MASTER

UR-553

AEC RESEARCH AND
DEVELOPMENT REPORT

A THEORETICAL AND EXPERIMENTAL INVESTIGATION OF THE TEMPERATURE
RESPONSE OF PIG SKIN EXPOSED TO THERMAL RADIATION

by

Thomas P. Davis

THE UNIVERSITY OF ROCHESTER
ATOMIC ENERGY PROJECT
ROCHESTER, NEW YORK

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

U N C L A S S I F I E D

UR-553

Biology and Medicine
TID-4500, (15th Ed.)

THE UNIVERSITY OF ROCHESTER
Atomic Energy Project
P. O. Box 287, Station 3
Rochester 20, New York

* * *

Contract W-7401-eng-49 between the U. S. Atomic
Energy Commission and the University of Rochester,
administered by the Department of Radiation Biology
of the School of Medicine and Dentistry.

A THEORETICAL AND EXPERIMENTAL INVESTIGATION OF THE TEMPERATURE

RESPONSE OF PIG SKIN EXPOSED TO THERMAL RADIATION

by

Thomas P. Davis

Division: Special Programs

Division Head: H. A. Blair

Section: Flash Burns

Section Head: J. R. Hinshaw

Date Completed: 6/10/59

Date of Issue: 7/8/59

U N C L A S S I F I E D

A THEORETICAL AND EXPERIMENTAL INVESTIGATION OF THE TEMPERATURE
RESPONSE OF PIG SKIN EXPOSED TO THERMAL RADIATION

ABSTRACT

This study was undertaken as a part of the problem of predicting the irreversible thermal injury of skin. Of first importance in this problem is the determination of the temperature-time-depth history of skin subjected to an arbitrary thermal insult. Although this paper deals with radiant energy heating of skin, it is intended that the results may be applied to other modes of energy input, as well.

The theoretical analysis of heat flow in irradiated skin is considered first, and the predicted responses of two models are presented. Methods are developed for the comparison of these theoretical predictions with experimental results; these comparison schemes define the experimental procedures and materials which are next described in detail. Finally, the experimental results are presented, together with an evaluation of the constants of skin, whenever possible.

While the experimental results failed to supply all the information necessary to compute the temperature response of skin to any arbitrary energy input, several important conclusions were reached. A most interesting result of the theoretical analysis is that the absorption pattern of radiation in skin may be determined directly by temperature measurements, and the data may be tested rigorously for their validity. Also, it is now possible to plan future experiments which will provide the necessary data which this study failed to supply. Most important is the indication that the desired prediction of thermal injury of skin may be achieved.

TABLE OF CONTENTS

	Page
Abstract	ii
List of Figures and Tables	viii
CHAPTER I. Introduction	
1.1 Genesis of the Problem	1
1.2 Review of Previous Work	3
1.3 The Method of Approach	5
1.4 Original Contributions	6
1.5 General Conclusions	7
Footnotes	8
CHAPTER II. General Aspects of the Theoretical Analysis	
2.1 The Method of Analysis	9
2.2 Derivation of the Heat Conduction Equation in a Substance at Rest	9
2.3 Further Consideration of the Heat Conduction Equation	12
2.4 The Model to be Analyzed--General Assumptions	16
2.5 Summary	19
Footnotes	20
CHAPTER III. Analysis of the Opaque Solid	
3.1 Definition of the Model	21
3.2 The Solution of the Heat Conduction Equation for the Opaque Solid	21
3.3 Specialization of the Solution for Rectangular Irradiance Function	24
3.4 The Fitting of Experimental Results to the Normalized Solution	26
3.5 The Opaque Composite Solid	32
3.6 The Opaque Solid Subjected to a Trapezoidal Irradiance Pulse	38

3.7	Summary of Predictions from Opaque Solid Theory	44
3.8	Utility of the Opaque Solid Model	49
	Footnotes	50
CHAPTER IV. Analysis of the Diathermanous Solid		
4.1	Definition of the Model	51
4.2	General Consideration of the Absorption of Radiation in Skin	52
4.3	Experimental Determinations of the Absorption of Radiation in Skin	57
4.4	The Solution of the Heat Conduction Equation for the Diathermanous Solid with Double Exponential Absorption	63
4.5	Experimental Evaluation of the Constants of the Diathermanous Solid	78
4.6	Direct Experimental Determination of the Pattern of Absorption of Radiation in Skin	80
4.7	Summary	85
	Footnotes	85
CHAPTER V. The Influence of the Quality of Radiation on the Diathermanous Solid Response		
5.1	Development of the Problem	87
5.2	Superposition of Solutions for the Diathermanous Solid	87
5.3	Superposition of the Normalized Solutions for Double Exponential Absorption	90
5.4	Values of the Constants of Skin Obtained from the Literature	91
5.5	The Influence of the Quality of Radiation on the Initial Time Rate of Change of Temperature	101
5.6	Direct Determination of the Absorption Pattern of Heterochromatic Radiation in Skin	104
5.7	Correction of Initial Slopes for Finite Irradiance Rise Time	106
5.8	Summary	107
	Footnotes	108

CHAPTER VI. Experimental Materials and Methods

6.1	Introduction	110
6.2	Biological	110
6.3	Physical--General Arrangement	112
6.4	Radiation Source	112
6.4.1	Shutter System	115
6.5	Animal Holder	117
6.6	Temperature Sensing Elements	118
6.6.1	Wollaston Wire Thermocouples	122
6.6.2	Butt-welded Thermocouples	123
6.6.3	Plated Thermocouples	124
6.6.4	Soldered Thermocouples	125
6.6.5	Calibration of Thermocouples	126
6.6.6	Placement of Thermocouples	127
6.6.7	Electrical Connections to the Thermocouples	128
6.7	Depth Measuring Device	129
6.8	Reference Elements	132
6.8.1	Temperature Regulated Water Bath	133
6.9	Calibration Circuits	133
6.10	Preamplifiers	134
6.11	Recording Assembly and System Response	137
6.12	Pulse Measuring Assembly	142
6.13	Timing Marker Generator	143
6.14	Exposure Procedure	145
6.15	Reduction of Data	146
6.16	Summary	147
	Footnotes	148

CHAPTER VII. Experimental Results; Opaque Skin

7.1	Scope of the Experimental Work	150
7.2	Animal No. 1389	150
7.3	Animal No. 1418	151
7.4	Animal No. 1428	156
7.5	Animals No. 1445 and 1455	159
7.6	Summary of Results; Average Thermal Constants of Pig Skin	163
7.7	Discussion of Results	165
7.8	Summary	169
	Footnotes	169

CHAPTER VIII. Experimental Results; Bare Skin

8.1	Scope of the Experimental Work	171
8.2	Reduction of Data	171
8.3	Summary of Temperature Response Measurements	173
8.4	Summary of Initial Slope Measurements	180
8.5	Comparison of Experimental Results with Theoretical Predictions	180
8.6	The Source of Error in the Temperature Measurements	185
8.6.1	Variation of Heat Capacity per Unit Volume with Depth	186
8.6.2	Location of Zero Time and Extrapolation of Responses to Zero Time	188
8.6.3	Direct Action of Radiation on the Thermocouples	189
8.7	Further Consideration of the Initial Slope Measurements	190
8.8	Summary	194
8.9	Location of Data	196
	Footnotes	196

CHAPTER IX. Summary of the Study	
9.1 Discussion of the Theoretical Analysis	197
9.2 Discussion of the Experimental Results	203
9.3 Consideration of the Direct Measurement of the Absorption Pattern of Radiation in Skin	206
9.4 Conclusion	208
Footnotes	209
Bibliography	210
APPENDIX I. Nomenclature	214
APPENDIX II. The Application of Heat Flow Theory and Chemical Kinetics to the Prediction of Irreversible Thermal Injury of Skin	216
APPENDIX III. The "Opaque Solid Function," $R(x)$	229
APPENDIX IV. Solution of the Composite Opaque Solid Model	231
APPENDIX V. Solution of the Diathermanous Solid Model, Double Exponential Absorption	240
ACKNOWLEDGEMENTS	260

LIST OF FIGURES AND TABLES

	Page
Figure 3-1. Predicted Response of the Opaque Solid	27
Figure 3-2. Predicted Response of the Opaque Solid	28
Figure 3-3. Factor for Correcting Trapezoidal Pulse Response to Ideal Step-Function Response, for Opaque Solid Surface	45
Figure 4-1. Predicted Response of the Diathermanous Solid, Single Exponential Absorption. $\lambda_2 = 1$	71
Figure 4-2a. Predicted Response of the Diathermanous Solid, Double Exponential Absorption. $\lambda_2 = 3, \lambda_1 = 0.1$	72
Figure 4-2b. Predicted Response of the Diathermanous Solid, Double Exponential Absorption. $\lambda_2 = 3, \lambda_1 = 0.5$	73
Figure 4-2c. Predicted Response of the Diathermanous Solid, Double Exponential Absorption. $\lambda_2 = 3, \lambda_1 = 1.0$	74
Figure 4-3a. Predicted Response of the Diathermanous Solid, Double Exponential Absorption. $\lambda_2 = 5, \lambda_1 = 0.1$	75
Figure 4-3b. Predicted Response of the Diathermanous Solid, Double Exponential Absorption. $\lambda_2 = 5, \lambda_1 = 0.5$	76
Figure 4-3c. Predicted Response of the Diathermanous Solid, Double Exponential Absorption. $\lambda_2 = 5, \lambda_1 = 1.0$	77
Figure 5-1. Relative Spectral Energy Absorption, $(1-R_\lambda) \cdot J_\lambda$. Chester White Pig Exposed to Carbon Arc Image Furnace	95
Figure 5-2. Linear Absorption Coefficients for Shallow (λ_1) and Deep (λ_2) Tissue	97
Table 5-1. Numerical Values for Solution of the Normalized Response Equation	99
Figure 5-3. Predicted Temperature Response of Bare Pig Skin Exposed to Carbon Arc Image Furnace	100
Figure 5-4. Predicted Initial Time Rate of Change of Temperature Response of Bare Pig Skin Exposed to Carbon Arc Image Furnace	103
Figure 6-1. Block Diagram of the Physical Equipment	113
Figure 6-2. The Carbon Arc Image Furnace	114

Figure 6-3.	Animal Holder, Open	119
Figure 6-4.	Animal Holder, Closed	120
Figure 6-5.	Animal Holder, in Exposure Position	121
Figure 6-6.	Amplifier-Recorder System	135
Figure 7-1.	Measured Response, Animal No. 1418 (Opaque Skin)	154
Figure 7-2.	Measured Response, Animal No. 1418 (Opaque Skin)	155
Figure 7-3.	Measured Response, Animal No. 1428 (Opaque Skin)	158
Figure 7-4.	Measured Response, Animals No. 1445 and 1455 (Opaque Skin)	160
Figure 7-5.	Measured Response, Animals No. 1445 and 1455 (Opaque Skin)	162
Figure 7-6.	Normalized Measured Response, Animals No. 1445 and 1455 (Opaque Skin)	164
Table 8-1.	Scope of the Experimental Work on Bare Skin	172
Table 8-2.	Summary of Bare Skin Temperature Responses	174
Figure 8-1a.	Measured Response, Bare Skin Exposures	176
Figure 8-1b.	Measured Response, Bare Skin Exposures	177
Figure 8-1c.	Measured Response, Bare Skin Exposures	178
Figure 8-2.	Average Measured Response, Bare Skin Exposures	179
Figure 8-3.	Measured Initial Time Rate of Change of Tempera- ture Response, Bare Skin Exposures	181
Figure 8-4.	Comparison of Predicted and Measured Bare Skin Response	183

A THEORETICAL AND EXPERIMENTAL INVESTIGATION OF THE TEMPERATURE
RESPONSE OF PIG SKIN EXPOSED TO THERMAL RADIATION*

CHAPTER I. INTRODUCTION

1.1 Genesis of the Problem

The expanding utilization of nuclear energy has posed many problems, not the least of which are in the field of medicine. In particular the existence of nuclear weapons as possible instruments of war demands planning on an unprecedented scale for treatment of mass casualties; yet these casualties may differ not only in number but in kind from anything known heretofore. There has been, therefore, considerable acceleration in research in many branches of the biological sciences aimed at defining the basic traumata involved, and thus providing a reasonable basis for the necessary planning.

In this laboratory the flash burn has been under investigation for over ten years (1, 2). This is the lesion resulting from the exposure of an individual to a brief pulse of visual and infrared radiation as released by an atomic detonation. The importance of this problem is indicated by the estimate of Leroy (3) that of the atomic bomb casualties in Japan, 65 to 85 percent were burned; over 90 percent of the lesions were of the primary, or flash-burn, type. Modern high-yield weapons have intensified this problem. Estimates from The Effects of Nuclear Weapons (4) indicate that 1° burns on bare skin will be sustained 30 miles from a 10 megaton blast; this radius enclosed an area of almost 3000 square miles. These estimates have been partly derived from, and are completely supported by work in this laboratory (5). When it is recalled that the hospitals of Boston and the surrounding area were severely taxed by the relative handful of casualties

*The material in this paper was submitted to the University of Rochester in partial fulfillment of the requirements for the Doctor of Philosophy degree in Biophysics.

from the Cocoanut Grove fire (6), the magnitude of the problem may be appreciated.

While the reason for existence of this laboratory is thus perfectly clear, it is also obvious that mere routine testing under specified conditions can never supply the information needed in the planning for treatment of mass casualties. New and untested situations continually arise for which no data are available. Further it would be highly desirable to inter-compare results obtained with various modes of energy input; this would make results from the older literature useful, and would aid in integrating the findings of various laboratories. Therefore, this laboratory has attempted to focus major attention on the more basic problems of the flash burn, in the hope of developing principles rather than specific results.

With the continual development and refining of experimental materials and methods it has become increasingly clear that highly precise tools are now available for the study, not merely of the atomic bomb burn, nor even of the more general radiant energy burn, but of the burn, per se. That is, it has now become feasible and most desirable to attempt to elucidate the fundamental mechanisms of irreversible tissue damage produced by a hyper-thermic episode. This problem may be termed "the prediction of irreversible thermal injury of skin"; the desideratum is the ability to define "damage" mathematically in terms of the characteristics of the thermal insult.

This over-all problem has been treated by the writer in a paper prepared several years ago. This report (from which some of the foregoing material has been taken) is included as Appendix II of this paper; while many of the ideas have since been discarded, the general approach therein described is still valid. In brief, the prediction problem is specialized to exclude photochemical reactions; it then follows that "damage" is a function of the temperature of the tissue involved, while the temperature,

in turn, is dependent upon the characteristics of the energy input. In the terminology of Appendix II:

$$\mathcal{D} = \mathcal{D}(\tau)$$

where $\tau = \tau(q'', x, t, \eta, \dots)$.

(All symbols are defined in Appendix I.)

Thus, the prediction of thermal injury of skin is now broken down into two sub-problems: first the prediction of the temperature response of skin for an arbitrary energy input, and second the prediction of irreversible tissue injury based on this temperature response. It is with the first problem, the determination of the temperature response of skin that this study is concerned.

It has been the purpose of the above remarks to define the rationale of this study. It is, clearly, but a small part in the total problem of burn prediction; however extension of the results to the latter will not be attempted here, but must await further research.

1.2 Review of Previous Work

An extensive historical review of the literature will not be given here; rather, citations of pertinent reports will be made as appropriate throughout this paper. However, two particularly interesting studies will be mentioned, since they have served as guides for much of the research here presented.

Apparently, the first worker in this field to attempt a mathematical formulation of thermal injury was F. C. Henriques, Jr., working with A. R. Moritz and others. His work is described briefly in Appendix II, and in full in a series of papers, two of which (7, 8) are of particular interest here. The procedure was as outlined above: first the temperature response of skin (to long-time application of hot water) was investigated theoretically and

experimentally (7). These data were then used in predicting certain burn thresholds by means of a "punishment integral," which was based on reaction kinetics considerations (8). The agreement between theory and experiment is extraordinarily good; too good, one might suspect, to be completely fortuitous. However, application of Henriques' punishment integral to radiant energy burns leads to predictions (9, 10), which are seriously at variance with recent data from this laboratory. It is possible that the difficulty lies in the method used in computing tissue temperatures (see Appendix II).

The second study is that of Buettner, who utilized Henriques' punishment integral in a slightly altered form, and estimated skin temperatures by a numerical analysis (11). This very thorough analysis culminates in a table giving the predicted depth of thermal injury for various radiant energy exposures (12). The postulated experimental conditions were such that they could be fulfilled quite precisely by the equipment of this laboratory. An experiment was therefore carried out to test these predictions (13); in all cases, the actual lesions were considerably less severe than predicted. Here again, one may question the accuracy of the temperature predictions.

These two studies would appear to be the only ones in the literature which attempt to predict injury from radiant energy exposures. Since neither has been fully in agreement with existent experimental data, it is clear that further work is necessary. As indicated in the above section, the proper starting point is the determination of the correct temperature response of skin. Indeed, it may be that when this information is obtained, the discrepancies in Henriques' or Buettner's work may disappear, and the ultimate goal of the prediction of irreversible thermal injury will have been achieved.

1.3 The Method of Approach

The problem of predicting the temperature response of skin is completely solved when one has specified the correct differential equation describing the heat flow in skin, and determined the numerical values of the constants of the system. If the equation is completely general, the predictions will hold for any general conditions; if the equation is restricted to certain cases, the predictions are likewise restricted.

General solutions of the classical heat flow equation are not obtainable, as pointed out in the next chapter; it is necessary to specialize this equation if solutions are to be obtained. The method adopted here, then, is the so-called model approach. First a simplified model of skin and the tissue-energy interaction is proposed which will allow the general heat flow equation to be so specialized as to be amenable to formal mathematical attack. Analytical solutions of this specialized equation are next obtained, which solutions are then normalized, or expressed in dimensionless variables, so as to be applicable to any system satisfying the basic requirements of the model. The normalizing factors will, in general, involve the unknown constants of the system.

These normalized solutions (which may be presented in numerical form) must now be examined for possible methods of comparison with experimental data. When such comparison schemes are developed, they will define the appropriate experimental procedures to be followed. When these experiments have been completed and the necessary data obtained, the comparisons are made, and if the agreement be satisfactory, then the normalizing factors--i.e., the unknown constants of the system--may be evaluated, and the normalized solutions specialized to the particular system under investigation; namely, skin. If the agreement be unsatisfactory, then one must either alter the original model or the experimental methods, whichever is at fault.

In a later chapter it will be shown that a most interesting result of the theoretical analysis permits one to interpret the experimental data independently of any choice of a model; indeed the data themselves can define the model. This is one of the most important results of this study, and will be treated in some detail.

1.4 Original Contributions

The derivation of the heat conduction equation presented in the next chapter is obviously not original, and has been given only for comparison with the more detailed derivation of Section 2.3, which, so far as the author is aware, has not been given before, and is of interest primarily for its consideration of the enthalpy function. Also, as indicated by the references cited, the solution presented in Chapter III for the opaque solid model is not original, nor is the method selected for evaluation of the thermal inertia. However, the development of a method for the evaluation of thermal diffusivity is an original contribution, as is the treatment of the surface response of the composite opaque solid, together with the correction for finite rise time of the irradiance pulse.

The diathermanous solid model with double exponential absorption, as defined in Chapter IV, was first proposed by Hardy, but the analytical solution for this model has apparently not been given before. Likewise, it is believed that the extensive treatment of the initial time rate of change of temperature response of the diathermanous solid, and the utilization of this factor in the analysis of experimental results is original with this study, and is a contribution of some value. The extension of the solution to the case of heterochromatic radiation depends directly upon superposition of solutions, which is so obvious that it would be embarrassing to claim originality for this development, although it seems not to have been treated

explicitly in the literature. In any event, the resultant curves of Figures 5-3 and 5-4 are original with this study, and, it is hoped, represent the best predictions to date of the response of bare pig (and human?) skin to solar-like radiation.

While the experimental materials and methods used in this study were quite straightforward, the simultaneous recording of the irradiance pulse and the temperature response is an interesting contribution of some merit. An extensive discussion of the experimental results will be found in later chapters, but it might be mentioned here that the clear demonstration of errors in the measurements, and the lengthy investigation of the source of these errors is a unique feature of this study.

1.5 General Conclusions

For the measurement of the so-called "thermal constants" of skin, the theoretical predictions and experimental results were in good agreement, which permitted these constants to be evaluated with considerable confidence. However, for the determination of the "optical" constants, or more specifically, the absorption pattern of radiation in the skin, the experimental measurements were seriously in error. Thus, the completely general prediction of the temperature response of skin to an arbitrary thermal insult has not been achieved by this study; on the other hand, the results are of great value in preparing specific plans for the work which remains to be done.

Aside from the demonstration that the combination of the theoretical and experimental approaches is not only reasonable but necessary, a most important contribution of this study is the indication that the future research is practicable, and the goal of the prediction of irreversible thermal injury of skin may be achieved.

FOOTNOTES

- (1) Pearse, H. E., J. T. Payne, and L. Hogg, The Experimental Study of Flash-Burns, *Ann. of Surg.* 130, 774-789 (1949).
- (2) Pearse, H. E. and H. D. Kingsley, Thermal Burns from the Atomic Bomb, *Surg. Gyn. and Obst.* 98, 385-394 (1954).
- (3) Leroy, G. V., Medical Sequelae of the Atomic Bomb Explosion, *J. Am. Med. Assoc.* 134, 1143 (1947).
- (4) Glasstone, S., ed., The Effects of Nuclear Weapons, 299, Fig. 7.47 (U. S. Government Printing Office, Washington, D.C., 1957).
- (5) Lerman, B. and J. R. Hinshaw, Relationships Between Burn Severity and the Simulated Thermal Pulses of Various Nuclear Weapons, University of Rochester Atomic Energy Project Report UR-533 (1958).
- (6) Pearse, H. E., Thermal Injuries of Nuclear Warfare, *Military Medicine* 118, 274 (1956).
- (7) Henriques, F. C., Jr. and A. R. Moritz, Studies of Thermal Injury I. The Conduction of Heat to and Through Skin and the Temperatures Attained Therein. A Theoretical and Experimental Investigation, *Am. J. Path.* 23, 531-549 (1947).
- (8) Henriques, F. C., Jr., Studies of Thermal Injury V. The Predictability and the Significance of Thermally Induced Rate Processes Leading to Irreversible Epidermal Injury, *Arch. Path.* 43, 489-502 (1947).
- (9) Henriques, F. C., Jr. and R. A. Maxwell, The Applicability of the Skin Damage Integral to the Prediction of Flash Burn Injury Thresholds, Including those Caused by Atomic Detonations, Technical Operations, Inc., Report TOI 54-19 (1956). (SECRET).
- (10) The Thermal Data Handbook, Armed Forces Special Weapons Publication AFSWP 700 (1954). (SECRET).
- (11) Buettner, K., Effects of Extreme Heat and Cold on Human Skin III. Numerical Analysis and Pilot Experiments on Penetrating Flash Radiation Effects, *J. Appl. Physiol.* 5, 207-220 (1952).
- (12) Ibid. 216.
- (13) Davis, T. P., Unpublished research (1956).

CHAPTER II

GENERAL ASPECTS OF THE THEORETICAL ANALYSIS

2.1 The Method of Analysis

The general procedure in the theoretical analysis of the conduction of heat in skin is determined by the choice of a model approach as described previously. Of first importance in this approach is the obtaining of a formal solution, in dimensionless terms, of the system. Obviously, then, one cannot employ numerical methods; indeed, the constants necessary for such methods are precisely the unknowns of the problem. The only suitable procedure is a formal, analytical attack on the classical differential equation of heat conduction.

2.2 Derivation of the Heat Conduction Equation in a Substance at Rest

The derivation of this equation is based upon the experimentally deduced equation of Fourier (1) which states that the rate of heat conduction, q , between two plates separated by some specific material is directly proportional to the area of the path connecting the plates, A , and their temperature difference, ΔT ; and inversely proportional to the separation of the plates, Δx . The constant of proportionality is termed the heat conductivity, k , and may be considered a basic property of the material separating the plates. Explicitly:

$$q = k A \frac{\Delta T}{\Delta x} \quad (2-1)$$

This equation may be written in differential form by allowing Δx to decrease to the infinitesimal length dx , and considering the process only during the infinitesimal time interval dt . The differential heat conducted in the x direction in this time is then

$$dQ = -k A \frac{dT}{dx} \cdot dt \quad (2-2)$$

where the term dT/dx is called the temperature gradient, and the minus sign is introduced to maintain the positive sense of heat flow in the positive x direction. (Heat flows from a region of higher temperature to one of lower temperature; i.e., in the direction of a negative temperature gradient.)

Now, following the standard procedure employed in the derivation of the equation of heat conduction (2), one selects as a system a differential volume element $dV = dx dy dz$ in the interior of a body through which heat is flowing, and considers the components of heat flow in the directions x , y , and z . While the presence of distributed heat sources or sinks within the body is permitted, no movement of parts of the body, as flow of fluid, for example, is allowed. Temperature is considered a function of space coordinates and time; i.e., $T = T(x, y, z, t)$.

In the time interval dt , the heat entering this system from the positive x direction, $dQ_{1,x}$, will be, from equation (2-2):

$$dQ_{1,x} = -k_x(dy \cdot dz) \frac{\partial T(x, y, z, t)}{\partial x} dt,$$

where k_x is the thermal conductivity in the x direction at the point x .

Now, the value of k_x will, in general, depend upon x and T , independently, or $k_x = k_x(x, T)$. But the latter is a function of the former, so that one may consider k_x as a function of x alone, that is:

$$dk_x = \frac{\partial k_x}{\partial x} \cdot dx$$

not

$$dk_x = \frac{\partial k_x}{\partial x} dx + \frac{\partial k_x}{\partial T} dT.$$

The heat flow out of the differential volume in the positive x-direction, $dQ_{2,x}$, at $x + dx$, is given by

$$dQ_{2,x} = -\left(k_x + \frac{\partial k_x}{\partial x} dx\right)(dy \cdot dz) \left[\frac{\partial}{\partial x} [T(x, y, z, t)] + \frac{\partial T(x, y, z, t)}{\partial x} dx \right].$$

Hence, the net heat flow into the system from the x direction is

$$\begin{aligned} dQ_{1,x} - dQ_{2,x} &= -k_x (dy \cdot dz) \frac{\partial T(x, y, z, t)}{\partial x} dt \\ &\quad + k_x (dy \cdot dz) \frac{\partial T(x, y, z, t)}{\partial x} dt \\ &\quad + k_x (dx \cdot dy \cdot dz) \frac{\partial^2 T(x, y, z, t)}{\partial x^2} dt \\ &\quad + \frac{\partial k_x}{\partial x} (dx \cdot dy \cdot dz) \frac{\partial T(x, y, z, t)}{\partial x} dt \\ &\quad + \frac{\partial k_x}{\partial x} (dx \cdot dy \cdot dz) \frac{\partial^2 T(x, y, z, t)}{\partial x^2} dx \cdot dt \\ &= (dx \cdot dy \cdot dz) \frac{\partial}{\partial x} \left[k_x \frac{\partial T(x, y, z, t)}{\partial x} \right] dt \\ &\quad + (dx \cdot dy \cdot dz) \frac{\partial k_x}{\partial x} \frac{\partial^2 T(x, y, z, t)}{\partial x^2} dx \cdot dt. \end{aligned}$$

Exactly similar expressions may be written for the component of heat flow in the y and z directions, giving the net heat flow into the system:

$$\begin{aligned} dQ_1 - dQ_2 &= (dx \cdot dy \cdot dz \cdot dt) \left\{ \left[\frac{\partial}{\partial x} \left(k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial T}{\partial z} \right) \right] \right. \\ &\quad \left. + \frac{\partial k_x}{\partial x} \frac{\partial^2 T}{\partial x^2} dx + \frac{\partial k_y}{\partial y} \frac{\partial^2 T}{\partial y^2} dy + \frac{\partial k_z}{\partial z} \frac{\partial^2 T}{\partial z^2} dz \right\} \quad (2-3) \end{aligned}$$

Continuing with the usual procedure, a heat storage factor dQ_3 is defined as the product of the total system heat capacity at constant pressure, C_p , and the average temperature rise, dT :

$$dQ_3 = C_p dT.$$

Now, $C_p = mc_p$, where m is the mass of the system and c_p the specific heat capacity, while $m = \rho dV$, where ρ is the density of the material. Further

$dT = \frac{\partial T}{\partial t} dt$, hence

$$dQ_3 = \rho c_p (dx \cdot dy \cdot dz) \frac{\partial T}{\partial t} dt. \quad (2-4)$$

Finally, the influence of distributed heat sources or sinks is considered by defining the heat production in the system, dQ_4 , as

$$dQ_4 = q''' \cdot (dx \, dy \, dz) \cdot dt \quad (2-5)$$

where q''' is the rate of heat production per unit volume. (The problem of a point source, which in essence implies infinite temperature, will not be considered here.)

Now, taking a "heat balance," one states that heat storage dQ_3 (equation 2-4), must be equal to the net heat input $dQ_1 - dQ_2$ (equation 2-3) plus the internal heat generation dQ_4 (equation 2-5), or

$$\begin{aligned} \rho c_p (dx \, dy \, dz \, dt) \frac{\partial T}{\partial t} = (dx \cdot dy \cdot dz \cdot dt) \left\{ \frac{\partial}{\partial x} (k_x \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y} (k_y \frac{\partial T}{\partial y}) \right. \\ \left. + \frac{\partial}{\partial z} (k_z \frac{\partial T}{\partial z}) + \frac{\partial k_x}{\partial x} \frac{\partial^2 T}{\partial x^2} dx + \frac{\partial k_y}{\partial y} \frac{\partial^2 T}{\partial y^2} dy \right. \\ \left. + \frac{\partial k_z}{\partial z} \frac{\partial^2 T}{\partial z^2} dz + q''' \right\}. \end{aligned}$$

If the "space-time" volume element $dV = dx \, dy \, dz \, dt$ is cancelled on each side of the above equation, then in the limit, as dx , dy and dz all approach zero, one obtains the classic heat conduction equation:

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} (k_x \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y} (k_y \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z} (k_z \frac{\partial T}{\partial z}) + q'''. \quad (2-6)$$

2.3 Further Consideration of the Heat Conduction Equation

With the basic equation of heat conduction now established (equation 2-6) one may proceed to the specification of suitable geometrical configurations and boundary conditions which will allow analytical solutions of the equation to be obtained. Before doing so, however, it is of interest to consider the derivation in some detail.

At the outset, it should be noted that Fourier's papers go back to at least 1807 (although publication was delayed until 1822), while the crucial

experiments of Joule were not performed until 1840. Thus, Fourier's work antedated by some 30 years the firm establishment of the mechanical theory of heat and the exact expression of the First Law of Thermodynamics.

It should not be too surprising, therefore, to note that all statements in the previous section are quite correct if one substitutes everywhere the word "caloric" for "heat," and if one is willing to renounce the First Law in favor of the old theory, assuming that caloric is a massless fluid. (It will be shown later that it would be convenient to endow caloric with inertia, however.) Actually, this should not be too distressing, since the science of heat transfer is quite distinct from thermodynamics, and it is a real simplification for workers in the former field to consider the thermodynamicist's "heat" as a fluid, strictly analogous to the "electric fluid" popular with early workers in electricity.

Obviously, it is not impossible, nor particularly difficult, to phrase the derivation of equation (2-6) in agreement with proper thermodynamic concepts. The first change necessary is to note that Q is not a function of thermodynamic coordinates, but depends upon the path traversed by a particular system; hence an infinitesimal amount of heat is not an exact differential, dQ , but an inexact differential, here represented by δQ . With this rather trivial change in nomenclature, equation (2-3) follows precisely as derived above, with the left-hand side reading $\delta Q_1 - \delta Q_2$.

The next factor considered, the "heat storage," requires somewhat more extensive consideration. Quoting from Zemansky (3) "Heat is energy in transit....When the flow of heat has ceased, there is no longer any occasion to use the word heat." The term "heat storage," then, is simply devoid of meaning.

Now the system being considered is of differential volume $dx dy dz$, and processes occur in a differential time interval dt . Recognizing that these differentials may independently be made as small as desired, one may be assured that the system will always be infinitesimally near equilibrium states (i.e., the system undergoes infinitesimal quasistatic processes); hence thermodynamic coordinates may be written for the system as a whole. It is then permissible to speak of the enthalpy, H , of the system. The enthalpy, in turn, may be considered to be a function of any two of the three thermodynamic coordinates of this closed system, temperature, T , pressure, P , and volume, V . Selecting the first two:

$$H = H(T, P).$$

Hence

$$dH = \left(\frac{\partial H}{\partial T}\right)_P dT + \left(\frac{\partial H}{\partial P}\right)_T dP,$$

and for an isobaric process ($dP = 0$)

$$dH = \left(\frac{\partial H}{\partial T}\right)_P dT,$$

or

$$dH = C_P dT, \quad (2-7)$$

from the definition of C_P .

Now, the change in temperature, dT , in equation (2-7) is a change in time only, since spatial dimensions are not thermodynamic coordinates.

(The system is always infinitesimally near equilibrium states.) Hence, equation (2-7) may be rewritten as

$$dH = C_P \frac{\partial T}{\partial t} dt$$

or, as in equation (2-4):

$$dH = \rho C_P (dx dy dz) \frac{\partial T}{\partial t} dt. \quad (2-8)$$

Finally, the "distributed heat source" factor (equation 2-5) must be considered. The terminology again is poor, for what is actually being described is the performance of work, δW , on the system. This is clear,

since this process can change the state of the system by other than a heat flow process. This is not ordinary "P dV" work, but at the molecular level, represents increased molecular kinetic energy due to dissipative processes. Hence, one may write

$$-\delta W = q''' \cdot (dx \, dy \, dz \, dt) \quad (2-9)$$

in direct analogy with equation (2-5). The minus sign is used to indicate that work is performed on the system.

The heat flow equation is now obtained by substitution in the differential expression of the First Law:

$$dE = \delta Q - \delta W',$$

where the $\delta W'$ term includes all work elements including P dV, and E is the internal energy of the system. Letting $\delta W' = P \, dV + \delta W$, where δW represents the "non-P dV" work elements, one obtains

$$dE = \delta Q - P \, dV - \delta W.$$

From the definition of the enthalpy function

$$H = E + P V$$

one may write

$$dE = dH - P \, dV - V \, dP,$$

which upon substitution in the expression for the First Law gives

$$dH - P \, dV - V \, dP = \delta Q - P \, dV - \delta W$$

or

$$dH = \delta Q - \delta W + V \, dP.$$

Now, for an isobaric process the V dP term is zero, and substitution of equation (2-8) for dH, equation (2-3) for $\delta Q = \delta Q_1 - \delta Q_2$, and equation (2-9) for δW , yields, in the limit:

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial T}{\partial z} \right) + q''' ,$$

which is in exact agreement with equation (2-6).

Thus, the heat conduction equation may be derived without recourse to such barbarous terms as "heat storage" or "heat generation," nor to such a questionable concept as a "heat balance" which implies some sort of conservation of "heat fluid." This is admittedly pedantic, but it is of real interest to note that a careful derivation of this equation has introduced no new limitations on its validity beyond the stated requirement that no movement of parts of the body is allowed, and the restriction to a constant pressure process implied in writing equation (2-4). (For a constant volume process, one need only replace c_p by c_v . This follows from the definition of $C_v = (\frac{\partial E}{\partial T})_v$, whence $dE = C_v dT$ for $dV = 0$.) Also noteworthy is the point, brought out in this derivation, that equation (2-6) is valid in the neighborhood of a discontinuity, but not "on" the discontinuity, since here one cannot define the thermodynamic coordinates of a differential volume element.

2.4 The Model to be Analyzed -- General Assumptions

A general solution of the heat conduction equation does not exist. In order to proceed, one must set up a model within the framework of which the equation can be so specialized as to be amenable to formal or numerical methods of attack. This model must specify the geometrical configuration of the system, and a sufficient number of boundary conditions (including initial conditions).

The ability of equation (2-6) to predict temperatures in agreement with those which actually occur in a real, physical system is completely dependent upon how faithfully the model describes the true situation. In even rather ideal conditions, the solution of this equation is not simple, particularly in the non-steady state (i.e. $\partial T / \partial t \neq 0$). Here, however,

it is desired to predict the temperature rise in skin exposed to brief pulses of radiant energy. If one were to require that the model reflect with high precision the detailed anatomy of the skin and the peculiarly difficult processes of radiant energy absorption therein, an overwhelming complex system would result. Clearly, for any progress, the model must embody rather severely simplifying assumptions.

First, the skin will be replaced by a so-called semi-infinite solid. This is a solid which extends infinitely far in both positive and negative y and z directions, and in the positive x direction. (As in the above sections, x, y, and z are mutually perpendicular Cartesian coordinates.) By definition, the surface of the solid coincides with the y-z plane, or the plane $x = 0$; this choice has been made simply as a matter of convenience. Next, it will be assumed that the thermal conductivity in the x-direction, k_x , the density, ρ , and the specific heat capacity, c_p , are all independent of y and z. Initially the solid is to be at uniform temperature throughout, and at zero time the surface is exposed uniformly to an arbitrary pulse of radiant energy. The absorption pattern of this radiation, whatever it may be, is likewise assumed independent of y and z.

With these assumptions only, considerable simplification of equation (2-6) has already been achieved. Of first importance is the result that the isothermals will be plane surfaces parallel to the surface of the solid; i.e., the temperature will be independent of y and z. The heat conduction equation thus reduces to the so-called "unidimensional" form

$$\rho c_p \frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[k_x(x) \frac{\partial T(x,t)}{\partial x} \right] + q'''(x,t), \quad (2-10)$$

$$x \geq 0, \quad t \geq 0.$$

Actually, this is an equation in two dimensions, x and t, and the functional dependence has been explicitly indicated. The equation is defined in the

region $x \geq 0$ and $t \geq 0$, which has also been indicated explicitly.

In addition, two boundary conditions have been established. First, since the solid is initially at uniform temperature throughout, one may write:

$$T(x, 0) = T_0$$

Secondly, it is physically reasonable that as x tends to infinity, the temperature will be unaffected by any energy input at the surface of the solid; i.e.

$$\lim_{x \rightarrow \infty} T(x, t) = T_0$$

This latter condition will hold so long as the radiant energy input, per unit area of surface of the solid, is finite. If one considers a column of this solid, of unit cross-sectional area, and extending in the positive x -direction, the total heat capacity of this column will tend to infinity as the length of the column tends to infinity. For finite energy input to this column, the average temperature rise, over all x , must then be zero, or

$$\lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x [T(\xi, t) - T_0] d\xi = 0$$

It follows then that $[T(x, t) - T_0]$ can be different from zero for a finite range of values of x , only, which establishes the validity of the condition given above.

As a matter of convenience, a new temperature scale will be defined by

$$U = T - T_0$$

so that the symbol U means temperature rise (in Kelvin or Centigrade degrees) above initial temperature. The symbol T is reserved for absolute temperature (in degrees Kelvin). Also, the terms ρ and c_p will always appear as the product (ρc_p); hence the symbol ν will be defined by $\nu = \rho c_p$. Equation (2-10) and the boundary conditions thus far established now become:

$$\nu \frac{\partial U(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[k_x(x) \frac{\partial U(x,t)}{\partial x} \right] + q'''(x,t), \quad x \geq 0, t \geq 0 \quad (2-11)$$

$$U(x,0) = 0 \quad (2-12)$$

$$U(\infty,t) = 0 \quad (2-13)$$

using the abbreviated notation $U(\infty,t)$ for $\lim_{x \rightarrow \infty} U(x,t)$.

One additional assumption will now be made. In the derivation of the heat conduction equation, the final expression was obtained by substitution in the differential form of the First Law: $dH = \delta Q - \delta W + V dP$. The left-hand side is given by equation (2-8) as

$$dH = \rho c_p (dx dy dz dt) \frac{\partial T}{\partial t}.$$

Thus far, no assumptions concerning the constancy of ρ and c_p have been made since only a differential temperature change ($dT = \frac{\partial T}{\partial t} dt$) is being considered. In the final expression the differential space-time volume element ($dx dy dz dt$) is cancelled out, leaving one apparently free to consider any temperature rise. Such treatment is allowed only if suitable analytical expressions for ρ and c_p as functions of temperature are used. In the following sections, the assumption will be made that the product $(\rho c_p) = \nu$ is a constant.

One may take a slightly different view of this assumption by recognizing that ν is, in fact, one of the unknowns of the system. Hence, the value which will be determined by experiment is an average over the temperature range utilized.

2.5 Summary

While the various assumptions made above have resulted in considerable simplification of the basic equation of heat conduction, the resultant

equation (2-11) is still not sufficiently specialized to permit a formal solution to be obtained. The functional forms of both $k_x(x)$ and $q'''(x,t)$ must be specified, and one additional boundary condition given. In the following chapters, two specific models will be proposed which will provide this needed additional information. Normalized solutions of the heat conduction equation will then be obtained, and methods for the experimental evaluation of the various constants of the models will be developed. The way will then be prepared for a logical presentation of the experimental materials and methods which were employed in this investigation.

FOOTNOTES

- (1) Jakob, M., Heat Transfer, Vol. I, 2 (John Wiley and Sons, Inc., New York, 1949).
- (2) Ibid., 9.
- (3) Zemansky, M. W., Heat and Thermodynamics, Third Edition, 63 (McGraw-Hill Book Company, Inc., New York, 1951).

CHAPTER III

ANALYSIS OF THE OPAQUE SOLID

3.1 Definition of the Model

A particularly simple and quite useful model is the isotropic semi-infinite opaque solid. Since the solid is opaque, the impinging radiation is either absorbed at the mathematical surface ($x = 0$), or is reflected, the reflectance being considered a constant. The conductivity is assumed to be independent of temperature and position ($k_x(x) = k$), and it is also assumed that there are no distributed sources or sinks ($q'''(x,t) = 0$). Additionally, the surface ($x = 0$) is considered to be insulated against all heat losses, so that the only heat flow across the surface is due to the absorbed radiant energy. From the equation of Fourier, as expressed by equation (2-2), this last statement implies:

$$\lim_{x \rightarrow 0^+} \frac{\partial U(x,t)}{\partial x} = -\frac{1}{k} (1-R) H(t),$$

where the irradiance, $H(t)$, is the rate of radiant energy incident per unit area, and R is the reflectance of the surface. Since one always considers the absorbed irradiance, or $(1-R)H(t)$, the symbol $H_a(t) = (1-R)H(t)$ will be used for convenience.

