42, 49, 56, 64, and 71, it can be seen that the strain rate in the bar tends to approach a constant value at both ends. Since the axial strain rate is dependent on both the radial and the axial stress resultants, there is some oscillation present. The shear stress on the end reduces the magnitude of these oscillations, as may be seen in Figs. 49 and 71. The strain rate for the "step" impact velocity is nearly equal to that for the finite rise impact velocity where there is no shear force on the supported end, (Figs. 42 and 49) but the magnitude of the strain for the

"step" impact velocity is greater due to the initial instantaneous straining. In general, the resultant axial strain tends to assume a linear distribution, (Figs. 39, 46, 53, 60, 68, and 75) but the largest strain is always on the impacted end with the smallest on the supported end.

The axial particle velocities all have curves which closely resemble one another after the initial velocity rise time (Figs. 36, 43, 50, 57, 65, and 72). The shear force on the end of the bar tends to reduce the oscillations present in the records of the particle velocities on the supported ends. (Figs. 51 and 73.)

Although values have been shown for the behavior of the quantities only on the ends, with the exception of the finite rise impact on a 1-in. specimen with no shear force on the supported end, the stress-distribution and strain-distribution curves may be employed to obtain information about the behavior at other points. The general tendency seems to be toward a state of dynamic equilibrium in which the bar is being strained at a constant rate under the action of a uniform axial stress of constant magnitude.

CHAPTER IV

COMPARISON AND EVALUATION OF RESULTS

In order to compare the calculated results for the axial stress and strain, these quantities have been converted to physical dimensions by assuming that the axial stress and strain are uniformly distributed over the bar cross section. The results are then plotted on the same graph with experimental measurements. The experimental measurements were taken from Ref. 26. Figs. 76 and 77 show results for the 1/2-in. - length specimens, and Figs. 78 and 79 show the results for the 1-in. - length specimens. All of the results were obtained with an impacting velocity for the cylinder of 780 in./sec and both theoretical analyses were made with a strain-rate constant of $\bar{k} = 60 \times 10^4 \text{ sec}^{-1}$. Particle velocity rise times of 3.73 and 3.32 μsec were used at the impacted end, respectively, for the elementary and modified theories.

Comparison of Theoretical Results

Comparison of the results from the elementary strain-rate theory with those from the modified strain-rate theory shows that a higher value of axial stress is predicted by the elementary theory than by the modified theory. The axial stress predicted by the modified theory varies in an oscillatory manner, while that of the elementary theory is comparatively smooth.

Fig. 76. COMPARISON OF EXPERIMENTAL AND THEORETICAL STRAIN-TIME RECORDS FOR 1/2 in. COPPER CYLINDER

Fig.77. COMPARISON OF EXPERIMENTAL AND THEORETICAL STRESS-TIME RECORDS AT
SUPPORTED END OF 1/2 in. COPPER CYLINDER

Fig. 78. COMPARISON OF THEORETICAL AND EXPERIMENTAL STRAIN-TIME RECORDS FOR 1 in. COPPER CYLINDER

Fig. 79. COMPARISON OF THEORETICAL AND EXPERIMENTAL STRESS-TIME RECORDS AT SUPPORTED
END OF 1 in. COPPER CYLINDER

The axial strains on the impacted end of the specimen are greater for the elementary theory than for the modified theory, while results from both theories agree very closely on the supported end. From Figs. 76 and 78, it can be seen that the record for the elementary theory starts at a later time than that for the modified theory. This is due to the slower propagational velocity predicted by the elementary theory. If this shift in time is accounted for, the resultant strain records are nearly identical. Both theories predict that a constant strain rate will be reached and that a linear strain distribution, varying from a maximum at the impacted end to a minimum at the supported end, will be attained.

The primary effects, then, of allowing the bar to move in the radial direction and, hence, including the effects of shear and inertia associated with this motion, is then found to be:

1. A reduction in the magnitude of the predicted axial quantities, stress, strain, and particle velocity at a given time.
2. A more oscillatory history for the magnitude of the axial quantities due to the influence of the associated radial quantities.
3. The appearance of a dilatational and a distortional propagational velocity, instead of only the "bar" velocity.

Comparison of Theoretical and Experimental Results

Comparisons between the theoretical and experimental results show that in all cases the predictions of the elementary theory are of larger magnitude than the experimental measurements.