3.2 The Solution of the Heat Conduction Equation for the Opaque Solid (1)

Under the assumptions above concerning $k_x(x)$ and $q'''(x,t)$, the right hand side of equation (2-11) becomes simply

$$\frac{\partial}{\partial x} \left[k \frac{\partial U(x,t)}{\partial x} \right] + 0 = k \frac{\partial^2 U(x,t)}{\partial x^2}.$$

Using the definition of thermal diffusivity, α ,

$$\alpha = \frac{k}{\rho c_p} = \frac{k}{\nu},$$

the heat conduction equation reduces for this model, to:

$$\frac{\partial U(x,t)}{\partial t} = \alpha \frac{\partial^2 U(x,t)}{\partial x^2}, \quad x \geq 0, \quad t \geq 0 \quad (3-1)$$

$$U(x,0) = 0, \quad x \geq 0 \quad (3-2)$$

$$U(\infty, t) = 0, \quad t \geq 0 \quad (3-3)$$

$$\frac{\partial U(0,t)}{\partial x} = -\frac{1}{k} H_a(t), \quad t \geq 0. \quad (3-4)$$

The solution of this boundary value problem is readily obtained by application of the Laplace transformation (2). This standard method reduces the partial differential equation (3-1) in x and t to an ordinary differential equation in x . Denoting the Laplace transformation with respect to t of $U(x,t)$ by $u(x,s)$ and that of $H_a(t)$ by $h_a(s)$; i.e.:

$$\mathcal{L}\{U(x,t)\} = u(x,s)$$

$$\mathcal{L}\{H_a(t)\} = h_a(s),$$

then taking the transform of both sides of (3-1) and using condition (3-2), one obtains:

$$s u(x,s) = \alpha \frac{d^2 u(x,s)}{dx^2} \quad (3-5)$$

while (3-3) and (3-4) become

$$u(\infty, s) = 0 \quad (3-6)$$

$$\text{and } \frac{du(x,s)}{dx} = -\frac{1}{k} h_a(s). \quad (3-7)$$

Rearranging (3-5) slightly

$$\left(\frac{d^2}{dx^2} - \frac{s}{\alpha}\right) u(x,s) = 0,$$

and the solution of this simple differential equation is

$$u(x,s) = A e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}} + B e^{+\frac{x}{\sqrt{\alpha}} \sqrt{s}},$$

where A and B are the two constants of integration. In view of condition

(3-6), B must be identically zero, while, from (3-7):

$$-\frac{1}{k} h_a(s) = -\sqrt{\frac{s}{\alpha}} A,$$

or

$$A = \sqrt{\frac{\alpha}{s}} \cdot \frac{h_a(s)}{k}.$$

Now, by definition $\alpha = k/\nu$, hence

$$\frac{\sqrt{\alpha}}{k} = \sqrt{\frac{k}{\nu}} \cdot \frac{1}{k} = \frac{1}{\sqrt{k\nu}}.$$

A new constant, the "thermal inertia," (3) μ , is now defined as

$$\mu = k \nu = k \cdot \rho c_p,$$

whence

$$A = \frac{h_a(s)}{\sqrt{\mu}} \cdot \frac{1}{\sqrt{s}},$$

and the solution for $u(x,s)$ becomes:

$$u(x,s) = \frac{h_a(s)}{\sqrt{\mu}} \cdot \frac{e^{-\frac{x}{\sqrt{\alpha}}\sqrt{s}}}{\sqrt{s}}. \quad (3-8)$$

Using the convolution integral and tabulated transforms (4), one may immediately obtain the inverse transform of $u(x,s)$ as

$$U(x,t) = \frac{1}{\sqrt{\pi\mu}} \int_0^t H_a(t-\lambda) \frac{e^{-\frac{x^2}{4\alpha\lambda}}}{\sqrt{\lambda}} d\lambda, \quad (3-9)$$

where λ is the dummy variable of integration. This form allows the calculation of the temperature rise, $U(x,t)$, for any arbitrary irradiance pulse $H(t)$, subject only to the restriction that $H(t)$ must be zero for time less than zero. (This is equivalent to the statement that the input pulse commences at zero time.)

The important point in the development so far is that equation (3-9) contains but two constants, μ and α , which may be called the "thermal constants" of the solid. Thus, by rendering any solid opaque, say by simply covering the surface with a thin opaque layer, these thermal constants of

the solid may be determined by measuring the temperature response as a function of depth and time. (The surface reflectance must be determined by separate means, as with a spectrophotometer fitted with a diffuse reflectance head.)

3.3 Specialization of the Solution for Rectangular Irradiance Function

To reduce the temperature response equation (3-9) to a somewhat simpler form, a specific function, the step function, will be selected for $H(t)$.

Thus:

$$\begin{aligned} H_a^{step}(t) &= 0, t < 0, \\ &= H_a, t \geq 0, \end{aligned}$$

where H_a is a constant. The integration indicated in equation (3-9) may now be performed, or the transform of the above-defined functional form for $H_a(t)$ may be substituted in equation (3-8) and the inverse transform taken directly. Following the latter course:

$$\mathcal{L}\{H_a^{step}(t)\} = h_a^{step}(s) = \frac{H_a}{s};$$

hence,

$$u^{step}(x, s) = \frac{H_a}{\sqrt{\mu}} \cdot \frac{e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}}}{s^{3/2}}$$

and (5),

$$U^{step}(x, t) = \frac{2H_a}{\sqrt{\pi}} \left[\sqrt{\frac{t}{\pi}} e^{-\frac{x^2}{4\alpha t}} - \frac{x}{\sqrt{4\alpha}} \operatorname{erfc}\left(\sqrt{\frac{x}{4\alpha t}}\right) \right], \quad (3-10)$$

where $\operatorname{erfc}(x)$ is the complementary error function defined by

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\lambda^2} d\lambda.$$

(Here, $\operatorname{erf}(x)$ is the error function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\lambda^2} d\lambda).$$

Now, the step function form for $H(t)$ requires that the irradiance must continue indefinitely at a constant level, while any actual pulse must

be terminated at some time. It will be convenient to consider a rectangular pulse:

$$\begin{aligned} H_a^{rect}(t) &= 0, \quad t < 0 \\ &= H_a, \quad 0 \leq t \leq \eta \\ &= 0, \quad t > \eta; \end{aligned}$$

i.e. a pulse of constant amplitude over the exposure time, η . Clearly, this form may be synthesized from two step functions as defined above,

$$H_a^{rect}(t) = H_a^{step}(t) - H_a^{step}(t - \eta).$$

The linearity of the original differential equation insures that superposition of solutions is allowed, hence

$$\begin{aligned} U^{rect}(x, t) &= U^{step}(x, t), \quad 0 \leq t \leq \eta, \\ &= U^{step}(x, t) - U^{step}(x, t - \eta), \quad t > \eta, \end{aligned}$$

or

$$\begin{aligned} U^{rect}(x, t) &= \frac{2H_a}{\sqrt{\pi}} \left[\sqrt{\frac{t}{\pi}} e^{-\frac{x^2}{4\alpha t}} - \frac{x}{\sqrt{4\alpha}} \operatorname{erfc}\left(\sqrt{\frac{x}{4\alpha t}}\right) \right], \quad 0 \leq t \leq \eta \\ &= \frac{2H_a}{\sqrt{\pi}} \left[\sqrt{\frac{t}{\pi}} e^{-\frac{x^2}{4\alpha t}} - \frac{x}{\sqrt{4\alpha}} \operatorname{erfc}\left(\sqrt{\frac{x}{4\alpha t}}\right) \right. \\ &\quad \left. - \sqrt{\frac{t-\eta}{\pi}} e^{-\frac{x^2}{4\alpha(t-\eta)}} + \frac{x}{\sqrt{4\alpha}} \operatorname{erfc}\left(\sqrt{\frac{x}{4\alpha(t-\eta)}}\right) \right], \\ &\quad t > \eta \end{aligned}$$

It is of some convenience to generalize these results by defining the dimensionless variables:

$$\tau = \frac{x}{\sqrt{4\alpha\eta}},$$

and

$$\tau = \frac{t}{\eta}.$$

By slight rearrangement, equation (3-10) then becomes:

$$\begin{aligned} U^{rect}(\sqrt{4\alpha\eta} \tau, \eta \tau) &= \frac{2H_a\sqrt{\eta}}{\sqrt{\pi\mu}} \sqrt{\tau} \left\{ \sqrt{\tau} \left[\frac{1}{\sqrt{\pi}} e^{-\frac{\tau^2}{\tau}} - \frac{\tau}{\sqrt{\tau}} \operatorname{erfc}\left(\sqrt{\frac{\tau}{\tau}}\right) \right] \right\}, \quad 0 \leq \tau \leq 1 \\ &= \frac{2H_a\sqrt{\eta}}{\sqrt{\pi\mu}} \sqrt{\tau} R\left(\frac{\tau}{\tau}\right), \quad 0 \leq \tau \leq 1, \end{aligned}$$

where

$$R(x) = \sqrt{\pi} \left[\frac{1}{\sqrt{\pi}} e^{-x^2} - x \operatorname{erfc}(x) \right] .$$

The function $R(x)$ is defined so that $R(0)$ is equal to unity. Thus, when ψ is zero (i.e. on the surface of the solid) the temperature rise is simply:

$$U^{\text{rect}}(0, \eta \tau) = \frac{2 H_a \sqrt{\eta}}{\sqrt{\pi \mu}} \sqrt{\tau} , \quad 0 \leq \tau \leq 1 \quad (3-11)$$

which is maximal at $\tau = 1$, or

$$U^{\text{rect}}(0, \eta) = \frac{2 H_a \sqrt{\eta}}{\sqrt{\pi \mu}} \quad (3-12)$$

The temperature response equation may now be further simplified by defining a normalized response, Γ , as

$$\Gamma(\psi, \tau) = \frac{U^{\text{rect}}(\sqrt{4\alpha\eta}\psi, \eta\tau)}{U^{\text{rect}}(0, \eta)}$$

whence the complete expression becomes:

$$\begin{aligned} \Gamma(\psi, \tau) &= \sqrt{\tau} R\left(\frac{\psi}{\sqrt{\tau}}\right) , \quad 0 \leq \tau \leq 1 \\ &= \sqrt{\tau} R\left(\frac{\psi}{\sqrt{\tau}}\right) - \sqrt{\tau-1} R\left(\frac{\psi}{\sqrt{\tau-1}}\right) , \quad \tau > 1 . \end{aligned} \quad (3-13)$$

Equation (3-13) presents the complete temperature-time-depth history of the opaque solid subjected to a rectangular heat input, in compact form. A brief table of the values of the function $R(x)$ is given in Appendix III. The accompanying curves display $\Gamma(\psi, \tau)$ as a function of dimensionless time, τ , with dimensionless depth, ψ , as a parameter (Figure 3-1), and as a function of ψ with τ as a parameter (Figure 3-2).

3.4 The Fitting of Experimental Results to the Normalized Solution

While equation (3-13) possesses a measure of mathematical satisfying elegance, the directly obvious physical variables have been somewhat lost in the normalization. Now the normalizing constants are essentially four: (1) the absorbed irradiance, H_a , (2) the exposure time, η , (3) the thermal inertia, μ , and (4) the thermal diffusivity, α . In any experimental

Figure 3-1. Predicted Response of the Opaque Solid

Figure 3-2. Predicted Response of the Opaque Solid

measurements, the first two, the absorbed irradiance and the exposure time, will depend upon the arrangement of the exposure source (and the independently measured surface reflectance), and must be known with as high an accuracy as possible. The latter two, the thermal inertia and thermal diffusivity, are the "unknowns" of the material; it is precisely the purpose of the experiment to yield numerical values for these constants. A logical question, then, is how one might best use the theoretical predictions in conjunction with experimental data to determine these values.

It is of interest, although probably not of profound theoretical significance, that the thermal inertia and the thermal diffusivity can be determined quite independently of each other. The evaluation of the former follows quite directly from equation (3-11), which gives the expected surface temperature rise as a function of dimensionless time, τ . The only unknown involved is μ . It is convenient to reduce the experimental results to temperature rise per unit absorbed irradiance, or $U(x,t)/H_a$; for brevity, this quantity is denoted by the symbol $U^*(x,t)$. Thus, if the measured values of $U^*(x,t)$ are divided by $\sqrt{\eta}$, and plotted against $\sqrt{t/\eta}$, then a straight line should result, passing through the origin, with a slope of $2/\sqrt{\pi\mu}$; i.e., from equation (3-11):

$$\frac{U^{*rect}(0,t)}{\sqrt{\eta}} = \left(\frac{2}{\sqrt{\pi\mu}} \right) \sqrt{\frac{t}{\eta}} \quad (3-14)$$

This will hold, of course, only for times such that $0 \leq t/\eta \leq 1$. The numerical value of μ can thus be found from this slope.

For the evaluation of the diffusivity, α , two methods present themselves. First, one may plot the logarithm of experimental values of T ; i.e. $U^*(x,t)/U^*(0,\eta)$ against $\log$ of $x/\sqrt{\eta}$, with t/η as a parameter. (The actual determinations of x and η will be described in detail in a later chapter.) The resultant family of curves should be identical to Figure 3-2, except for a simple translation of the abscissa. This employs the convenient

property of a logarithmic scale that a multiplicative constant causes a simple linear shift, only. Thus, if the experimental curves are shifted over the theoretical curves until the "best" fit is obtained, the multiplicative constant can be readily determined. Here, this constant is simply $1/\sqrt{4\alpha}$; hence α may be evaluated directly.

The second method employs a somewhat different approach. Inspection of Figure 3-1 reveals that for any non-zero value of dimensionless depth, ψ , the dimensionless temperature reaches a "smooth" maximum (i.e. the first derivative of T with respect to τ becomes zero) at a value of τ (always greater than unity) which increases with increasing ψ . Letting τ_{max} be the value of the normalized time at which such a maximum response occurs, the functional dependence of ψ on τ_{max} will now be deduced.

Returning to the original form of $U^{rect}(x,t)$ it will be recalled that this solution was obtained by superposition of two step function responses as

$$U^{rect}(x,t) = U^{step}(x,t) - U^{step}(x,t-\eta)$$

Hence, it follows that

$$\begin{aligned} \frac{\partial U^{rect}(x,t)}{\partial t} &= \frac{\partial}{\partial t} [U^{step}(x,t) - U^{step}(x,t-\eta)] \\ &= \frac{\partial U^{step}(x,t)}{\partial t} - \frac{\partial U^{step}(x,t-\eta)}{\partial t} \end{aligned}$$

The derivatives on the right are readily found from equation (3-8), utilizing the useful property of the Laplace transform (6) that

$$\frac{\partial U(x,t)}{\partial t} = \mathcal{L}^{-1} \left\{ s u(x,s) - U(x,0) \right\};$$

hence

$$\frac{\partial U^{step}(x,t)}{\partial t} = \frac{H_a}{\sqrt{\pi \mu}} \cdot \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4\alpha t}}$$

Now the extrema are at values such that:

$$\left. \frac{\partial U^{rect}(x, t)}{\partial t} \right|_{t=t_{max}} = 0$$

or

$$\frac{Ha}{\sqrt{\pi \alpha t}} \left[\frac{1}{\sqrt{t_{max}}} e^{-\frac{x^2}{4\alpha t_{max}}} - \frac{1}{\sqrt{t_{max} - \eta}} e^{-\frac{x^2}{4\alpha(t_{max} - \eta)}} \right] = 0$$

Rearranging slightly, and introducing γ and τ_{max} as previously defined

$$e^{+\gamma^2 \left(\frac{1}{\tau_{max} - 1} - \frac{1}{\tau_{max}} \right)} = \sqrt{\frac{\tau_{max}}{\tau_{max} - 1}}$$

or

$$\frac{\gamma^2}{\tau_{max}(\tau_{max} - 1)} = \frac{1}{2} \ln \frac{\tau_{max}}{\tau_{max} - 1},$$

and finally:

$$\gamma = \sqrt{\frac{\tau_{max}(\tau_{max} - 1)}{2} \ln \frac{\tau_{max}}{\tau_{max} - 1}} \quad (3-15)$$

Equation (3-15) gives the expected value of γ ($= x / \sqrt{4\alpha\eta}$) for which a temperature maximum will occur at a specified value of τ_{max} ($= t_{max}/\eta$).

The application of this relationship to the experimental determination of α is straightforward. For each subsurface temperature measurement, one notes the time, t_{max} , at which the maximum temperature was reached. Then plotting $x/\sqrt{\eta}$ (again, these values must be known) against

$$\sqrt{\frac{\frac{t_{max}}{\eta} \left(\frac{t_{max}}{\eta} - 1 \right)}{2} \ln \left(\frac{\frac{t_{max}}{\eta}}{\frac{t_{max}}{\eta} - 1} \right)}$$

should yield a straight line, passing through the origin, with slope of $\sqrt{4\alpha}$. This follows from the trivial rearrangement of equation (3-15)

to

$$\frac{x}{\sqrt{\eta}} = \sqrt{4\alpha} \sqrt{\frac{\frac{t_{max}}{\eta} \left(\frac{t_{max}}{\eta} - 1 \right)}{2} \ln \left(\frac{\frac{t_{max}}{\eta}}{\frac{t_{max}}{\eta} - 1} \right)} \quad (3-16)$$

Since the subsurface temperature maxima always occur at times such that

$\frac{t_{max}}{\eta} > 1$, there is no problem with negative or zero values in the

denominator of the logarithm, or with negative values under the radical.

In reviewing the above methods for the experimental evaluation of the thermal constants, μ and α , it may again be pointed out that they are determined quite independently of each other, with surface temperature measurements only being used for μ and subsurface measurements for α . In fact, it is clear that surface measurements, which are certainly the simplest to make, can give information concerning μ , only, and do not permit an evaluation of α to be made. For the latter constant, subsurface measurements, complete with accurate values of the depth of placement of the temperature sensing element, must be made.

It might also be mentioned that the thermal inertia, μ , and diffusivity, α , may be considered "derived" constants, defined in terms of the "fundamental" constants, thermal conductivity, k , and heat capacity per unit volume, ν , by

$$\mu = k \cdot \nu,$$

$$\alpha = k / \nu.$$

Obviously, the latter two constants may be found from the former by

$$k = \sqrt{\mu \cdot \alpha},$$

$$\nu = \sqrt{\mu / \alpha};$$

hence, it would seem to be a matter of complete indifference which set-- μ and α , or k and ν --be considered "fundamental," and which "derived."

3.5 The Opaque Composite Solid

The foregoing prescriptions for determining the thermal constants from experimental data will, of course, be valid only if the experimental material satisfies the assumptions made in deriving the various equations involved. It is reasonable now to ask what changes in the experimental results might be expected if certain of these assumptions are not satisfied,

in particular the constancy of ν and k . The model selected for this investigation is the opaque composite semi-infinite solid. Here, the various assumptions in Chapter II and Section 3.1 are still applicable except that the thermal conductivity is assumed to be a constant value k_1 for the values of x from zero to a value b , and a constant value k_2 for x greater than b . Likewise the heat capacity per unit volume is taken as ν_1 for x from zero to b and ν_2 for x greater than b . Equations (3-1) through (3-4) are now replaced by

$$\nu_1 \frac{\partial U(x,t)}{\partial t} = k_1 \frac{\partial^2 U(x,t)}{\partial x^2}, \quad 0 \leq x < b, \quad t \geq 0 \quad (3-17)$$

$$\nu_2 \frac{\partial U(x,t)}{\partial t} = k_2 \frac{\partial^2 U(x,t)}{\partial x^2}, \quad x > b, \quad t \geq 0 \quad (3-18)$$

$$U(x, 0) = 0, \quad x \geq 0 \quad (3-19)$$

$$U(\infty, t) = 0, \quad t \geq 0 \quad (3-20)$$

$$\frac{\partial U(0,t)}{\partial x} = -\frac{1}{k_1} H_a(t), \quad t \geq 0 \quad (3-21)$$

$$U(b-0, t) = U(b+0, t) \quad (3-22)$$

$$k_1 \frac{\partial U(b-0, t)}{\partial x} = k_2 \frac{\partial U(b+0, t)}{\partial x} \quad (3-23)$$

Equations (3-17) and (3-18) follow directly from the definition of the composite solid; equations (3-19) through (3-21) are identical to (3-2) through (3-4) which were discussed previously; equations (3-22) and (3-23) are "continuity" conditions. The first, which is more correctly written:

$$\lim_{x \rightarrow b^-} U(x,t) = \lim_{x \rightarrow b^+} U(x,t), \quad t \geq 0$$

simply states that the temperature response function suffers no jump at the break point b . Such a jump, in view of the absence of internal sources or sinks, is a physical impossibility. Equation (3-23) expresses the physically

necessary condition that heat flow into the break point from the left must be equal to the flow out to the right. Here again the notation is an abbreviation for

$$\lim_{x \rightarrow b^-} \left[k_1 \frac{\partial U(x,t)}{\partial x} \right] = \lim_{x \rightarrow b^+} \left[k_2 \frac{\partial U(x,t)}{\partial x} \right], \quad t \geq 0.$$

Letting $\alpha_1 = k_1/\nu_1$, and $\alpha_2 = k_2/\nu_2$, and using condition (3-19), the Laplace transforms of (3-17) and (3-18) give two ordinary differential equations analogous to (3-5), which are readily solved to yield:

$$u(x,s) = A e^{-\frac{x}{\alpha_1} \sqrt{s}} + B e^{\frac{x}{\alpha_1} \sqrt{s}}, \quad 0 \leq x < b,$$

and

$$u(x,s) = C e^{-\frac{x}{\alpha_2} \sqrt{s}} + D e^{\frac{x}{\alpha_2} \sqrt{s}}, \quad x > b,$$

where four constants of integration, A, B, C, and D must be evaluated. From the transform of (3-20), $u(\infty, s) = 0$, it follows that D must be identically zero. Further, from the transform of condition (3-21) it follows that

$$-\frac{1}{k_1} h_a(s) = -\sqrt{\frac{s}{\alpha_1}} A + \sqrt{\frac{s}{\alpha_1}} B,$$

or

$$B = A - \frac{1}{\sqrt{\mu_1}} \cdot \frac{h_a(s)}{\sqrt{s}}$$

Hence the solutions in $u(x,s)$ become

$$u(x,s) = A(e^{-\frac{x}{\alpha_1} \sqrt{s}} + e^{\frac{x}{\alpha_1} \sqrt{s}}) - \frac{h_a(s)}{\sqrt{\mu_1}} \cdot \frac{e^{-\frac{x}{\alpha_1} \sqrt{s}}}{\sqrt{s}}, \quad 0 \leq x < b,$$

$$u(x,s) = C e^{-\frac{x}{\alpha_2} \sqrt{s}}, \quad x > b.$$

The constants A and C must be evaluated from the transforms of (3-22) and (3-23), namely:

$$u(b-0, s) = u(b+0, s)$$

and

$$k_1 \frac{du(b-0, s)}{dx} = k_2 \frac{du(b+0, s)}{dx}$$

The evaluation of these constants is quite straightforward, and is presented in detail in Appendix IV. Upon substitution of the complete expressions for A and C, the complete solution in $u(x,s)$ becomes:

$$u(x, s) = \frac{h_a(s)}{\sqrt{\mu_1}} \left\{ \frac{e^{\frac{b-x}{\sqrt{\alpha_1}} \sqrt{s}} (1 + \sqrt{\frac{\mu_2}{\mu_1}}) + e^{-\frac{b-x}{\sqrt{\alpha_1}} \sqrt{s}} (1 - \sqrt{\frac{\mu_2}{\mu_1}})}{\sqrt{s} [e^{\frac{b}{\sqrt{\alpha_1}} \sqrt{s}} (1 + \sqrt{\frac{\mu_2}{\mu_1}}) - e^{-\frac{b}{\sqrt{\alpha_1}} \sqrt{s}} (1 - \sqrt{\frac{\mu_2}{\mu_1}})]} \right\} \quad 0 \leq x < b; \quad (3-24)$$

$$u(x, s) = \frac{h_a(s)}{\sqrt{\mu_1}} \left\{ \frac{2 e^{-\frac{x-b}{\sqrt{\alpha_2}} \sqrt{s}}}{\sqrt{s} [e^{\frac{b}{\sqrt{\alpha_1}} \sqrt{s}} (1 + \sqrt{\frac{\mu_2}{\mu_1}}) - e^{-\frac{b}{\sqrt{\alpha_1}} \sqrt{s}} (1 - \sqrt{\frac{\mu_2}{\mu_1}})]} \right\} \quad x > b. \quad (3-25)$$

In Appendix IV, these expressions are reduced to a somewhat more manageable form, and the inverse transforms taken to give general formulas for the temperature response $U(x, t)$. Here, however, it will be sufficient to consider only the surface response to a step function. It is convenient to make the substitution $x = 0$ in the transformed function $u(x, s)$, rather than in the general equation for $U(x, t)$, since the problem of taking the inverse transform is greatly simplified. As stated previously,

$$h_a(s) = \frac{H_a}{s}$$

for the step function form of $H_a(t)$, hence equation (3-24) becomes

$$u(0, s) = \frac{H_a}{\sqrt{\mu_1}} \cdot \frac{e^{\frac{b}{\sqrt{\alpha_1}} \sqrt{s}} (1 + \sqrt{\frac{\mu_2}{\mu_1}}) + e^{-\frac{b}{\sqrt{\alpha_1}} \sqrt{s}} (1 - \sqrt{\frac{\mu_2}{\mu_1}})}{s^{3/2} [e^{\frac{b}{\sqrt{\alpha_1}} \sqrt{s}} (1 + \sqrt{\frac{\mu_2}{\mu_1}}) - e^{-\frac{b}{\sqrt{\alpha_1}} \sqrt{s}} (1 - \sqrt{\frac{\mu_2}{\mu_1}})]}$$

The dimensionless constant λ is now defined as:

$$\lambda = \frac{1 - \sqrt{\mu_2/\mu_1}}{1 + \sqrt{\mu_2/\mu_1}}$$

from which it is clear that λ^2 must always be less than unity. The expression above for $u(0, s)$ may now be written as:

$$u(0,s) = \frac{H_a}{\sqrt{\mu_1}} \cdot \frac{i}{s^{3/2}} \cdot \frac{1 + \lambda e^{-\frac{2b}{\alpha_1} \sqrt{s}}}{1 - \lambda e^{-\frac{2b}{\alpha_1} \sqrt{s}}}$$

or

$$u(0,s) = \frac{H_a}{\sqrt{\mu_1}} \left[\frac{1}{s^{3/2}} + 2 \sum_{n=1}^{\infty} \lambda^n \frac{e^{-\frac{2b n}{\alpha_1} \sqrt{s}}}{s^{3/2}} \right] \quad (3-26)$$

Equation (3-26) is in such form that the inverse Laplace transform can be taken. However, it will be simpler to consider what form a plot of $U(0,t)$ against $\sqrt{t}$ will take, i.e. to derive the function $\frac{\partial U(0,t)}{\partial \sqrt{t}}$. From equation (3-11) it is seen that for the isotropic opaque solid

$$\frac{\partial U(0,t)}{\partial \sqrt{t}} = \frac{2 H_a}{\sqrt{\pi \mu_1}}$$

and the question here is what deviation from this simple relationship will occur in the case of the composite solid.

Now,

$$\frac{\partial U}{\partial \sqrt{t}} = 2 \sqrt{t} \frac{\partial U(0,t)}{\partial t}$$

while as mentioned previously

$$\frac{\partial U(0,t)}{\partial t} = \mathcal{L}^{-1} \{ s u(0,s) \}$$

Hence

$$\begin{aligned} \frac{\partial U(0,t)}{\partial t} &= \frac{H_a}{\sqrt{\mu_1}} \mathcal{L}^{-1} \left\{ \frac{1}{\sqrt{s}} + 2 \sum_{n=1}^{\infty} \lambda^n \frac{e^{-\frac{2b n}{\alpha_1} \sqrt{s}}}{\sqrt{s}} \right\} \\ &= \frac{H_a}{\sqrt{\mu_1}} \left[\frac{1}{\sqrt{\pi t}} + 2 \sum_{n=1}^{\infty} \lambda^n \frac{e^{-\frac{b^2 n^2}{\alpha_1^2 t}}}{\sqrt{\pi t}} \right] \end{aligned}$$

and, from above

$$\frac{\partial U(0,t)}{\partial \sqrt{t}} = \frac{2 H_a}{\sqrt{\pi \mu_1}} \left[1 + 2 \sum_{n=1}^{\infty} \lambda^n e^{-\frac{b^2 n^2}{\alpha_1^2 t}} \right] \quad (3-27)$$

While the form of equation (3-27) is not immediately obvious, the limiting

values are quite simple. Letting t approach zero, the exponential will likewise approach zero, hence

$$\frac{\partial U(0,0)}{\partial \sqrt{t}} = \frac{2H_a}{\sqrt{\pi\mu_1}}$$

The initial slope of the temperature response plotted against the square root of time thus should be $2H_a/\sqrt{\pi\mu_1}$ from which μ_1 can be obtained directly.

Now, letting t increase without bound (i.e. $t \rightarrow \infty$) the exponential term will approach unity, and thus

$$\begin{aligned} \frac{\partial U(0,\infty)}{\partial \sqrt{t}} &= \frac{2H_a}{\sqrt{\pi\mu_1}} \left[1 + 2 \sum_{n=1}^{\infty} \lambda^n \right] \\ &= \frac{2H_a}{\sqrt{\pi\mu_1}} \left[-1 + 2 \sum_{n=0}^{\infty} \lambda^n \right] \\ &= \frac{2H_a}{\sqrt{\pi\mu_1}} \left[-1 + \frac{2}{1-\lambda} \right] \\ &= \frac{2H_a}{\sqrt{\pi\mu_1}} \left[\frac{1+\lambda}{1-\lambda} \right]. \end{aligned}$$

But

$$\frac{1+\lambda}{1-\lambda} = \frac{1 + \frac{1 - \sqrt{\mu_2/\mu_1}}{1 + \sqrt{\mu_2/\mu_1}}}{1 - \frac{1 - \sqrt{\mu_2/\mu_1}}{1 + \sqrt{\mu_2/\mu_1}}} = \sqrt{\frac{\mu_1}{\mu_2}}$$

Hence

$$\frac{\partial U(0,\infty)}{\partial \sqrt{t}} = \frac{2H_a}{\sqrt{\pi\mu_2}}$$

or the plot of $U(0,t)$ vs $\sqrt{t}$ should tend to reach a straight line with slope $2H_a/\sqrt{\pi\mu_2}$, after a sufficient length of time. From this limiting slope, the value of μ_2 may be obtained. If the experimental data indicate that the slope of the temperature response vs the square root of time is constant for all values of time, then the implication is that $\mu_1 = \mu_2$. This does not "prove" that the solid is isotropic, since compensating variations in k and ν could result in this constancy of μ ; however it

will be shown later that in the case of skin such compensation is highly unlikely. Thus, the constancy of the slope of $U(0,t)$ vs $\sqrt{t}$ may be considered sufficient justification for the assumption of constancy of k and ν .

3.6 The Opaque Solid Subjected to a Trapezoidal Irradiance Pulse

The considerations of the previous section focus attention on the initial temperature response of the opaque solid. In any actual experiment, considerable care must be exercised in order to obtain reliable data at these very early times, since several factors other than variation of μ with depth may affect the results. Certainly, some means must be provided to indicate, on the actual temperature recordings, the precise moment of initiation of the exposure. In addition, the transient response of the thermal sensing element-amplifier-recorder system must be known in order to establish the form of the true temperature response from the recorded response. Both of these matters will be dealt with in following chapters. There is another factor, rather easily overlooked, which should be considered now; this concerns the actual irradiance pulse shape delivered to the test specimen.

It is, unfortunately, a complete physical impossibility to deliver a true mathematical step function in any real situation. While the rise time of the irradiance pulse may be made very small, it can never become precisely zero, as required for a true step function. For any given situation, it is probably always possible to design a shuttering mechanism which will provide a rise time so short that it may be considered a true step function with negligible error. With an irradiance source such as a carbon arc furnace, however, this approach poses several unique problems. The shutter must control extremely high values of radiant power, must be completely opaque when closed, and should be completely transparent when open for wavelengths from the near ultraviolet out to about 2.6μ in the infrared. Thus both light-weight,

fast-acting mechanical shutters and the vastly more rapid Kerr cell types are not particularly suitable, although admittedly they probably could be adapted for this use.

A simpler approach would seem to be to use a robust mechanical shutter, with the fastest practicable opening time, and then apply mathematical corrections to the resultant data to account for the finite rise time. This is the procedure selected for this investigation.

Investigation revealed that the opening characteristic of the shutter employed (which will be described fully in a later chapter) could be approximated quite closely by a linearly rising segment (ramp function) followed by a "flat-topped" segment. The closing characteristic is similar, yielding a trapezoidal rather than a rectangular pulse. Graphically this is represented as follows:

The absorbed irradiance pulse is thus given by

$$\begin{aligned}
 H_a^{\text{trap}}(t) &= 0, \quad t \leq 0 \\
 &= \left(\frac{H_a}{\delta}\right) t, \quad 0 \leq t \leq \delta \\
 &= H_a, \quad \delta \leq t \leq \eta \\
 &= H_a - \left(\frac{H_a}{\delta}\right) t, \quad \eta \leq t \leq \eta + \delta \\
 &= 0, \quad t \geq \eta + \delta
 \end{aligned}$$

This pulse may also be synthesized from four ramp functions, as follows:

$$H_a^{+ramp}(t) = H_a^{ramp}(t) - H_a^{ramp}(t-\delta) \\ - H_a^{ramp}(t-\eta) + H_a^{ramp}(t-\eta-\delta),$$

where

$$H_a^{ramp}(t) = 0, \quad t < 0 \\ = \left(\frac{H_0}{\delta}\right)t, \quad t \geq 0$$

Since the temperature response to the trapezoidal pulse form can be obtained by superposition of solutions, it is only necessary to derive the temperature response to a ramp function. Thus

$$\begin{aligned} U^{+ramp}(x,t) &= U^{ramp}(x,t), \quad 0 \leq t \leq \delta \\ &= U^{ramp}(x,t) - U^{ramp}(x,t-\delta), \quad \delta \leq t \leq \eta \\ &= U^{ramp}(x,t) - U^{ramp}(x,t-\delta) - U^{ramp}(x,t-\eta), \\ &\quad \eta \leq t \leq \eta + \delta \end{aligned}$$

For the isotropic opaque solid, the ramp function solution can be found directly from equation (3-9), the general solution for this model. It is extraordinarily simple, however, to attack this problem in a more general way, after which the special case of the isotropic opaque solid will be considered.

Turning to the transformed equations in $u(x,s)$, it will be noted that for both the isotropic and the composite opaque solid, $u(x,s)$ is directly proportional to $h_a(s)$, the transform of the irradiance function. It will be seen in the next chapter that this same relationship holds true for the diathermanous solid as well. In fact, this must hold for any model, so long as ν and k are independent of temperature; i.e. the governing differential equation is linear. Certainly, for any linear model subjected to an irradiance step function, if the absorbed irradiance be doubled, then the temperature response will likewise be doubled; hence one can state that

$$U^{step}(x,t) = H_a \cdot F(x,t),$$

where $F(x,t)$ depends only upon the particular model involved. Taking the Laplace transform of this expression gives

$$\begin{aligned} u^{step}(x,s) &= H_a \mathcal{L}\{F(x,t)\} \\ &= \frac{H_a}{s} \cdot s \mathcal{L}\{F(x,t)\}. \end{aligned}$$

Clearly, the term H_a/s is simply the specialization of the transformed general irradiance function, $h_a(s)$, for the particular case of a step function; since this transformed response equation must be of the same form for any (transformed) irradiance function, it follows that in general

$$u(x,s) = h_a(s) \cdot s \mathcal{L}\{F(x,t)\}.$$

Now consider the ramp function:

$$\begin{aligned} H_a^{ramp}(t) &= 0, \quad t < 0 \\ &= \left(\frac{H_a}{\delta}\right) \cdot t, \quad t \geq 0. \end{aligned}$$

The Laplace transform is given by

$$\begin{aligned} h_a^{ramp}(s) &= \mathcal{L}\{H_a^{ramp}(t)\} \\ &= \frac{H_a}{\delta} \cdot \frac{1}{s^2} \end{aligned}$$

and the transform of the temperature response by

$$u^{ramp}(x,s) = \frac{H_a}{\delta} \cdot \frac{1}{s^2} \cdot s \mathcal{L}\{F(x,t)\}.$$

This last expression may be rearranged slightly to give

$$u^{ramp}(x,s) = \frac{1}{\delta} \cdot \frac{1}{s} \left[\frac{H_a}{s} \cdot s \mathcal{L}\{F(x,t)\} \right],$$

where the term in brackets is recognized as simply $u^{step}(x,s)$, or

$$u^{ramp}(x,s) = \frac{1}{\delta} \cdot \frac{1}{s} u^{step}(x,s).$$

Now, using the property of the Laplace transform (7) that if $f(s)$ is the transform of $F(t)$ then

$$\mathcal{L}^{-1}\left\{\frac{1}{s} f(s)\right\} = \int_0^t F(\tau) d\tau,$$

the inverse of the above expression may be written immediately as

$$U^{\text{ramp}}(x, t) = \frac{1}{\delta} \int_0^t U^{\text{step}}(x, \tau) d\tau.$$

Using the notation $U^*(x, t)$ as the response for unit absorbed irradiance, one may write finally:

$$U^{\text{ramp}}(x, t) = \left(\frac{H_a}{\delta}\right) \int_0^t U^{*\text{step}}(x, \tau) d\tau, \quad (3-28)$$

where H_a/δ is the slope of the ramp function, and $U^{*\text{step}}(x, t)$ may be recognized as the indicial transfer function. Equation (3-28) is a general relationship, valid for any linear model. (It is, in fact, a special case of the superposition integral, commonly used in communications and network analysis. The term "indicial transfer function" comes from this latter field.)

One can now return to the question of the influence of the finite shutter opening time on the initial temperature response of the opaque solid. The surface temperature rise per unit absorbed irradiance of the isotropic solid is given by

$$U^{*\text{step}}(0, t) = \frac{2}{\sqrt{\pi\mu}} \sqrt{t}$$

from equation (3-11). Hence the ramp function response is

$$U^{\text{ramp}}(0, t) = \frac{2H_a}{\delta\sqrt{\pi\mu}} \int_0^t \sqrt{\tau} d\tau,$$

and considering only the early phases of the previously defined trapezoidal pulse,

$$\begin{aligned}
 U^{trap}(0,t) &= \frac{2Ha}{\delta\sqrt{\pi\mu}} \int_0^t \sqrt{\tau} d\tau, \quad 0 \leq t \leq \delta \\
 &= \frac{2Ha}{\delta\sqrt{\pi\mu}} \left[\int_0^t \sqrt{\tau} d\tau - \int_0^{t-\delta} \sqrt{\tau} d\tau \right], \quad \delta \leq t \leq \eta
 \end{aligned}$$

The latter expression may be simplified by noting that

$$\int_0^t f(\tau) d\tau - \int_0^{t-\delta} f(\tau) d\tau = \int_{t-\delta}^t f(\tau) d\tau$$

hence

$$\begin{aligned}
 U^{trap}(0,t) &= \frac{2Ha}{\delta\sqrt{\pi\mu}} \int_0^t \sqrt{\tau} d\tau, \quad 0 \leq t \leq \delta \\
 &= \frac{2Ha}{\delta\sqrt{\pi\mu}} \int_{t-\delta}^t \sqrt{\tau} d\tau, \quad \delta \leq t \leq \eta.
 \end{aligned}$$

Performing the indicated integrations gives

$$\begin{aligned}
 U^{trap}(0,t) &= \frac{4Ha}{3\delta\sqrt{\pi\mu}} (t)^{3/2}, \quad 0 \leq t \leq \delta \\
 &= \frac{4Ha}{3\delta\sqrt{\pi\mu}} [t^{3/2} - (t-\delta)^{3/2}], \quad \delta \leq t \leq \eta.
 \end{aligned}$$

A correction function F^{trap} , is now defined, which when multiplied by the trapezoidal pulse surface response will give the "corrected" step function surface response; i.e.

$$\begin{aligned}
 F^{trap}(t) &= \frac{U^{step}(0,t)}{U^{trap}(0,t)} \\
 &= \frac{3}{2} \left(\frac{\delta}{t} \right), \quad 0 \leq t \leq \delta \\
 &= \frac{3}{2} \frac{\left(\frac{\delta}{t} \right)}{1 - \left(1 - \frac{\delta}{t} \right)^{3/2}}, \quad \delta \leq t \leq \eta.
 \end{aligned}$$

The last expression, for time greater than δ , may be cleared up by expanding the denominator as

$$\begin{aligned}
\frac{1}{1 - (1 - \frac{\delta}{t})^{3/2}} &= \frac{1}{1 - \left[1 - \frac{3}{2} \left(\frac{\delta}{t} \right) + \left(\frac{3}{2} \right) \left(\frac{1}{2} \right) \frac{1}{2!} \left(\frac{\delta}{t} \right)^2 + \left(\frac{3}{2} \right) \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \frac{1}{3!} \left(\frac{\delta}{t} \right)^3 + \dots \right]} \\
&= \frac{1}{\frac{3}{2} \left(\frac{\delta}{t} \right) \left[1 - \frac{1}{2} \cdot \frac{1}{2!} \left(\frac{\delta}{t} \right) - \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \frac{1}{3!} \left(\frac{\delta}{t} \right)^2 - \left(\frac{1}{2} \right) \left(\frac{1}{2} \right) \left(\frac{3}{2} \right) \left(\frac{1}{4!} \right) \left(\frac{\delta}{t} \right)^3 - \dots \right]} \\
&= \frac{\frac{2}{3} \left(\frac{t}{\delta} \right)}{1 - \frac{1}{2!} \left(\frac{\delta}{2t} \right) - \frac{1}{3!} \left(\frac{\delta}{2t} \right)^2 - \frac{1 \cdot 3}{4!} \left(\frac{\delta}{2t} \right)^3 - \dots}
\end{aligned}$$

The correction function may now be expressed as

$$\begin{aligned}
F^{\text{trap}}(t) &= \frac{3}{2} \left(\frac{\delta}{t} \right), \quad 0 \leq t \leq \delta, \\
&= \frac{1}{1 - \frac{1}{2!} \left(\frac{\delta}{2t} \right) - \frac{1}{3!} \left(\frac{\delta}{2t} \right)^2 - \frac{1 \cdot 3}{4!} \left(\frac{\delta}{2t} \right)^3 - \dots}, \quad \delta \leq t \leq \eta.
\end{aligned}$$

A plot of F^{trap} vs the dimensionless time ratio (t/δ) is given in Figure 3-3, from which it can be seen that the correction factor approaches unity rapidly as (t/δ) increases. With directly measured values of δ , the irradiance rise time, and using the curve of Figure 3-3, it is now a simple matter to correct a measured temperature response to a true step function response. The measurement of δ will be described in a subsequent chapter.

3.7 Summary of Predictions from Opaque Solid Theory

The various relations now established enable one to examine in some detail the temperature response of the opaque semi-infinite solid. For convenience, the assumptions involved in the analysis and the predictions based on these assumptions will now be briefly summarized.

Figure 3-3. Factor for Correcting Trapezoidal Pulse Response to Ideal Step-Function Response, for Opaque Solid Surface

a. General assumptions

1. The receiver is a semi-infinite solid at rest, initially at uniform temperature, T_0 , throughout.
2. The heat capacity per unit volume, ρ , and the thermal conductivity, k , are independent of lateral position and time (or temperature). They may vary with depth, x , only.
3. At zero time, the surface of the solid, in the plane $x = 0$, is to be exposed uniformly to a pulse of radiation.
4. The receiver is opaque; i.e. radiation is absorbed at the surface, $x = 0$. The reflectance, R , is constant in lateral position and time.

b. Specific model I

1. The solid is isotropic; i.e. k and ρ are constant.
2. The irradiance pulse is rectangular, of duration η seconds.
3. Statement of the problem:

$$\frac{\partial U(x,t)}{\partial t} = \alpha \frac{\partial^2 U(x,t)}{\partial x^2}, \quad x \geq 0, \quad t \geq 0$$

$$U(x,0) = 0, \quad x \geq 0$$

$$U(\infty, t) = 0, \quad t \geq 0$$

$$\begin{aligned} \frac{\partial U(0,t)}{\partial x} &= 0, \quad t < 0, \quad t > \eta \\ &= -\frac{1}{k} H_a, \quad 0 \leq t \leq \eta \end{aligned}$$

4. Normalized solution:

$$T(\gamma, \tau) = \frac{U^{rect}(\sqrt{4\alpha\eta}\gamma, \eta\tau)}{U^{rect}(0, \eta)} = \sqrt{\tau} R\left(\frac{\gamma}{\sqrt{\tau}}\right), \quad 0 \leq \tau \leq 1$$

$$= \sqrt{\tau} R\left(\frac{\gamma}{\sqrt{\tau}}\right) - \sqrt{\tau-1} R\left(\frac{\gamma}{\sqrt{\tau-1}}\right), \quad \tau \geq 1,$$

where $\gamma = \frac{x}{\sqrt{4\alpha\eta}}$, $\tau = \frac{t}{\eta}$,

and $R(x) = \sqrt{\pi} \left[\frac{1}{\sqrt{\pi}} e^{-x^2} - x \operatorname{erfc}(x) \right]$.

5. Determination of thermal constants:

Thermal inertia, μ , is evaluated from

$$\frac{U^{*rect}(0, t)}{\sqrt{\eta}} = \frac{2}{\sqrt{\pi\mu}} \sqrt{\frac{t}{\eta}}, \quad 0 \leq \frac{t}{\eta} \leq 1.$$

Thermal diffusivity, α , is evaluated from

$$\frac{x}{\sqrt{\eta}} = \sqrt{4\alpha} \sqrt{\frac{\frac{t_{\max}}{\eta}(\frac{t_{\max}}{\eta} - 1)}{2} \ln\left(\frac{\frac{t_{\max}}{\eta}}{\frac{t_{\max}}{\eta} - 1}\right)},$$

where x is the depth for which the response $U^{rect}(x, t)$ reaches a maximum at time $t_{\max}$.

c. Specific model II

1. The solid is composite; i.e.

$$\begin{aligned} V &= V_1, \quad 0 \leq x < b, \\ &= V_2, \quad x > b, \\ k &= k_1, \quad 0 \leq x < b, \\ &= k_2, \quad x > b. \end{aligned}$$

2. The irradiance pulse is a step function
3. Statement of the problem:

$$\frac{\partial U(x,t)}{\partial t} = \alpha_1 \frac{\partial^2 U(x,t)}{\partial x^2}, \quad 0 \leq x < b, \quad t \geq 0$$

$$\frac{\partial U(x,t)}{\partial t} = \alpha_2 \frac{\partial^2 U(x,t)}{\partial x^2}, \quad x > b, \quad t \geq 0$$

$$U(x,0) = 0, \quad x \geq 0$$

$$U(\infty, t) = 0, \quad t \geq 0$$

$$\begin{aligned} \frac{\partial U(0,t)}{\partial x} &= 0, \quad t < 0 \\ &= -\frac{1}{k} H_a, \quad t \geq 0 \end{aligned}$$

$$U(b-0, t) = U(b+0, t), \quad t \geq 0$$

$$k_1 \frac{\partial U(b-0, t)}{\partial x} = k_2 \frac{\partial U(b+0, t)}{\partial x}, \quad t \geq 0$$

4. Slope of surface response vs $\sqrt{t}$:

$$\frac{\partial U^{step}}{\partial \sqrt{t}} = \frac{2H_a}{\sqrt{\pi\mu_1}} \left[1 + 2 \sum_{n=1}^{\infty} \lambda^n e^{-\frac{b^2 n^2}{\alpha_1 t}} \right]$$

5. Evaluation of thermal inertias:

μ_1 evaluated by

$$\lim_{t \rightarrow 0^+} \frac{\partial U^{step}(0,t)}{\partial \sqrt{t}} = \frac{2}{\sqrt{\pi\mu_1}}$$

μ_2 evaluated by

$$\lim_{t \rightarrow \infty} \frac{\partial U^{step}(0,t)}{\partial \sqrt{t}} = \frac{2}{\sqrt{\pi\mu_2}}$$

d. Correction for finite rise time of irradiance pulse

1. Define trapezoidal pulse as

$$\begin{aligned} H_a^{trap}(t) &= 0, \quad t < 0 \\ &= \left(\frac{H_a}{\delta}\right)t, \quad 0 \leq t \leq \delta \\ &= H_a, \quad \delta \leq t \leq \eta \\ &= H_a - \left(\frac{H_a}{\delta}\right)t, \quad \eta \leq t \leq \eta + \delta \\ &= 0, \quad t \geq \eta + \delta \end{aligned}$$

2. In general, for any model:

$$\begin{aligned} U^{trap}(x, t) &= \left(\frac{H_a}{\delta}\right) \int_0^t U^{step}(x, \tau) d\tau, \quad 0 \leq t \leq \delta \\ &= \frac{H_a}{\delta} \int_{t-\delta}^t U^{step}(x, \tau) d\tau, \quad \delta \leq t \leq \eta \\ &\dots \end{aligned}$$

3. For surface response of isotropic opaque solid

$$\begin{aligned} F^{trap}(t) &= \frac{U^{step}(0, t)}{U^{trap}(0, t)} = \frac{3}{2} \left(\frac{\delta}{t}\right), \quad 0 \leq t \leq \delta \\ &= \frac{1}{1 - \frac{1}{2!} \left(\frac{\delta}{2t}\right) - \frac{1}{3!} \left(\frac{\delta}{2t}\right)^2 - \frac{1 \cdot 3}{4!} \left(\frac{\delta}{2t}\right)^3 - \dots}, \quad \delta \leq t \leq \eta \end{aligned}$$

3.8 Utility of the Opaque Solid Model

It requires no more than a casual observation to convince one that normal living skin is far from opaque, but rather is remarkably translucent. This is true even for rather heavily pigmented skins, as attested to by transmission measurements in the visual and infrared spectral regions (8). It is logical to ask, then, what possible utility the foregoing analysis of the opaque solid may have in elucidating the temperature response of normal skin.

First, it should be pointed out that simply by coating skin with an extremely thin layer of an opaque material, it can be made to approximate an opaque material quite closely. Thus there is no difficulty in altering the experimental material to fit the theory, but this does not explain the reasons for such a procedure.

This question could best be answered after the development, in the next chapter, of the diathermanous solid theory. For the present, it will only be pointed out that the solution for the temperature response of this model is quite complex, in contradistinction to the relatively simple expressions derived for the opaque solid. In particular, the "thermal" and

"optical" constants are so entangled, that their experimental evaluation becomes extraordinarily difficult. Now, a surface treatment to render the skin opaque will not alter the thermal constants (k and ν) of the skin; hence these may be determined by the methods developed previously, whereupon they now become knowns, not unknowns, in the diathermanous solid solution. The evaluation of the remaining unknowns then becomes not only possible, but quite straightforward.

Before leaving this matter, some consideration should be given to the assertion that the surface treatment to render the skin opaque will not alter the thermal constants of the material. While this would certainly seem to be true in theory, in actual practice it may not have been so, as will be discussed in some detail in later chapters. The possible alteration of the thermal constants by surface treatment remains one of the unresolved problems of the present investigation.

FOOTNOTES

- (1) Coulbert, C. D., W. F. MacInnes, T. Ishimoto, et al., Temperature Response of Infinite Flat Plates and Slabs to Heat Inputs of Short Duration at One Surface, University of California at Los Angeles, Department of Engineering Report (1951).
- (2) Churchill, R. V., Modern Operational Mathematics in Engineering, Chapters IV and VII (McGraw-Hill Book Company, Inc., New York 1944).
- (3) The term "thermal inertia for surface heating" was coined by J. D. Hardy. See Section 7.7, and literature citations therein.
- (4) Churchill, R. V., Op. cit., 37, and 299, Transform No. 84
- (5) Ibid., 299, Transform No. 85
- (6) Ibid., 294, Operation No. 3
- (7) Ibid., 294, Operation No. 5
- (8) Hardy, J. D., H. T. Hammel, and D. Murgatroyd, Spectral Transmittance and Reflectance of Excised Human Skin, J. Appl. Physiol. 2, 259-264 (1956).

CHAPTER IV

ANALYSIS OF THE DIATHERMANOUS SOLID

4.1 Definition of the Model

The semi-infinite diathermanous solid model attempts to quantitate the influence of the penetration of radiation into the skin. The choice of this term is rather unfortunate, since it implies the ability to transmit heat (literally "heat through"), and the term "diathermic" is so used in thermodynamics; hence the opaque solid is diathermanous, in this sense. Complying with more or less accepted usage, however, the term will be employed here in the restricted sense of describing a material which can be penetrated by radiant energy.

In defining this model, the basic assumptions of Chapter II will be followed, and in addition k and ν will be considered constant. The heat conduction equation thus reduces to

$$\frac{\partial U(x,t)}{\partial t} = \alpha \frac{\partial^2 U(x,t)}{\partial x^2} + \frac{1}{\nu} q'''(x,t), \quad x \geq 0, \quad t \geq 0 \quad (4-1)$$

with boundary conditions

$$U(x,0) = 0, \quad x \geq 0 \quad (4-2)$$

$$U(\infty, t) = 0, \quad t \geq 0. \quad (4-3)$$

The model will then be completely defined by specifying the function $q'''(x,t)$ and providing one additional boundary condition.

The latter problem will be considered first. As before, the surface of the solid is assumed to be insulated against heat losses, so that the only energy crossing this boundary is the incoming radiation. Here, however, absorption occurs in depth; i.e. the radiation must traverse a finite thickness of material to suffer a finite absorption and conversion into internal energy. Equation (3-4) for the opaque solid is therefore to be replaced by

the homogeneous boundary condition

$$\frac{\partial U(0,t)}{\partial x} = 0, \quad t \geq 0 \quad (4-4)$$

(This is the same condition which applies in the analagous problem of internal Joulean heating of an electrical conductor, with insulated surface.)

The only remaining matter is the specification of the function $q'''(x,t)$. This is equivalent to requiring that the absorption pattern of the radiation in depth be defined, and is one of the major problems of this investigation. This will now be considered in some detail.

4.2 General Consideration of the Absorption of Radiation in Skin

Before dealing with the precise form of $q'''(x,t)$, certain relations which the function must satisfy will be developed. It is important to note, first, that skin is a highly scattering medium, even a thin layer producing almost complete diffusion of an initially collimated beam. Therefore, the usual concepts of transmission and reflection do not apply, and one must speak of total forward and back scatter. If a beam of radiation, of irradiance H_0 , is incident on this scattering medium, a certain fraction, RH_0 , will be back scattered (into the total back hemisphere), where the symbol R may be used in analogy with the surface reflectance of the opaque solid. Similarly, the irradiance absorbed in the material must be $(1-R)H_0$, which again will be represented by H_a . Note that H_0 (and H_a) may be a function of time, but, as before R will be assumed constant.

Now, if a nonscattering diathermanous material (colored glass, for example) is placed in a beam of collimated radiation, it is quite clear that any diminution in the radiation with depth must be due solely to absorption. If the irradiance at the depth x is $H(x,t)$, then the absorption of radiation at this depth in a volume element of unit cross section and thickness dx must be simply $H(x,t) - H(x+dx,t)$. This, however, will be precisely the same as

the "heat generation" per unit volume, $q'''(x,t)$, times the volume of the element, dx . Hence

$$\begin{aligned} q'''(x,t) dx &= H(x,t) - H(x+dx, t) \\ &= H(x,t) - \left[H(x,t) + \frac{\partial H(x,t)}{\partial x} \cdot dx \right] \\ &= - \frac{\partial H(x,t)}{\partial x} dx, \end{aligned}$$

or

$$q'''(x,t) = - \frac{\partial H(x,t)}{\partial x}$$

It is also clear that the absorption (or "heat generation") per unit volume at the depth x must be proportional to the irradiance at that depth, or

$$q'''(x,t) = \gamma(x) H(x,t),$$

where the proportionality constant (the linear absorption coefficient), γ , may be a function of x , as noted. The assumption that γ is independent of x leads immediately to the familiar exponential attenuation pattern

$$H(x,t) = H_0(t) e^{-\gamma x},$$

from which it follows that

$$q'''(x,t) = \gamma H_0(t) e^{-\gamma x}.$$

In the more general case of $\gamma = \gamma(x)$ integration of the equations above leads to

$$H(x,t) = H_0(t) e^{-\int_0^x \gamma(\xi) d\xi},$$

and

$$q'''(x,t) = \gamma(x) H_0(t) e^{-\int_0^x \gamma(\xi) d\xi}.$$

(In all of the above, the small loss by reflection at the interface has been neglected.) Here, then, the determination of the linear absorption coefficient as a function of depth completely specifies the function $q'''(x,t)$.

In the case of the scattering diathermanous material, the situation is, unfortunately, more complex. Plane parallel radiation incident on such a body will soon become quite diffused due to multiple internal scattering.

It is quite possible to define, for any position and time, a net irradiance in the positive x direction, say $H_+(x,t)$, in analogy to the term $H(x,t)$ for the nonscattering case, above. However the absorption per unit volume $q'''(x,t)$ will not depend directly on this net forward irradiance, since a given volume of the material can absorb radiation from any direction. At each level, one must consider the total irradiance, $H_T(x,t)$, available for absorption; given a plane surface of unit cross-section, and at the depth x , the irradiance available for absorption at this depth is the arithmetic sum of all the radiant flux which passes through the plane from any direction. Thus at the surface this total irradiance will be equal to the sum of the incident irradiance, $H_0(t)$, and the total scattered back out of the material, which from the definition of R above, is simply $RH_0(t)$. Explicitly:

$$H_T(0,t) = (1+R) H_0(t).$$

Now, can one still claim that any reduction in $H_T(x,t)$ with depth is due solely to absorption, and hence

$$q'''(x,t) = - \frac{\partial H_T(x,t)}{\partial x} ?$$

It is not difficult to show that the answer must be no. Consider a scattering, but perfectly non-absorbing semi-infinite solid. Since there will be no temperature rise in the material, the back scattered radiation must be equal to the incident radiation, which means that R must be unity. It follows from the discussion above, that, in this case $H_T(0,t) = 2H_0(t)$. If a decrease in $H_T(x,t)$ with depth were due to absorption only, then since there is no absorption in this ideal material, $H_T(x,t)$ must equal $2H_0(t)$ for all values of x . However, at least some part of the back scattered radiation comes from a differential surface layer, hence in the immediately subadjacent layer there cannot be the same total amount of radiation. Repeating this argument for successively deeper layers leads to the conclusion that

$H_T(x, t)$ must tend to zero as x increases without bound, whether absorption occurs or not; i.e.

$$H_T(\infty, t) = 0.$$

Thus, the reduction in $H_T(x, t)$ is not dependent upon absorption, only, and

$$q'''(x, t) \neq \frac{\partial H_T(x, t)}{\partial x}.$$

It is convenient at this point to recognize that the time dependence of $q'''(x, t)$ is simply that of the absorbed irradiance pulse H_a , which depends only upon the time, t , while the variation of $q'''(x, t)$ with depth is independent of the irradiance pulse shape, and hence depends only upon some function of x , say $F(x)$. Clearly, $q'''(x, t)$ may be expressed as the product

$$q'''(x, t) = H_a(t) \cdot F(x).$$

Now let the function $V(x)$ be such that

$$-\frac{dV(x)}{dx} = F(x), \quad x \geq 0.$$

$V(x)$ is thus defined to within an additive constant; for convenience, let $V(\infty) = 0$. Substitution of this relation in the expression above for $q'''(x, t)$ gives:

$$q'''(x, t) = -H_a(t) \cdot \frac{dV(x)}{dx}$$

or

$$q'''(x, t) = -\frac{\partial}{\partial x} [H_a(t) \cdot V(x)], \quad (4-5)$$

from which $V(x)$ may be recognized as the absorption pattern of radiation in the material, and $H_a(t) \cdot V(x)$ might be thought of as the net "absorbable" irradiance at given values of t and x . It will be useful now to develop a necessary condition which will provide a limiting value of $V(x)$.

For the uniformly irradiated semi-infinite scattering solid, the integral of all absorption per unit volume in a column of unit cross section and extending indefinitely far in depth must equal the radiation absorbed

for this unit cross section, or

$$\int_0^{\infty} q'''(x, t) dx = (1-R) H_0(t) = H_a(t) \quad (4-6)$$

This relation holds, since any lateral scatter out of this column will be exactly compensated for by scatter into the column from adjacent regions.

Equation (4-6) is simply a statement of conservation of energy. Substituting equation (4-5) in (4-6), and noting that $V(\infty) = 0$ leads to the desired relation.

$$\begin{aligned} H_a(t) &= - \int_0^{\infty} \frac{\partial}{\partial x} [H_a(t) \cdot V(x)] dx \\ &= - H_a(t) \int_{V(0)}^{V(\infty)} dV \\ &= H_a(t) [V(0) - V(\infty)] \end{aligned}$$

$$\text{or} \quad V(0) = 1 \quad (4-7)$$

Now, in analogy with the non-scattering diathermanous solid, one can define, with complete generality, a linear absorption coefficient, $\gamma(x)$, such that:

$$q'''(x, t) = \gamma(x) [H_a(t) \cdot V(x)] .$$

From equation (4-5), then

$$\frac{dV(x)}{dx} = - \gamma(x) V(x),$$

or

$$\ln \frac{V(x)}{V(0)} = - \int_0^x \gamma(\xi) d\xi,$$

whence from the value of $V(0)$ as found above

$$V(x) = e^{-\int_0^x \gamma(\xi) d\xi} .$$

Substitution back in equation (4-5) gives, finally:

$$q'''(x, t) = \gamma(x) H_a(t) e^{-\int_0^x \gamma(\xi) d\xi} \quad (4-8)$$

It must be realized that nothing of great theoretical significance has been obtained in going from equation (4-5) to (4-8) and, in fact, the latter relation could have been obtained at the outset simply by defining the depth

dependence of $q'''(x,t)$, $F(x)$, as

$$F(x) = \gamma(x) e^{-\int_0^x \gamma(\xi) d\xi},$$

although such a definition might have been a trifle startling. It is pertinent to ask, however, whether this development has resulted in defining a depth function which can be evaluated experimentally, since the form of $q'''(x,t)$ must be determined by some sort of experimental procedure. This last statement must be qualified by noting that the most elegant method of determining $q'''(x,t)$ would be to solve the transport equation, applying all the known characteristics of skin. This would be inordinately complex, and probably not enough is known about the detailed tissue-radiant energy interaction to justify the procedure. Hence, at the present time at least, $q'''(x,t)$ must be determined experimentally.

4.3 Experimental Determinations of the Absorption of Radiation in Skin

In the previous section three expressions for $q'''(x,t)$ were given as:

$$q'''(x,t) = H_a(t) \cdot F(x)$$

$$q'''(x,t) = - \frac{\partial}{\partial x} [H_a(t) \cdot V(x)]$$

$$q'''(x,t) = \gamma(x) H_a(t) e^{-\int_0^x \gamma(\xi) d\xi}.$$

While these relationships are in reality completely interchangeable, they serve to typify the different approaches to the problem of specifying $q'''(x,t)$.

The first equation,

$$q'''(x,t) = H_a(t) \cdot F(x)$$

would seem to be too noncommittal to permit direct experimental determination. One of the most important results of this study is that such direct determination is possible. This will be considered in some detail in a later section

of this chapter; here it will merely be stated that one can, in theory at least, determine the precise form of $F(x)$ (which, of course, is all that is necessary, since $H_a(t)$ is under the control of the experimenter) directly from temperature measurements.

Another method of approach involves the concept of a net "absorbable" irradiance, with the function $q'''(x,t)$ determined according to the second equation above. The pattern of this absorbable radiation in depth--i.e. the form of $V(x)$ --may simply be assumed, or it may be based on experimental evidence. An interesting example of the former is a linear absorption pattern (1), proposed apparently to secure a simple form for $q'''(x,t)$.

Here,

$$\begin{aligned} V(x) &= \left(1 - \frac{x}{L}\right), \quad 0 \leq x \leq L \\ &= 0, \quad x > L, \end{aligned}$$

from which

$$\begin{aligned} q'''(x,t) &= \frac{1}{L} H_a(t), \quad 0 \leq x \leq L \\ &= 0, \quad x > L. \end{aligned}$$

It is simple to show that for this particular pattern, the linear absorption coefficient, $\gamma(x)$, must be given by

$$\begin{aligned} \gamma(x) &= \frac{1}{L \left(1 - \frac{x}{L}\right)}, \quad 0 \leq x \leq L \\ &= \infty, \quad x \geq L \end{aligned}$$

Thus, the absorption coefficient increases steadily from a minimum value of $1/L$ at the surface ($x = 0$), and becomes infinite at the depth $x = L$. This particular variation of $\gamma(x)$ with x is almost surely precisely the opposite of the true situation, since the best measurements to date (which will be discussed below) indicate a decrease in the linear absorption coefficient with depth.

Another interesting example of the employment of the concept of "absorbable" irradiance is contained in a paper by Buettner (2), with a

curve of $V(x)$ deduced from various pieces of experimental evidence gleaned from a thorough literature review. The slope of this curve (apparently obtained graphically) gives $q'''(x,t)$, which is then used in a numerical analysis of heat flow. Of interest in Buettner's analysis is the treatment of heavily pigmented Negro skin ("limited penetration") with a resultant sharp "spike" in $q'''(x,t)$ at the melanin layer.

In the third expression above for $q'''(x,t)$ the dependence of this function on depth is described in terms of a linear absorption coefficient, $\nu(x)$. If, for a scattering material, this coefficient bears some close resemblance to that of a non-scattering medium, where its physical significance is quite clear, then one should be able to determine the form of $q'''(x,t)$ by transmittance and reflectance (i.e., forward and back scatter) measurements on skin sections. It is, in the writer's opinion, extremely difficult either to confirm or deny the propriety of such optical measurements on theoretical grounds. Essentially, the problem is to make an intelligent appraisal of the effect of the necessary process of removing sections of the material to determine variation in forward and back scatter with thickness; the scatter from the material thus removed can now no longer contribute to absorption in that remaining.

While there may be some question as to how (or whether) measurements of scatter may be employed in specifying the volume absorption of radiation in intact skin, at least there are now no doubts as to the proper experimental methods which must be followed in order that the measurements themselves be valid. The work of Hardy and his collaborators, particularly in the last decade, has furnished the best information available at present on the scatter of visible and infrared radiation by skin, and also has established quite clearly the procedures necessary for the obtaining of such reliable and consistent data (3, 4, 5, 6). Briefly summarized, the requirements are as

follows:

1. Forward scatter ("transmission"), back scatter ("reflection"), and absorption measurements must all be made on the same sample. (See also Buettner (2), bottom of p. 207).
2. These measurements must be made on an absolute basis, that is, not based on comparison with some questionable reflectance standard.
3. The scattered and absorbed components must be measured over the desired spectral range with monochromatic radiation of moderate purity. Because of the pronounced influence of wavelength on absorption coefficient, the use of broad spectral bands--isolated, say, by filters--is inadvisable.
4. It is highly desirable, although probably not necessary, to establish not only the total scatter into each hemisphere, but also the polar distribution of this scatter, again as a function of wavelength. Such data yield important clues as to the details of the tissue-energy interaction.
5. The thickness of the sample under investigation should be known accurately.
6. In addition, it would seem desirable to follow the procedure established by Hardy and vary the thickness from one sample to another by removing the deeper tissues, leaving always the intact surface to be exposed to the incident monochromatic radiation.

The instrument used by Hardy, a goniometer spectrophotometer (7), satisfies most elegantly the first four requirements above. For example, with respect to the second of these, the summation of scattered radiation for a non-absorbing material will be within a fraction of one percent of the measured incident radiation. A very slight amount of absorption can thus be measured with good accuracy.

The tissues analyzed in this instrument were obtained from surgical specimens or from autopsy, and kept in saline or a moist chamber at all

times. Samples were cut to thicknesses of from 0.3 to 2.1 mm, after which each specimen of suitable thickness was mounted between microscope slides and sealed with stop-cock grease. From the thickness of this assembly, measured with micrometer calipers, the thickness of the two microscope slides was deducted to give the tissue thickness accurately. While Hardy himself stresses the artificiality of these conditions, he points out that this mode of preparation is necessary to obtain consistent results (8). Further, while one must assume that the scattering properties of these preparations have been altered, it cannot be claimed a priori that they are drastically dissimilar from those of living, intact skin. Hence, it seems reasonable to accept the results of Hardy et al. as the best current estimates of the absorption of radiation in skin.

A preliminary observation might first be noted. Measurements on stacks of ground glass plates led to the result that the increase in absorption of radiation with stack thickness (i.e. number of plates) followed an exponential form in exact accordance with what one would expect for a non-scattering medium (9). This may provide some justification for expressing the absorption pattern in terms of a linear absorption coefficient. With skin samples, in the spectral region from 1.0μ out to the limit studied, 2.4μ , exactly the same situation prevailed; i.e. the absorption of radiation could be characterized by a single absorption coefficient, independent of depth, although dependent on wavelength. For a short wavelength span, then, equation (4-8) may be written

$$q'''(x, t) = \gamma H_a(t) e^{-\gamma x}.$$

For wavelengths shorter than 1.0μ , however, the data were best fit by a "double exponential" form with one absorption coefficient, γ_1 , applicable for the superficial layers, and another, γ_2 , applying for the deeper tissue.

Again, both γ_1 and γ_2 depended upon wavelength, and in all cases, γ_1 was greater than γ_2 ; i.e. the absorption was more pronounced near the surface. Now, equation (4-8) must be expressed (for a narrow wavelength interval) as:

$$\begin{aligned} q'''(x,t) &= \gamma_1 H_a(t) e^{-\gamma_1 x}, \quad 0 \leq x < b \\ &= \gamma_2 H_a(t) e^{-\gamma_1 b} e^{-\gamma_2(x-b)}, \quad x > b, \end{aligned} \quad (4-9)$$

where b is the depth of the "break" in the absorption pattern. Clearly, the first equation above, involving only a single coefficient, γ , is a special case of the second set, where $\gamma_1 = \gamma_2$. Thus, one may take equation (4-9) as the general expression for $q'''(x,t)$, allowing γ_1 and γ_2 to assume any (positive) values, including equality. This pattern of $q'''(x,t)$, for brevity termed the double exponential pattern, is the one selected for the theoretical analysis of the diathermanous solid, which will follow in the next section.

The objective of this and the previous section was to establish a functional form of $q'''(x,t)$ for substitution in the governing differential equation (4-1). This objective has been achieved with the statement of equation (4-9). These sections have been quite extensive for the purpose of offering some justification for the double exponential pattern; however, it is definitely not claimed that this form has now been rigorously established as being "true"; i.e. representing the actual state of affairs in skin. Rather, equation (4-9) having now been presented, it should simply be regarded as another assumption of the diathermanous solid model, which assumption can be validated only by experiment. With this qualification, the main task of establishing the temperature response of the diathermanous solid model will now be resumed.

4.4 The Solution of the Heat Conduction Equation for the Diathermanous Solid with Double Exponential Absorption

Collecting the various equations presented previously, one may now state the problem in full as:

$$\frac{\partial U(x,t)}{\partial t} = \alpha \frac{\partial^2 U(x,t)}{\partial x^2} + \frac{\gamma_1}{\nu} H_{\nu_1}(t) e^{-\gamma_1 x}, \quad 0 \leq x < b, \quad t \geq 0 \quad (4-10)$$

$$\frac{\partial U(x,t)}{\partial t} = \alpha \frac{\partial^2 U(x,t)}{\partial x^2} + \frac{\gamma_2}{\nu} H_{\nu_2}(t) e^{-\gamma_1 b} e^{-\gamma_2(x-b)}, \quad x > b, \quad t \geq 0 \quad (4-11)$$

$$U(x,0) = 0, \quad x \geq 0 \quad (4-2)$$

$$U(\infty, t) = 0, \quad t \geq 0 \quad (4-3)$$

$$\frac{\partial U(0,t)}{\partial x} = 0, \quad t \geq 0 \quad (4-4)$$

$$U(b-0, t) = U(b+0, t), \quad t \geq 0 \quad (4-12)$$

$$\frac{\partial U(b-0, t)}{\partial x} = \frac{\partial U(b+0, t)}{\partial x}, \quad t \geq 0 \quad (4-13)$$

Equations (4-12) and (4-13) are continuity conditions analagous to equations (3-22) and (3-23) of Chapter III. As stated in the previous section, both

γ_1 and γ_2 are dependent upon wavelength, hence the above set of equations is applicable only in a wavelength interval sufficiently short that the coefficients are approximately constant. Extension of the results to a broad spectral range will be considered later.

Once again, the problem will be attacked by the method of the Laplace transform, with the consequent reduction of equations (4-10) and (4-11) to second order ordinary differential equations. Thus, taking the transform of these equations, and utilizing condition (4-2), gives

$$\frac{d^2 U(x,s)}{dx^2} - \frac{s}{\alpha} U(x,s) = - \frac{\gamma_1}{\alpha \nu} h_{\nu_1}(s) e^{-\gamma_1 x}, \quad 0 \leq x < b \quad (4-14)$$

and

$$\frac{d^2 u(x,s)}{dx^2} - \frac{s}{\alpha} u(x,s) = \frac{\gamma_2}{\alpha \nu} h_a(s) e^{-\gamma_1 b} e^{-\gamma_2(x-b)}, \quad x > b, \quad (4-15)$$

where, as before, $u(x,s)$ and $h_a(s)$ are the transforms of $U(x,t)$ and $H_a(t)$, respectively. The remaining boundary and continuity conditions become:

$$u(\infty, s) = 0 \quad (4-16)$$

$$\frac{du(0,s)}{dx} = 0 \quad (4-17)$$

$$u(b-0, s) = u(b+0, s) \quad (4-18)$$

$$\text{and} \quad \frac{du(b-0, s)}{dx} = \frac{du(b+0, s)}{dx} \quad (4-19)$$

It is interesting to note that in this problem, the transform of the irradiance function, $h_a(s)$, enters in the non-homogeneous differential equation, while the boundary values are homogeneous, in contradistinction to the opaque solid problem where the differential equation is homogeneous, and $h_a(s)$ enters via a non-homogeneous boundary condition.

The homogeneous solution of equation (4-14), say $u_H(x,s)$, is

$$u_H(x,s) = A e^{-\frac{s}{\alpha} x} + B e^{+\frac{s}{\alpha} x}, \quad 0 \leq x < b,$$

precisely as in the opaque solid case. To this solution must be added the particular integral, $u_p(x,t)$ which may be assumed to have the form

$$u_p(x,s) = M e^{-\gamma_1 x}.$$

Substituting this expression in (4-14) gives

$$(\gamma_1^2 - \frac{s}{\alpha}) M e^{-\gamma_1 x} = - \frac{\gamma_1}{\alpha \nu} h_a(s) e^{-\gamma_1 x},$$

or

$$M = \frac{\gamma_1}{\nu} \frac{h_a(s)}{(s - \gamma_1^2 \alpha)}.$$

The condition (4-17) requires that

$$-\sqrt{\frac{s}{\alpha}} A + \sqrt{\frac{s}{\alpha}} B - \frac{r_1^2}{\nu} \frac{h_a(s)}{(s - r_1^2 \alpha)} = 0,$$

from which the constant of integration B may be evaluated in terms of A to give the complete solution of (4-14) as:

$$u(x,s) = A \left(e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}} + e^{+\frac{x}{\sqrt{\alpha}} \sqrt{s}} \right) + \frac{r_1}{\nu} \frac{h_a(s)}{(s - r_1^2 \alpha)} \left(e^{-r_1 x} + r_1 \sqrt{\frac{\alpha}{s}} e^{+\frac{x}{\sqrt{\alpha}} \sqrt{s}} \right),$$

$$0 \leq x < b$$

Similar considerations lead to the complete solution of (4-15) as:

$$u(x,s) = C e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}} + \frac{r_2}{\nu} \frac{h_a(s) e^{-r_2 b}}{(s - r_2^2 \alpha)} e^{-r_2(x-b)}, \quad x > b,$$

where the limiting value (4-16) has been used to evaluate one of the constants of integration. The two constants of integration remaining in the above equations must now be evaluated by the continuity conditions (4-18) and (4-19). The procedure is given in Appendix V, where the detailed solution for the diathermanous solid model is presented. With the appropriate expressions for these two constants substituted in the equations above, one obtains finally the transformed response for this model as

$$u(x,s) = \frac{h_a(s)}{\nu} \left\{ \frac{r_1 e^{-r_1 x}}{s - r_1^2 \alpha} - \frac{r_1^2 \sqrt{\alpha}}{\sqrt{s} (s - r_1^2 \alpha)} e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}} + \frac{e^{-r_1 b}}{2} \left(\frac{r_2}{\sqrt{s} (\sqrt{s} + r_2 \sqrt{\alpha})} - \frac{r_1}{\sqrt{s} (\sqrt{s} + r_1 \sqrt{\alpha})} \right) \left(e^{-\frac{b+x}{\sqrt{\alpha}} \sqrt{s}} + e^{-\frac{b-x}{\sqrt{\alpha}} \sqrt{s}} \right) \right\}, \quad 0 \leq x < b, \quad (4-20)$$

and

$$u(x,s) = \frac{h_a(s)}{\nu} \left\{ \frac{r_2 e^{-r_2 b} e^{-r_2(x-b)}}{s - r_2^2 \alpha} - \frac{r_1^2 \sqrt{\alpha}}{\sqrt{s} (s - r_1^2 \alpha)} e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}} + \frac{e^{-r_1 b}}{2} \left[\left(\frac{r_2}{\sqrt{s} (\sqrt{s} + r_2 \sqrt{\alpha})} - \frac{r_1}{\sqrt{s} (\sqrt{s} + r_1 \sqrt{\alpha})} \right) e^{\frac{x+b}{\sqrt{\alpha}} \sqrt{s}} - \left(\frac{r_2}{\sqrt{s} (\sqrt{s} - r_2 \sqrt{\alpha})} - \frac{r_1}{\sqrt{s} (\sqrt{s} - r_1 \sqrt{\alpha})} \right) e^{-\frac{x-b}{\sqrt{\alpha}} \sqrt{s}} \right] \right\}, \quad x > b. \quad (4-21)$$

Letting the bracketed terms in equations (4-20) and (4-21) be $f_1(x,s)$ and $f_2(x,s)$ with inverse transforms $F_1(x,t)$ and $F_2(x,t)$, respectively, one may write the solution for $U(x,t)$ by use of the convolution as

$$\begin{aligned} U(x,t) &= \int_0^t H_a(t-\tau) \frac{1}{V} F_1(\tau) d\tau, \quad 0 \leq x < b \\ &= \int_0^t H_a(t-\tau) \frac{1}{V} F_2(\tau) d\tau, \quad x > b, \end{aligned}$$

for any general irradiance pulse form. It is more useful, however, to consider again the step function:

$$\begin{aligned} H_a(t) &= 0, \quad t < 0 \\ &= H_a, \quad t \geq 0; \end{aligned}$$

whereupon

$$h_a(s) = \frac{H_a}{s}$$

The determination of the inverse transforms of (4-20) and (4-21) is straightforward but quite lengthy, and has been relegated to Appendix V. As is shown there, the final expression for the temperature response, $U^{\text{step}}(x,t)$ can be normalized by defining the dimensionless terms

$$\begin{aligned} \xi &= \gamma_1 x, & \lambda_1 &= \gamma_1 b \\ \theta &= \gamma_1^2 \alpha t, & \lambda_2 &= \gamma_1 / \gamma_2 \end{aligned}$$

and

$$\Psi(\xi, \theta) = \frac{k \gamma_1 U^{\text{step}}(\xi/\gamma_1, \theta/\gamma_1^2 \alpha)}{H_a}$$

whereupon the normalized response becomes

$$\begin{aligned} \Psi(\xi, \theta) = & \frac{2}{\sqrt{\pi}} \sqrt{\theta} e^{-\frac{\xi^2}{4\theta}} - \xi \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\theta}}\right) - e^{-\xi} \\ & + \frac{e^{\theta}}{2} \left[e^{\xi} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\xi}{2\sqrt{\theta}}\right) + e^{-\xi} \operatorname{erfc}\left(\sqrt{\theta} - \frac{\xi}{2\sqrt{\theta}}\right) \right] \\ & + \frac{e^{-\lambda_1}}{2} \left[\left(\operatorname{erfc}\left(\frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) + \operatorname{erfc}\left(\frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) \right) (1 - \lambda_2) \right. \\ & \quad - e^{\lambda_1 + \xi} e^{\theta} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) + \lambda_2 e^{\frac{\lambda_1 + \xi}{\lambda_2}} e^{\frac{\theta}{\lambda_2}} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) \\ & \quad \left. - e^{\lambda_1 - \xi} e^{\theta} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) + \lambda_2 e^{\frac{\lambda_1 - \xi}{\lambda_2}} e^{\frac{\theta}{\lambda_2}} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) \right], \end{aligned}$$

$$\theta \geq 0, \quad 0 \leq \xi < \lambda_1$$

(4-22)

and

$$\begin{aligned} \Psi(\xi, \theta) = & \frac{2}{\sqrt{\pi}} \sqrt{\theta} e^{-\frac{\xi^2}{4\theta}} - \xi \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\theta}}\right) - e^{-\lambda_1} \left[\lambda_2 \left(e^{\frac{\lambda_1 - \xi}{\lambda_2}} - 1 \right) + 1 \right] \\ & + \frac{e^{\theta}}{2} \left[e^{\xi} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\xi}{2\sqrt{\theta}}\right) + e^{-\xi} \operatorname{erfc}\left(\sqrt{\theta} - \frac{\xi}{2\sqrt{\theta}}\right) \right] \\ & + \frac{e^{-\lambda_1}}{2} \left[\left(\operatorname{erfc}\left(\frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) + \operatorname{erfc}\left(\frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) \right) (1 - \lambda_2) \right. \\ & \quad - e^{\lambda_1 + \xi} e^{\theta} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) + \lambda_2 e^{\frac{\lambda_1 + \xi}{\lambda_2}} e^{\frac{\theta}{\lambda_2}} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) \\ & \quad \left. - e^{\lambda_1 - \xi} e^{\theta} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) + \lambda_2 e^{\frac{\lambda_1 - \xi}{\lambda_2}} e^{\frac{\theta}{\lambda_2}} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) \right], \end{aligned}$$

$$\theta \geq 0, \quad \xi > \lambda_1$$

(4-23)

This function thus depends upon the dimensionless variables ξ and θ , proportional to depth and time, respectively, and the dimensionless parameters λ_1 and λ_2 .

These two expressions are seen to be identical except that the term $-e^{-\xi}$ in (4-22) is replaced by $-e^{-\lambda_1} [\lambda_2 (e^{\frac{\lambda_1 - \xi}{\lambda_2}} - 1) + 1]$ in (4-23). This, incidentally makes it quite simple to check the continuity conditions (4-18) and (4-19). Thus

$$\lim_{\xi \rightarrow \lambda_1^-} (-e^{-\xi}) = -e^{-\lambda_1},$$

and
$$\lim_{\xi \rightarrow \lambda_1^+} \left\{ -e^{-\lambda_1} [\lambda_2 (e^{\frac{\lambda_1 - \xi}{\lambda_2}} - 1) + 1] \right\} = -e^{-\lambda_1};$$

further
$$\lim_{\xi \rightarrow \lambda_1^-} \frac{\partial}{\partial \xi} (-e^{-\xi}) = e^{-\lambda_1}$$

while
$$\lim_{\xi \rightarrow \lambda_1^+} \frac{\partial}{\partial \xi} \left\{ -e^{-\lambda_1} [\lambda_2 (e^{\frac{\lambda_1 - \xi}{\lambda_2}} - 1) + 1] \right\} = e^{-\lambda_1}.$$

Hence the continuity equations must be satisfied. It should also be noted that if $\lambda_2 = 1$ (i.e. $\gamma_1 = \gamma_2$) both equations reduce to

$$\begin{aligned} \Psi(\xi, \theta) = & \frac{2}{\sqrt{\pi}} \sqrt{\theta} e^{-\frac{\xi^2}{4\theta}} - \xi \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\theta}}\right) - e^{-\xi} \\ & + \frac{e^{\theta}}{2} \left[e^{\xi} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\xi}{2\sqrt{\theta}}\right) + e^{-\xi} \operatorname{erfc}\left(\sqrt{\theta} - \frac{\xi}{2\sqrt{\theta}}\right) \right], \end{aligned}$$

which is the expression for "single exponential" absorption. Finally, the first two terms in the equations may be expressed as

$$\frac{2}{\sqrt{\pi}} \sqrt{\theta} e^{-\frac{\xi^2}{4\theta}} - \xi \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\theta}}\right) = \frac{2}{\sqrt{\pi}} \sqrt{\theta} R\left(\frac{\xi}{2\sqrt{\theta}}\right),$$

where the function $R(x)$ was defined in Chapter III (see Appendix III). This does not imply that equation (4-22) or (4-23) will reduce to the solution for the opaque solid for some limiting situation; this is clearly impossible since the homogeneous boundary condition (4-4) can never reduce to the

non-homogeneous expression (3-4), which latter condition is a unique feature of opaque solid theory.