For the 1/2-in. specimens, the predictions of the modified theory are in good agreement with the experimental measurements with the exception of the strain at the supported end. In Fig. 76, the strain at the impacted end, as predicted by the modified theory for the 1/2-in. specimens agrees very well with the experimental results after the disturbance has traversed the bar one time. However, the agreement between the predicted and measured strains on the supported end is very poor. This is surprising since Fig. 77 shows that the magnitudes of the stress at the supported end for the theoretical calculations and the experimental results are in good agreement. The parabolic function of time, assumed for the particle velocity on the impacted end in the calculations, appears to be in error. A function agreeing more nearly with that shown by the experiments would have given a better agreement between the stresses.

Although the agreement is not as close, similar results are found for the 1-in. specimens. It appears that in the case of the 1-in. specimens, some other strain-rate law may give better agreement between the calculated strains and the measured strains during the first 20 micro-sec. After this time, reasonably good agreement between the calculated and measured strains at the impact end is found, but there is still a discrepancy between the strains at the

supported end. The stresses at the supported end, Fig. 79, show a general quantitative agreement, but the rise-time difference still occurs.

There are two possible explanations for the difference in the calculated and measured stress-time records. One is the above-mentioned rise-time difference. The steeper rise time present in the calculated case would lead to larger oscillations in the calculated results. The second is that the measurements from which the stress record was obtained were made in the steel bar near the impacted end. The response of the steel bar probably has some influence on the measured record.

One possible reason for the odd shape of the experimental curve during the "rise" period is that there may be a cushion of highly compressed air building up between the impacting cylinder and the specimen. Such conditions would start a disturbance traveling down the bar before the impacting cylinder actually made contact with the specimen.

Conclusions and Observations

For finite specimens of a strain-rate-dependent material:

1. The elementary strain-rate theory predicts a magnitude for the axial stress and the axial strain that is greater than the magnitudes of these quantities predicted by the modified strain-rate theory or measured by experimental methods.

2. The modified strain-rate theory is better able to predict the experimental results than the elementary theory and for the 1/2-in. specimens gives very good agreement with the experimental results, with the exception of the strain at the supported end.

3. There is better agreement between the experimental and theoretical results obtained for the 1/2-in. specimens than for the 1-in. specimens.

4. The rise time of the particle velocity on the impacted end has no effect on the final stress and strain rate, but it does affect the value of the strain; the shorter rise times will give a greater strain after any given time interval. The interval of time before the bar reaches this final stress and strain rate will be longer in the modified strain-rate theory than in the elementary strain-rate theory.

5. In the modified strain-rate theory, the application of a shear stress in the radial direction reduces the magnitudes of the oscillations in the other variables and gives better agreement with the experimental results. Shear stresses should be included on both ends of the specimen for the best approximation.

6. Comparisons of the predictions of the strain-time history by the modified strain-rate theory with the experimental results leads to the conclusion that the linear plastic strain-rate law is inadequate. A law that allows a greater decrease in excess stress after a given interval of time could allow a high

enough strain rate in the early time history to agree with the experimental measurements during this period. The exponential strain-rate assumption could be one such law.

Recommendations for Further Study

1. The experimental and theoretical discrepancies in the strain-time records at the supported end should be considered in an additional study.

2. Experimental records at points other than the two ends should be obtained to determine the agreement between theoretical and experimental results at points other than the ends.

3. A possibility for formulating a more accurate theory would be to allow the cross sections originally plane and perpendicular to the neutral axis to take on a parabolic shape. This could be obtained by adding another function to the assumed expression for the displacement in the x-direction, i. e., let

$$U(\bar{x}, \bar{t}) = U_0(\bar{x}, \bar{t}) + \frac{l\bar{r}^2}{2\bar{a}^2} U_2(\bar{x}, \bar{t})$$

This would allow the bar to have an additional degree of freedom. This procedure could also be applied to displacements in the radial direction. However, it probably would not be as significant a correction as that applied to the axial displacement.

4. It is also known that the actual plastic flow rates depend on the existing stress state. It would be very desirable to reformulate the problem using a more realistic material description, i. e.,

allowing the material to flow under a state of combined stresses.

5. Studies of the applicability of this theory to the propagation of a plastic disturbance in a semi-infinite rod should be made. A better understanding of the phenomenon could be obtained, since there would be no reflections to influence the results.

6. The influence of the boundaries could best be obtained by including in the problem the elastic disturbance set up in the steel bar and in the impacting cylinder. The only restriction imposed would be that the boundary conditions would have to be compatible at the points where the two steel mediums touched the copper specimen.

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