Other than the few comments above, it is difficult to make any general statements about equations (4-22) and (4-23). Their complexity makes it virtually impossible to gain any intuitive "feel" of the form of the solutions, and one is forced to employ numerical computations and point-by-point curve drawing. Even this procedure is not simple; using a desk calculator, it requires from 30 minutes to one hour to determine and check a single value of $\Psi(\xi, \theta)$ for one set of values of ξ , θ , λ_1 , and λ_2 . Accordingly, the solutions were programmed for numerical computation on the University Computing Center's IBM 650 digital computer, and values of $\Psi(\xi, \theta)$ determined for rather wide ranges of the above-mentioned variables and parameters. (It is interesting to note that the computation time per point was about 30 seconds, which is long for machine time, but an improvement by about a factor of 100 over average desk calculator time.) The programming, debugging, and actual machine operation were all done by Dr. A. M. Dutton, of the Departments of Radiation Biology and Mathematics; his labor of love is here acknowledged with profound gratitude.

The numerical values obtained upon print-out of the computer answer cards may be thought of as representing points on a hyper-surface in five dimensions, Ψ , ξ , θ , λ_1 , and λ_2 . To be useful, these must be reduced to two-dimensional plots, with the other variates represented as parameters. While the data have been plotted in various ways, the most immediately appealing scheme is to present Ψ , the normalized response, as a function of θ , the normalized time; this secures a close resemblance to experimental curves, where temperature was recorded against time. Accordingly, this is the procedure followed in the following sets of curves. In Figure 4-1, the normalized temperature rise is plotted against the

normalized time, with normalized depth as a parameter; the ratio of linear absorption coefficients, λ_2 , is unity, hence this plot represents the results for "single exponential" absorption. As noted above, the results in this case are independent of λ_1 , the normalized depth of the break in the absorption pattern. Figures 4-2a, b, and c present the same information with λ_2 equal to 3, and λ_1 equal to 0.1, 0.5, and 1.0. In Figures 4-3a, b, and c, λ_2 is 5, and λ_1 is 0.1, 0.5, and 1.0. It will be noted in several of the figures that values for ξ equal to 10 do not appear. These points were not calculated, since it seemed likely that this represented a depth far below any which would actually be observed.

A feature of some interest in these curves is that for the smaller values of ξ , at least, the initial slope (i.e. $\lim_{\theta \rightarrow 0^+} \frac{\partial T(\xi, \theta)}{\partial \theta}$) is finite; it will be shown below that this holds for all values of ξ . This is in marked contrast to the solutions for the opaque solid. In this latter case, the initial time rate of change of the temperature response is infinite on the surface ($x = 0$), and zero for all subsurface positions ($x > 0$). An immediate consequence of this last fact is that when one obtains the response of the opaque solid for a rectangular pulse by superposition there will be no immediate downward break in the subsurface response curves, and the temperature will continue to rise to a "smooth" maximum at some time following the termination of the exposure.

Consider now this same superposition to obtain the rectangular pulse response of the diathermanous solid. Because of the initially finite slope of temperature response for all depths, it follows that there will be a break in all curves at the termination of the exposure. Inspection of the accompanying figures suggests that for some depths (values of ξ up to 2.0, at least), the slope of the temperature response curves will actually be negative immediately following the termination of the pulse, which is

Figure 4-1. Predicted Response of the Diathermanous Solid, Single Exponential Absorption

Figure 4-2a. Predicted Response of the Diathermanous Solid, Double Exponential Absorption

Figure 4-2b. Predicted Response of the Diathermanous Solid, Double Exponential Absorption

Figure 4-2c. Predicted Response of the Diathermanous Solid, Double Exponential Absorption

Figure 4-3a. Predicted Response of the Diathermanous Solid, Double Exponential Absorption

Figure 4-3b. Predicted Response of the Diathermanous Solid, Double Exponential Absorption

Figure 4-3c. Predicted Response of the Diathermanous Solid, Double Exponential Absorption

simply another way of stating that the temperature reaches its maximum value at the end of the exposure, and thereafter declines back to its initial value. This is a very characteristic feature of the diathermanous solid response; not only on the surface, but also for some distance into the material, the temperature decreases abruptly upon termination of a rectangular pulse.

It might be mentioned here in passing, that the solutions for the diathermanous solid response, equations (4-22) and (4-23) have been presented for an irradiance step function, only. For a rectangular pulse of duration η seconds, one may write, for the normalized response,

$$\begin{aligned}\Psi^{rect}(\xi, \theta) &= \Psi^{step}(\xi, \theta) \quad , \quad 0 \leq \theta \leq \tau_1^2 \alpha \eta \\ &= \Psi^{step}(\xi, \theta) - \Psi^{step}(\xi, \theta - \tau_1^2 \alpha \eta) \quad , \quad \theta > \tau_1^2 \alpha \eta.\end{aligned}$$

If one writes these out in full, performing the indicated substitutions in (4-22) and (4-23), one quickly becomes convinced that nothing of a general nature can be obtained from the lengthy and unwieldy expressions.

4.5 Experimental Evaluation of the Constants of the Diathermanous Solid

While equations (4-22) and (4-23) represent an interesting exercise in the solution of a particular partial differential equation, they will be of value in the present study only if some means can be found for experimental evaluation of the various unknowns of the equations. Two of the constants which appear are thermal conductivity, k , and thermal diffusivity, α , which occur in the normalizing factors for temperature rise and time, respectively. These, however, are "thermal" constants, which may be evaluated directly from opaque solid studies, and hence may be considered as known quantities in this present case. (As explained in Chapter III, it is precisely for this reason that one would study "opaqued" skin response.) The normalizing factors thus contain but one unknown "optical" constant,

γ_1 , the linear absorption coefficient for the superficial layers. The other unknowns which must be determined, then, are γ_2 , the linear absorption coefficient for the deeper material, and b , the depth of the break between γ_1 and γ_2 .

If one compares the curves of Figures 4-1, 4-2, and 4-3, it is immediately obvious that these sets of curves are all quite similar. Thus, it is virtually impossible to determine the unknown constants simply by comparing experimental curves with these predicted ones. There are simply no outstanding characteristics to serve as a guide in making such comparisons, and further, all variables--temperature, depth, and time--have been stretched by unknown factors.

Considerable progress can be made if one turns to an alternative procedure developed from consideration of the initial slopes of the $\bar{Y}(\varepsilon, \theta)$ vs θ curves. As noted before, these values all appear to be finite, which is a distinctive characteristic of the diathermanous solid. It is of interest, then, to develop an analytical expression for these slopes.

It will be convenient at this point to revert to the solutions for $U(x,t)$ before normalization. In Appendix V, the general expression for $\partial U(x,t)/\partial t$ is presented, and when one takes the limit as t approaches zero, the following is obtained:

$$\begin{aligned} \frac{\partial U(x,t)}{\partial t} &= \frac{\gamma_1}{\nu} H_a e^{-\gamma_1 x} , \quad 0 \leq x < b \\ &= \frac{\gamma_2}{\nu} H_a e^{-\gamma_2 b} e^{-\gamma_2 (x-b)} , \quad x > b . \end{aligned} \quad (4-24)$$

Now, these extraordinarily simple equations suggest a most obvious method for determining γ_1 and γ_2 , and hopefully also b . Consider first the initial time rate of change of the surface temperature. This will be simply

$$\frac{\partial U(0,0)}{\partial t} = \frac{\gamma_1 H_a}{\nu} ,$$

or, using the previously defined U^* as the response per unit absorbed irradiance:

$$\frac{\partial U^*(0,0)}{\partial t} = \frac{\gamma_1}{V}.$$

With the thermal constant V known, γ_1 is immediately given by simply measuring this initial slope from the recorded data. Now consider the logarithmic form of equation (4-24):

$$\begin{aligned} \ln \frac{\partial U^*(x,0)}{\partial t} &= \ln \frac{\gamma_1}{V} - \gamma_1 x, \quad 0 \leq x < b \\ &= \ln \frac{\gamma_2 e^{-(\gamma_1 - \gamma_2)b}}{V} - \gamma_2 x, \quad x > b. \end{aligned}$$

Thus, plotting the logarithm of the measured initial slopes of temperature at various depths against depth, x , one should obtain two straight lines, one with a slope of $-\gamma_1$ for $0 \leq x < b$, and the other with a slope of $-\gamma_2$ for $x > b$. This procedure, subject to the practical difficulties of achieving accurate measurements of the initial slope, offers a straightforward means of determining the unknown constants γ_1 , γ_2 , and b .

4.6 Direct Experimental Determination of the Pattern of Absorption of Radiation in Skin

Equation (4-24) has been derived for the special case of an irradiance step function, i.e.

$$\begin{aligned} H_a(t) &= 0, \quad t < 0 \\ &= H_a, \quad t \geq 0 \end{aligned}$$

Let this functional form for $H_a(t)$ now be substituted in the expression for $q'''(x,t)$ selected for this analysis (equation 4-9):

$$\begin{aligned} q'''(x,t) &= 0, \quad t < 0, \quad x \geq 0 \\ &= \gamma_1 H_a e^{-\gamma_1 x}, \quad t \geq 0, \quad 0 \leq x < b \\ &= \gamma_2 H_a e^{-\gamma_1 b} e^{-\gamma_2 (x-b)}, \quad t \geq 0, \quad x > b, \end{aligned}$$

and determine, in particular, the value of $q'''(x,t)$ at $t = 0$. Obviously, this is:

$$\begin{aligned} q'''(x,0) &= \gamma_1 H_a e^{-\gamma_1 x}, \quad 0 \leq x < b \\ &= \gamma_2 H_a e^{-\gamma_1 b} e^{-\gamma_2(x-b)}, \quad x > b. \end{aligned}$$

Comparison of this last equation with equation (4-24) leads to the surprising result that in this special case of double exponential absorption and step-function input:

$$\frac{\partial U(x,0)}{\partial t} = \frac{1}{\nu} q'''(x,0) \quad (4-25)$$

Since double exponential absorption contains single exponential as a special case, for this latter pattern the relation (4-25) will be satisfied as well.

It is natural to ask whether this relationship is valid only for an exponential-type form of $q'''(x,t)$. From the solutions for the linear absorption pattern, briefly mentioned in Section 4.3, it may be shown that for the step function form of $H_a(t)$, equation (4-25) is satisfied. (These solutions may be found in Appendix II; since this form of absorption is known to be incorrect, the derivation of the solutions has not been presented.) In addition, this relation was found to hold for several textbook problems. If one is willing to adopt the definition of $q'''(x,t)$ for the opaque solid as

$$\begin{aligned} q'''(x,0) &= \infty, \quad x = 0 \\ &= 0, \quad x > 0, \end{aligned}$$

then (4-25) is valid for this model, as well.

May equation (4-25) be accepted as a general relationship, valid not only for various forms of the absorption pattern, but also for arbitrary irradiance input functions? If so, it follows that one has at hand not merely a method of checking some particular assumed absorption pattern, but rather a means of actually establishing the pattern itself directly from experimental

data. In Section 4.2, it was pointed out that $q'''(x,t)$ may be expressed with complete generality as a product

$$q'''(x,t) = H_a(t) \cdot F(x),$$

where the depth dependence of $q'''(x,t)$, $F(x)$, may be considered as the general absorption pattern of the radiation. Substituting this expression in equation (4-25) leads to

$$\frac{\partial U(x,0)}{\partial t} = \frac{1}{\nu} H_a(0) \cdot F(x)$$

or, if $H_a(0)$ is non-zero:

$$F(x) = \frac{\nu}{H_a(0)} \frac{\partial U(x,0)}{\partial t} \quad (4-26)$$

All terms on the right-hand side of this expression may be determined by experiment; hence this relation establishes the previously mentioned claim that one can define the absorption pattern of radiation in skin, $F(x)$, directly from temperature measurements. Equation (4-26) represents one of the most important results of this study; it is, so far as the writer can determine, an original contribution to the problem of determining the interaction of radiation with a diathermanous material. In theory, at least, this procedure could be useful in studies of microwaves, or, more generally, any system governed by the diffusion (heat conduction) equation.

Now, all of this development clearly depends upon the general validity, as yet unestablished, of equation (4-25). It might be noted here that substitution of equation (4-25) in equation (4-1) yields the equivalent statement:

$$\frac{\partial^2 U(x,0)}{\partial x^2} = 0, \quad x \geq 0 \quad (4-27)$$

A review of several texts on heat transfer and mathematics failed to yield an analytical proof of either equation (4-25) or (4-27); also the writer was unable after some time and effort to establish such a proof, and hence

was forced to fall back on a physical argument which indicated the probable generality of these expressions. This argument may be phrased as follows:

In the first instant after the irradiation commences, the temperature rise at any position will be governed solely by the absorption of radiant energy at that point, and will be uninfluenced by the temperature of surrounding regions; from this, equation (4-25) follows. Now, the term

$\partial^2 U(x,t) / \partial x^2$ in the heat conduction equation represents the heat flow term; equation (4-27), then, follows from the argument that no heat will flow until some infinitesimal temperature gradient is established. This is equivalent to the claim above that initially, the temperature at any given point is independent of that of surrounding regions.

While this argument may be intuitively appealing, it is quite unacceptable as proof of the general validity of equation (4-25) or (4-27). First, it provides no clue as to the class of systems wherein these relations are valid. Secondly, it employs, only thinly disguised, the concept of heat as a fluid (caloric); indeed, this fluid, while still massless, must be endowed with inertia. What is needed is a rigorous proof of these equations, whereupon the interesting concepts in the paragraph above become consequences, not "proofs" of the relations.

Accordingly, attention must again be turned to the problem of constructing an analytical proof of either (4-25) or (4-27). Consider the case of unidirectional heat flow in a semi-infinite isotropic body initially at uniform temperature throughout, and with insulated surface. Two boundary conditions may be given immediately:

and
$$U(x,0) = 0, \quad x \geq 0 \quad (4-2)$$

$$\frac{\partial U(0,t)}{\partial x} = 0, \quad t \geq 0 \quad (4-4)$$

Now, define the Laplace transform of $U(x,t)$ with respect to x as

$$\mathcal{L}_x \{ U(x,t) \} = u(s,t);$$

equation (4-2) above thus becomes

$$u(s,0) = 0. \quad (4-28)$$

From the appropriate property of the Laplace transform (10) one can write

$$\mathcal{L}_x \left\{ \frac{\partial^2 U(x,t)}{\partial x^2} \right\} = s^2 u(s,t) - s U(0,t) - \frac{\partial U(0,t)}{\partial x}.$$

The last term in this equation is zero, by condition (4-4). Now evaluate this transform for zero time, i.e. $t = 0$:

$$\mathcal{L}_x \left\{ \frac{\partial^2 U(x,0)}{\partial x^2} \right\} = s^2 u(s,0) - s U(0,0).$$

But, from (4-28) and (4-2), both terms on the right are zero, hence

$$\mathcal{L}_x \left\{ \frac{\partial^2 U(x,0)}{\partial x^2} \right\} = 0$$

Since $\mathcal{L}^{-1}\{0\} = 0$, it follows that

$$\frac{\partial^2 U(x,0)}{\partial x^2} = 0$$

which establishes equation (4-27), and hence (4-25) also. This proof may be unduly restrictive; i.e. these relations may be valid in a much broader class of systems than here defined. (In fact, the proof has been presented here in detail in the hope that someone with more adequate mathematical background than that of the writer will develop a more general statement.) However, equation (4-25) will certainly hold for the model defined by equations (4-1) through (4-4), which is all that is necessary for the present study.

One additional relation of considerable importance will now be derived. If equation (4-25) is substituted in equation (4-6), one obtains

$$\int_0^\infty \frac{\partial U(x,0)}{\partial t} dx = \frac{H_a(0)}{\nu} \quad (4-29)$$

This is a necessary condition, independent of any assumptions regarding the form of the absorption pattern; one then has a valuable check on the accuracy with which the initial slopes have been determined, since the terms on the right hand side will be known, and the integral may be evaluated graphically.

4.7 Summary

The foregoing treatment of the diathermanous solid has been rather lengthy; probably unreasonably so. However, many of the problems considered have not been adequately treated in the literature, nor is it claimed that they have been adequately treated here. It is only hoped that these extensive discussions will suggest the proper questions which must be answered before the over-all problem of the interaction of radiation with tissue may be considered solved.

Turning to the solutions for the double exponential absorption pattern (equations 4-22 and 4-23), one important problem remains. As previously stated, these solutions apply only over a wavelength interval sufficiently narrow that the linear absorption coefficients may be regarded as constants. The superposition of solutions to account for variation of these coefficients will be considered in the following chapter. The theoretical development will then be sufficiently complete so that attention may be turned to the experimental phase of this study.

FOOTNOTES

- (1) A Study of the Physical Basis of Burn Production with Applications to the Defensive Reactions to an Atomic Bomb Air Burst, Medical Research and Development Board, Office of the Surgeon General, 34 (Author and date of publication unknown.).
- (2) Buettner, K., Effects of Extreme Heat and Cold on Human Skin III. Numerical Analysis and Pilot Experiments on Penetrating Flash Radiation Effects, J. Appl. Physiol., 5, 209 (1952).

- (3) Hardy, J. D. and C. Muschenheim, Radiation of Heat from the Human Body IV. The Emission, Reflection, and Transmission of Infrared Radiation by the Human Skin, J. Clin. Invest. 13, 817-831 (1934).
- (4) Hardy, J. D. and C. Muschenheim, Radiation of Heat from the Human Body V. The Transmission of Infrared Radiation Through Skin, J. Clin. Invest. 15, 1-9 (1936).
- (5) Hardy, J. D. Annual Progress Report for 1951 (to Chief of Naval Research, Attn. Biology Branch).
- (6) Hardy, J. D., H. T. Hammel, and D. Murgatroyd, Spectral Transmittance and Reflectance of Excised Human Skin. J. Appl. Physiol. 9, 257-264 (1956).
- (7) Clark, C., R. Vinegar, and J. D. Hardy, Goniometric Spectrometer for the Measurement of Diffuse Reflectance and Transmittance of Skin in the Infrared Spectral Region, J. Opt. Soc. Am. 43, 993-998 (1953).
- (8) Hardy, J. D., H. T. Hammel, and D. Murgatroyd, op. cit., 257.
- (9) Ibid., 260.
- (10) Churchill, R. V., Modern Operational Mathematics in Engineering, 294, Operation No. 4 (McGraw-Hill Book Company, Inc., New York, 1944).

CHAPTER V
THE INFLUENCE OF THE QUALITY OF RADIATION ON THE
DIATHERMANOUS SOLID RESPONSE

5.1 Development of the Problem

In the analysis of the opaque solid, the incident radiation was considered only as so much energy or power, and no consideration of the quality (wavelength distribution) was necessary. Indeed, the general solution of the opaque solid is equally applicable to the case of contact heating with the absorbed irradiance, $H_a(t)$, replaced by the more general power input per unit area, $q''(t)$, which involves a suitable heat transfer coefficient.

In contrast to this situation, the formal solution for the diathermanous solid is valid only so long as the linear absorption coefficients are constant. It has already been pointed out that these coefficients are strongly dependent upon wavelength; hence the diathermanous solid response depends upon both the quantity and the quality of the incident radiation. It will be the purpose of this chapter to develop methods for extending the previously established relations to allow for this variation of the "optical" constants with wavelength.

Specifically, the problem is to establish the validity of superposition of the formal solutions for the diathermanous solid. Rules will then be developed for superposition of the normalized solutions, and the results of machine computations, based on constants taken from the literature, will be given. Some consideration will also be given to the influence of quality of radiation on the initial time rate of change of the temperature response.

5.2 Superposition of Solutions for the Diathermanous Solid

Let the relative spectral energy distribution of the incident

irradiance be divided into n contiguous segments, within each of which the linear absorption coefficients, γ_1 and γ_2 , and the backscatter factor, R , may be considered essentially constant. Now, imagine a diathermanous solid exposed to substantially monochromatic radiation, of wavelength corresponding to the i^{th} wavelength interval. At any position and time, the "internal heat generation" or absorption per unit volume will be given by $q'''_i(x,t)$. Similarly, the absorption per unit volume due to monochromatic radiation corresponding to the j^{th} interval will be $q'''_j(x,t)$. If both of these monochromatic beams are now directed on the solid simultaneously, the absorption per unit volume at x and t will be

$$q'''_i(x,t) + q'''_j(x,t)$$

provided that the receiver is passive. (Note that this relation will hold even if the time dependence of $q'''_i(x,t)$ is different from that of $q'''_j(x,t)$.) By obvious extension of this argument, it follows that the total absorption per unit volume at any position and time, $q'''(x,t)$, due to the simultaneous action of all n wavelength intervals of incident irradiance is

$$q'''(x,t) = \sum_{i=1}^n q'''_i(x,t). \quad (5-1)$$

The temperature response, $U(x,t)$, due to this total volume absorption, $q'''(x,t)$, will be given by the solution of the differential equation (4-1):

$$\frac{\partial U(x,t)}{\partial t} = \alpha \frac{\partial^2 U(x,t)}{\partial x^2} + \frac{1}{\rho} q'''(x,t), \quad x \geq 0, \quad t \geq 0.$$

Rearranging slightly, and substituting (5-1), this relation becomes

$$\left(\frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial x^2} \right) U(x,t) = \frac{1}{\rho} \sum_{i=1}^n q'''_i(x,t), \quad x \geq 0, \quad t \geq 0. \quad (5-2)$$

Also $U(x,t)$ must satisfy the boundary conditions

$$U(x,0) = 0, \quad x \geq 0 \quad (5-3)$$

$$U(\infty, t) = 0, \quad t \geq 0 \quad (5-4)$$

$$\text{and } \frac{\partial U(0,t)}{\partial x} = 0, \quad t \geq 0 \quad (5-5)$$

Now consider the temperature response, $U_i(x, t)$, of the diathermanous solid exposed to monochromatic radiation of wavelength λ_i . There will be n such functions, corresponding to n wavelength intervals of the incident heterochromatic radiation, each of which must satisfy the following set of equations:

$$\left(\frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial x^2}\right) U_i(x, t) = \frac{1}{V} q'''_i(x, t), \quad x \geq 0, \quad t \geq 0, \quad i = 1, 2, \dots, n \quad (5-6)$$

$$U_i(x, 0) = 0, \quad x \geq 0, \quad i = 1, 2, \dots, n \quad (5-7)$$

$$U_i(\infty, t) = 0, \quad t \geq 0, \quad i = 1, 2, \dots, n \quad (5-8)$$

$$\frac{\partial U_i(0, t)}{\partial x} = 0, \quad t \geq 0, \quad i = 1, 2, \dots, n \quad (5-9)$$

If α is a constant, strictly independent of x and t , then the term

$$\left(\frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial x^2}\right)$$

may be recognized as a linear operator. As a consequence of this, if one takes the sum over all n of both sides of equation (5-6);

$$\sum_{i=1}^n \left[\left(\frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial x^2}\right) U_i(x, t) \right] = \sum_{i=1}^n \frac{1}{V} q'''_i(x, t),$$

then the order of summation and differentiation may be interchanged to give

$$\left(\frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial x^2}\right) \sum_{i=1}^n U_i(x, t) = \frac{1}{V} \sum_{i=1}^n q'''_i(x, t). \quad (5-10)$$

Further, it is obvious from the homogeneity of conditions (5-7) through (5-9) that

$$\sum_{i=1}^n U_i(x, 0) = 0, \quad x \geq 0 \quad (5-11)$$

$$\sum_{i=1}^n U_i(\infty, t) = 0, \quad t \geq 0 \quad (5-12)$$

$$\text{and } \sum_{i=1}^n \frac{\partial}{\partial x} U_i(0, t) = \frac{\partial}{\partial x} \sum_{i=1}^n U_i(0, t) = 0, \quad t \geq 0. \quad (5-13)$$

Comparing equations (5-2) through (5-5) with (5-10) through (5-13), it follows that

$$U(x,t) = \sum_{i=1}^n U_i(x,t), \quad x \geq 0, \quad t \geq 0, \quad (5-14)$$

and the validity of superposition of solutions is established.

For the special case of double exponential absorption and step-function irradiance input, as treated in the previous chapter, the expression for $q'''_i(x,t)$ becomes:

$$\begin{aligned} q'''_i(x,t) &= 0, \quad t < 0 \\ &= \gamma_{1i} H_{0i} e^{-\gamma_{1i} x}, \quad t \geq 0, \quad 0 \leq x < b, \\ &= \gamma_{2i} H_{0i} e^{-\gamma_{1i} b} e^{-\gamma_{2i}(x-b)}, \quad t \geq 0, \quad x > b, \\ &\quad i = 1, 2, \dots, n. \end{aligned} \quad (5-15)$$

Note that the depth of the break in the absorption curve, b , is assumed independent of wavelength. Substitution of this relation for $q'''_i(x,t)$ in (5-6) will yield n sets of equations, each set identical to (4-10) through (4-13) of the previous chapter, with the trivial addition of the subscript "i" (where $i = 1, 2, \dots, n$) to $U(x,t)$, γ_1 , and γ_2 . The formal (non-normalized) solution of each set will give a value of $U_i(x,t)$, and the response for the total wavelength distribution, may then be found by superposition, as demonstrated by (5-14), above.

5.3 Superposition of the Normalized Solutions for Double Exponential Absorption

The procedure outlined above for determining the temperature response of the diathermanous solid to heterochromatic radiation is quite straightforward. For the special case under consideration, double exponential absorption, the solutions have been obtained in normalized form, $\Psi(\xi, \theta)$,

for convenience in computation, and here it is not true that

$$\Psi(\xi, \theta) = \sum_{i=1}^n \Psi_i(\xi, \theta).$$

Simple superposition does not hold for these normalized solutions because the normalizing factors for Ψ , ξ , and θ all involve γ_i , and this latter coefficient is wavelength dependent.

In order to add these normalized solutions, then, it is necessary to transform each back to real temperature rise, and add terms corresponding to the same values of real depth and time. From the definitions of Ψ , ξ , and θ given in Chapter IV; i.e.:

$$\xi = \gamma_i x$$

$$\theta = \gamma_i^2 \alpha t$$

$$\Psi = \frac{k \gamma_i}{H_a} U$$

it follows that the total temperature response $U(x,t)$, is given by

$$U(x,t) = \sum_{i=1}^n \frac{H_{ai}}{k \gamma_{ii}} \Psi_i(\gamma_{ii} x, \gamma_{ii}^2 \alpha t). \quad (5-16)$$

The implication of equation (5-16) is rather discouraging: it is, in general, impossible to obtain normalized solutions for the diathermanous solid exposed to any arbitrary heterochromatic radiation. In order to compare theoretical and experimental responses, it is necessary first to have at hand numerical values of the constants of the diathermanous solid; it is these very values, however, which the experiment is supposed to provide. Thus the procedures developed in Section 4.5 of the previous chapter are valid only if monochromatic radiation is employed, and have questionable utility for heterochromatic irradiation.

5.4 Values of the Constants of Skin Obtained from the Literature

The most direct way out of this dilemma is to select a particular

set of conditions for the experimental study; this will include a choice of radiant energy source, and an experimental animal. Now by direct measurement, where possible, and literature review, the constants and parameters appropriate to these conditions will be estimated, and numerical solutions of equation (5-16) obtained for suitable ranges of depth and time. This predicted response may now be compared with the experimental results; if these are in reasonable agreement, then at least one can claim that the form of the equations and the constants from the literature are not contradicted by experiment. If the agreement is poor, then the model, the constants, and/or the experimental results may be incorrect, and a different experimental approach would be necessary to decide which is at fault.

The experimental conditions will be described in detail in a later chapter, but for present purposes it will be sufficient to specify the following:

1) Source -- a carbon arc image furnace (1, 2). The relative spectral energy distribution (3) and the total irradiance output in absolute units (4) have been accurately determined.

2) Exposure pulse -- a rectangular pulse (i.e. trapezoidal with very rapid rise time) of 0.5 second duration. For time less than 0.5 second, this is equivalent to a step function, and it will be so considered, hereafter.

3) Experimental animals -- young Chester White pigs. The similarity in structure of the skin of this animal and human skin has been described (5). The spectral reflectance of the skin of the Chester White and fair human skin is also quite similar (6, 7). Finally, closely comparable lesions are produced in pig and fair human skin by very nearly the same radiant exposures (8, 9). These points provide some justification for the necessity of using some constants derived from measurements on human skin.

The simplest term to evaluate (and the one least likely to be in error) is the absorbed irradiance in the i^{th} wavelength interval, $H_{ai}(t)$. Since the incident irradiance, $H_o(t)$, is a step function, and the spectral distribution of the source is constant in time, it follows that $H_{ai}(t)$ is likewise a step function:

$$\begin{aligned} H_{ai}(t) &= 0, \quad t < 0 \\ &= H_{ai}, \quad t \geq 0. \end{aligned}$$

Now, if R_λ is the spectral reflectance (or back-scatter factor) of the skin (whence $(1 - R_\lambda)$ is the spectral absorptance), and one defines λ_{i-1} and λ_i as the lower and upper limits, respectively, of the i^{th} wavelength interval, then H_{ai} is given by

$$\begin{aligned} H_{ai} &= \int_{\lambda_{i-1}}^{\lambda_i} (1 - R_\lambda) H_{o\lambda} d\lambda \\ &= H_o \int_{\lambda_{i-1}}^{\lambda_i} (1 - R_\lambda) \left(\frac{H_{o\lambda}}{H_o} \right) d\lambda, \end{aligned}$$

where $H_{o\lambda}$ is the absolute spectral distribution of the incident irradiance, and H_o is the total incident irradiance. Now the relative spectral energy distribution, J_λ , is proportional to the ratio $(H_{o\lambda} / H_o)$, i.e.

$$J_\lambda = A \left(\frac{H_{o\lambda}}{H_o} \right),$$

where the proportionality constant is simply the total area under the relative distribution curve, or

$$A = \int_0^\infty J_\lambda d\lambda.$$

Now, from the above expression, the total absorbed irradiance, H_a , is

$$H_a = H_o \int_0^\infty (1 - R_\lambda) \left(\frac{H_{o\lambda}}{H_o} \right) d\lambda.$$

Hence

$$\begin{aligned} \frac{H_{ai}}{H_a} &= \frac{H_o \int_{\lambda_{i-1}}^{\lambda_i} (1-R_\lambda) \left(\frac{H_{o\lambda}}{H_o} \right) d\lambda}{H_o \int_0^\infty (1-R_\lambda) \left(\frac{H_{o\lambda}}{H_o} \right) d\lambda} \\ &= \frac{\int_{\lambda_{i-1}}^{\lambda_i} (1-R_\lambda) J_\lambda d\lambda}{\int_0^\infty (1-R_\lambda) J_\lambda d\lambda}, \end{aligned}$$

or $H_{ai} = H_a K_i$ (5-17)

where $K_i = \frac{\int_{\lambda_{i-1}}^{\lambda_i} (1-R_\lambda) J_\lambda d\lambda}{\int_0^\infty (1-R_\lambda) J_\lambda d\lambda}$ (5-18)

Values of $(1 - R_\lambda) J_\lambda$ have been determined from the measurements of R_λ for the Chester White pig (10), and J_λ on the carbon arc furnace (11), and are shown in Figure 5-1 as a function of wavelength. For any choice of wavelength intervals, values of K_i may be obtained from this curve by numerical integration or planimetry.

The next step is the evaluation of the linear absorption coefficients. No information is available for pig skin, and it is here that one must turn to values for human skin; some justification for this procedure has been given above. While considerable information on this latter tissue is available, the task of selecting accurate values for the absorption coefficients is not simple, since values cited in the literature vary widely (12), and no single report covers the desired wavelength span.

As mentioned previously, the recent report of Hardy, et al. (13) contains the most reliable data available. Careful evaluation of ϵ_1 and ϵ_2 are presented only at four wavelengths; for interpolation, only rather rough transmission curves are given covering the spectrum from 0.7 to 2.4 μ . For

Figure 5-1. Relative Spectral Energy Absorption, $(1-R_\lambda) \cdot J_\lambda$. Chester White Pig Exposed to Carbon Arc Image Furnace

shorter wavelengths, the values obtained by Hansen (14) on human autopsy skin were selected, as these seemed to converge toward reasonable agreement with the data of Hardy. Figure 5-2 presents the smoothed curves of the absorption coefficients vs wavelength. It must be stressed that this figure is compounded of about equal parts of freehand art and literature research; it is hoped, however, that it represents the best estimates available as to the true values of these coefficients.

It will be noted that the absorption coefficients vary quite rapidly with wavelength. This would suggest that, for accuracy, the wavelength intervals must be made small. In Fig. 5-2, the eight intervals selected are shown, and average values of γ_1 and γ_2 indicated for each interval by a horizontal bar. These wavelength segments are not short, and the variation of absorption coefficients in several is large. The increased precision which would be gained by selecting a larger number of shorter intervals is hardly justified in view of the dubious accuracy of the curves, and the resultant increase in computation time.

Reference to equation (5-15) reveals that only one additional constant must be evaluated; namely the depth of the break in the absorption pattern, b . Hardy has suggested a value of 0.04 cm (15), based, however, not on anatomical considerations, but rather on the thinnest section which could be conveniently handled in his goniometer spectrophotometer. It would seem more reasonable to expect that a break in the absorption pattern would be associated with some more or less abrupt change in the structure of skin, the epidermal-dermal junction being the most likely candidate. This would set the value of b equal to 0.01 cm, which is a rough average of epidermal thickness in the flank areas of young swine. If this value be correct, then the values of γ_1 given by Hardy are low to the extent that his superficial tissue samples contained some material with a lower absorption coefficient (γ_2).

Figure 5-2. Linear Absorption Coefficients for Shallow (γ_1) and Deep (γ_2) Tissue

Now, by employing equation (5-17) for H_{ai} , equation (5-16) may be written in terms of the temperature response per unit absorbed irradiance, $U^*(x,t)$ as

$$U^*(x,t) = \sum_{i=1}^n \frac{K_i}{k \gamma_{ii}} \Psi_i(\gamma_{ii} x, \gamma_{ii}^2 \alpha t). \quad (5-19)$$

The numerical values necessary for the solution of Ψ_i in each interval, and the "weighting factor" $K_i/k \gamma_{ii}$, are tabulated below. The numerical values employed for the thermal constants, k and α , are

$$k = 9.8 \times 10^{-4} \text{ cal cm}^{-1} \text{ deg}^{-1} \text{ sec}^{-1}$$

$$\alpha = 8.2 \times 10^{-4} \text{ cm}^2 \text{ sec}^{-1}$$

which were determined from "opaqued" pig skin.

As noted previously, the weighted values of Ψ_i must be added at values of ξ and θ corresponding to the same real depth and time. In real dimensions, the following values were selected:

$$x = 0, 0.02, 0.04, \text{ and } 0.08 \text{ cm};$$

$$t = 0.1, 0.2, 0.3, 0.5, 0.7, 1.0, \text{ and } 1.5 \text{ sec}$$

(There are thus $4 \times 7 = 28$ points in each interval, or a total of $8 \times 28 = 224$ points to be determined.) Numerical values of the normalized coordinates corresponding to the above real variables were calculated, and, together with the appropriate values of the parameters λ_1 and λ_2 , inserted into the previously mentioned program for the IBM 650 computer. The resultant values of normalized response were properly weighted and summed, with the final values obtained being presented as a set of curves of $U^*(x,t)$ vs t in Figure 5-3. These curves, then, represent the predicted temperature response of skin based on available constants. Comparison with experimental results will be given in a later chapter.

Table 5-1

Numerical Values for Solution of the Normalized Response Equation

Inter- val	Wavelength (μ)	K	γ_1 (cm^{-1})	γ_2 (cm^{-1})	λ_1	λ_2	$\gamma_1^2 \alpha$	$\frac{K}{k \gamma_1}$
1	0-0.46	0.240	150	25	1.50	6.0	18.45	1.626
2	0.46-0.54	0.117	60	10	0.60	6.0	2.952	1.982
3	0.54-0.70	0.167	25	10	0.25	2.5	0.512	6.789
4	0.70-1.00	0.142	15	10	0.15	1.5	0.184	9.621
5	1.00-1.34	0.112	12	12	0.12	1.0	0.118	9.485
6	1.34-1.86	0.117	25	25	0.25	1.0	0.512	4.756
7	1.86-1.98	0.015	80	80	0.80	1.0	5.248	0.191
8	1.98- ∞	0.090	40	40	0.40	1.0	1.312	2.287

Figure 5-3. Predicted Temperature Response of Bare Pig Skin Exposed to Carbon Arc Image Furnace

5.5 The Influence of the Quality of Radiation on the Initial Time Rate of Change of Temperature

The subjective comparison of theoretical and experimental results, as outlined in the previous section, is clearly most unsatisfactory, and represents a rather low order of research. It is difficult to decide a priori just what will constitute "good" agreement. More important, if the agreement be judged "poor," there is no suggestion as to the cause or causes of this failure. If one wishes to examine critically the literature values tabulated above, or suggest other values as determined by experiment, a more rigorous scheme must be devised.

Considerable aid in this direction is obtained if one turns to the interesting relations concerning the initial time rate of change of temperature, as developed in Chapter IV. It is only necessary to note that the systems of equations defining the "interval" response, $U_i(x,t)$, and the total response, $U(x,t)$ are included in the class of systems, as defined in the previous chapter, for which

$$\frac{\partial U(x,0)}{\partial t} = \frac{1}{V} q'''(x,0) \quad (5-20)$$

and also

$$\frac{\partial U_i(x,0)}{\partial t} = \frac{1}{V} q'''_i(x,0). \quad (5-21)$$

Now with the superposition of (non-normalized) solutions established by equation (5-14), it is an obvious step to note that

$$\frac{\partial U(x,t)}{\partial t} = \frac{\partial}{\partial t} \sum_{i=1}^n U_i(x,t).$$

Interchange of differentiation and summation for this finite sum of well-behaved functions gives

$$\frac{\partial U(x,t)}{\partial t} = \sum_{i=1}^n \frac{\partial U_i(x,t)}{\partial t},$$

and in particular, in the limit as t approaches zero from above:

$$\frac{\partial U(x, 0)}{\partial t} = \sum_{i=1}^n \frac{\partial U_i(x, 0)}{\partial t} . \quad (5-22)$$

(This equation is obtained in more direct fashion by substitution of equations (5-20) and (5-21) in the initial relation (5-1).) Superposition thus holds also for the initial time rates of change.

If one now accepts the form of $q'''_1(t)$ as defined by (5-15), then, using expression (5-17) for H_{a1} ,

$$\begin{aligned} \frac{\partial U^*(x, 0)}{\partial t} &= \frac{1}{V} \sum_{i=1}^n \gamma_{1i} K_i e^{-\gamma_{1i} x} , \quad 0 \leq x < b \\ &= \frac{1}{V} \sum_{i=1}^n \gamma_{2i} K_i e^{-\gamma_{1i} b} e^{-\gamma_{2i} (x-b)} , \quad x > b , \end{aligned} \quad (5-23)$$

where, again, the response per unit absorbed irradiance, U^* , has been used. The sum of exponential decay curves is not itself an exponential curve, hence the plot of the logarithm of the initial slopes against depth, x , will not consist of straight line segments. The expected form is shown in Figure 5-4, where the product of the heat capacity per unit volume, V , and the initial slope $\partial U^*(x, 0)/\partial t$, has been plotted on a logarithmic scale against linear depth in centimeters. Numerical values for substitution in equation (5-23) were taken from Table 5-1.

Now consider the comparison of the theoretical predictions embodied in Figure 5-4 with experimental results. While it is still difficult to decide a priori just what will constitute "good" agreement, at least considerable improvement has been made in the problem of selecting the probable causes of manifestly "poor" agreement. First it will be noted that equation (5-23) is expressed in real dimensions, and there are no thermal constants of questionable accuracy "buried" in normalizing or weighting factors; the only thermal constant which appears, V , enters as a multiplicative factor, only.

Figure 5-4. Predicted Initial Time Rate of Change of Temperature Response of Bare Pig Skin Exposed to Carbon Arc Image Furnace

Secondly, at the depth of the break in the absorption pattern, b , there appears a finite discontinuity in the predicted curve; if such a break appears, but at a different depth, in the experimental curve, the error in the selection of the numerical value of b is obvious, and can be immediately corrected.

The most important point, however, is that in following this method, the experimental data are in a sense self-checking. It was established previously that a necessary condition on $q'''(x,t)$, independent of any assumptions regarding the depth dependence of this function, is:

$$\int_0^{\infty} q'''(x,t) dx = H_a(t).$$

It follows from equation (5-20) that for the irradiance step function here being considered,

$$\int_0^{\infty} \nu \cdot \frac{\partial U^*(x,0)}{\partial t} dx = 1. \quad (5-24)$$

Hence, the area under the experimentally determined curve of the initial slopes multiplied by ν plotted against depth, x , must be equal to unity, and if this is not so, errors must exist in the measurements, themselves. It may be noted in passing that equation (5-23) satisfies this condition since:

$$\begin{aligned} \int_0^{\infty} \nu \frac{\partial U^*(x,0)}{\partial t} dx &= \int_0^b \sum_{i=1}^n \gamma_{1i} K_i e^{-r_{1i}x} dx \\ &\quad + \int_b^{\infty} \sum_{i=1}^n \gamma_{2i} K_i e^{-r_{1i}b} e^{-\gamma_{2i}(x-b)} dx \\ &= \sum_{i=1}^n K_i \\ &= 1. \end{aligned}$$

5.6 Direct Determination of the Absorption Pattern of Heterochromatic Radiation in Skin

The previous sections have treated the problems of comparing experimental results with predictions from the pre-selected model; namely the

diathermanous solid with double exponential absorption. Indeed it is this comparison which is the crux of the model approach, as discussed in the first chapter. As was pointed out in the last chapter, however, the relation

$$\frac{\partial U(x, 0)}{\partial t} = \frac{1}{V} q'''(x, 0)$$

provides a means of establishing the desired depth dependence of the volume absorption function, $q'''(x, t)$, independently of any hypothesized model.

Since this relation still holds with heterochromatic radiation, it is in theory possible, for any particular spectral distribution, to obtain a graphical, if not analytical, expression for $q'''(x, t)$, but this expression will hold only for the specific spectral distribution. Thus, if one wishes to predict the response to, say, tungsten lamp illumination, then the form of $q'''(x, t)$ deduced for solar illumination will be of little or no value.

While it is not appropriate here to indulge in speculation, it might be mentioned that this method of direct determination of the absorption pattern suggests an intelligent application of "subtractive" experiments, where sharp cut-off optical filters are used to absorb wavelengths shorter than the cut-off wavelength, while the remainder are transmitted to the test animal (16). Extensive series of such filters are available commercially, with rather closely spaced cut-off wavelengths. Noting that (5-20) can be written as

$$V \frac{\partial U(x, 0)}{\partial t} = \sum_{i=1}^n q'''_i(x, 0)$$

it follows that one can determine first $\sum_{i=1}^n q'''_i(x, t)$, then by removal of, say, the first interval, $\sum_{i=2}^n q'''_i(x, t)$. The difference is simply $q'''_1(x, t)$. Filtering out the second band will give $\sum_{i=3}^n q'''_i(x, t)$, from which one can determine $q'''_2(x, t)$, and so on. The demands on the accuracy of such a procedure are extreme, but there would appear to be no theoretical objections to the method.

5.7 Correction of Initial Slopes for Finite Irradiance Rise Time

It was pointed out in Chapter III, that no real device can produce an ideal irradiance step function, and the actual irradiance pulse is more closely approximated by superposition of ramp functions, or a trapezoidal pulse. A general solution for a trapezoidal pulse response was developed as

$$\begin{aligned}
 U^{trap}(x, t) &= \left(\frac{H_a}{\delta}\right) \int_0^t U^{step}(x, \tau) d\tau, \quad 0 \leq t \leq \delta \\
 &= \left(\frac{H_a}{\delta}\right) \int_{t-\delta}^t U^{step}(x, \tau) d\tau, \quad \delta \leq t \leq \eta \\
 &\dots
 \end{aligned}$$

where

$$\begin{aligned}
 H_a^{trap}(t) &= 0, \quad t < 0 \\
 &= \left(\frac{H_a}{\delta}\right)t, \quad 0 \leq t \leq \delta \\
 &= H_a, \quad \delta \leq t \leq \eta \\
 &\dots
 \end{aligned}$$

For the surface response of the opaque solid, a factor, F^{trap} , was then derived, which allowed for correction of trapezoidal pulse response to ideal step-function response. A brief examination of the solutions for the diathermanous solid response shows that this analytical method becomes hopelessly complex; indeed, for heterochromatic radiation, no general correction factor can be given.

Numerical integration of predicted response curves indicates that the trapezoidal pulse response quickly becomes indistinguishable from the step-function response, for the shutter opening times employed in this study. The procedure followed, then, was that of simply extrapolating the measured response curves back to the very carefully determined zero time. Since the extrapolations were in all cases quite short, this procedure is subject to little error.

5.8 Summary

At this point, the theoretical analysis of the temperature response of skin to radiant energy may be considered to be sufficiently complete for present purposes. Predicted responses for bare (diathermanous) and opaque skin have been presented, and wherever possible, methods of comparing experiment with theory have been developed. It is now appropriate to consider the experimental methods and results of this study; these will be treated in the following chapters.

It is important to realize that many severely simplifying assumptions have been embodied in the models proposed, assumptions which in some cases have been only briefly mentioned or even simply implied. As an example, the "necessary" relation (5-24) is valid only for the uniformly irradiated semi-infinite solid, as here stated, and it will be seen that the radiation was actually delivered to a sharply circumscribed area of a definitely finite animal. Whenever possible, these assumptions will be justified by experimental evidence, but (as stated in Appendix II) it must be admitted that the complex process of a radiant energy exposure of living skin is extraordinarily resistant to formal mathematical attack. In spite of this, the theoretical analysis is essential, if experimental findings are to be interpreted intelligently.

The experiments described in the succeeding chapters are by no means complete; they have posed almost as many questions as they have answered. It is hoped, however, that the foregoing theoretical analysis is sufficiently thorough to serve as a guide for future experimentation for some time to come. It is for this reason that this analysis has been placed first not only in presentation, but in over-all importance in this study.

FOOTNOTES

- (1) Davis, T. P., L. J. Krolak, R. M. Blakney, and H. E. Pearse, Modification of the Carbon Arc Searchlight for Producing Experimental Flashburns, J. Op. Soc. Am. 44, 766-769 (1954).
- (2) Davis, T. P., "The Carbon Arc Image Furnace" in Proceedings of the Symposium on High Temperature -- A Tool for the Future, 10-15 (Stanford Research Institute, Menlo Park, California, 1956).
- (3) Krolak, L. J. and T. P. Davis, A Universal Spectrophotometer for the Measurement of the Relative Spectral Distribution of the Carbon Arc Source, University of Rochester Atomic Energy Project Report UR-367 (1954).
- (4) Davis, T. P., A Standard Radiation Calorimeter, M.S. Thesis, Department of Physics, The University of Rochester, 51, et seq. (1954).
- (5) Moritz, A. R. and F. C. Henriques, Jr., Studies of Thermal Injury II. The Relative Importance of Time and Surface Temperature in the Causation of Cutaneous Burns, Am. J. Path. 23, 698 et seq. (1947).
- (6) Kuppenheim, H. F., J. M. Dimitroff, P. M. Melotti, I. C. Graham, and D. W. Swanson, Spectral Reflectance of the Skin of Chester White Pigs in the Ranges 235-700 mμ and 0.707-2.6 μ, J. App. Physiol. 9, 75-78 (1956).
- (7) Krolak, L. J., The Measurement of Diffuse Reflectance of Pig Skin, Titanium Dioxide Paint, and India Ink; the Transmittance of Titanium Dioxide and India Ink, University of Rochester Atomic Energy Project Report UR-439 (1956).
- (8) Perkins, J. B., H. E. Pearse, and H. D. Kingsley, Studies on Flash Burns: The Relation of Time and Intensity of Applied Thermal Energy to the Severity of Burns, University of Rochester Atomic Energy Project Report UR-217, 21-25 (1952).
- (9) Hinshaw, J. R. and T. P. Davis, Unpublished research to be incorporated in a paper on the prediction of burn severity from the thermal pulse of nuclear weapons.
- (10) Krolak, L. J., op. cit., 13.
- (11) Krolak, L. J. and T. P. Davis, op. cit., 33.
- (12) Hardy, J. D., Annual Progress Report for 1951 (to Chief of Naval Research, Attn. Biology Branch).
- (13) Hardy, J. D., H. T. Hammel, and D. Murgatroyd, Spectral Transmittance and Reflectance of Excised Human Skin. J. Appl. Physiol. 9, 262 (1956).
- (14) Hansen, K. G., On the Transmission through Skin of Visible and Ultraviolet Radiation, Ph.D. Thesis, The University of Copenhagen (Published as Supplementum LXXI, Acta Radiologica), 96-98 (1948).

- (15) Hardy, J. D., H. T. Hammel, and D. Murgatroyd, op. cit., 261.
- (16) Berkley, K. M., T. P. Davis, and H. E. Pearse, Study of Flash Burns: The Effect of Spectral Distribution on the Production of Cutaneous Burns, University of Rochester Atomic Energy Project Report UR-336 (1954).

CHAPTER VI

EXPERIMENTAL MATERIALS AND METHODS

6.1 Introduction

In the previous chapters, extensive consideration has been given to the problem of comparing theoretical and experimental results. The comparison schemes developed define the general types of experimental information which must be obtained; procedures were thus selected so as to provide these necessary data in proper form and with as high accuracy as possible. In the following sections both the experimental materials and the methods of their use will be described. In addition, wherever possible the necessity of including a particular item of equipment will be explained on the basis of the demands imposed by the theoretical analysis.

6.2 Biological

As mentioned in the last chapter, the experimental animals employed were young Chester and Yorkshire White pigs. Most animals were of the former breed, and only a few of the latter were used. No differences in burn response between these breeds have ever been noted in this laboratory, and no differences in temperature response were found in this study. For these purposes the breeds seem to be identical.

The animals available were weanlings with body weights between 9 and 12 kg. Prior to an experiment, a specimen was selected which was of suitable size and without obvious abnormalities in skin appearance. Both males and females were used. Eighteen to twenty-four hours before preparation, food and water were withdrawn from the animal selected. Preparation consisted of anesthetization with Dial in Urethane (Ciba) administered intraperitoneally in a dosage of about 65 mg per kg of fasted body weight. Except for the slightly lower dosage, the procedure was as developed by Kingsley, et al. (1).

In addition, 1.5 mg per kg of chlorpromazine (Thorazine, Smith, Kline and French) was administered intramuscularly to suppress shivering (2). The hair on both flanks was clipped and the remaining stubble removed with an ordinary electric razor (Norelco or Schick). The entire animal was thoroughly hosed and the shaved flanks washed thoroughly with detergent and water.

A pig thus prepared is not able to maintain normal body temperature in a cool environment. To avoid any difficulties with severe hypothermia, and to achieve fairly constant skin temperatures (3), the temperature of the room in which all experimental work was done was held at about 30°C (86°F). Each animal was usually prepared at about 8:00 A.M., and satisfactory anesthesia and immobility were maintained in most cases until about 4:00 P.M., at which time the shiver reflex had returned, and the animal began to exhibit "walking" motions. Experimental work was then terminated and the animal returned to his cage. By the next morning, twenty-four hours after preparation, most pigs were able to gain their feet, but their gait was most uncertain. After forty-eight hours, recovery was apparently complete. None of the fifteen animals used were lost.

On a few animals, the lesions were biopsied twenty-four hours after burning. (For this brief procedure, anesthetization was by veterinary sodium pentobarbital, with three to five ml administered intravenously.) Biopsies were taken across the burned areas so that normal tissue was included at each end of the sample. The tissues were fixed in 10% formalin and embedded in paraffin. Duplicate sections were stained with hematoxylin and eosin, and by a modification of Verhoeff's elastic tissue stain as developed by Hinshaw (4). The excellent workmanship of Mrs. Fredette and Mrs. VanWinkle, of the Histology laboratory of this Project, in the preparation of these sections is gratefully acknowledged.

The average reflectance of the skin of the Chester White pig to the

carbon arc furnace employed in these studies is 0.40, as determined in this laboratory (5). For the exposures on "opaque" skin, the skin was painted with India ink, to which was added a few drops of a liquid detergent. This covering has also been investigated in this laboratory (6); the reflectance to the carbon arc is 0.10, and the transmittance of a thin film of this material is no greater than 0.11. Thus very little radiation can penetrate into the skin. In this study, absolutely no evidence of a "diathermanous-type" response was ever observed with the India ink painted skin. The thickness of a typical ink film was determined by covering half of a sheet of cellophane with the material, and measuring the thickness of the cellophane, and the cellophane-plus-ink with micrometer calipers. The value obtained by difference in these readings was 0.0005", or about 13 microns. The film was probably thinner on the skin, since the ink penetrated into the tissue somewhat.

6.3 Physical -- General Arrangement

The physical equipment used in this study may be divided into several major component groups, as indicated schematically in Figure 6-1. The various items were all assembled in the temperature-controlled room mentioned above. In the following sections the pertinent details of the component groups will be presented, approximately in the order indicated in Figure 6-1.

6.4 Radiation Source

The radiation source used for all exposures was an arc imaging furnace, constructed around a surplus 24 inch Corps of Engineers searchlight, Model 1942 (7) (Fig. 6-2). The primary source of radiation in this furnace is a high current carbon arc (8), with a 10 mm positive carbon (National Carbon Co. "Ultrex") operated at about 145 amperes (current density of 185 amps per cm^2). Operating power for the unit is provided by a rotary converter consisting of a 50 HP three-phase induction motor directly coupled to a 30 kw

Figure 6-1. Block Diagram of the Physical Equipment

Figure 6-2. The Carbon Arc Image Furnace

(300 amperes at 100 volts) dc generator.

The standard steel parabolic mirror of the searchlight has been replaced by glass first-surfaced aluminized ellipsoidal reflector, manufactured by the Bausch & Lomb Optical Co., with first and second focal lengths of 11 and $52\frac{1}{2}$ inches, respectively. The positive carbon crater of the arc is placed at the first focal point of this mirror, whereupon it is imaged, with a magnification of about five, at the second focus; the actual exposure plane is located slightly inside this second focal point.

The spatial distribution of irradiance in the exposure plane displays approximate radial symmetry about the optic axis of the unit, and follows quite closely a Gaussian function:

$$H(r) = H(0) e^{-\pi a r^2}$$

where $H(r)$ is the irradiance at a radial distance r from the optic axis, and $H(0)$ is the irradiance on this axis (9). The maximum "full open" value of $H(0)$ for this furnace is $35 \text{ cal cm}^{-2}\text{sec}^{-1}$; while $a = 0.13 \text{ cm}^{-2}$. At a radial distance of 0.564 cm the irradiance has dropped to about 88% of this central maximum, while at 0.9 cm it has decreased to about 70% of maximum. The irradiances cited in the remainder of this paper are always average values over the central circular area of 1.0 cm^2 (i.e. the area within a circle of 0.564 cm radius); as noted above, over this area the irradiance may be considered reasonably constant. This spatial average irradiance may be varied by a diaphragm and screens from a full open value of $33 \text{ cal cm}^{-2}\text{sec}^{-1}$ down to slightly below $0.1 \text{ cal cm}^{-2}\text{sec}^{-1}$. During the period covered by this study, the calibration of the furnace was checked frequently to insure the stability of the output.

6.4.1 Shutter System

The shutter normally used on this furnace employs two light aluminum

vanes, one for opening, the other for closing, actuated by rotary electrical solenoids. Each solenoid is powered through a thyatron; a synchronous motor driven cam controls the thyatron grids, and provides precise timing. Exposure times of from 0.1 second to as long as desired may be obtained.

For the present study, this shutter presented two difficulties. The opening and closing times are about 0.015 second, which is quite satisfactory for routine work. However, as pointed out previously, it is desirable to obtain the shortest possible irradiance rise times, in order to secure a reasonable approximation to a true step function, and to avoid large corrections of the initial temperature response. A much more serious difficulty was the electrical pulse generated in the low-level temperature measuring circuits each time a rotary solenoid was actuated. Although the shutter had been expressly designed to keep these unwanted pulses to a minimum, they could not be eliminated entirely; further, this noise occurred at the initiation of the exposure, precisely where the maximum accuracy was demanded by theoretical considerations.

Accordingly, an alternative arrangement was devised, using the pulse-shaping wheel seen prominently in Figure 6-2 (10). This wheel was altered so that over half of its periphery an opaque aluminum-foil screen extended into and completely obscured the converging beam of radiation from the source; over the remaining half of the wheel's periphery, the radiation could pass unobstructed. Between the pulse-shaping wheel and the exposure plane, and as close to the former as possible, an opaque metal screen was mounted, with a vertically disposed, $1\frac{1}{4}$ cm width slit through the center of the screen. This wheel-and-slit system was used in conjunction with the previously described vane-type shutter in the following way. The wheel was rotated at one second per revolution, with this speed accurately measured by an electronic tachometer of special design. When it was desired to make an exposure, a

hand operated asbestos-board screen was swung out of the beam, thus energizing the shutter control circuits. As soon as the wheel rotated so that the leading edge of the opaque half had swept over the slit, the opening solenoid on the vane shutter was actuated. Approximately one-half second later, well after any electrical disturbance due to the solenoid had damped out, the trailing edge of the opaque section of the wheel swept past the slit, thus initiating the exposure. After one-half second, the leading edge of the opaque section again swept over the slit, terminating the exposure, and shortly thereafter the second rotary solenoid was fired, closing the vane shutter. A single half-second pulse was thus delivered to the test animal; measured rise time was about 0.005 second, or some three times faster than the vane shutter. Further, at the crucial opening point, no electrical noise was injected into the measuring circuits. Synchronization between wheel and vane shutter was automatic, and the system performed without malfunction through the entire study. While only a single exposure time of 0.5 second was used, this was no disadvantage for this investigation.

6.5 Animal Holder

In the usual investigation in this laboratory, the experimental animal is simply hand-held behind a fixed water-cooled aperture plate which delimits the exposure area (11). The circular aperture is centered on the optic axis of the furnace, and is about 1.8 cm in diameter. The water circulated through the aperture plate is held at about 35°C.

For the present study, this simple arrangement was not adequate. In order to make an exposure, it is necessary to have the exposed surface in the vertical exposure plane, and only a few inches behind the vane shutter; in this position it is impossible to insert fine temperature sensing elements into the burn site. To circumvent this problem, a hinged animal holder

was constructed as shown in Figures 6-3 (open), 6-4 (closed), and 6-5 (in exposure position).

The animal was first placed in this holder, flank up, and fixed with blocks, wedges, and straps. The hinged aperture plate was swung down over the animal's flank, and the limits of the intended burn site marked with ink. This plate was then swung out of the way, and the "thermocouple pattern" inked on the skin, as shown in Figure 6-3. After installation of the thermoelements (to be described below), the aperture plate was again closed over the animal, and the entire holder rotated to bring the selected site into a vertical plane. The holder was so fixed to the furnace exposure table that the aperture would now be in the exposure plane and precisely centered about the optic axis of the source.

6.6 Temperature Sensing Elements

The search for a satisfactory temperature sensing element was the first project undertaken in this investigation. This search was initiated in 1952, and has by no means ended. It would not be appropriate here to present a detailed history of this program, but in view of the somewhat unusual materials employed, a brief review is in order.

The primary considerations were small size, coupled with best possible sensitivity, fast response time, adequate strength, and lack of tissue reaction. It was decided quite early that thermistors would not be suitable, since even the unmounted bead types were larger than desirable. In addition, there were problems of insertion, insulation of lead wires, and the not particularly impressive response time constants available. Attention was therefore directed to thermocouples, which in spite of their meager output voltages have the advantages of rapid response and ease of insertion (with proper construction). Further, the only lower limit on their size is set by

Figure 6-3. Animal Holder, Open

Figure 6-4. Animal Holder, Closed

Figure 6-5. Animal Holder, in Exposure Position

Figure 6-4. Animal Holder, Closed

Figure 6-5. Animal Holder, in Exposure Position

considerations of strength and feasibility of construction.

6.6.1 Wollaston Wire Thermocouples

This last consideration, that of ease of construction of the thermoelements, was by no means of minor importance, since it was anticipated that a large number of elements would be needed for this investigation. An elegant solution to this problem seemed to be the use of Wollaston process wire in the production of fine thermocouples. This wire is a composite material, formed by placing a sleeve of one metal around a rod of another; the assembly is then drawn through dies to reduce its diameter as though it were a single wire (12). Normally, the Wollaston process is used to obtain wires of very small diameter, by simply dissolving away the unwanted outer jacket in a suitable reagent, thus freeing the fine inner core. For thermocouple production, this procedure is revised slightly. A suitable length of the composite wire is selected, and the jacket dissolved away for only half the length of the wire. If the resistances of jacket and core are selected properly, the shunting effect of the latter in the former is negligible, and one has easily formed a thermocouple at the jacket-core junction.

After extensive discussions with a leading supplier of Wollaston process wire (13), one suitable pair of metals was selected: a jacket of fine silver over a pure palladium core. Although the thermoelectric power of this combination was given by tables as only 10 microvolts per degree, the metals would be expected to be quite benign in tissue, which was important in order to avoid large circulating currents due to electrochemical action. Calculations indicated that a jacket to core diameter ratio of four to one would yield insignificant shunting of the palladium within the silver; hence an order for material with a jacket diameter of 0.002 inch, and a core diameter of 0.0005 inch was placed. On advice of the supplier, an order for

silver and palladium extension lead wires and foils for connecting clamps was placed at the same time, with all these materials to be taken from the same melts of the respective metals. This precaution was deemed necessary to avoid unwanted thermal emf's generated at the junction of two supposedly identical metals from different melts. At the same time, extensive development of an amplifying-recording system matching the indicated thermoelectric power was begun.

Unfortunately, the supplier was unable to furnish the Wollaston wire in the desired size; the smallest ratio of diameters which could be achieved was ten to one. The wire obtained, then, had a 0.003 inch diameter silver jacket over a 0.0003 inch diameter palladium core. Although the jacket was somewhat larger, and the core much smaller than desired, about two dozen thermocouples were constructed of this material.

This construction was fully as simple as had been hoped. First, an eight inch length of 0.005 inch diameter spring steel wire was sharpened on one end, and butt-welded (using a capacitor discharge) on the other to a four inch length of the Ag-Pd Wollaston wire. This steel wire served as a leader to introduce the thermocouple into the skin. The terminal two inches of the Wollaston wire were dipped into nitric acid, which quickly removed the silver jacket. The completed assembly was slipped into a fine metal tube for protection until use. The palladium proved so fragile, however, that not one of the elements constructed was ever successfully inserted into an animal and connected to the electric circuit.

6.6.2 Butt-welded Thermocouples

Undoubtedly, in other hands or for other purposes, the Wollaston wire thermocouples described above could be most satisfactory. However, after it appeared that breakage would never be much below 100%, attention was turned

other means of fabrication. In view of the extensive investment of time and money in a system compatible with silver-palladium junctions, the decision was made to continue using these metals. (Immersion of thermocouples in Ringer's solution had shown that electrochemical effects were completely negligible.)

The first alternative procedure was the butt-welding, by capacitor discharge, of two inch lengths of silver and palladium wires, each with a diameter of 0.001 inch. The Job-like patience in carrying out this method of Mrs. Marjory Pecora Pawley, formerly of this laboratory, is gratefully acknowledged. After three months, about ten feet of each of these wires were consumed in the net production of six assemblies, complete with steel leaders, as described above. This procedure was then abandoned.

6.6.3 Plated Thermocouples

The ease of thermocouple construction from Wollaston wire suggested the opposite procedure: rather than dissolve the silver jacket from the composite wire, one could plate silver on a fine palladium wire. This project was turned over to Peter Hudson, formerly of this laboratory; his labor is gratefully acknowledged. The procedure developed was as follows: a four inch length of 0.001 inch diameter palladium wire was suspended vertically under slight tension in a plastic holder. The lower two inches of this wire was dipped into a preliminary strike bath for twenty seconds; this same section of wire was then immersed in a standard silver cyanide bath for plating (14). The plating current was adjusted initially to 0.2 milliamperes, corresponding to a current density of about 5 milliamperes per cm^2 ; this current was gradually increased to a maximum value of 1.0 milliamperes as the plating increased in thickness. In a total plating time of two hours, a silver coating of about 0.004 inch could be obtained. Eight

plating cells were connected in series, so that eight thermocouples were formed at the same time.

From electrical conductivity measurements, the silver coating was judged to be uniform and dense, although under low power magnification, the surface appeared rather grainy. The measured temperature-emf relation of these couples was almost identical to those of the Wollaston wire and butt-welded elements described above. In general, these were quite satisfactory thermocouples except for the large diameter of the silver jacket necessary to eliminate shunting effects of the palladium core. Further work on this method was halted with the successful development of the procedure which will be described next.

6.6.4 Soldered Thermocouples

The writer is indebted to Dr. J. B. Hursh, of this department, for recommending that soldered thermojunctions be investigated. The procedure which was developed yielded completely satisfactory elements of surprising strength.

Construction was carried out under a stereomicroscope, with the wires held in a special manipulator constructed by J. A. Basso, of this laboratory. Again, silver and palladium wires were used, each of two inch length, but of 0.002 inch diameter. First, one end of each wire was flattened slightly over about 1 mm, and these two flattened portions carefully brought side by side, with a film of a mild rosin flux on the mating surfaces. A stick of ordinary soft solder was wiped over the clean tip of a fine soldering pencil, and the trace of solder adherent to the tip transferred to the junction; the molten solder was immediately pulled up between the flattened wires by capillary action. The finished junction was but slightly larger than the wires themselves, and in most cases, it could not be located by the unaided

eye. Assembly was completed by soldering an eight inch length of spring steel leader (both 0.005 and 0.010 inch diameters were used) to the free end of the silver wire; the entire unit was inserted in a length of fine plastic tubing for protection until use. The writer was able to fabricate about five such assemblies per hour.

The majority of the thermocouples used in this study were constructed by Miss Marilyn Aldrich, of this laboratory. Her patience and extraordinary skill are deeply appreciated.

6.6.5 Calibration of Thermocouples

The temperature-emf relations of several of the soldered thermocouples were determined, and were found to be so nearly identical that it was felt that individual calibration of each couple would be quite unnecessary. In all, a total of five complete calibration runs were made, covering the temperature range of 35°C to 135°C.

Each thermocouple was placed in a well-stirred mineral oil bath, provided with an external tape heater. The temperature was raised slowly to its highest value, and then returned slowly to the starting point. Temperatures were read on an A. H. Thomas specification, etched stem, mercury-in-glass thermometer, graduated from -5 to 250°C by 0.5°C. This instrument was pointed for total immersion, and emergent stem corrections were applied to all readings. Thermo-emf was measured with the amplifier-recorder system to be described below. This system, rather than a potentiometer, was used so that any loading by the amplifier on the couple would be automatically accounted for.

The resultant emf-temperature curves were essentially linear from 35°C to 100°C. From the least squares fit over this range, the calibration factor for these thermocouples (the inverse of the thermoelectric power)

was found to be 0.0875°C per microvolt, with a standard deviation of about 0.002°C per microvolt.

6.6.6 Placement of Thermocouples

The placement of thermocouples for subsurface temperature measurements was quite simple. The sharpened end of the steel wire leader was inserted into the skin a few millimeters from the edge of the exposure site (which was marked as noted in Section 6.5). When the tip of the leader was at what was judged to be the desired depth, it was then guided laterally through the skin, and directly under the center of this marked site. Radial lines inked on the skin, and visible in Figure 6-3, were helpful in maintaining proper alignment. From the surface, the progress of the tip of the leader could easily be felt, and when it had passed completely under the exposure area, it was urged back out to the surface by folding the skin over the sharp point. The leader was then pulled completely through, thus bringing the attached thermocouple into position; the leader was then clipped off and discarded. The 0.010 inch diameter leaders were used for deep placement, and the lighter 0.005 inch for shallow.

It was now necessary to adjust the thermocouple so that the junction itself was precisely on the center line of the exposure site. During construction of each thermocouple, a small drop of red paint was placed on the palladium wire a known distance (12.5 mm) from the junction. A simple paste-board scale was made with two marks separated by this distance. With the couple now inserted in the animal, the scale was placed with one mark on the center of the exposure site, and the thermocouple pulled back and forth until the red "tag" was aligned with the other mark. Since the steel leader had been carefully centered, one was now assured that the thermojunction was exactly on the center line of the exposure area. The thermocouple was fixed

in this position with tiny tabs of cellophane tape, while subsequent couples were installed. The measurement of the depth of the junctions will be considered in the next section.

The emplacement of surface thermocouples was, surprisingly, more difficult than the subsurface elements, the major difficulty being the maintaining of good contact between skin and wire. After considerable effort, a satisfactory method was developed. In order to secure and maintain good contact it was found to be necessary to align the thermocouple with the fold lines of the skin. With the "grain" direction of the skin determined, the tip of the leader was pushed into and immediately back out of the skin a few millimeters from the edge of the exposure site. This process was repeated on the opposite side of the site, and, as before, the thermocouple pulled into approximate position and the leader clipped off. Now, using a magnifying glass, the junction was centered accurately, and with the skin slightly compressed with one hand, each end of the thermocouple was tied in a simple knot around the tab of skin through which that end passed. When the skin was released, the thermocouple was placed under gentle tension due to the elasticity of the tissue. With care, thermocouple breakage was held to about one in five in this procedure. (In contrast, only rarely did breakage occur in the placement of subsurface elements.)

In the early phase of this study, four thermocouples--one surface and three subsurface--were installed in each exposure site. This was later reduced to three--one surface and two subsurface. This will be discussed in more detail subsequently.

6.6.7 Electrical Connections to the Thermocouples

Screw clamps with silver and palladium foil clamping surfaces were used to secure simple, rapid, and reliable electrical connections to the

thermocouples. Four pairs of clamps were mounted on (but electrically insulated from) a metal ring, each pair consisting of one silver and one palladium clamp on opposite sides of the ring. This ring was mounted on one end of a supporting tube, which was composed of a steel pipe (for magnetic shielding) securely pinned within a copper pipe (for electrical shielding). Silver and palladium extension lead wires were soldered to the clamping foils, and these wires then passed through the supporting tube to a reference junction assembly. The lead wires, of 0.010 inch diameter, were insulated with fine polyethylene tubing.

6.7 Depth Measuring Device

For determination of the thermal diffusivity and the absorption pattern of skin, it is necessary that one know the real depth, x , at which a temperature measurement is made. Of the various schemes considered for measuring the depth of placement of a thermocouple, the one finally selected was patterned quite closely after that developed by Schilling, working with Ross and Moritz at Western Reserve (15). The procedure, as modified for this study, utilized the ferro-magnetic properties of the spring steel leader used to insert the thermocouples, with a tape recorder erase head employed as the sensing element.

The electrical circuitry was quite straightforward. The erase head was shunted by a variable capacitor, and this parallel L-C circuit placed in one leg of a unity ratio resistive Wheatstone bridge, with the adjacent resistance leg also variable. The bridge was powered by a General Radio Company 1000 cps tuning fork oscillator through a shielded bridge transformer, and the bridge output measured with a DuMont type 403 cathode ray oscilloscope and a Heathkit ac vacuum tube voltmeter. The bridge output was also loaded by a General Radio Company narrow pass filter tuned to

1000 cps, to suppress the second harmonic content in the output voltage.

With no magnetic material near the erase head pickup, the resistor and capacitor could be adjusted to secure minimum bridge output; these adjustments were made while observing the CRO presentation. Now, when any magnetic material was brought near the gap in the erase head, the inductance of the latter was increased, the bridge was therefore unbalanced, and the unbalanced voltage could be read on the vacuum tube voltmeter. The magnitude of this output was dependent upon the separation between the erase head and the magnetic material. The instrument was calibrated by placing slips of paper of various thicknesses between the pickup head and a sample of either the 0.005 or 0.010 inch diameter wire used for the leaders. The head was carefully positioned over the wire to secure maximum output voltage which was then plotted against the paper thickness as measured by micrometer calipers.

To measure the depth of a subsurface thermocouple, the circuit was first balanced by an assistant while the erase head was well isolated from any magnetic material. The assistant then noted the bridge output voltage for zero separation, by holding the head directly on a sample of the appropriate steel leader; this one-point calibration check proved to be a highly reliable indication of any drifts in the system. During this checking procedure, the experimenter was inserting a thermocouple in an anesthetized animal as described above, proceeding only to the point where the steel leader had been passed completely under the exposure site, with the tip protruding back out through the skin surface. If the calibration check was satisfactory, the erase head was passed to the experimenter, who positioned the pickup precisely over the leader, using a simple plastic centering jig. Using very light pressure to avoid compressing the skin or flexing the leader, the head was "rocked" for maximum bridge output voltage. This

procedure was repeated until exact agreement was achieved in two successive trials, and this value of output voltage recorded on the master data sheet. These voltage values were later translated into depths from the calibration curve, taking care to add the radius of the leader so as to obtain the depth to the center of the leader.

The leader was then pulled on through, bringing the attached thermocouple into position, as has been previously described. (It was assumed, and must certainly be true, that the thermocouple would be at the same depth as the leader.) The silver and palladium wires being non-magnetic, the depth of another leader in the same site could now be measured without interference from the first thermocouple.

It will be noted that the medium between the erase head and the steel wire was paper for the calibration, and skin (or skin plus minute amounts of silver and palladium) in actual use. The magnetic susceptibilities of these materials are so small that the error introduced by this procedure is completely negligible.

The over-all error in these depth measurements is difficult to assess. The calibration data are accurate to about 0.05 mm, but it is doubtful if the thermocouple depths were measured to that accuracy. An estimate of 0.1 mm is probably not unreasonable. Actually, since the temperature gradients through the skin are very steep, an error in depth measurement of this magnitude can be quite serious. Up to the present time, however, no better measuring system has been devised, so one must hope that by taking many measurements, these errors will be smoothed out. This is hardly a satisfactory situation, and certainly more consideration must be given to this problem in the future.

6.8 Reference Elements

As mentioned in subsection 6.6.7, the extension lead wires from the thermocouple clamping ring were brought through the composite iron and copper support tube to a reference (or cold) junction assembly. This consisted of an open bottom steel box, securely fastened to the distal end of the support tube, with two, four-terminal tie-point strips mounted within the box. The silver and palladium leads from each thermocouple were brought to adjacent tie points, and there fastened securely to the tinned copper leads of a two conductor cable. This assembly is visible in Figure 6-5, with the four shielded cables coming from the side of the box.

When the animal holder was rotated into exposure position, the open side of the steel box was down. A brass vessel filled with water was slipped up inside this box so as to immerse completely the reference junctions. Water from a constant temperature bath was circulated rapidly around the junctions. In theory, the entire amplifying-recording system could now be considered an "in-between" metal, so that only the potential difference of Ag-Pd junctions at the reference and measured temperatures would be recorded. In practice, all parasitic thermal emf's could not be eliminated, although in several determinations the resultant zero off-set never exceeded 2°C equivalent temperature difference. In fact, from the consistent values of initial temperatures observed throughout the experiment, it was inferred that the zero off-set probably did not exceed 0.5°C.

It will be recalled that in the theoretical development of this program, only the temperature rise was considered, with the initial temperature throughout the material assumed constant. This assumption is not strictly true for skin, since a slight initial temperature gradient does exist through this tissue. However, this may be corrected for fairly accurately by measuring only the temperature rise above the initial temperature at each depth. Hence,

one is not particularly concerned with accurate measurements of the initial temperatures, and zero off-set due to parasitic emf's is not serious.

6.8.1 Temperature Regulated Water Bath

The bath used to supply constant temperature water for the reference junctions consisted of a 10 inch by 10 inch cylindrical Pyrex jar containing an E. H. Sargent Company circulating and heating tower. This device provides completely turbulent flow throughout the entire volume of the bath, and is one of the most satisfactory units commercially available. The temperature was controlled by a Prince "Magnet" thermoregulator, operating a 200 watt heater through a sensitive electronic relay (16). The bath temperature was held at $35.0 \pm 0.02^\circ\text{C}$. Water was pulled from this bath by a small centrifugal pump, and forced at rather high velocity across the reference junctions. This water returned to the bath by gravity flow.

6.9 Calibration Circuits

A voltage calibrating circuit was placed in each thermocouple channel, ahead of the initial preamplification. Insertion of these circuits in this position contributed to the input noise of the system, but this increase in noise was justified by the high accuracy obtained by introducing the calibrating voltages directly in series with the thermocouples.

The four calibrating circuits were essentially identical. A 10 ohm $\pm 1\%$ wire wound resistor was placed in one preamplifier input lead, with an adjustable current driven through this resistor from a Wheatstone bridge type circuit. This current was measured directly by a 100-0-100 microampere, center zero panel meter. Since the input resistance of the following preamplifier was about 7000 ohms, the shunting effect on the 10 ohm resistor amounted to only about $1/7$ of 1% , which is negligible. Hence the calibration voltage was given simply by Ohm's law as ten times the measured current. Each

calibrating circuit was powered individually by a Mallory No. RM-4R mercury cell. The circuits were placed on the same chassis which housed the preamplifiers.

In use, a calibration signal of suitable magnitude was recorded in each channel just prior to an exposure, and a second signal of the same magnitude inserted at the end of the recording. The average of these two calibration deflections was divided into the calibration voltage to give the voltage sensitivity, in microvolts per millimeter, of that channel for that particular exposure. This value was then multiplied by the known calibration factor of the Ag-Pd thermocouples to give the temperature sensitivity of the channel in centigrade degrees per millimeter of deflection. It will be recalled that in the theoretical analysis, the temperature response per unit absorbed irradiance was the quantity of primary concern; hence, the temperature sensitivity was divided by the value of absorbed irradiance pertaining to that particular exposure to give a "scale factor" as:

$$\frac{\text{degrees temperature rise per unit absorbed irradiance}}{\text{millimeters of chart deflection}} .$$

This scale factor multiplied by the deflections recorded during the exposure gave the temperature response in the desired form. The scale factor was determined for each thermocouple channel for every exposure made in this study.

6.10 Preamplifiers

The preamplifiers used in each thermocouple channel were low-level, dc instruments designed and constructed by the writer. Four identical amplifiers were placed on a common chassis, with operating voltages obtained from common power supplies. The complete assembly is shown in Figure 6-6.

The amplifiers were of the chopper-input chopper-output type, with heavy dc feedback around the complete circuits. The low-level input signal was first chopped at 60 cps by a Stevens-Arnold A-11 chopper, and the square-

Figure 6-6. Amplifier-Recorder System

wave ac signal passed to an input transformer. The latter was a special unit made by Southwestern Industrial Electronics, with a turns ratio of 1:265 and a reflected primary impedance of 40 ohms center tap to grid. This transformer drove two conventional stages of R-C coupled, balanced amplification, utilizing 12AX7 twin triodes. All tube heaters were powered by well-filtered dc. The second stage was followed by a balanced cathode follower which provided a low impedance drive for the rectifying output chopper, also a Stevens-Arnold A-11 type. The pulsating dc output was smoothed by a low-pass R-C filter. Current feedback around the entire circuit was provided by 5000 ohm and 10 ohm resistors connected in series to ground, with the low side of the preamplifier input (the center tap of the input transformer primary) connected to the junction between these elements.

Since the closed-loop gain of each amplifier was determined almost entirely by the ratio of these feedback resistors, low noise, low temperature coefficient, metal film types were used. Although the four preamplifiers were supposedly identical, their performance characteristics differed slightly from one to another. Typical specifications are: voltage gain, 500; input resistance, 7000 ohms; equivalent input noise with 10 ohms input resistance, 3 microvolts peak-to-peak, predominantly 120 cps ripple. This last characteristic was the one most subject to variation from one channel to another. The peak-to-peak equivalent input noise of the best channel was less than 1 microvolt, while that of the worst was 5 microvolts. Since this noise was for the most part the 120 cps ripple remaining in the pulsating dc output after filtering, its magnitude increased with increasing output voltage. Thus, for the noisiest channel mentioned above, the equivalent input noise increased to about 12 microvolts peak-to-peak at an output of 200 millivolts. Since this corresponds to a 400 microvolt input signal, the signal-to-noise ratio here is 33, which is quite acceptable.

The frequency response of a typical unit was fairly clean from dc to 40 cps, and within ± 3 db to about 85 cps, with, however, considerable heterodyning around 60 cps, the chopping frequency. The distortion at frequencies above 40 cps was severe; at 200 cps, where there was still a measurable response, the output waveform was approximately triangular. Thus, the bandwidth here specified cannot be employed blindly in calculating the amplifier response to an arbitrary input signal. Actually, these response data are not particularly meaningful, the important measurements being those of the complete thermocouple-preamplifier-recorder system. These will be discussed in the next section.

6.11 Recording Assembly and System Response

A Sanborn Model 150 four-channel recorder was used, with Model 150-1300 dc coupling preamplifiers in all four channels, providing a basic sensitivity in each channel of 50 millivolts per cm of stylus deflection. The channels used for temperature measurements were, of course, driven by the output of the chopper-input preamplifiers described above. This additional gain of 500 increased the system gain to 10 microvolts per mm, as measured from the input terminals of the preamplifiers. (These latter should probably be termed "pre-preamplifiers," since the Sanborn assembly contains "preamplifiers.")

No attempt was made to adjust the gain of each channel to secure precisely this sensitivity; rather, as noted in Section 6.9, the calibration circuits preceding the chopper-input preamplifiers were used to establish an exact scale factor for each channel on every exposure. However, the sensitivities were always very nearly 10 microvolts per mm, so one can define an average temperature sensitivity from the calibration factor for the Ag-Pd couples; this factor, as stated previously, was found to be 0.0875°C

per microvolt output. The temperature sensitivity, then, was 0.875°C per mm of stylus deflection, or, inversely, 1.14 mm per $^{\circ}\text{C}$. Since only the central 40 mm section of the Sanborn chart was used, full scale response for full recorder gain was 35°C rise, or 70°C maximum. With the Sanborn pre-amplifier gain attenuated by a factor of 2 (X-2 setting), the full scale temperature rise was 70°C , or a maximum temperature of 105°C . As mentioned in subsection 6.6.5, the linearity of the Ag-Pd thermocouples was satisfactory up to 100°C , and it is only over the range 35°C to 100°C that the stated calibration factor is valid. Thus, by never utilizing attenuations greater than 2, one automatically avoided operating beyond the linear range of the thermocouples. If, as occasionally happened in the early exposures, a stylus was driven off scale, the data were simply lost, and the furnace output was reduced for the next exposure. This means, of course, that at no time in this investigation were skin temperatures in excess of 100°C ever studied.

In the previous chapters, considerable discussion was devoted to the measurement of temperature response at, or immediately after, the initiation of the exposure. This, in turn focusses attention on the time response of the thermocouple-amplifier-recorder system to arbitrary temperature input functions, and specifically to a step function. Three procedures were used to investigate this system response; two employed electrical input signals to the chopper-input preamplifier, while the third utilized a "temperature input" to the entire system. These procedures will now be described briefly.

The first test consisted of applying voltages to the amplifier-recorder system which increased exponentially with time. The form of the predicted response curves of the diathermanous solid is vaguely suggestive of an exponential form; hence it was felt that an exponential voltage input function would give some indication of the ability of the system to follow the expected

response form. These test voltages were generated in a simple resistance-capacitance circuit with initial time rates of change of stylus deflection selected to be approximately equal to those expected on the basis of the predicted temperature response. The output voltage of this circuit was developed across a simple, resistive, 500 to 1 voltage divider; full output voltage (with maximum value of about 200 millivolts) was applied directly to one channel of the Sanborn recorder, while a signal with precisely the same form, but attenuated by a factor of 500 was applied to an adjacent recorder channel through one of the chopper-input preamplifiers. Since the latter had a gain of 500, the two recorded signals should have been precisely the same, and any differences could be ascribed to distortion introduced by the preamplifier. Further, if either or both recordings did not follow an exponential form, this would suggest that the system probably could not faithfully follow the predicted response form.

The results of this test were most encouraging. Not only did the recordings follow an exponential form with considerable accuracy, but the two channels showed almost superimposable responses, showing that the chopper-input preamplifiers could indeed handle this type of signal with satisfactory fidelity.

The second test consisted of applying a voltage step function, of about 200 microvolt amplitude, to a preamplifier input, and recording the resultant response on one channel of the Sanborn recorder. While there was some variability in results, the typical response was that of a slightly underdamped system; the recorder stylus, after an initial rapid upscale deflection, oscillated briefly before coming to its new rest position. The first upward excursion overshoot the final deflection by about 10%, with the peak reached about 7 milliseconds after applying the step function input. Using these values, one may express the response to a unit step function

input as

$$A(t) = S \left[1 - e^{330t} (\cos 450t + 0.73 \sin 450t) \right], \quad (6-1)$$

where $A(t)$ (with units of, say, millimeters of deflection per microvolt input) is by definition the indicial transfer function, and S is the system sensitivity (with the same units as $A(t)$). Now, by the superposition integral (17) the recorder deflection, say $Y(t)$, due to any arbitrary input voltage, $E(t)$, may be expressed as

$$Y(t) = E(0) A(t) + \int_0^t \frac{dE(\lambda)}{d\lambda} A(t-\lambda) d\lambda. \quad (6-2)$$

Now, in principle, if one has a particular form of $Y(t)$, one can determine the system input voltage, $E(t)$, which gave this recording, although the inversion of equation (6-2) to obtain $E(t)$ in terms of $Y(t)$ is by no means simple, and analogue computation is definitely indicated. There is, however, an even more fundamental objection to the use of this equation. One is, after all, interested in correcting the recordings to allow for any deficiencies in the system response to an arbitrary temperature change; equation (6-2) allows one to correct the thermocouple output voltage, $E(t)$, but it is possible, and even likely, that the principal lag in the system would be in the thermocouple itself.

The third method of testing the system response was designed to provide an over-all indicial transfer function, in terms of recorder stylus deflection per unit temperature change in the immediate environment of a thermocouple. The procedure was quite simple. A soldered Ag-Pd thermocouple, as described in subsection 6.6.4, was mounted in the thermocouple clamping ring (6.6.7), and this latter assembly then plunged rapidly into an oil bath heated about 10°C above ambient. This provided a temperature step function input to the system.

Unfortunately, the results of this test were so variable that no general indicial transfer function could be obtained. In about half the tests, the response was quite smooth, corresponding to a simple "resistance-capacitance" type system. In the remainder of the tests, some oscillation in stylus position occurred, although the stylus did not always overshoot its final position. Possibly these oscillations--which were much slower than those observed with the voltage step function input--were due to temperature gradients in the oil bath. One might also suspect "movement artifacts," since the thermocouple was subjected to considerable stress in entering the bath.

The one factor which was quite reproducible from one test to another was the time required for the stylus to reach 67% ($1 - e^{-1}$) of its final value. This rise time was about 10 milliseconds, which is satisfactorily short. If the over-all system were indeed analogous to a resistance-capacitance network, this rise time would suggest a bandwidth of about 16 cps; it would be most interesting if a "temperature sine-wave generator" could be devised which could check this prediction. To date, the highest "generator" frequency the writer has been able to achieve is about 1 cps, using an intentionally unstable feed-back control system. Further work in this area would be desirable.

In view of this failure to derive a rigorous scheme for correcting the initial recorded response to the "true" temperature response, it was necessary to employ the classic method of extrapolation to zero time. Since any transients (or lag) introduced by the thermocouple-amplifier-recorder system would be damped out within 30 or 40 milliseconds, at most, this extrapolation actually could introduce very little error, providing the location of initial time on each recording was known with high accuracy. Two pieces of equipment were used to achieve this, a pulse measuring assembly and a

timing marker generator; these will be described in the following sections.

6.12 Pulse Measuring Assembly

The pulse measuring assembly was designed to place, on one channel of the four channel recorder, a recording of the exact irradiance pulse form delivered to the experimental animal for each exposure. The pickup element was a small plane mirror (visible in Figure 6-4) located just in front of the water-cooled aperture plate of the animal holder. When this holder was rotated into exposure position the mirror was below and slightly to one side of the optic axis of the source, and inclined at 45° to this axis. A lens and type 929 vacuum phototube were mounted on an optical bench placed on a table beside the carbon arc furnace (Figure 6-6). During an exposure, some of the radiation which was back-scattered from the animal's skin was picked up by the plane mirror and directed to the lens which then focussed it onto a tiny aperture in an aluminum foil shield around the phototube envelope. This simple phototube photometer thus measured, with essentially no delay, the exact radiation pulse delivered to the animal. The phototube was operated well below saturation, to assure linear response. To provide a low-impedance drive for the recorder, the phototube output was coupled directly to a 12AU7 cathode follower, located immediately beneath the phototube socket. (The other triode unit of the 12AU7 was used to provide quiescent voltage balancing, and to insert timing marks on the recording as will be described below.) The total voltage swing impressed on the recorder for most exposures was about 2 volts (considerably less than this for India ink covered skin); hence this particular recorder channel was operated at attenuations of 10 or 20, giving sensitivities of 0.5 or 1.0 volt per centimeter, respectively.

By placing a recording of the irradiance pulse directly on one channel of the four-channel recorder, highly accurate determinations of zero time

and irradiance rise time were made possible, but at the expense of losing one channel for temperature recording. Thus, temperature information was traded for time information.

6.13 Timing Marker Generator

The pulse measuring assembly allowed one to determine the location of zero time quite accurately only on the pulse recording channel. It was ascertained that there could be relative shifts between the four styli of the recorder of as much as 0.3 mm along the printed time scale of the chart; these shifts seemed to be due to slight errors in chart paper tracking, or possibly to slight errors in printing the paper. In the Sanborn instrument, the styli write on the chart as the latter passes over a sharp edge of metal; possibly this edge was not precisely normal to the direction of chart motion. In any event, it was necessary to determine, for each recording, the precise values of these relative displacements in order to locate the zero time position on the three temperature recording channels.

This was accomplished by inserting marker signals on all four styli simultaneously, and at regular intervals. Thus, if for some exposure, it was found that the scale distance from one marker to the initiation of the pulse (on the pulse recording channel) was, say, 34.6 mm, then the distance from this same marker to the zero time point on the other three channels was likewise 34.6 mm. The essential point is that the marker signals were introduced on all four styli simultaneously; hence, one did not need to depend upon the accuracy of alignment of the printed time lines on the chart paper. In addition, since the marker signals were applied at accurately known intervals, one could determine the true chart speed for each recording.

The time base for the unit was the power line frequency; this is held within a fraction of one percent of 60 cps, which is quite accurate enough

for this purpose. This alternating current was used to drive a Leeds and Northrup chopper, with the resultant 60 contacts-per-second signal fed to a divide-by-30 unit. The latter was a Model 160A Potter predetermined electronic counter, set for a count of 30; on reaching full count, a Western Electric SPDT mercury relay, incorporated in the counter, was switched from one pole to the other. With this relay on one side, a 2 μ fd capacitor was charged from a $1\frac{1}{2}$ volt dry cell, and when the relay was switched to the other pole, the capacitor discharged rapidly through a 5000 ohm potentiometer. This cycle was repeated once per second, thus giving one short voltage pulse every second across the potentiometer.

The dc coupling preamplifiers of the Sanborn recorder feature balanced, or differential, input, while the outputs of the chopper-type preamplifiers which drove the three temperature measuring channels (A, B, and D) were single-ended. Thus, an additional, low-side input was available on each of these channels. These three low-side inputs were placed in parallel and driven by a voltage picked off the 5000 ohm potentiometer of the timing marker circuit. This voltage was adjusted to give styli deflections of 1 or 2 millimeters, depending upon recorder attenuator setting (X-2 or X-1, respectively). The irradiance pulse recording channel (C) was operated at considerably lower gain, as noted above; hence it was necessary to use a larger marker voltage for this channel, which was obtained across the full 5000 ohm potentiometer. This signal was applied to the grid of the cathode follower providing quiescent voltage balancing to the low-side input of channel C.

Timing marker pips were thus recorded simultaneously on each channel, as desired. Each pip had an abrupt downward-going leading edge followed by a rapid return to initial position, and occupied about 10 milliseconds, or 1% of the total time between markers. Thus, there was only about one chance

in a hundred that a pip would occur at the beginning of an exposure, and hence obliterate the all-important initial response. Naturally, this occurred with somewhat higher frequency than predicted, although it was usually possible to smooth out the recordings by eye, and thus salvage the data.

6.14 Exposure Procedure

For each exposure, the thermocouples were installed, with their depths measured, as described previously. The electrical connections were then made, and circuit continuity checked; often it was necessary to eliminate accidental shorts or grounds by insulating the offending sections of thermocouple wire with tiny tabs of cellophane tape. The animal holder was then rotated into exposure position, the reference bath raised into place, and the location of the animal behind the water-cooled aperture and the contact of the surface thermocouple given a final check.

While an assistant struck and adjusted the carbon arc furnace, the recorder was started at a slow chart speed with all channels switched to off position; this placed an initial baseline on each channel. The recorder inputs were then switched on, and calibration signals, from the circuits described in Section 6.9, recorded. These signals were usually either 200 or 400 microvolts, depending upon recorder attenuation (X-1 or X-2, respectively).

The recorder chart speed selector was then set to the highest chart speed available, 100 mm per second, and the timing marker generator set into operation. The chart drive was turned on and one second later, the assistant initiated the exposure cycle described in subsection 6.4.1. The chart drive was left on high speed for about 10 seconds, then shifted to a low speed, 1.0 mm per second, and the cooling phase followed out to about 120 seconds. Since time markers were lost during this shift, the timing of this final

110 second section of chart was obtained with sufficient accuracy from a stop watch which was started by hand at approximately zero time. During this time, the animal was left undisturbed in exposure position.

By 120 seconds after the initiation of the pulse, the styli had all returned to very nearly their initial positions. The recorder inputs were again switched off to obtain a record of the final styli baseline positions; then, with the inputs returned to the on position, final calibration signals were recorded.

The animal was then rotated back out of exposure position, and the thermocouples carefully removed. The subsurface couples were completely unharmed and could be re-used after soldering them to another length of steel leader; some of these elements were used many times. Because of the knots tied in the surface thermocouples, the wires were badly fatigued and liable to breakage upon re-use; they were therefore discarded.

This procedure was repeated through the day until the decrease in the depth of anesthesia of the animal made further thermocouple emplacement impossible. On one of the last animals used in this study, seven exposures were made in one day, each involving one surface and two subsurface thermocouples, and of these 21 thermocouples, only one was broken. Undoubtedly, with further experience, this record can be improved.

6.15 Reduction of Data

All recorder charts were read under a low power microscope, with tracing positions estimated to 0.1 mm. The records were first read by Miss Marilyn Aldrich, of this laboratory, and then re-read by the writer; the two estimates never differed by more than 0.1 mm. This precision is quite remarkable when it is recalled that in some cases the noise (120 cps ripple) was 1.2 mm, peak-to-peak.

The chart time scale positions of the one second timing marks were first noted, and from these, the actual chart speed determined. This factor was used to reduce all time scale readings to true time. The zero time position on each channel was reckoned as outlined in Section 6.13. Next, the initial and final calibration signals on each of the three temperature recording channels were averaged, and these averages, together with the known value of absorbed irradiance for the particular exposure, used to compute a scale factor for each channel, as defined in Section 6.9. Finally, all chart deflections were corrected for any shift in recorder baseline, assuming a linear drift with time. Chart readings were then converted to temperature rise per unit absorbed irradiance, at true time after initiation of the pulse. Since differences in tracing positions were always involved, the precision of these differences was about 0.2 mm. At a chart speed of 100 mm per second, this corresponds to a precision in zero time location of about 2 milliseconds. This latter figure also represents the accuracy of the time measurements, since the time base (the power line frequency) was considerably more accurate than this.

The over-all system temperature sensitivity at full recorder gain was given previously as 0.875°C per mm of stylus deflection. This would lead to a precision in measuring the temperature rise of about 0.2°C. This is not a good estimate of the accuracy of the determinations, since thermocouple calibration errors, differences between thermocouples, calibration circuit errors, and other unknown errors could affect the accuracy. It is suggested that ten times this figure, or 2°C, is a safely pessimistic estimate of the accuracy.

6.16 Summary

In the foregoing sections, the more important items of equipment used

in this study have been described, and their functions briefly sketched. The general disposition of several of these items can be seen in the accompanying photographs. The experimental procedures which have been outlined above evolved only after considerable experience. In particular, in the early exposures, all four channels were used for temperature recording, and no attempt was made to obtain accurate placement on the recordings of the initiation of the irradiance pulse. Also, after the pulse measuring assembly was added, several exposures were made before the necessity of employing a cathode follower coupling stage was appreciated.

In the experimental results presented in the following chapters, these earlier measurements have been, for the most part, discarded. As a result, of the 15 animals employed in this study, the results from only 10 have been used. It should be reemphasized that the experimental work is far from complete, and much remains to be done.

FOOTNOTES

- (1) Kingsley, H. D., L. Hogg, Jr., J. T. Payne, and J. H. Morton, Studies in Prolonged Anesthesia in Swine and Dogs, University of Rochester Atomic Energy Project Report UR-152, 65-75 (1951).
- (2) Berkley, K. M., H. E. Pearse, and T. P. Davis, Studies of Flash Burns: The Influence of Skin Temperature in the Production of Cutaneous Burns in Swine, University of Rochester Atomic Energy Project Report UR-338, 9 (1954).
- (3) Ibid., 10.
- (4) Hinshaw, J. R. and H. E. Pearse, Histologic Techniques for the Differential Staining of Burned and Normal Tissues, Surg. Gyn. and Obs. 103, 726-730 (1956).
- (5) Krolak, L. J., The Measurement of Diffuse Reflectance of Pig Skin, Titanium Dioxide Paint, and India Ink; the Transmittance of Titanium Dioxide and India Ink, University of Rochester Atomic Energy Project Report UR-439, 12 (1956).
- (6) Ibid., 12.
- (7) Davis, T. P. and H. E. Pearse, The Use of the Carbon Arc and Burning Magnesium as Thermal Sources for Experimental Burns, Ann. Surg. 144, 68 (1959).

- (8) Finkelburg, W., The High Current Carbon Arc, FIAT Final Report No. 1052 (1946).
- (9) Swope, G. A. and F. C. Henriques, Thermal Pulse Properties of Five High Intensity Laboratory Sources, Technical Operations, Inc., Report No. TOI 54-8, 6 (1954).
- (10) Mixter, G., Jr. and T. P. Davis, A Method of Shaping Thermal Energy Pulses from a Carbon Arc Source, University of Rochester Atomic Energy Project Report UR-387 (1955).
- (11) Mixter, G., Jr. and L. J. Krolak, Critical Energy of Fabric as Indicated by its Persistence Time Under Thermal Radiation, University of Rochester Atomic Energy Project Report UR-260, 34-36 (1953).
- (12) Strong, J., Procedures in Experimental Physics, 542 (Prentice-Hall, Inc., New York, 1938).
- (13) Baker & Co., Inc., Newark, N.J. In these discussions, George Mixter, Jr., M.D., formerly of this laboratory, made many valuable contributions. His interest and assistance is gratefully acknowledged.
- (14) Personal communication from Mr. Edwin Wallin, of Stromberg-Carlson Co.
- (15) Ross, O. A., A. E. Axelrod, and A. R. Moritz, Progress Report of Project No. G-3645 to Morphology and Genetics Study Section, Division of Research Grants, Department of Health, Education, and Welfare, for July 16, 1952 to August 31, 1954, p. 8. Also personal visit by the author to this laboratory.
- (16) Davis, T. P. An Electronic Relay, University of Rochester Atomic Energy Project Report UR-348, 9-14 (1954).
- (17) Arguimbau, L. A., Vacuum-Tube Circuits, 153-161 (John Wiley & Sons, Inc., New York, 1948).

CHAPTER VII

EXPERIMENTAL RESULTS; OPAQUE SKIN

7.1 Scope of the Experimental Work

A total of five animals were used for this phase of the experimental work, with 36 exposures made and 63 thermocouples emplaced. Of these totals, the results of 42 thermocouples from 24 exposures on four animals were used. Of these 42 thermocouples, 24 were surface (for evaluation of the thermal inertia, μ), and 18 were subsurface (for evaluation of the thermal diffusivity, α). Since the procedures varied from one animal to another, the exposures and results for each pig will be described briefly in the following sections.

7.2 Animal No. 1389

Six exposures were made on this 12.4 kg pig on February 11, 1958. This was the first experiment on opaque skin; flat black Krylon spray enamel was used, rather than India ink. This produced film thicknesses ranging from 50 μ to over 100 μ . One might suspect that such thick films would be quite unsatisfactory, and the experimental results showed that this was the case. Further, these exposures employed the vane-type shutter, rather than the pulse wheel, and were made before installation of the pulse measuring assembly or the timing marker generator. For these reasons, the data are not included in the final averages.

It is interesting to consider the effect of the thick film of black enamel. For the first two exposures, the exposure site was painted before installation of the thermocouples, so the surface couple was "over" the enamel. On the third exposure, the thermocouples were installed first, and then the paint was applied; here the surface element was "under" the Krylon. For each of the remaining three exposures, two surface couples were used, one

under and one over the paint. On the average, the temperature rise for the five elements over the enamel was higher than that for the four couples under the film. Approximate values of thermal inertia were found to be: from the thermocouples over the paint, $11 \times 10^{-4} \text{ cal}^2 \text{ cm}^{-4} \text{ deg}^{-2} \text{ sec}^{-1}$, and from those under the paint, $19 \times 10^{-4} \text{ cal}^2 \text{ cm}^{-4} \text{ deg}^{-2} \text{ sec}^{-1}$. The average for all seven elements was 14×10^{-4} , with a truly enormous spread from a minimum of 6.4×10^{-4} to a maximum of 27×10^{-4} , all in the same units as above. Part of this spread in values of μ was due to the difficulty in locating true zero time on the records, and it was this set of exposures which led to the design and installation of the pulse measuring photometer and the timing marker circuit.

Only three good subsurface temperature records were obtained from these exposures, but they are of dubious value because of the large and unknown thickness of enamel added after the depth readings were made. No attempt was made, therefore, to obtain values of thermal diffusivity, α , from these measurements. All in all, the principal contribution of these exposures was to illustrate the deficiencies in the experimental techniques employed.

7.3 Animal No. 1418

Eight exposures were made on this 10.2 kg animal on March 20, 1958. The pulse measuring assembly and the timing marker generator were both utilized throughout. This series of exposures was designed to determine whether the thermal constants of the skin were independent of depth, or whether the skin should be treated as a composite solid. It will be recalled that in Section 3.5 of Chapter III, it was demonstrated that the linearity of the surface temperature rise vs square root of time is the crucial test of the constancy, with depth, of the thermal inertia, μ . Hence, on this animal,

only surface temperatures were measured, and no subsurface elements were emplaced. The skin was rendered opaque by application of India ink plus detergent. The thermocouples were tied into position and oriented with the fold lines of the skin, as described in the previous chapter.

The superficial layer of skin is often described as "dessicated," and while this term is probably incorrect, it is certainly true that the water content is well below that of the dermis, and even the deeper strata of the epidermis. One might suspect, then, that the heat capacity per unit volume, ν , of the superficial layer would be less than that of the deeper tissue. It would seem reasonable that the thermal conductivity, k , would likewise be smaller for the surface material. Thus, if the thermal inertia μ ($\mu = k \nu$) were, indeed, depth dependent, it would be logical that the value for the superficial layer would be smaller than that for the deeper material, or, in the terms of Section 3.5, $\mu_1 < \mu_2$. Assuming that the skin can be considered as the simple composite solid treated in the above-mentioned section, it follows that the slope of the curve of surface temperature rise vs square root of time would initially be proportional to $(\mu_1)^{-\frac{1}{2}}$, while after some time had elapsed, this slope would become proportional to $(\mu_2)^{-\frac{1}{2}}$; this initial slope would then be larger than that at a later time. Since the surface layer is quite thin, the transition between these two slopes would be expected to occur very soon after the initiation of the pulse; hence it was necessary in the exposures here described to investigate carefully the temperature response in the first few hundredths of a second.

In the first five exposures made on this animal, the pulse wheel and vane shutter combination described in subsection 6.4.1 was used; the pulse duration was therefore about $\frac{1}{2}$ second, and the irradiance rise time (from zero to maximum value) about 5 milliseconds. In order to obtain large recorder

deflections and hence improve the precision of the measurements, a rather high value of incident irradiance was used, $5.3 \text{ cal cm}^{-2} \text{ sec}^{-1}$; this leads to a value of $4.8 \text{ cal cm}^{-2} \text{ sec}^{-1}$ for the absorbed irradiance. Because of this, the surface temperatures reached 100°C in about $\frac{1}{4}$ second, and hence the recorder stylus was driven off scale; in the first $\frac{1}{4}$ second, however, accurate determinations of temperature response could be made.

The last three exposures on this animal were made at a greatly reduced furnace output, with the absorbed irradiance being only about 460 millical $\text{cm}^{-2} \text{ sec}^{-1}$. The vane-type shutter only was used, giving irradiance rise times of about 15 milliseconds, and the exposures were of some 7 seconds duration. Electrical noise from the vane shutter obscured the early phases of the recorded responses, and these exposures were primarily useful to determine if there was a transition between $(\mu_1)^{-\frac{1}{2}}$ and $(\mu_2)^{-\frac{1}{2}}$ response dependence at times much later than expected.

The data from all eight exposures for times out to about $\frac{1}{4}$ second ($\sqrt{\text{time}} = 0.5$) are plotted in Figure 7-1, and the data for the three long-time exposures in Figure 7-2. In all cases, the surface temperature rise per unit absorbed irradiance, $U^*(0,t)$, has been corrected for the irradiance rise time, as explained in Section 3.6. Of all the response curves, only that for exposure No. 6 displays the change in curvature expected if the thermal inertia were a function of depth, while that for exposure No. 8 has precisely the opposite curvature. For the five exposures with accurate temperature measurements at short times, no suggestion of this predicted curvature is seen. From this, one may state that within the precision of these experiments, the thermal inertia of the skin is independent of depth. It is highly unlikely that compensating errors in the thermal conductivity and the heat capacity per unit volume could produce this constancy of thermal inertia; hence one may claim with some confidence that these thermal constants

Figure 7-1. Measured Response, Animal No. 1418 (Opaque Skin)

Figure 7-2. Measured Response, Animal No. 1418 (Opaque Skin)

are likewise independent of depth, or at least that the data do not support the suggestion that they are depth dependent.

One disturbing feature of these results is that the best-fitting straight line drawn in Figure 7-1 does not pass through the origin, as would be expected from theory. There would seem to be an error in the location of true zero time of almost 3 milliseconds, which error is quite consistent from one exposure to another. This consistency strongly suggests that this zero time error arose from a time delay in the thermocouple-amplifier-recorder system. There exists some evidence that this delay is in the chopper-input preamplifier, although this conclusion is not yet definitely established. In any event, in future work, the timing marker signals will be introduced at the inputs of these preamplifiers, so that any delay introduced by them will be corrected for automatically.

From the expression derived in Chapter III for the surface temperature response of the opaque solid to a step function input,

$$U^*(0,t) = \frac{2}{\sqrt{\pi\mu}} \sqrt{t},$$

it follows that the slope of the $U^*(0,t)$ vs $\sqrt{t}$ plot of Figure 7-1 is equal to $2(\pi\mu)^{-\frac{1}{2}}$. The value obtained from the least squares fit is

$$\mu = 10.0 \times 10^{-4} \text{ cal}^2 \text{ cm}^{-4} \text{ deg}^{-2} \text{ sec}^{-1}.$$

7.4 Animal No. 1428

Six exposures were made on this 11.6 kg pig on April 10, 1958. The experimental details were precisely as described above for pig No. 1418, with the exception that a commercial low-level dc amplifier (Allegany Instrument Co. Model 220) was substituted for the chopper-type preamplifier in the thermocouple channel. The commercial instrument has a band width extending from dc to 20 kc, which is broader by a factor of about 500 than that of the shop-made unit, although the zero stability of the latter is

much better. The principal aim of these exposures was to see if the use of a broad band preamplifier would eliminate the zero point error noted in the data from pig No. 1418. Again, only surface thermocouples were employed. Calibration signals were obtained by bridging from the calibration circuit preceding channel D of the chopper-type preamplifiers.

In Figure 7-3, the data from this animal are presented as the surface temperature rise per unit absorbed irradiance, $U^*(0,t)$, vs the square root of time after initiation of the irradiance pulse. As in Figures 7-1 and 7-2, the data have been corrected for the finite irradiance rise time to give true step function response, as described in Chapter III. The regression line, obtained by the method of least squares, now crosses the square root of time axis at a value of $0.029 \text{ sec}^{\frac{1}{2}}$, which leads to a zero time error of 0.0009 second, or a little less than 1 millisecond. This is certainly small, but the consistency from one exposure to another suggests that the error is real.

Comparison of the zero point error for these two sets of exposures (on animals 1418 and 1428) would point rather clearly to a small time lag in the thermocouples themselves, and a much larger time lag in the chopper-input preamplifiers. However, in another set of exposures, to be described below, a zero point error considerably smaller than 1 millisecond was obtained, using one of these same chopper-type preamplifiers. In short, this matter is still unresolved. It is highly unlikely, however, that this zero point error, whatever its origin, affects the accuracy of the determinations of the thermal inertia, ρc , since, except for this slight time shift, the temperature responses do follow the predicted square root of time functional form most satisfactorily.

Before considering the next sets of exposures, one additional aspect of the data from animal No. 1428 should be mentioned. Twice during the

Figure 7-3. Measured Response, Animal No. 1428 (Opaque Skin)

course of these exposures, mistakes in procedure were made which rendered the exposure data valueless. Rather than remove the thermocouples and prepare a new site immediately, the animal was left in position for about five minutes, and a second exposure made on the same site. The data from these "double burns" are included in Figure 7-3 as exposures No. 2 and No. 3a; the points are clearly in no way atypical, and in fact are quite close to the best-fitting line. This result is in agreement with the findings of Lipkin and Hardy (1), who showed, however, that two prior exposures can increase the value of μ by a factor of four, apparently by causing local vasodilatation. Further investigation of this phenomenon has high priority in plans for future work in this laboratory.

From the slope of the best-fitting straight line in Figure 7-3, the value of thermal inertia was found to be

$$\mu = 12.5 \times 10^{-4} \text{ cal}^2 \text{ cm}^{-4} \text{ deg}^{-2} \text{ sec}^{-1}$$

7.5 Animals No. 1445 and 1455

Identical procedures were used with these two animals, burned on July 14, and September 8, 1958, respectively. Eleven surface and subsurface temperature measurements were made on pig No. 1445 in four exposures; and 17 such measurements in six exposures on No. 1455. The chart reading procedure was altered slightly so as to obtain directly the time in the normalized form suggested by the analysis of Chapter III. This normalized time, τ , is defined as the ratio of real time, t , to the exposure time, η . The latter value was obtained directly from the recorded pulse form. The surface temperature responses for each exposure were read at specified values of

τ , and then averaged for each animal. These averaged data are presented in Figure 7-4 as "reduced" temperature response, $U(0,t)/H_a \sqrt{\eta}$, vs the square

Figure 7-4. Measured Response, Animals No. 1445 and 1455
(Opaque Skin)

root of normalized time, $\sqrt{\tau}$. From opaque solid theory,

$$U(0,t) = \frac{2Ha}{\sqrt{\pi\mu}} \cdot \sqrt{t};$$

hence

$$\frac{U(0,t)}{Ha \sqrt{\eta}} = \frac{2}{\sqrt{\pi\mu}} \sqrt{\tau},$$

and the slopes of the best-fitting straight lines shown in Figure 7-4 will be equal to $2(\pi\mu)^{-\frac{1}{2}}$. The values of thermal inertia, μ , thus determined are: for pig No. 1445

$$\mu = 10.8 \times 10^{-4} \text{ cal}^2 \text{ cm}^{-4} \text{ deg}^{-2} \text{ sec}^{-1},$$

and for pig No. 1455

$$\mu = 13.9 \times 10^{-4} \text{ cal}^2 \text{ cm}^{-4} \text{ deg}^{-2} \text{ sec}^{-1}.$$

Note that the zero time error for the data from the former animal is about $\frac{1}{2}$ millisecond (in real time), while from the latter it is slightly less than 2 milliseconds. Thus, these errors are comparable to that obtained when using the broad-band preamplifier described previously, which argues against the introduction of an appreciable time delay by the chopper-input preamplifiers.

For each subsurface thermocouple the value of normalized time at which the temperature response reached its maximum, $\tau_{\max}$, was determined, as well as the depth, x , to the center of the thermocouple as measured by the depth reading device described in the previous chapter. These pairs of values are presented in Figure 7-5 as $x/\sqrt{\eta}$ plotted against

$$\sqrt{\frac{\tau_{\max}(\tau_{\max}-1)}{2}} \ln \frac{\tau_{\max}}{\tau_{\max}-1}.$$

From the development of Section 3.4,

$$\frac{x}{\sqrt{\eta}} = \sqrt{4\alpha} \sqrt{\frac{\tau_{\max}(\tau_{\max}-1)}{2}} \ln \frac{\tau_{\max}}{\tau_{\max}-1};$$

hence the slopes of the best-fitting straight lines in Figure 7-4 are equal

Figure 7-5. Measured Response, Animals No. 1445 and 1455
(Opaque Skin)

to $\sqrt{4\alpha}$. The values of thermal diffusivity obtained are: for pig No. 1445

$$\alpha = 8.65 \times 10^{-4} \text{ cm}^2 \text{ sec}^{-1}$$

and for pig No. 1455:

$$\alpha = 8.25 \times 10^{-4} \text{ cm}^2 \text{ sec}^{-1}$$

The scatter in these data is surprisingly small, and in general, they seem to follow the predicted linear relationship most satisfactorily. Here again, the best-fitting straight lines do not pass through the origin, as required by theory, but the errors are small, and it is hardly possible to claim on the basis of these two curves that the error is consistent, rather than random.

Considerable variation from one exposure to another was noted in the measured values of temperature rise, so that the "curve matching" procedure developed in Section 3.4 for the determination of α was found to be quite impractical. This scatter is seen clearly in Figure 7-6 which presents a profile of normalized temperature response, $\overline{T}$, as defined in Chapter III, vs "reduced" depth, $x/\sqrt{\eta}$, for a single value of normalized time, $\tau = 1.0$. The curve which has been drawn in was computed on the basis of the average value of thermal diffusivity, α , as given below, and is not necessarily a best-fitting curve through these particular plotted points.

7.6 Summary of Results: Average Thermal Constants of Pig Skin

To obtain a single set of estimates of the thermal constants of pig skin, the various values above were averaged, with each weighted according to the number of exposures involved in its determination. Thus, for the thermal inertia, μ ,

$$\mu = \frac{(8 \times 10.0) + (6 \times 12.5) + (4 \times 10.8) + (6 \times 13.9)}{24} \times 10^{-4},$$

while

$$\alpha = \frac{(7 \times 8.65) + (11 \times 8.25)}{18} \times 10^{-4}.$$

Figure 7-6. Normalized Measured Response, Animals No. 1445 and 1455 (Opaque Skin)

By definition

$$\mu = k \cdot \nu,$$

and

$$\alpha = k/\nu;$$

hence the thermal conductivity, k , is

$$k = \sqrt{\mu \cdot \alpha},$$

while the heat capacity per unit volume, ν , is

$$\nu = \sqrt{\mu/\alpha}$$

Performing the indicated operations gives

$$\mu = 11.7 \times 10^{-4} \text{ cal}^2 \text{ cm}^{-4} \text{ deg}^{-2} \text{ sec}^{-1},$$

$$\alpha = 8.4 \times 10^{-4} \text{ cm}^2 \text{ sec}^{-1},$$

and

$$k = 9.9 \times 10^{-4} \text{ cal cm}^{-1} \text{ deg}^{-1} \text{ sec}^{-1},$$

$$\nu = 1.18 \text{ cal cm}^{-3} \text{ deg}^{-1}.$$

7.7 Discussion of Results

One of the most important results of the measurements described above is that apparently one may consider the thermal constants of skin as true constants, independent of depth. It has been pointed out that the constancy of μ with depth, which follows from the close adherence to a simple square root of time relationship of the surface temperature rise, is a strong indication that all the thermal constants are likewise independent of depth. The excellent fit to a linear relation of the data presented in Figure 7-5 gives independent confirmation of this claim, since this simple linear relationship will hold only if α is independent of depth, and if both μ and α are true constants, then k and ν must be, also.

This conclusion must be qualified to this extent: it is true if and only if the surface treatment (painting with India ink) in no way altered

the thermal constants of the skin. Now, it was pointed out in Section 7.3 that because of the relative dessication of the superficial layer of the epidermis, the value of the heat capacity per unit volume, ρ , might well be considerably lower for this region than for the deeper, more hydrated tissues. It follows that painting the surface with India ink, which is an aqueous suspension, might so increase the water content of the superficial layer that the heat capacity per unit volume of this region would be increased to a value more nearly equal to that obtaining in the deeper layers. The conclusion that ρ is independent of depth would thus be true only if the skin surface were artificially wetted.

While this may seem to be a trivial matter, it actually is of considerable importance in the interpretation of certain measurements on bare, or diathermanous, skin. This will be considered further in the next chapter, where it will be shown that the value of ρ for non-wetted superficial tissue could be as low as $0.85 \text{ cal cm}^{-3} \text{ deg}^{-1}$, although this is a most approximate estimate, and probably represents a lower limit. All in all, however, it is possible that the painting with India ink does alter slightly the thermal constants of the surface layers of the skin, simply by increasing the water content of these tissues.

Turning now to the numerical values of the thermal constants of pig skin, it would be of some interest to compare the estimates given in the previous section with those presented by others. One of these constants which has been studied extensively is the thermal inertia, μ ; Hardy (who coined the term "thermal inertia for surface heating") and his associates have presented several papers discussing the measurement of this constant (2, 3, 4). This authority has developed quite elegant procedures for determining surface temperature by means of an infrared photometer. The latest and undoubtedly best measurements reported (4) were made on human skin, using

non-penetrating infrared radiation to heat the skin; this eliminated the necessity of using India ink to render the skin opaque. The average value of μ obtained in this study was $10.8 \times 10^{-4} \text{ cal}^2 \text{ cm}^{-4} \text{ deg}^{-2} \text{ sec}^{-1}$, with a standard deviation of $\pm 0.8 \times 10^{-4}$. It is tempting to claim that this close agreement confirms the accuracy of the measurements of the present study, but the fact that different species are involved in this comparison raises the possibility of compensating errors yielding accidental agreement. The writer, however, is inclined to believe that the demonstrated similarities of pig and human skin may well extend to similarity in thermal constants; hence this agreement is not fortuitous, but does, in fact, provide confirmation of the accuracy of the surface temperature measurements here reported.

In order to evaluate any one of the other thermal constants of skin, it is necessary to make subsurface temperature measurements, as in this study, or to employ completely different procedures than measurement of the temperature response of the skin to radiant energy exposures. So far as the writer can determine, this study is the only one in which a direct experimental determination of thermal diffusivity, α , was attempted; more commonly, the thermal conductivity, k , or heat capacity per unit volume, ν , has been the constant determined, frequently in vitro. Buettner has presented a rather extensive discussion of various measurements of k (5) and has suggested, as a compromise of the available data, the expression

$$k = 7 (1 + 3x) \times 10^{-4} \text{ cal sec}^{-1} \text{ deg}^{-1} \text{ cm}^{-1}$$

(for x in cm) during the heating phase (with penetrating radiation) and

$$k = 10^{-3} \text{ cal sec}^{-1} \text{ deg}^{-1} \text{ cm}^{-1}$$

during the cooling phase. To the writer, this procedure is not entirely clear. In any event, these values are in excellent agreement with the value, $9.9 \times 10^{-4} \text{ cal sec}^{-1} \text{ deg}^{-1} \text{ cm}^{-1}$, reported here.

The value of heat capacity per unit volume, ν , given in the last

section, $1.18 \text{ cal cm}^{-3} \text{ deg}^{-1}$, is quite interesting, being one of the highest estimates the writer has seen. This constant can be evaluated quite directly in vitro by employing the classic method of mixtures to measure the specific heat capacity, c_p , and determining the density, ρ , by mass and volume measurements; the value of ν is then, by definition, the product of c_p and ρ . Henriques has reported such measurements, with the value obtained being $0.69 \text{ cal cm}^{-3} \text{ deg}^{-1}$, for the epidermis only; the density of the epidermis was assumed to be 0.8 gm cm^{-3} (6). In contrast to this rather low value, Buettner (7) and Lawson, Thomas, and Simms (8) have assumed a value of unity for this constant. Chen and Jensen (9) have selected a value of 0.8 for "standard" skin, as read from Figure 3 of the referenced report. On this same figure, Chen and Jensen have quoted data from Hardy for "moist skin in summer" as

$$k = 10^{-3} \text{ cal sec}^{-1} \text{ deg}^{-1} \text{ cm}^{-1}$$

and
$$\mu = 12.5 \times 10^{-4} \text{ cal}^2 \text{ cm}^{-4} \text{ deg}^{-2} \text{ sec}^{-1}.$$

These would lead to a value of ν of $1.25 \text{ cal cm}^{-3} \text{ deg}^{-1}$, which is the only value the writer has found which exceeds that obtained in this study.

In general, it would seem that most authors have simply assumed a value of ν , and there appears to be a remarkable paucity of careful in vivo determinations of this constant. Hence, the writer tends to accept the value here determined, $1.18 \text{ cal cm}^{-3} \text{ deg}^{-1}$, as more nearly representative of living pig skin than the more commonly cited values of 0.8 to $1.0 \text{ cal cm}^{-3} \text{ deg}^{-1}$. In any event, the values of thermal inertia, μ , and thermal conductivity, k , as determined in this study seem to be in excellent agreement with careful determinations reported in the literature, and if these two values be accepted then the values of thermal diffusivity, α , and heat capacity per unit volume, ν , are, of course, fixed by the defining equations for these constants.

7.8 Summary

In view of the above considerations, it may be suggested that the values of the thermal constants of pig skin presented in Section 7.6 are probably accurate to about $\pm 10\%$; the accuracy may be better than this for μ , and poorer for α , since the latter involves the difficult and not overly precise measurement of depth. Actually, a precise statement of accuracy is not necessary for present purposes, inasmuch as these thermal constants are only needed for the analysis of the more important case of diathermanous skin. In this connection, it should be pointed out that, due to a slight error in the initial analysis of the opaque skin data, the values of k and α used in Section 5.4 for the numerical solution of the diathermanous solid response are respectively 1% and 2% below the final values presented in Section 7.6. It will become clear in the presentation of the following chapter that errors of this magnitude, and even errors of 10%, are quite insignificant when compared to the large discrepancies which were found between predicted and measured responses for diathermanous skin.

FOOTNOTES

- (1) Lipkin, M. and J. D. Hardy, Measurement of Some Thermal Properties of Human Tissues, *J. Appl. Physiol.* 7, 216 (1954).
- (2) Hardy, J. D., Method for the Rapid Measurement of Skin Temperature During Exposure to Intense Thermal Radiation, *J. Appl. Physiol.* 5, 559-566 (1953).
- (3) Lipkin, M. and J. D. Hardy, *op. cit.*, 212-217.
- (4) Hendler, E., R. Crosbie, and J. D. Hardy, Measurements of Skin Heating During Exposure to Infrared Radiation, *J. Appl. Physiol.* 12, 177-185 (1958).
- (5) Buettner, K., Effects of Extreme Heat and Cold on Human Skin III. Numerical Analysis and Pilot Experiments on Penetrating Flash Radiation Effects, *J. Appl. Physiol.* 5, 210-211 (1952).
- (6) Henriques, F. C., Jr. and A. R. Moritz, Studies of Thermal Injury I. The Conduction of Heat to and through Skin and the Temperatures Attained Therein. A Theoretical and an Experimental Investigation, *Am. J. Path.* 23, 538-539 and 546 (1947).

- (7) Buettner, K., op. cit., 212.
- (8) Lawson, D. I., P. H. Thomas, and D. L. Simms, The Thermal Properties of Skin, Fire Research Station (Boreham Wood, Herts., England) F. R. Note 224 / 1955, 7 (1955).
- (9) Chen, N. Y. and W. P. Jensen, Skin Simulants with Depth Magnification, Massachusetts Institute of Technology Fuels Research Laboratory Technical Report No. 5, A-11 Figure 3 (1957).

CHAPTER VIII

EXPERIMENTAL RESULTS; BARE SKIN

8.1 Scope of the Experimental Work

For the investigation of the response of bare, or diathermanous skin, ten animals were employed, with 39 exposures made and 103 thermocouples emplaced. Of these totals, the results from six animals, 25 exposures, and 68 thermocouples were usable; of the last, 21 were on the surface and 47 were subsurface. The data from the first four animals employed were discarded since techniques were poorly developed, and the pulse measuring photometer and timing generator had not yet been installed. Table 8-1 summarizes the scope of this phase of the experiment.

It should be noted that for the exposures on animals 1402 and 1403 the pulse measuring assembly was not yet in the form described in Chapter VI, the phototube being coupled directly to the Sanborn recorder, without the intervening cathode follower. As a result, the photometer-recorder system had a long time constant, and measurements of irradiance pulse rise time were completely inaccurate. However, the point of initiation of the pulse was still recorded accurately, so the data from these exposures may be included.

Four of the five exposures on animal No. 1446 were made with the skin surface wetted with water plus a few drops of detergent. The reason for this "sham painting" will be described below. Except for these four exposures, the experimental techniques were exactly as described in Chapter VI for the six animals 1402 through 1450.

8.2 Reduction of Data

The recorded responses were reduced as previously explained, with the temperature rise per unit absorbed irradiance, $U^*(x,t)$, determined at various

Table 8-1
Scope of the Experimental Work on Bare Skin

Pig	Date	Expo- sures	Thermo- couples	Pulse Recorded	Timing	Irradiance	
						Incident $\frac{\text{cal}}{\text{cm}^2\text{sec}}$	Absorbed $\frac{\text{cal}}{\text{cm}^2\text{sec}}$
1370	1-20-58	4	7	No	No	3.9	2.4
1373	1-23-58	4	7	No	No	3.9	2.4
1375	1-28-58	4	14	No	No	5.0	3.0
1392	2-13-58	2	7	No	No	5.0	3.0
1402	2-25-58	2	5	Yes ^(a)	Yes	5.0	3.0
1403	2-27-58	5	13	Yes ^(a)	Yes	5.0	3.0
1408	3-4-58	4	10	Yes	Yes	5.3	3.2
1444	6-5-58	5	15	Yes	Yes	5.2	3.1
1446	7-16-58	5 ^(b)	13	Yes	Yes	5.2	3.1
1450	9-18-58	4	12	Yes	Yes	5.2	3.1
10		39	103			Total	
6		25	68			Usable	

(a) No cathode follower

(b) The surface was wetted on four exposures

values of real time. Measured depths were corrected for leader wire radius to obtain the true depth to the center line of each thermocouple. The data from each exposure were plotted simply as $U^*(x,t)$ vs time, with one curve for each thermocouple employed for that particular exposure. Inasmuch as the primary concern was comparison with the theoretical step-function response, the records were analyzed only through the heating phase--i.e., during the exposure pulse. As noted previously, it was not practical to derive an analytical expression for correcting the recorded response to true step-function response; hence the initial portion of each curve was obtained by smooth extrapolation to zero time. Because of the uncertainty in the source and average magnitude of the time lag noted in the opaque skin results, no attempt was made to correct for this lag, and the zero time position on each recording was located as described in Section 6.13.

8.3 Summary of Temperature Response Measurements

From these replotted response curves, values of $U^*(x,t)$ were read at times of 0.05, 0.10, 0.15, 0.20, 0.30, 0.40, and 0.50 second after initiation of the irradiance pulse. These data, together with the corresponding depths, are given in Table 8-2, which is thus a complete summary of all the usable bare skin exposures made in this study. These tabulated values are also presented graphically in Figures 8-1a, 8-1b, and 8-1c as profiles of $U^*(x,t)$ vs depth, x , for the seven values of time mentioned above. Note that for $x = 0$, only the arithmetic means of the 21 surface temperature measurements were plotted for each time.

Finally, values of $U^*(x,t)$ were read from these seven smoothed curves at values of depth of 0, 0.02, 0.04, 0.08, and 0.16 cm, and again plotted, reverting to the $U^*(x,t)$ vs time presentation. This final set of doubly-smoothed curves, Figure 8-2, is then a graphical representation of the

Table 8-2

Summary of Bare Skin Temperature Responses

No.	Exp. No.	Corrected Depth, cm	$\frac{\partial U^*(x,0)}{\partial t}$	$U^*(x,t)$						
				0.05	0.10	0.15	0.20	0.30	0.40	0.50
				sec	sec	sec	sec	sec	sec	sec
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1	4-1408	0	264	8.51	10.34	11.58	12.70	14.39	15.68	16.74
2	3-1408	0	232	5.50	6.88	7.94	8.74	10.13	11.37	12.47
3	1-1450	0	218	7.50	9.86	11.59	12.92	14.83	16.36	17.64
4	2-1408	0	207	4.16	5.33	6.01	6.71	7.83	8.78	-
5	4-1444	0	206	3.62	4.85	5.88	6.73	8.17	9.43	10.60
6	5-1444	0	197	4.70	6.34	7.49	8.39	10.00	11.59	13.14
7	3-1402	0	190	5.35	7.87	9.20	10.36	12.27	13.78	15.16
8	4-1402	0	183	5.90	8.73	10.18	11.50	13.47	15.20	16.86
9	4-1446	0	175	5.73	7.72	8.83	9.87	11.78	13.51	15.13
10	2-1403	0	172	4.27	5.85	6.97	7.86	9.45	10.96	12.33
11	3-1450	0	167	5.03	6.65	7.73	8.72	10.34	11.82	12.97
12	5-1403	0	163	5.40	7.32	8.62	9.60	11.43	12.73	14.07
13	3-1403	0	160	3.92	5.60	6.88	7.96	9.63	11.43	12.96
14	4-1403	0	158	4.85	6.48	7.60	8.40	9.92	11.23	12.48
15	1-1403	0	155	4.80	6.65	7.87	8.83	10.34	11.68	12.91
16	2-1450	0	151	6.27	9.48	11.50	12.88	14.73	16.14	17.32
17	4-1450	0	150	5.88	8.29	9.62	10.81	12.54	13.81	14.85
18	3-1446*	0	145	3.53	5.14	6.38	7.53	9.43	11.14	12.72
19	6-1446*	0	144	4.10	5.67	6.82	7.78	9.31	10.72	11.89
20	5-1446*	0	124	2.62	4.16	5.62	6.81	8.88	10.62	12.18
21	2-1446*	0	120	2.76	4.44	5.80	7.13	9.26	11.25	13.02
22	1-1444	0.015	150	3.93	6.33	8.10	9.50	11.94	13.92	-
23	1-1450	0.016	62.0	2.37	3.77	5.05	6.06	8.02	9.59	11.04
24	2-1444	0.019	105	2.56	3.84	4.83	5.83	7.48	8.94	-
25	2-1450	0.020	80.0	2.48	3.77	4.93	5.86	7.66	9.16	10.45
26	1-1403	0.021	69.6	2.07	3.25	4.24	5.01	6.51	8.03	9.22
27	4-1450	0.021	64.7	2.00	3.13	4.08	4.84	6.24	7.53	8.52
28	5-1444	0.021	60.3	1.72	2.76	3.75	4.60	6.24	7.76	9.06
29	3-1402	0.023	63.6	2.00	3.44	4.48	5.50	7.34	8.97	10.43
30	4-1444	0.024	72.2	1.59	2.46	3.18	3.86	5.22	6.57	-
31	3-1444	0.025	104	2.32	3.47	4.53	5.58	7.48	9.18	-
32	3-1403	0.026	112	2.37	3.82	5.05	6.05	7.91	9.45	11.00
33	2-1446*	0.027	75.2	2.32	3.62	4.73	5.57	7.22	8.41	9.83
34	2-1408	0.029	80.5	2.22	3.19	3.84	4.42	5.52	6.61	-
35	5-1446*	0.029	59.1	1.82	3.00	4.01	5.02	6.83	8.39	9.76
36	3-1408	0.030	81.2	1.43	2.15	2.85	3.48	4.76	5.91	7.02
37	4-1408	0.031	90.9	1.83	2.59	3.19	3.77	4.93	6.08	7.18
38	2-1403	0.031	73.7	2.22	3.37	4.24	5.03	6.36	7.73	8.94
39	3-1446*	0.032	90.3	2.43	3.65	4.63	5.53	7.02	8.32	9.40
40	1-1408	0.035	85.2	1.73	2.42	3.02	3.58	4.72	5.83	6.95

Table 8-2

Continued

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
41	4-1446	0.039	81.0	2.50	3.77	4.86	5.86	7.68	9.40	11.02
42	3-1450	0.040	68.2	2.58	3.79	4.76	5.57	6.98	8.14	9.34
43	3-1444	0.045	90.9	1.86	2.79	3.63	4.32	5.71	6.99	8.20
44	4-1444	0.056	45.4	1.06	1.47	1.84	2.16	2.73	3.30	3.82
45	4-1403	0.061	68.2	1.83	2.61	3.10	3.56	4.36	5.09	5.82
46	1-1444	0.070	71.0	1.80	2.43	2.78	3.11	3.77	4.29	-
47	2-1444	0.076	77.2	2.07	2.81	3.30	3.74	4.48	5.04	-
48	5-1444	0.081	42.0	0.95	1.39	1.70	1.99	2.46	2.87	3.24
49	1-1444	0.083	54.7	1.75	2.67	3.32	3.83	4.86	5.71	-
50	2-1446*	0.098	53.4	1.57	2.23	2.72	3.11	3.79	4.37	4.89
51	3-1444	0.098	34.7	0.69	0.97	1.15	1.33	1.60	1.84	2.12
52	2-1450	0.100	52.2	1.60	2.43	2.94	3.37	4.00	4.54	5.02
53	1-1450	0.101	48.1	1.56	2.37	2.90	3.40	4.08	4.70	5.23
54	1-1403	0.113	38.2	1.44	2.14	2.56	2.88	3.41	3.80	4.16
55	3-1450	0.118	30.5	1.23	1.90	2.40	2.80	3.44	3.91	4.37
56	3-1408	0.120	44.3	0.82	1.12	1.41	1.62	2.03	2.35	2.64
57	5-1446*	0.122	53.2	1.52	2.19	2.77	3.17	3.91	4.47	5.03
58	4-1446	0.122	50.2	1.63	2.57	3.13	3.68	4.55	5.29	5.99
59	3-1403	0.122	40.8	1.02	1.48	1.84	2.17	2.63	3.00	3.41
60	4-1450	0.124	26.3	0.79	1.13	1.38	1.60	1.94	2.29	2.51
61	2-1403	0.127	38.2	0.83	1.13	1.34	1.45	1.73	1.92	2.13
62	4-1408	0.132	26.7	0.82	1.21	1.52	1.73	2.13	2.41	2.68
63	3-1446*	0.137	35.6	1.21	1.77	2.16	2.46	3.16	3.53	4.00
64	2-1444	0.137	32.0	0.89	1.24	1.49	1.68	1.94	2.20	-
65	1-1408	0.143	40.0	1.10	1.51	1.74	1.95	2.34	2.67	2.93
66	4-1402	0.143	32.4	0.87	1.27	1.52	1.73	2.07	2.34	2.58
67	4-1403	0.155	37.7	0.85	1.22	1.47	1.67	2.05	2.34	2.60
68	3-1402	0.169	28.7	0.98	1.53	1.85	2.16	2.56	2.82	3.12

*Surface wetted with water

Figure 8-1a. Measured Response, Bare Skin Exposures

Figure 8-1b. Measured Response, Bare Skin Exposures

Figure 8-1c. Measured Response, Bare Skin Exposures

Figure 8-2. Average Measured Response, Bare Skin Exposures

"average" bare skin response, as determined by this experiment.

8.4 Summary of Initial Slope Measurements

For each of the replotted response curves, described in Section 8.2, the initial slope, $(\partial U^*(x,0)/\partial t)$, was estimated by eye, simply by adjusting a straightedge to be tangent to the initial portion of the curve (1). These values are also tabulated in Table 8-2, and are shown graphically in Figure 8-3 on a logarithmic scale against a linear depth scale. The values for $x = 0$ (surface measurements) are plotted as points except for the four determinations with wetted surface, which are shown by X's. The indicated average value for the surface initial slope, $185 \text{ deg cal}^{-1} \text{ cm}^2$, was calculated as the arithmetic mean of the 17 "normal" values, and does not include these four wetted-skin measurements.

The line drawn in Figure 8-3 was determined by the method of least squares as the best-fitting straight line through the values for x greater than zero; i.e. for subsurface measurements. This line, which may be expressed as

$$\frac{\partial U^*(x,0)}{\partial t} = 96.8 e^{-7.22x} \text{ deg cal}^{-1} \text{ cm}^2$$

(where x is in centimeters), is thus valid only for non-zero values of x , and is not intended to fit the surface ($x = 0$) values of initial slope.

8.5 Comparison of Experimental Results with Theoretical Predictions

The most notable feature of Table 8-2 is the remarkable variability in the data. This scatter is even more noticeable in Figures 8-1a, b, and c, and, in fact, the smooth profile curves shown in these figures are based more on free-hand art work than one would consider desirable. (The zero slope of these curves at $x = 0$ was, of course, dictated by the boundary condition 4.4 of the diathermanous solid model. In order to decide experimentally whether

Figure 8-3. Measured Initial Time Rate of Change of Temperature Response, Bare Skin Exposures

this slope actually is zero or non-zero, it would be necessary to introduce temperature sensing elements, of probably less than 10μ diameter, at depths of about 10 to 20μ . This would be a difficult task and for present purposes, at least, not justified.) The regularity of the points defining the "average" response curves of Figure 8-2 masks, but obviously has not eliminated, the variability in the original data. Thus, these curves must be regarded as indicating the central tendency of the experimental data, and not as a precise presentation of highly reproducible results. Undoubtedly, the curves are better than order of magnitude estimates, but it would probably be optimistic to claim that they represent the "true" average response with an accuracy better than a factor of two.

This same sort of variability in experimental results was encountered in the opaque skin studies described in Chapter VII, but in that case rather elegant schemes were available for comparing experiment with theory; hence, it was possible to compute the unknown constants of the theoretical model with a fairly high level of confidence. As pointed out in Chapter V, no such comparison schemes are possible for bare, or diathermanous, skin exposed to heterochromatic radiation, and one is forced to fall back on the manifestly unsatisfactory procedure of gross comparison of theoretical predictions and experimental results. It would now seem that the scatter in the experimental data would make such comparisons meaningless.

Such, however, is not the case. If one compares the experimental response curves of Figure 8-2 with the theoretical predictions shown in Figure 5-3, one can see clearly that the agreement is poor, but it is not at all clear whether the level of disagreement is such that one would reject the hypothesis that the responses are, in fact, equal. Certainly, there are no order of magnitude differences between these two sets of curves. A much more revealing comparison, however, is shown in Figure 8-4 where the

Figure 8-4. Comparison of Predicted and Measured Bare Skin Response

responses are plotted on logarithmic scales against a linear time scale; the ordinates of the two sets of curves--experimental and theoretical--have been shifted arbitrarily so as to bring into coincidence the experimental and predicted surface responses at 0.5 second. Now it can be seen that unquestionably theory and experiment do not agree, and the disagreement cannot be removed simply by applying a multiplicative "correction factor" (which corresponds to a relative shift of ordinates in Figure 8-4) to either set of curves. Not only do the two sets of curves disagree in form, but the spacing between isodepth curves is completely different, and no amount of free-hand redrawing of the profiles of Figures 8-1a, b, and c, can reduce this difference significantly.

Having thus concluded that the theoretical predictions and experimental results do not agree, one is faced with the task of locating the source, or sources, of this disagreement. At first thought, one might point to the variability in the experimental data, and suggest that this raises serious doubts as to the validity of the entire set of experimental results. It must be recalled, however, that the so-called theoretical predictions are based on literature values of linear absorption coefficients, some of which values may be little better than order of magnitude estimates. One is thus forced to choose between two sets of not overly reliable data.

Real progress in this problem can be made by turning to the experimentally determined values of the initial slope of the temperature response. While again, there is considerable variability in these data, there can be no doubt that the general trend shown in Figure 8-3 is real, and that the indicated line fits the data reasonably well. Now if one assumes, for the moment, that this same line may be extended from $x = 0$ to $x \rightarrow \infty$, then it follows that

$$\begin{aligned}
 \int_0^{\infty} \gamma \frac{\partial U^*(x,0)}{\partial t} dx &= (1.18) (96.8) \int_0^{\infty} e^{-7.22x} dx \\
 &= \frac{(1.18) (96.8)}{7.22} \\
 &= 16. \qquad (8-1)
 \end{aligned}$$

Yet, it has been pointed out that this integral, as a consequence of the conservation of energy, must necessarily be equal to unity! Here there can be no question; the measured values of initial slope are too high, on the average, by a factor of about 16. Even if one assumed that the curve of initial slope dropped to zero for depths greater than 0.17 cm (the deepest measurement made), the area under the curve would still be about 11. The actual area must be greater than this latter figure, since it is completely unrealistic to assume that the initial slope of the temperature response will be zero for depths greater than 0.17 cm (which is equivalent to claiming that the tissue is opaque at these depths), and further the true curve must lie above the simple exponential shown in Figure 8-3 for small values of x , since the measured value of the initial slope at $x = 0$ is about 185 deg sec^{-1} for unit absorbed irradiance, and not 96.8. Thus there is unquestionably a real systematic error in the measured temperature responses, which leads to these high values of initial slope. A major part of the experimental work done in this study was devoted to the investigation of possible sources of this error. Several of the more pertinent results of this work will be described in the next section.

8.6 The Source of Error in the Temperature Measurements

Several possible sources of error in the temperature measurements, or specifically in the measurements of initial slope, were studied, including variation in the value of the heat capacity per unit volume, γ , with depth,

errors in zero time location or extrapolation of the response curves to zero time, and direct interaction of the thermocouples with the radiation penetrating through the skin. These possibilities will be discussed and evaluated in the following subsections.

8.6.1 Variation of Heat Capacity per Unit Volume with Depth

For the case of the diathermanous solid, with a double exponential absorption pattern, and exposed to monochromatic radiation, it was established in Chapter IV that

$$\frac{\partial U^*(0,0)}{\partial t} = \frac{\gamma_1}{\nu}$$

where the left-hand side is the initial slope of the surface response, and

γ_1 is the linear absorption coefficient for the superficial layer of the solid. The analysis of Chapter V demonstrated that for heterochromatic radiation, this expression becomes

$$\frac{\partial U^*(0,0)}{\partial t} = \frac{1}{\nu} \sum_{i=1}^n \gamma_{ii}$$

where γ_{ii} applies to the i^{th} wavelength interval. From the literature values of the linear absorption coefficients tabulated in Chapter V, one predicts that

$$\left[\frac{\partial U^*(0,0)}{\partial t} \right]_{\text{theory}} = \frac{58}{\nu} \text{ deg cal}^{-1} \text{ cm}^2.$$

The very earliest measurements made in this study indicated that the value of this initial slope of surface temperature response was about $200 \text{ deg cal}^{-1} \text{ cm}^2$; the final average value was found to be

$$\left[\frac{\partial U^*(0,0)}{\partial t} \right]_{\text{experimental}} = 185 \text{ deg cal}^{-1} \text{ cm}^2.$$

Focussing attention only on these values for the surface response it is clear that theory and experiment would be in exact agreement if the value of the heat capacity per unit volume, ν , for the superficial layers were

$0.32 \text{ cal cm}^{-3} \text{ deg}^{-1}$. It was this consideration which led to the investigation of the composite opaque solid as presented in Section 3.5, and the development of experimental tests to determine if the value of ν could indeed be this low for the surface layers. These tests have been presented in the previous chapter; the results showed that ν was independent of depth, with a value of $1.18 \text{ cal cm}^{-3} \text{ deg}^{-1}$.

However, as pointed out in Section 7.7 of the previous chapter, there is the possibility that wetting of the surface by the India ink coating could have increased a low value of ν for the surface layers, to a value more nearly equal to that obtaining in the highly hydrated deeper tissue. This would, of course, result in the false conclusion that ν was depth-independent. Because of this possibility, the four sham-painting exposures on animal No. 1446 were made; here, the skin was "painted" with tap water with a few drops of detergent added. If the India ink coating had increased the value of ν for the superficial tissues from a true value of 0.32 to a value of $1.18 \text{ cal cm}^{-3} \text{ deg}^{-1}$, then the wetting of the surface with water should have decreased the initial slope of the surface response from 185 to $50 \text{ deg cal}^{-1} \text{ cm}^2$. From the data for these four exposures as given in Table 8-2, the average surface initial slope is about $133 \text{ deg cal}^{-1} \text{ cm}^2$. If this latter value is the correct one for a surface ν of $1.18 \text{ cal cm}^{-3} \text{ deg}^{-1}$, then the "dry skin" value of $185 \text{ deg cal}^{-1} \text{ cm}^2$ for initial slope would suggest a "true" value of ν for the surface layers of about $0.85 \text{ cal cm}^{-3} \text{ deg}^{-1}$, which is still almost three times larger than the value of 0.32 necessary to secure agreement between predicted and measured values of initial slope of the surface response.

It is most important to note that in these four cases, the skin was thoroughly wetted just prior to the exposure, and almost no drying time was permitted. This is in sharp contrast to the procedure when India ink was

applied to the skin; here, about two minutes elapsed between the painting and the exposure. Hence, the experimental conditions were quite different, and the value of ν for the surface layers of dry skin (namely, $0.85 \text{ cal cm}^{-3} \text{ deg}^{-1}$) which was obtained by intercomparing results from these experiments, must be regarded with considerable skepticism. The lowered value of initial slope of the surface response with wetted skin may have been caused simply by additional energy being required to heat a finite film of water on the surface, and not to an increase in tissue heat capacity. The essential point is that this reduction in initial slope was far less than would be required to secure agreement between the predicted and the experimental values.

It should also be made clear that this suggested variation of ν with depth, even if true, would fail to explain the real problem: the area under the curve of initial slope vs depth is greater by a factor of 16 than that demanded by consideration of conservation of energy. Certainly, it is completely inadmissible to suggest that the value of ν , for all depths, must be less by a factor of 1/16 than that measured. This would lead to a value of about $0.07 \text{ cal cm}^{-3} \text{ deg}^{-1}$, which is an order of magnitude below the lower estimates in the literature.

8.6.2 Location of Zero Time and Extrapolation of Responses to Zero Time

In the previous chapter, considerable discussion was devoted to the apparent existence of time lags in the thermocouple-amplifier-recorder system which lead to a zero time error of several milliseconds. This error was particularly troublesome in analyzing the opaque skin data because of the necessity of presenting temperature responses against a square root of time scale; such a transformation expands the initial portion of the scale, and hence "magnifies" the zero point error. In contrast to this, examination of the curves of Figure 8-2 (where the response is plotted against a

linear time scale) reveals that a time lag of one or two milliseconds would have only a very slight effect on the form of the curves near zero time. Actually, the effect would be to increase the values of initial slopes slightly, which is precisely the opposite effect from that necessary to reduce the error in the measurements.

A much more important potential source of error is in the extrapolation of the replotted response curves to zero time. Usually the first point plotted was at about 10 to 20 milliseconds after zero time; obviously, by rather heroic pencil manipulation, one could bring any response curve into the origin with as small a slope as desired, including zero. It is just as obvious that such a procedure would result in a pronounced inflection in the curve, and both intuition and theory (as shown in Figure 5-3) reject such an inflection. Certainly, the necessity of employing this extrapolation is most unfortunate; one would like to have an ideal, infinite band-width temperature measuring and recording system (or, at least be able to correct a non-ideal response to approximate closely the ideal) so as to obtain accurate measurements arbitrarily close to zero time. However an extrapolation over the initial ten or twenty milliseconds can really be made with considerable confidence, and it is hardly possible that consistent errors would have been introduced by this process which would yield estimates of the initial slopes an order of magnitude too high.

8.6.3 Direct Action of Radiation on the Thermocouples

In addition to the above possible sources of error, there is the obvious question of error in the temperature or depth calibrations. This can be dismissed immediately, since the same instruments and techniques used for these bare skin studies were also used for the opaque skin measurements, and the latter are in good agreement with results in the literature.

Thus, it seems that one is forced to conclude that only one possibility remains; namely, the direct interaction of the thermocouples with the radiation striking and penetrating the skin. This is unquestionably the most likely source of error; indeed, it will be shown shortly that to some extent it is a necessary error.

At the same time, this is an extremely unfortunate situation, since if the thermocouples were heated directly by the radiation, it follows that the temperature of these elements was always higher, by an unknown amount, than the temperature of the surrounding tissue. It would seem then that one is forced to admit that the bare skin measurements are of dubious value unless and until one makes some evaluation of the temperature difference between a thermoelement and its surroundings. Actually the situation is not quite as desperate as this; considerable useful information can be extracted from these measurements by employing once more the concept of initial slope, which, interestingly enough, is the very concept which was used originally to establish firmly the fact that the measurements are in error.

8.7 Further Consideration of the Initial Slope Measurements

From the analysis of Chapter IV, in any region at a depth x ,

$$\frac{\partial U(x,0)}{\partial t} = \frac{1}{v} q'''(x,0),$$

where $q'''(x,0)$ is the initial radiant power absorption per unit volume. Now if one takes a thermocouple as the region under consideration, then this relation should hold for the thermocouple itself irrespective of the composition or temperature of the surrounding material. One would expect, then, that the initial response of a thermocouple in tissue should be exactly the same as the initial response in any other medium, for example, air. From this reasoning follows the statement above that the initial slope of skin response as determined by the thermocouples, must be in error. As a check on this line

of reasoning, the following simple experiment was performed.

One of the standard 0.002 inch diameter Ag-Pd thermocouples was suspended horizontally in air just behind and below the aperture in the water-cooled aperture plate of the animal holder (Section 6.5). The thermocouple was arranged to be within but not touching a petri dish, and a plane mirror was placed just behind the aperture and inclined at an angle so that radiation passing through the aperture would be reflected onto the thermocouple. With the latter in air, several exposures were made and the carbon arc furnace output adjusted to secure about mid-scale deflection of the recorder. The petri dish was now filled with water so that the thermojunction was completely immersed; according to the argument above, the initial response of the couple should still be the same as when the element was in free air. Upon making repeated exposures, the thermoelement response was absolutely and unequivocally zero. (The length of exposure was not sufficient to produce any measurable temperature rise in the water.) Some of the water was now pipetted out until the thermocouple lay approximately in the water surface; the response upon exposure was again zero. Finally, more water was removed until only a very thin bridge of water was still held to the thermocouple wire by surface tension. Now, a response approximately equal to $1/3$ the noise level was observed; it was too small to attempt to determine the initial slope.

Now these results apparently contradict the principle which at the same time provides the only reasonable explanation of the known errors of skin temperature measurements. This is a most interesting situation, and is not yet completely understood; however the most likely explanation of the contradiction follows from a consideration of the "fine structure" of the thermocouples themselves.

A thermocouple, being composed of metal, is an opaque solid. This means that, on the surface of the wire, the initial absorption per unit

volume, $q'''(x,0)$, will approach infinity, and thus the initial slope of the thermocouple temperature might be expected to be infinite also, even with the couple immersed in water. But from the description of the thermoelements as given in subsection 6.6.4, one notes that the actual thermojunction is a surface in the interior of this opaque solid, and thus, the indicated thermocouple temperature will not be that of the surface of the wire. Further, the energy absorbed at the thermocouple surface can be very quickly dissipated in a surrounding material such as water, since the heat transfer coefficient at the metal-water interface will be very large. The actual thermojunction surface will then not be heated at all. In tissue, however, where the heat transfer coefficient will not be so high, the thermojunction will be heated by the energy absorbed at the wire surface, and the indicated initial response will be that of a very shallow location in an opaque solid. Note that the subsurface response in an opaque solid has an initial slope of zero, and begins to rise only after a finite delay.

It now appears that this indicated initial thermocouple response will be a complex function of the heat transfer coefficient between the wire and its surroundings, the thermoelement geometry, and the orientation of its sensitive surface. Since these factors were, for this study, unknown, attempts to resolve this problem by an analytical approach have been unsuccessful. However, it is claimed that because of the extrapolation of the records back to zero time, an initial slope is obtained which is not the true value for either the thermojunction or the surrounding tissue, but which is, nevertheless, directly proportional to the radiant flux striking the thermocouple. This important assertion has not been proved, and is probably not susceptible of rigorous proof, but from a consideration of the linearity of the system it seems that it must be so.

Now, assuming that this claim is true, it follows that Figure 8-3 does

not represent $q'''(x,0)$, as hoped, but instead indicates the depth distribution of the total "absorbable" irradiance; i.e., employing the noncommittal symbol $S_0(x)$ for the measured initial slope at the depth x , for unit absorbed irradiance,

$$S_0(x) = K V(x), \quad (8-2)$$

where K is an unknown constant of proportionality, and $V(x)$ is the absorption pattern of radiation in the skin as defined in Section 4.2. Now, following the development of this latter section:

$$q'''(x,t) = - \frac{\partial}{\partial x} [H_a(t) \cdot V(x)]; \quad (4-5)$$

hence
$$q'''(x,t) = - \frac{\partial}{\partial x} \left[\frac{H_a(t)}{K} S_0(x) \right], \quad (8-3)$$

or
$$q'''(x,t) = - \frac{H_a(t)}{K} \cdot \frac{dS_0(x)}{dx}. \quad (8-4)$$

The constant of proportionality, K , can now be evaluated by the necessary condition

$$\int_0^{\infty} q'''(x,t) dx = H_a(t). \quad (4-6)$$

Thus:

$$\begin{aligned} H_a(t) &= \int_0^{\infty} q'''(x,t) dx \\ &= - \frac{H_a(t)}{K} \int_0^{\infty} \frac{dS_0(x)}{dx} \cdot dx \\ &= \frac{H_a(t)}{K} [S_0(0) - S_0(\infty)]. \end{aligned}$$

Now, since $V(x)$ must decline to zero as x increases without bound, it follows that

$$S_0(\infty) = 0,$$

and

$$K = S_0(0).$$

The complete expression for $q'''(x,t)$ now becomes

$$q'''(x,t) = \frac{Ha(t)}{S_o(0)} \cdot \frac{dS_o(x)}{dx}, \quad (8-5)$$

where $S_o(x)$ is the function graphed in Figure 8-3, and $S_o(0)$ is the intercept of this curve with the ordinate.

8.8 Summary

Unfortunately, the numerical value of this intercept cannot now be determined. The measured initial slopes are in some way dependent upon the heat transfer coefficient between thermocouple and surrounding tissue, and certainly this coefficient must have been considerably different for the surface couples than that for the subsurface elements; hence the value of K , the proportionality constant in equation (8-2), will likewise be different for the surface than for the subsurface measurements. Perhaps one should select the value of $S_o(0)$ as determined from the wetted-skin experiments, since here the heat transfer coefficients for surface and subsurface thermocouples might be more similar. However, the number of such exposures is few, and the influence of the film of water on the surface response unknown; thus it seems rather unreasonable to use these data.

Considering the positive results, the curve of Figure 8-3 suggests that for depths greater than about 0.02 cm, at least, the absorption of radiation from the carbon arc furnace can be approximated rather well by a simple exponential, with a linear absorption coefficient of about 7.2 cm^{-1} , which is considerably lower than the values cited in Table 5-1. For the superficial layers, the absorption is greater than this, as suggested by Hardy (see Section 4.3), although the value of the linear absorption coefficient in this region cannot be determined from these measurements. The depth to the break between these absorption coefficients is not shown by Figure 8-3, but it must lie above 0.02 cm, and is probably less than 0.015 cm.

This reinforces the supposition that this break coincides with the epidermal-dermal junction, which is at a depth of about 0.008 to 0.01 cm for the young Chester White pig.

Turning now to the comparison of predicted and measured temperature response, as shown in Figure 8-4, there is no doubt that the distinct difference in form between the two sets of curves is due to errors in measurement. Initially, the thermocouples were heated above the temperature of the tissue, so the measured response curves are too high at early times. It seems unlikely that this gross temperature difference persisted throughout the entire heating cycle, however, so that the measured temperatures may be reasonably close to actual tissue temperatures after a few tenths of a second.

In this connection it should be noted that in none of the biopsies taken was there ever any evidence of abnormally high temperatures around a thermocouple site. In fact, no evidence of the presence of a thermocouple was ever found at all, even though the sections were carefully studied in regions where it was known a couple had been located. Thus, at least the thermocouples were not heated to tissue-damaging temperature.

Considering the difference in spacing between the predicted and measured curves, for the various values of depth, the theoretical analysis indicates that this spacing is dependent on the value of linear absorption coefficient. Thus the evidence of this study indicates that here the predicted curves are in error, since the measured coefficient, for subsurface tissue, is considerably lower than predicted. Admittedly, the procedure used to evaluate this absorption coefficient is open to question, but at least this represents an in vivo determination, and the tissue was not in the highly artificial condition as in the in vitro studies upon which the predictions are based.

Finally, it is most clear that a great deal more work remains to be done in the measurement of bare skin response. In the writer's opinion, the results presented in this chapter definitely do not indicate that accurate temperature measurements in bare skin are impossible, but only that the procedures used here were not particularly suitable. In the next chapter, several schemes will be described briefly which, it is hoped, will lead to the desired end of accurate temperature determinations in a diathermanous material such as skin.

8.9 Location of Data

All original data pertaining to this study are on file in the physics laboratory of the Flash Burn Section of the University of Rochester Atomic Energy Project.

FOOTNOTES

- (1) We have recently investigated an ingenious "grating tangentimeter" described by T. E. Thompson, J. L. Oncley, and H. Svensson in Rev. Sci. Inst. 29, 977 (1958), with which angular measurements may be made with a precision of ± 0.1 degree, under favorable conditions. A simplified model of such an instrument, capable of a precision of ± 0.5 degree, has been set up in this laboratory, and used to check several of these eye estimates of initial slopes. The values from these two methods agree to within 20%. In the future, the grating tangentimeter will be used for all initial slope determinations.

CHAPTER IX

SUMMARY OF THE STUDY

9.1 Discussion of the Theoretical Analysis

It was stated earlier that of the two phases of this study, theoretical and experimental, the former is considered the more important. Considerable effort has been devoted to making the analyses as complete and general as possible, so as to provide a basis not only for the experimental work presented here, but also for additional research for some time to come. In spite of this effort, the analyses will be valid only if the basic assumptions employed are valid; therefore it will now be worthwhile to re-examine several of these assumptions, particularly in the light of certain of the experimental results.

Consider first the assumption that the skin could be considered a uniformly irradiated solid. It was pointed out in Chapter VI that the exposure site was delimited by a 1.8 cm diameter aperture in a water-cooled plate, and that at the edges of this aperture the irradiance was only about 70% of that at the center. While this would seem to be a gross violation of the assumption, the real question is whether sufficient lateral heat conduction occurred to negate the unidimensional form of the heat conduction equation. The results of the opaque skin experiments confirm the fact that skin is a rather poor conductor; hence if one considers only the central circular area of one square centimeter, where the irradiance is uniform within about 10%, the isothermals will be very nearly planes parallel to the skin surface. The tissue around this central column serves as a thermal guard, and prevents excessive lateral conduction into the skin behind the water-cooled shield. Hence this assumption is probably fairly well satisfied.

There remains the additional question of whether, in the case of bare

skin exposures, considerable radiation may not have been scattered out of this central, uniformly irradiated column. This would have two effects. First, the absorption of radiation in depth would appear to be more rapid than is actually the case, since part of the loss of radiation would be by scatter rather than true absorption. Second, the much-used condition that the integral over all depths of the radiant power absorption per unit volume must necessarily be equal to the absorbed irradiance would no longer hold, since this assumes that any scatter out of a column of tissue will be compensated for by scatter into the column from adjacent regions. If, in fact, the outgoing scatter is greater than the incoming scatter, then

$$\int_0^{\infty} q'''(x,t) dx < H_a(t).$$

This is a difficult matter to evaluate, although if one assumes a linear absorption coefficient of 7.2 cm^{-1} , then radiation proceeding laterally from the axis of the exposed column of skin to a point 0.564 cm off the axis (the edge of a circle of 1 cm^2 area) would be attenuated by over 98%. Further, scatter out of the central, uniformly irradiated column would be in part compensated by scatter into this column from the tissue around it but within the 1.8 cm diameter exposure site. Hence, it may be concluded that loss of radiation by lateral scatter was probably not serious.

Next consider the assumption that the skin could be considered infinitely thick. Since the skin is actually only about 0.18 cm thick, this would also appear to be a very poor assumption. Here, the crucial point is contained in the limiting condition based on this assumption; namely

$$\lim_{x \rightarrow \infty} U(x,t) = 0.$$

Thus, when a depth has been reached at which the temperature remains approximately unaffected by an exposure on the surface, then, by definition, this depth is "infinity." For the opaque skin exposures, it was found that a

thermocouple emplaced at the dermal-fat junction displayed almost no temperature rise until many seconds after the exposure, and the very slight rise seemed to be the result of local vasodilatation, not heat conduction. Thus opaque skin may be considered a very good approximation to a semi-infinite solid, with the reservation that the exposure times must be short, as in this study. The case of long exposures has been treated quite elegantly by Hendler, Crosbie, and Hardy (1).

Again, it is necessary to consider separately the bare skin exposures. Here, even for the deepest couples employed (dermal-fat interface), prompt and significant temperature responses were noted. This would be of little consequence if one could assume that the thermal and optical constants of the subadjacent fat, and the deeper fascia and muscle were similar to those of the dermis. Since this is probably incorrect, it is necessary to consider just what effect these deep layers, with their different thermal and optical properties, would have on the temperature response of the dermis. The straight analytical attack on this question is quite complex, and no useful results have been obtained as yet. One might suspect, however, that the insulating quality of the fatty tissue will produce higher deep dermal temperatures than would be predicted by the simple theory. This is an area in which the theoretical analysis must be extended.

Another basic assumption, which was originally thought to be exceedingly poor, is that the thermal constants of skin may be considered to be independent of depth. The excellent agreement between theory and experiment for the case of opaque skin indicates that the skin may indeed be considered to be a uniform, isotropic material. In view of the complex structure of skin, with stratification both laterally and in depth, this is a surprising conclusion.

It must again be pointed out that this uniformity of skin may be more

apparent than real, and be the result of the surface treatment employed to render the skin opaque. This has been discussed at length in Chapters VII and VIII. In this connection, it is interesting to note that in the measurements of surface temperature response made by Hendler and his collaborators (2) the skin was heated by non-penetrating infrared radiation which obviated the necessity of covering the surface with an opaque film. Since the temperature responses measured by these workers followed quite closely the simple square root of time relation predicted for a uniform solid, one might conclude that the noted uniformity of skin is not the result of surface treatment, but is, in fact, real. However, the response time of the infrared photometer used by Hendler was probably quite long; hence, inflections in the response curve characteristic of depth variations of the thermal constants could probably not have been observed in these experiments. It must be concluded that the possibility exists that there is some variation of thermal constants of skin with depth, but this variation, if it exists, is small.

Two additional basic assumptions will now be considered: first, that the skin is initially at uniform temperature throughout, and second, that the surface of the skin is insulated--that is, there are no conduction, convection, or radiation losses at this surface. Now the skin actually should be considered as a more or less thin lamina between a constant temperature core and the environment. Under normal conditions, heat is conducted from the core through the skin and dissipated at the surface to the cooler surroundings. Thus, there does exist an initial temperature gradient through the skin, and the surface is not perfectly insulated. While it is not difficult to set up the boundary value problem incorporating these facts, the solution, particularly for diathermanous skin, becomes rather complex. The above-mentioned analysis of Hendler, Crosbie, and Hardy treats this situation for opaque skin, and these authors conclude that the assumptions of uniform

initial temperature and insulated surface are justified for exposures of a few seconds duration, or less. The measurements on opaque skin reported in Chapter VII of this study confirm this conclusion, since the responses followed the theoretical predictions quite closely. It is reasonable that if these assumptions be true for opaque skin, then they must also hold for bare, or diathermanous skin, as well. Note, however, that if one considers the long cooling phase after a short exposure, the assumption that the surface is insulated becomes quite poor, since undoubtedly much of the energy absorbed in the skin is eventually dissipated to the environment rather than being lost solely by conduction into depth.

Finally, the assumption that the skin may be considered a passive solid must be investigated. It is necessary first to recall the position which the present study occupies in the over-all problem of the prediction of irreversible thermal injury of skin, as defined in Chapter I. It was there claimed that damage is a function of temperature, while the temperature in turn is a function of the characteristics of the thermal insult. The over-all problem was thus broken down into two problems: the determination of tissue temperatures, and the correlation of tissue damage with these temperatures.

It follows, then, that the assumption here being considered leads to the paradoxical situation of computing the temperatures of a passive material, and using these temperatures to predict the active degradation of the material. Thus, if this study is to have any utility, this assumption must be false, by definition. The degree of failure of the assumption will depend upon such factors as the activation energy of the damage process, or processes, which factors are precisely those which require accurate temperature data for their evaluation.

Now, in the present study, all exposures were designed to be so small

that no extensive tissue damage would occur; hence, it is probable that the measured temperatures are representative of passive skin. (However, one might question whether the high value of heat capacity per unit volume of skin may not reflect some energy "loss" in degrading dermal collagen.) The important question is how accurately these experimental results and theoretical predictions can be extended to exposures such that tissue damage does occur, whereupon the system becomes, by necessity, non-linear.

Further consideration of this question leads to two rather interesting conclusions. First, the division of the over-all problem of burn prediction into the two subproblems of tissue temperature determination, and temperature-damage correlation is probably improper and, strictly speaking, impossible. If one wishes to avoid the paradox of employing "passive" tissue temperature to predict "active" tissue damage, then temperatures and damage must be considered together. The second conclusion is that the very failure of this assumption of a passive--hence linear--receiver may provide a most fruitful method for the quantitation of the thermal damage process. Thus, consider the heat flow equation rephrased to include a distributed heat sink term which would account properly for the endothermic tissue degradation. The solution of the equation would predict non-linear (i.e. exposure-dependent) temperature response, where the non-linearity would be due to tissue damage. But this solution, with all unknowns evaluated by measuring temperatures for tissue-damaging exposures, would constitute the damage prediction equation; that is, it would contain the information necessary to predict tissue degradation as a function of the characteristics of the energy input.

No attempt has yet been made in this laboratory to follow up this line of reasoning, although it would seem to be most important to do so. It might be noted that this problem can be attacked using opaque skin,

since this case is vastly easier to handle both theoretically and experimentally. Eventually, of course, the study must be extended to bare skin, since this is the case of maximum interest. However, present plans call for a thorough study of the depth of damage of India ink painted skin, as related to duration and magnitude of radiant exposure, with concomitant temperature measurements. At the same time, the theoretical analysis of the non-linear, actively responding system will be attacked, employing the simple, opaque solid model. Hopefully, this approach may result in real progress in the predicting of irreversible tissue damage.

9.2 Discussion of the Experimental Results

The measurements of temperature response of opaque skin were, in general, most satisfactory. In the absence of penetrating radiation, it is highly probable that the thermocouples were following the true tissue temperatures quite faithfully, as suggested by the excellent agreement between theory and experiment. A disturbing feature of these measurements, however, was the significant variation in temperature response from one exposure to another. It is interesting to note that other workers in this laboratory have reported large variations in depth of damage (in bare skin) produced by identical exposures (3). Thus, it may be that the variability in temperature response is real, and due to the biologist's neutrino, "biological variation." The studies described in the paragraph above may help clarify this problem.

In contrast to the results with opaque skin, the bare skin measurements were unsatisfactory, so far as the accurate determination of the optical properties of living skin is concerned. This does not mean, however, that this phase of the experimental program was a total failure. The major difficulty in making these measurements, the direct action of the penetrating

radiation on the thermocouples, was by no means unexpected; it is obvious that this interaction may occur. So far as the writer is aware, however, this is the first study which proved conclusively that this effect does occur, which is some accomplishment, if a rather negative one.

Actually, one can point out several positive accomplishments of this work. First, by application of the results of the theoretical analysis, it was possible to quantitate the error caused by direct heating of the thermocouples, insofar as this affected the initial slope of the temperature response. From these considerations, it was possible to derive a value of the linear absorption coefficient for the dermis, by employing fairly reasonable assumptions. While this value, 7.2 cm^{-1} , is not precise, it is one which must be disproved before it can be discarded. Further, the results of this study indicate that the linear absorption coefficient for superficial tissue is higher than that for the deeper material, which is fully in agreement with the measurements of Hardy and his collaborators (4). The depth of the break in the absorption pattern, however, seems to be considerably less than that suggested by these authors, and apparently is near to or coincides with the epidermal-dermal interface.

Now, it is reasonable to ask whether accurate temperature measurements in radiated bare skin can ever be achieved. The answer to this question is almost surely in the affirmative. From the failures of the present study, one can see what steps must be taken to overcome these failures; indeed this is a most important accomplishment of this study.

First, one would like to have experimental confirmation of the important theoretical result expressed in equation (4-24):

$$\frac{\partial U(x,0)}{\partial t} = \frac{1}{\nu} q'''(x,0).$$

This will be done by immersing thermocouples in a strongly colored aqueous solution; the thermal and optical constants of such a diathermanous, non-scattering medium can be measured independently with considerable accuracy, so it will be possible to verify the above relation directly. From the interesting observations reported in Section 8.7 concerning the zero thermocouple response in clear water, it is inferred that these thermoelements can follow the temperature of an aqueous solution quite accurately. There will be no problem with convection currents in the solution, since only the initial response is of interest.

Secondly, it is abundantly clear that the skin surface temperature must be measured with an infrared radiometer, as recommended by Hardy (5). From the results of Hendler, et al. (6), it appears possible to measure the surface temperature of bare skin exposed to carbon arc radiation, simply by sampling those wavelengths of the re-emitted radiation for which the skin is very nearly opaque. From preliminary studies, it appears that the only problem is the practical one of obtaining an adequate signal-to-noise ratio in a broad band-width system. Such a photometer will permit accurate surface temperature measurements uncomplicated by direct interaction of the incident irradiation with the temperature sensing system. This development alone will permit accurate in vivo measurements of the linear absorption coefficient of the superficial tissue, γ_1 , as indicated by the expression developed in Chapter IV:

$$\frac{\partial U^*(0,0)}{\partial t} = \frac{\gamma_1}{\nu}$$

Finally, it will be necessary to reconsider the problem of measuring subsurface temperatures. Probably the most reasonable approach to this problem is to attempt to derive an analytical expression for the response of an opaque temperature sensing element imbedded in a scattering diathermanous medium such

as skin. If the dependence of the thermoelement temperature on that of the surrounding tissue can be found, then it will be a straightforward matter to correct the indicated responses to get the true tissue temperatures.

This analytical approach will be practical only if the temperature sensing element has such ideal geometry that the mathematical boundary value problem does not become unmanageably complex. Actually, such elements are already at hand. If one takes a suitable length of the Ag-Pd Wollaston wire described in subsection 6.6.1, and dissolves away the silver jacket over only a few millimeters in the center of this length, the remaining fine palladium filament connecting the two halves forms an elegantly small and sensitive resistance thermometer. Not only is the geometry of this element quite simple, but the heat transfer coefficient between thermometer and tissue can be determined in vivo by passing a known current through the palladium filament and measuring its equilibrium temperature rise. These elements are extremely fragile, so that placement will be difficult, although not impossible.

Other methods of obtaining accurate subsurface tissue measurements are also under consideration. For example, one may compare the results from thermocouples of various diameters--hence various surface-to-volume ratios--and extrapolate to the ideal zero surface-to-volume ratio element. Since such a thermoelement has an infinite diameter, this may be a rather questionable extrapolation. However, employing different diameter elements would be quite useful in checking theoretical predictions concerning this parameter.

9.3 Consideration of the Direct Measurement of the Absorption Pattern of Radiation in Skin

From the analysis of Chapter V it is easy to show that for a dia-

thermanous solid exposed to heterochromatic radiation the following relation holds:

$$\gamma \frac{\partial U(x, 0)}{\partial t} = \sum_{i=1}^n q'''_i(x, 0),$$

where $q'''_i(x, 0)$ is the initial rate of energy absorption per unit volume for the i^{th} wavelength interval of the incident radiation. Expressing $q'''_i(x, 0)$ as the product function

$$q'''_i(x, 0) = H_{a_i}(0) \cdot F_i(x),$$

where $H_{a_i}(0)$ is the initial absorbed irradiance function for the i^{th} wavelength interval, and $F_i(x)$ is the absorption pattern for this radiation, it follows that one can, in principle, determine this latter function by direct temperature measurement. By its very nature, this procedure offers the most direct and unambiguous method for the evaluation of the heating effect of penetrating radiation, which effect must be known for temperature prediction purposes. This underlines the importance of achieving accurate subsurface temperature measurements in bare skin.

Actually, the functions $F_i(x)$ must be known for each small wavelength interval over the spectral range of interest, if one is to be able to predict temperature responses to any arbitrary wavelength distribution. Thus, even accurate subsurface temperature measurements employing, say, carbon arc radiation are not enough; one must employ either monochromatic radiation or subtractive, sharp cut-off filters in order to evaluate the absorption pattern as a function of wavelength. Both of these methods present real experimental difficulties. Even with a fast monochromator, with a broad spectral band-pass, the output irradiance is low. It follows that temperature responses will be quite small, hence difficult to measure with accuracy. On the other hand, the use of sharp cut-off filters, discussed in Section 5.6,

implies that one must determine each $F_i(x)$ as

$$F_i(x) = \frac{1}{H_{a_i}(0)} \left[\sum_{j=i}^n q'''_j(x,0) - \sum_{j=i+1}^n q'''_j(x,0) \right].$$

In general, each $F_i(x)$ will be a small difference between two relative large numbers, which latter are subject to large experimental variability. Hence, with this method also, it will be difficult to achieve good accuracy.

Assuming that these practical problems can be solved, one is immediately faced by a much more difficult fundamental problem. An accurate statement of the absorption pattern of radiation in skin, as a function of wavelength, is primarily of use in predicting the temperature response of a linear passive system. If one wishes to predict tissue injury, and still avoid the "passive-active tissue" paradox discussed above, then the theory must be extended to include the effects of such injury. It would seem that the factor $q'''(x,t)$, in the heat flow equation, must now contain both a wavelength-dependent energy absorption term and a (presumably) wavelength-independent term which accounts for the endothermic tissue degradation process. If one demands that the damage predicting scheme be capable of handling an exposure to radiation of any arbitrary wavelength distribution, then it is clear that these two terms must be evaluated separately. Will the experimental procedures described above be of any utility in untangling these terms? If not, what procedures must be used? The writer is unable to answer these questions, and in fact does not hope to be able to do so until some progress has been made in the opaque skin studies described in Section 9.1.

9.4 Conclusion

The specific results given in this chapter and the preceding ones are not as extensive as one would desire. In all honesty, it must be admitted that the completely general prediction of the temperature response

of skin to any arbitrary thermal insult remains an unsolved problem. Still, this study does constitute a highly satisfactory demonstration of the utility and the necessity of combining the theoretical and experimental approaches to this problem.

If the above remarks seem to have dwelt more on plans for the future than on accomplishments of this research, this is not unreasonable. It is obvious that much work remains to be done, but the real contribution of this study is the conclusion that this future research is possible and practicable, and that the ultimate goal of predicting the irreversible thermal injury of skin may be achieved.

FOOTNOTES

- (1) Hendler, E., R. Crosbie, and J. D. Hardy, Measurement of Skin Heating During Exposure to Infrared Radiation, J. Appl. Physiol. 12, 177-185 (1958).
- (2) Ibid., 180.
- (3) Hinshaw, J. R., The Irradiance Dependency of Exposure Time as a Factor in Determining the Severity of Radiant Thermal Burns, University of Rochester Atomic Energy Project Report UR-451, 6 (1956).
- (4) Hardy, J. D., H. T. Hammel, and D. Murgatroyd, Spectral Transmittance and Reflectance of Excised Human Skin, J. Appl. Physiol. 9, 262 (1956).
- (5) Stoll, A. M. and J. D. Hardy, Direct Experimental Comparison of Several Surface Temperature Measuring Devices, Rev. Sci. Inst. 20, 678-686 (1949).
- (6) Hendler, E., R. Crosbie, and J. D. Hardy, op. cit. It is interesting to note that the instrumentation described in this report is now capable of measuring surface temperatures for bare skin exposed to penetrating radiation. The system undoubtedly has a narrow band-width, but if the indicial transfer function is determined so that measured responses may be corrected, then these authors may immediately determine the linear absorption coefficient for the surface layer of skin for a variety of spectral distributions.

BIBLIOGRAPHY

1. Arguimbau, L. B., Vacuum-Tube Circuits (John Wiley & Sons, Inc., New York, 1948).
2. Berkley, K. M., T. P. Davis, and H. E. Pearse, Study of Flash Burns: The Effect of Spectral Distribution on the Production of Cutaneous Burns, University of Rochester Atomic Energy Project Report UR-336 (1954).
3. Berkley, K. M., H. E. Pearse, and T. P. Davis, Studies of Flash Burns: The Influence of Skin Temperature in the Production of Cutaneous Burns in Swine, University of Rochester Atomic Energy Project Report UR-338 (1954).
4. Buettner, K., Effects of Extreme Heat and Cold on Human Skin III. Numerical Analysis and Pilot Experiments on Penetrating Flash Radiation Effects, J. Appl. Physiol. 5, 207-220 (1952).
5. Chen, N. Y. and W. P. Jensen, Skin Simulants with Depth Magnification, Massachusetts Institute of Technology Fuels Research Laboratory Technical Report No. 5 (1957).
6. Churchill, R. V., Modern Operational Mathematics in Engineering (McGraw-Hill Book Company, Inc., New York, 1944).
7. Clark, C., R. Vinegar, and J. D. Hardy, Goniometric Spectrometer for the Measurement of Diffuse Reflectance and Transmittance of Skin in the Infrared Spectral Region, J. Opt. Soc. Am. 43, 993-998 (1953).
8. Coulbert, C. D., W. F. MacInnes, T. Ishimoto, et al., Temperature Response of Infinite Flat Plates and Slabs to Heat Inputs of Short Duration at One Surface, University of California at Los Angeles, Department of Engineering Report (1951).
9. Davis, T. P., A Standard Radiation Calorimeter, M.S. Thesis, Department of Physics, The University of Rochester (1954).
10. Davis, T. P., "The Carbon Arc Image Furnace" in Proceedings of the Symposium on High Temperature -- A Tool for the Future, 10-15 (Stanford Research Institute, Menlo Park, California, 1956).
11. Davis, T. P., L. J. Krolak, R. M. Blakney, and H. E. Pearse, Modification of the Carbon Arc Searchlight for Producing Experimental Flashburns, J. Op. Soc. Am. 44, 766-769 (1954).
12. Davis, T. P. and H. E. Pearse, The Use of the Carbon Arc and Burning Magnesium as Thermal Sources for Experimental Burns, Ann. Surg. 149, 68-76 (1959).
13. Derksen, W. L., T. D. Murtha, and T. I. Monahan, Thermal Conductivity and Diathermanoy of Human Skin for Sources of Intense Thermal Radiation Employed in Flash Burn Studies, J. Appl. Physiol. 11, 205-210 (1957).

14. Finkelburg, W., The High Current Carbon Arc, FIAT Final Report No. 1052 (1946).
15. Glasstone, S., ed., The Effects of Nuclear Weapons (U. S. Government Printing Office, Washington, D.C., 1957).
16. Hansen, K. G., On the Transmission Through Skin of Visible and Ultra-violet Radiation, Ph.D. Thesis, The University of Copenhagen (Published as Supplementum LXXI, Acta Radiologica, 1948).
17. Hardy, J. D., Annual Progress Report for 1951 (to Chief of Naval Research, Attn. Biology Branch).
18. Hardy, J. D., Method for the Rapid Measurement of Skin Temperature During Exposure to Intense Thermal Radiation, J. Appl. Physiol. 5, 559-566 (1953).
19. Hardy, J. D., H. T. Hammel, and D. Murgatroyd, Spectral Transmittance and Reflectance of Excised Human Skin, J. Appl. Physiol. 9, 257-264 (1956).
20. Hardy, J. D. and C. Muschenheim, Radiation of Heat from the Human Body IV. The Emission, Reflection, and Transmission of Infrared Radiation by the Human Skin, J. Clin. Invest. 13, 817-831 (1934).
21. Hardy, J. D. and C. Muschenheim, Radiation of Heat from the Human Body V. The Transmission of Infrared Radiation Through Skin, J. Clin. Invest. 15, 1-9 (1936).
22. Hendler, E., R. Crosbie, and J. D. Hardy, Measurement of Skin Heating During Exposure to Infrared Radiation, J. Appl. Physiol. 12, 177-185 (1958).
23. Henriques, F. C., Jr., Studies of Thermal Injury V. The Predictability and the Significance of Thermally Induced Rate Processes Leading to Irreversible Epidermal Injury, Arch. Path. 43, 489-502 (1947).
24. Henriques, F. C., Jr. and R. A. Maxwell, The Applicability of the Skin Damage Integral to the Prediction of Flash Burn Injury Thresholds, Including those Caused by Atomic Detonations, Technical Operations, Inc., Report TOI 54-19 (1956) (SECRET).
25. Henriques, F. C., Jr. and A. R. Moritz, Studies of Thermal Injury I. The Conduction of Heat to and Through Skin and the Temperatures Attained Therein. A Theoretical and Experimental Investigation, Am. J. Path. 23, 531-549 (1947).
26. Hinshaw, J. R., The Irradiance Dependency of Exposure Time as a Factor in Determining the Severity of Radiant Thermal Burns, University of Rochester Atomic Energy Project Report UR-451 (1956).
27. Hinshaw, J. R. and H. E. Pearse, Histologic Techniques for the Differential Staining of Burned and Normal Tissues, Surg. Gyn. and Obs. 103, 726-730 (1956).
28. Jakob, M., Heat Transfer, Vol. I (John Wiley and Sons, Inc., New York, 1949).

29. Kingsley, H. D., L. Hogg, Jr., J. T. Payne, and J. H. Morton, Studies in Prolonged Anesthesia in Swine and Dogs, University of Rochester Atomic Energy Project Report UR-152 (1951).
30. Krolak, L. J., The Measurement of Diffuse Reflectance of Pig Skin, Titanium Dioxide Paint, and India Ink; the Transmittance of Titanium Dioxide and India Ink, University of Rochester Atomic Energy Project Report UR-439 (1956).
31. Krolak, L. J. and T. P. Davis, A Universal Spectrophotometer for the Measurement of the Relative Spectral Distribution of the Carbon Arc Source, University of Rochester Atomic Energy Project Report UR-367 (1954).
32. Kuppenheim, H. F., J. M. Dimitroff, P. M. Melotti, I. C. Graham, and D. W. Swanson, Spectral Reflectance of the Skin of Chester White Pigs in the Ranges 235-700 mμ and 0.707-2.6 μ, J. App. Physiol. 9, 75-78 (1956).
33. Lawson, D. I., P. H. Thomas, and D. L. Simms, The Thermal Properties of Skin, Fire Research Station (Boreham Wood, Herts., England) F.R. Note 224/1955 (1955).
34. Lerman, B. and J. R. Hinshaw, Relationships Between Burn Severity and the Simulated Thermal Pulses of Various Nuclear Weapons, University of Rochester Atomic Energy Project Report UR-533 (1958).
35. Leroy, G. V., Medical Sequelae of the Atomic Bomb Explosion, J. Am. Med. Assoc. 134, 1143 (1947).
36. Lipkin, M. and J. D. Hardy, Measurement of Some Thermal Properties of Human Tissues, J. Appl. Physiol. 7, 212-217 (1954).
37. Mixer, G., Jr. and T. P. Davis, A Method of Shaping Thermal Energy Pulses from a Carbon Arc Source, University of Rochester Atomic Energy Project Report UR-387 (1955).
38. Mixer, G., Jr. and L. J. Krolak, Critical Energy of Fabric as Indicated by its Persistence Time Under Thermal Radiation, University of Rochester Atomic Energy Project Report UR-260 (1953).
39. Moritz, A. R. and F. C. Henriques, Jr., Studies of Thermal Injury II. The Relative Importance of Time and Surface Temperature in the Causation of Cutaneous Burns, Am. J. Path. 23, 695-720 (1947).
40. Pearse, H. E., Thermal Injuries of Nuclear Warfare, Military Medicine 118, 274-278 (1956).
41. Pearse, H. E. and H. D. Kingsley, Thermal Burns from the Atomic Bomb, Surg. Gyn. and Obst. 98, 385-394 (1954).
42. Pearse, H. E., J. T. Payne, and L. Hogg, The Experimental Study of Flash-Burns, Ann. of Surg. 130, 774-789 (1949).

43. Perkins, J. B., H. E. Pearse, and H. D. Kingsley, Studies on Flash Burns: The Relation of Time and Intensity of Applied Thermal Energy to the Severity of Burns, University of Rochester Atomic Energy Project Report UR-217 (1952).
44. Ross, O. A., A. E. Axelrod, and A. R. Moritz, Progress Report of Project No. G-3645 to Morphology and Genetics Study Section, Division of Research Grants, Department of Health, Education, and Welfare, for July 16, 1952 to August 31, 1954.
45. Stoll, A. M. and J. D. Hardy, Direct Experimental Comparison of Several Surface Temperature Measuring Devices, Rev. Sci. Inst. 20, 678-686 (1949).
46. Strong, J., Procedures in Experimental Physics (Prentice-Hall, Inc., New York, 1938).
47. Swope, G. A. and F. C. Henriques, Thermal Pulse Properties of Five High Intensity Laboratory Sources, Technical Operations, Inc., Report No. TOI 54-8 (1954).
48. Zemansky, M. W., Heat and Thermodynamics, Third Edition (McGraw-Hill Book Company, Inc., New York, 1951).
49. A Study of the Physical Basis of Burn Production with Applications to the Defensive Reactions to an Atomic Bomb Air Burst, Medical Research and Development Board, Office of the Surgeon General (Author and date of publication unknown).
50. The Thermal Data Handbook, Armed Forces Special Weapons Publication AFSWP 700 (1954) (SECRET).

APPENDIX I

Nomenclature

Only the more important symbols are included in this list, and trivial constants, self-evident terms, and symbols adequately defined in context have been omitted.

- a - As subscript = "absorbed."
- c_p - Specific heat capacity ($\text{cal gm}^{-1} \text{deg}^{-1}$).
- h - Laplace transform of irradiance function, H.
- k - Thermal conductivity ($\text{cal sec}^{-1} \text{cm}^{-1} \text{deg}^{-1}$).
- q - Rate of heat flow (cal sec^{-1}).
- q'' - Rate of heat flow per unit area ($\text{cal cm}^{-2} \text{sec}^{-1}$).
- q''' - Rate of heat generation (absorption) per unit volume
($\text{cal cm}^{-3} \text{sec}^{-1}$).
- t - Time (sec).
- u - Laplace transform of temperature rise, U.
- x - Depth (cm).
- C_p - Total heat capacity (cal deg^{-1}).
- H - Irradiance, radiant power per unit area ($\text{cal cm}^{-2} \text{sec}^{-1}$).
- J_λ - Relative spectral energy.
- Q - Heat (cal), or Radiant exposure, the time integral of irradiance
(cal cm^{-2}).
- R - Reflectance, or back-scatter factor.
- T - Absolute temperature ($^{\circ}\text{K}$).
- U - Temperature rise (centigrade degrees).
- $\mathcal{D}$ - Damage symbol.
- α - Thermal diffusivity ($\text{cm}^2 \text{sec}^{-1}$).
- γ - Linear absorption coefficient (cm^{-1}).

- δ - Trapezoidal irradiance pulse rise time (sec).
- η - Rectangular irradiance pulse duration, exposure time (sec).
- θ - Dimensionless time, diathermanous solid model.
- μ - Thermal inertia = $k \cdot \nu$ ($\text{cal}^2 \text{ cm}^{-4} \text{ deg}^{-2} \text{ sec}^{-1}$).
- ν - Heat capacity per unit volume ($\text{cal cm}^{-3} \text{ deg}^{-1}$).
- ξ - Dimensionless depth, diathermanous solid model.
- ρ - Density (gm cm^{-3}).
- τ - Dimensionless time, opaque solid model.
- ψ - Dimensionless depth, opaque solid model.
- Π - Dimensionless temperature rise, opaque solid model.
- Ψ - Dimensionless temperature rise, diathermanous solid model.

APPENDIX II

The Application of Heat Flow Theory and Chemical Kinetics to the
Prediction of Irreversible Thermal Injury of Skin

Thomas P. Davis
Biophysics Seminar
November 13, 1956

I. INTRODUCTION

During the course of our study of radiant energy "flash" burns it has become increasingly evident that it would be highly desirable to be able to intercompare results obtained with different time-irradiance input pulses, with various spectral distributions, or even with different modes of energy input; i.e., conduction or convection. Similarly we have become increasingly aware of the necessity of extending laboratory findings to new unexplored situations; this latter problem is acute in the case of those attempting to predict military and civilian casualties in the event of a nuclear attack. The desideratum for these and related problems is the ability to express damage in terms of the characteristics of the thermal insult; that is, $\mathcal{D} = \mathcal{D}(q'', \eta, x, t \dots)$, where $\mathcal{D}$ is some undefined measure of irreversible thermal damage to skin.

Some empirical expressions do exist (see, for example, Davis, Hinshaw, and Pearse, A Comparison of the Effects on Bare Porcine Skin of Radiant Energy Delivered in the Forms of Square and Simulated Field Pulses, UR-418 (1955).), but they are only convenient summaries of existing laboratory data, and, having no theoretical significance, cannot be extended to new situations. Obviously to obtain the desired result, a generally applicable prediction formulation, a somewhat more fundamental approach is indicated.

The approach adopted here is as follows:

First, the study is restricted to exclude photochemical reactions; i.e., reactions of the type $X + h\nu = X^* = P$. With this restriction it is then claimed that damage is solely a function of temperature of the concerned tissue, or

$$D = D(T)$$

where

$$T = T(q'', \eta, x, t, \dots).$$

Thus, the problem is now broken down into two subproblems: First the prediction of the temperature-time-space history of the skin for any arbitrary energy input, and second, the prediction of irreversible tissue injury based on this temperature response. This rather formal presentation of what is probably an immediately obvious approach has been adopted for purposes of unifying the remaining presentation, which often seems far afield from the central matter.

The "damage symbol," D , still remains undefined. The problem of definition is actually an integral part of the problem of prediction, and further discussion of this matter will be deferred until after treatment of the problem of determining the temperature response.

$$\text{II. } T = T(q'', \eta, x, t, \dots)$$

The basic equation of heat conduction in a substance at rest is

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(k_x \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k_y \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial T}{\partial z} \right) + q'''$$

where the k 's may be functions of position and time. General solutions of this equation are not obtainable, and one must proceed to set up a model within the framework of which the equation can be so specialized as to obtain a solution. In proceeding to develop such a model the following assumptions will be made:

1. The receiver is semi-infinite, with the surface in the plane $x = 0$, and extending in the positive x direction.
2. The heat input to the system is uniform over the surface.
3. The receiver is isotropic and passive.
4. The surface is perfectly insulated (including no reradiation losses).
5. The receiver is initially at uniform temperature throughout.

These assumptions are presented in more detail, and fully discussed with respect to the actual system, skin, in reference (43)*.

Under these assumptions, isothermals will be planes parallel to the surface, and the heat flow equation becomes

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2} + \frac{q'''(x,t)}{\rho c_p}$$

The problem will now be specialized further by considering only radiant energy input to the system, and the following three models will be proposed:

1. Opaque solid.
2. Diathermanous solid, exponential-type absorption.
3. Diathermanous solid, linear-type absorption.

Solutions of the above specialized heat flow equation may now be obtained readily. These will be presented below:

Model 1 - opaque solid

Statement of the problem:

1. $T_t''(x,t) = \alpha T_{xx}''(x,t)$
2. $T''(x,0) = T_0$
3. $T''(\infty,t) = T_0$
4. $T_x''(0,t) = -\frac{1}{k}(1-R)H(t)$

*References throughout this appendix refer to the Bibliography preceding Appendix I.

Solution:

$$T^{(1)}(x, t) = \frac{1}{\sqrt{\pi k \rho c_p}} \int_0^t (1-R) H(t-\tau) \frac{e^{-\frac{x^2}{4\alpha\tau}}}{\sqrt{\tau}} d\tau + T_0$$

Model 2 - diathermanous solid, exponential-type absorption

Statement of the problem:

1. $T_t^{(2)}(x, t) = \alpha T_{xx}^{(2)}(x, t) + \frac{\gamma}{\rho c_p} (1-R) H(t) e^{-\gamma x}$
2. $T^{(2)}(x, 0) = T_0$
3. $T^{(2)}(\infty, t) = T_0$
4. $T_x^{(2)}(0, t) = 0$

Solution:

$$T^{(2)}(x, t) - T_0 = \frac{\gamma}{2\rho c_p} \int_0^t (1-R) H(t-\tau) e^{-\gamma^2 \alpha \tau} \left[2 \cosh \gamma x + e^{-\gamma x} \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha\tau}} - \gamma\sqrt{\alpha\tau}\right) - e^{+\gamma x} \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha\tau}} + \gamma\sqrt{\alpha\tau}\right) \right] d\tau.$$

Model 3 - Diathermanous solid, linear-type absorption

Statement of the problem:

1. $T_t^{(3)}(x, t) = \alpha T_{xx}^{(3)}(x, t) + \frac{(1-R)H(t)}{\rho c_p L}$, $0 \leq x < L$
 $= \alpha T_{xx}^{(3)}(x, t)$, $x > L$
2. $T^{(3)}(x, 0) = T_0$
3. $T^{(3)}(\infty, t) = T_0$
4. $T_x^{(3)}(0, t) = 0$
5. $T^{(3)}(L-0, t) = T^{(3)}(L+0, t)$
6. $T_x^{(3)}(L-0, t) = T_x^{(3)}(L+0, t)$

Solution:

$$T^{(3)}(x, t) - T_0 = \frac{1}{2\rho c_p L} \int_0^t (1-R) H(t-\tau) \left[\operatorname{erf}\left(\frac{L+x}{2\sqrt{\alpha\tau}}\right) + \operatorname{erf}\left(\frac{L-x}{2\sqrt{\alpha\tau}}\right) \right] d\tau,$$

$x \geq 0$

It may be noted that the solution for model 1 also describes the temperature response for Newtonian (contact) heating, with replacement of

$H(t)$ by a suitable heat input function $q''(t)$, involving the heat transfer coefficient.

The problem of selecting the "best" model to fit skin, and the determination of constants would seem to be a formidable experimental task. Some improvement may be obtained by further simplifying the heat input function, and introducing certain dimensionless parameters. For the input, we will consider the following function:

$$\begin{aligned} H(t) &= H_0, \quad 0 \leq t \leq \eta \\ &= 0, \quad t < 0, \quad t > \eta \end{aligned}$$

The term $(1 - R) H(t - \tau)$ may now be brought out from under the integral sign in the foregoing expressions, and solutions obtained immediately for limits from 0 to η . For times greater than η , the temperature equals $T^{(1)}(x, t) - T^{(1)}(x, t - \eta)$ by superposition. While this is some improvement, the expressions are still not easy to fit to experimental data. However, one may note the following relatively simple expressions:

1. $T^{(1)}(0, t) = \frac{2(1-R)H_0}{\sqrt{\pi k \rho c_p}} \sqrt{t} + T_0$
2. $T_t^{(2)}(0, 0) = \frac{(1-R)H_0}{\rho c_p} \cdot \gamma$
3. $T_t^{(3)}(0, 0) = \frac{(1-R)H_0}{\rho c_p} \cdot \frac{1}{L}$

Now, by making the skin opaque, say with India ink or black Krylon, one may obtain, from (1) above, the product $k \rho c_p$ from surface temperature measurements. The ρc_p product is about 0.9, hence one may obtain k , and from this $\alpha = k/\rho c_p$. Next, according to the above equations (2) and (3), from the initial slope of the surface temperature rise, one may obtain an experimental value, say $\sigma \text{ cm}^{-1}$; which will be equal to either γ or L^{-1} , depending upon which model, 2 or 3, is "correct." Now, let us define two dimensionless variables, in terms of this experimental constant σ :

$$\xi = \sigma x, \quad \theta = \sigma^2 \alpha t.$$

Now solutions for models 2 and 3 may be written as

$$\frac{k\sigma}{(1-R)H_0} \Delta T^{(2)}(\xi, \theta) = \frac{2}{\sqrt{\pi}} \sqrt{\theta} e^{-\frac{\xi^2}{4\theta}} - \xi \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\theta}}\right) - e^{-\xi} + \frac{e^{\theta}}{2} \left[e^{\xi} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\xi}{2\sqrt{\theta}}\right) + e^{-\xi} \operatorname{erfc}\left(\sqrt{\theta} - \frac{\xi}{2\sqrt{\theta}}\right) \right],$$

and

$$\frac{k\sigma}{(1-R)H_0} \Delta T^{(3)}(\xi, \theta) = \frac{1}{2} \int_0^{\theta} \left[\operatorname{erf}\left(\frac{1+\xi}{2\sqrt{\psi}}\right) + \operatorname{erf}\left(\frac{1-\xi}{2\sqrt{\psi}}\right) \right] d\psi$$

These are now being computed by Dr. Dutton on the IBM 650 for rather wide ranges of ξ and θ . These analytical expressions may be compared with experimental measurements, permitting a choice of the better model.

It may be noted that this approach is somewhat different from that of others (refs. 4, 49), who have obtained values of the optical and thermal constants of skin from in vitro measurements, and applied these to the direct solution of the heat flow equation, either by analytical or numerical methods. It is believed that the method here presented is preferable, but only in case a satisfactory model has been initially selected. Preliminary calculations, for $\xi = 0$ and $\xi = 1$, however, indicate that the temperature responses under models 2 and 3 are remarkably similar, and it may be that the system is not markedly sensitive to the absorption pattern.

It should also be stressed that both models 2 and 3 are valid only for very white skin. The presence of a heavily pigmented layer at the base of the epidermis will undoubtedly alter the temperature-depth profiles considerably. It is not planned, at the present time, to extend these investigations to cover this important situation.

The experimental measurements will be made by introducing thermocouples into the skin of Chester White pigs, these being the experimental animals used for virtually all work by this group. The thermocouples are silver-palladium, made by dissolving away the jacket of silver-palladium Wollaston

wire. The silver is of 0.003 inch diameter, and the palladium 0.0003 inch diameter. For introduction, the silver end of the couple is butt-welded to a length of 0.005 inch diameter spring steel wire, with the leading end of this wire sharpened to permit easy entry into the skin. The depth of the couple is measured by utilizing the ferro-magnetic properties of the steel leader, employing a tape recorder erase head as the sensing element. Silver and palladium extension wires lead from the couple to a reference junction maintained near normal skin temperature. The thermal emf is amplified and fed to a Sanborn recorder; the entire amplifier-recorder system has an upper frequency limit of about 40 cps, insuring adequate time response. Calibration circuits are also included ahead of the first amplifier, so that the over-all system response may be determined. The recorder available has four input channels, so that four couples at various depths may be used for each exposure.

At this point, it would be well to review the presentation to this point. With certain simplifying assumptions, the general heat flow equation has been specialized so as to be amenable to formal solution. Certain models for the interaction of radiation with skin have been proposed, and normalized temperature-time-depth functions derived. The solutions of these functions, presented as families of curves, will then permit selection of the "best" model, and determination of the constants involved. With this accomplished, one may then compute the temperature response of skin for any arbitrary heat input, including that by conduction and convection, where suitable heat transfer coefficients can be obtained. We are now prepared to consider the next phase of the problem, that of determining the relation of the temperature to irreversible tissue damage.

III. $\mathcal{D} = \mathcal{D}(\tau)$.

As mentioned previously the definition of damage is a central part of this problem, and hence will be considered first. In the Flash Burn Section, we have set up a scale of burn severity, ranging from 0 to 5+, based on gross observation, or "surface" appearance. This scale is defined as follows:

- 0 - no burn
- 1+ - red burn
- 2+ - patchy white burn
- 3+ - uniform white burn
- 4+ - blebbed white burn
- 5+ - carbonized burn

Subdivisions of mild, moderate, and severe within each major group yield a scale of 16 levels of severity. Experimental burn results are handled as quantal data to establish median effective exposures for a given level of severity, and are presented typically as iso-response curves on a plot of radiant exposure vs exposure time, for example.

Thus, it might seem at first sight that the desirable prediction scheme would be one which would yield median effective exposures for such iso- N+ curves; i.e., iso-response curves for the Nth level of grossly observable severity. The principal objection to this scheme is the paradoxical statement frequently made by workers in this laboratory that "one 2+ burn (say) is not the same as another 2+ burn." While such a statement is meaningless under the definition of the surface appearance scale, it nevertheless represents the undoubtedly correct subjective feeling as to the true severity of a particular lesion.

It has long been agreed that a more meaningful criterion of damage is the depth of tissue destruction. In spite of such agreement, the criterion

has been little used in the past because of difficulties of depth assessment. Recently, however, the development of certain staining methods by Hinshaw (27) has greatly eased the problem of differentiating damaged from normal tissue, and while certain problems still remain, notably those of shrinkage and swelling of the injured tissue, it would seem that accurate depth of damage determinations are now readily obtainable.

Now, using this depth of damage as the criterion of burn severity, one may design experiments to determine the median effective exposure for a given depth (quantal data), or the mean depth for a given exposure. While the experimental design would be quite different for these two approaches, it is claimed that the end results are the same providing that for a given radiant exposure, the distribution of depth of damage is symmetric; i.e., the mean equals the median. Hence the approach here employed is that of predicting the mean depth of damage (as displayed by staining techniques) produced by a given thermal insult.

Having now selected a definite quantity to be predicted, one must investigate means of accomplishing this. The first method which might be suggested is that of the "maximum temperature attainment" criterion. Here, one would postulate that irreversible tissue damage will result upon the attainment of some fixed critical temperature. This approach has the charm of maximum simplicity, but the disadvantage of probable failure. Experience with animals (23) and inanimate "skin simulants" (5) has indicated that the maximum temperature attainment criterion is inadequate to explain experimental results.

A more reasonable approach is that of Henriques and Moritz and summarized in the "punishment integral":

$$\Omega = P \int_0^t e^{-\frac{A E}{R T^{1.4}}} d\tau$$

While this is a frankly empirical approach, it does possess a reasonable basis, and further has been the method most frequently employed for prediction purposes. It would seem worthwhile, then, to digress briefly to discuss this formulation, a thorough exposition of which will be found in the references.

The punishment integral is based on work by Henriques and Moritz using hot water burns on pigs. The experiment consisted in the determination, for various water temperatures, of the minimum application times necessary to produce trans-epidermal necrosis (threshold A), and maximum application times which could be tolerated without irreversible epidermal damage (threshold B) (25).

The quantity Ω was then defined as "an arbitrary function of epidermal injury as determined by histological examination" (23), and it was postulated that this "injury" follows a relation

$$\frac{d\Omega}{dt} = P e^{-\frac{\Delta E}{RT}}$$

where the similarity to chemical kinetics is obvious. The temperature, T , is to be taken as that at the dermal-epidermal junction. In the case of long-time burns (time > 60 seconds) this temperature will be approximately equal to that of the applied hot water, $T = T_s$, a constant, so that one may write

$$\Omega = P t e^{-\frac{\Delta E}{RT_s}}$$

Since P is an arbitrary constant, Henriques selected a value of $\Omega = 1$ for threshold A, and by curve fitting obtained the values

$$P = 3.1 \times 10^{98} \text{ sec}^{-1}$$

$$\Delta E = 150,000 \text{ cal mole}^{-1}$$

Then, the expression

$$1 = 3.1 \times 10^{98} \int_0^t e^{-\frac{75000}{T(\tau)}} d\tau$$

fit the experimental data with high precision, not only in the steady-state-region, but also in the non-steady-state region down to application times of a few seconds. Further, with these same constants, a value of $\Omega = 0.53$

similarly followed the threshold B data.

Henriques has subsequently applied this punishment integral to radiant energy burn data from this laboratory, using the following procedure: the temperature at the dermal-epidermal junction is computed on opaque solid theory, using the radiant exposure for a median effective exposure for an N+ burn at, say, 1 second exposure time (square-wave pulse). The resulting value of Ω then obtained is used to predict the N+ median effective exposure for other exposure times, or other input pulse forms (24, 50).

When compared with recent data from this laboratory, these predictions are not notably good, except in the case of "opaqued" skin at the 2+ level, and normal skin at the 4+ and 5+ levels. In the former case, one might expect reasonable agreement since 2+ at all times considered corresponds rather well with trans-epidermal necrosis, and the model used for computing $T = T(x,t)$ is not grossly dissimilar to the actual system. In the latter case, no reasonable explanation is apparent, other than compensating errors.

Buettner (4) has followed what would appear to be a somewhat better procedure. He has used Henriques' formulation without change, but has proposed that $\Omega = 1$ be the criterion for irreversible injury and has then determined the depth at which this holds, for a given energy input. In his computation of temperature response, he has also used a diathermanous solid model, with constants obtained from a literature review and some direct determinations. The most interesting feature of his extraordinarily thorough analysis is a set of predictions giving the expected depth of irreversible injury for a variety of radiant exposures, all of two second duration square-wave form. Since his work is based on white skin, and uses the punishment integral which, in turn, is based on the Chester White pig as the experimental animal, we are in an excellent position to check directly his predictions. (It may be noted that Buettner presents only pilot experiment results, and

not a direct check.) Such an experiment has been performed recently, and while the results are not yet available, our prior work at exposure times near two seconds indicates that these predictions are in error.

The approach which the writer has developed follows quite closely those of Buettner and Henriques. First a mythical reaction $Y \rightarrow Z$ in the skin is hypothesized which follows a first-order reaction; i.e.,

$$-\frac{dc_Y}{dt} = k c_Y.$$

But, from Eyring's theory of absolute rates

$$k = \frac{RT}{N_0 h} e^{\frac{\Delta S_a}{R}} e^{-\frac{\Delta H_a}{RT}}.$$

Thus,

$$-\frac{dc_Y}{c_Y} = \frac{R}{N_0 h} e^{\frac{\Delta S_a}{R}} T e^{-\frac{\Delta H_a}{RT}} dt$$

or

$$\ln\left(\frac{c_Y^0}{c_Y}\right) = \frac{R}{N_0 h} e^{\frac{\Delta S_a}{R}} \int_0^t T(\tau) e^{-\frac{\Delta H_a}{RT(\tau)}} d\tau.$$

Now, assume that irreversible thermal injury, as defined by the above mentioned staining techniques, corresponds to a certain critical value of

c_Y^0/c_Y . Then, if a constant S is defined as proportional to, but not necessarily equal to $\frac{R}{N_0 h} e^{\frac{\Delta S_a}{R}}$, then one may arbitrarily select S such that at that value of x equal to the depth of irreversible injury, $\ln\left(\frac{c_Y^0}{c_Y}\right) = 1$

Thus, the supposition is that

$$1 = S \int_0^\infty T(x, t) e^{-\frac{\Delta H_a}{RT(x, t)}} dt$$

defines the value of x equal to the depth of irreversible tissue injury for a given hyperthermic episode. Note that the integration must be carried to infinity--i.e., to the time when the temperature has returned to normal--since one must consider the hypothetical reaction during the entire period of elevated temperatures, including the relaxation phase.

Since this formulation must be capable of predicting not only our data but also that of Henriques and Moritz, the values of the constants would

be expected to be approximately the same. However, we have seen that Buettner's careful analysis does not yield reliable depth estimates. Therefore, it will be necessary to investigate not only the question of the best possible temperature response calculations, but also the damage predicting scheme, per se. For instance, the possibility of reaction order other than first, competing reactions, or consecutive reactions should be considered.

IV. CONCLUSION

It will be realized that many severely simplifying assumptions have been made in this development. Insofar as possible, these assumptions must be checked by experiment, although it must be admitted that the extreme complexity of the burn process in living skin is extraordinarily resistant to formal mathematical attack. It is to be hoped particularly that the proposed hypothetical reaction $Y \rightarrow Z$ may eventually be translated into a meaningful statement involving demonstrable species, for then the entire damage prediction scheme could be based on solid experimental results. And, indeed, if the present investigation proves successful, the information obtained as to heats of activation may provide a clue as to the probable species involved, and thus aid in its own theoretical justification.

APPENDIX III

The "Opaque Solid Function," $R(x)$

The "opaque solid function," $R(x)$, was defined in Chapter III as

$$R(x) = \sqrt{\pi} \left[\frac{1}{\sqrt{\pi}} e^{-x^2} - x \operatorname{erfc}(x) \right].$$

Numerical values of this function can be obtained conveniently and rapidly using an ordinary desk calculator and the extensive Tables of Probability Functions, Vol. I (A. N. Lowan, Technical Director), prepared by the Federal Works Agency, Work Projects Administration for the City of New York, Sponsored by the National Bureau of Standards (1941). This table presents values of $\frac{2}{\sqrt{\pi}} e^{-x^2}$ and $\operatorname{erf}(x)$ (the former is the first derivative of the latter) at closely spaced values of the argument. For computation, it is convenient to rearrange the above expression to

$$R(x) = \sqrt{\pi} \left[\frac{1}{2} \left(\frac{2}{\sqrt{\pi}} e^{-x^2} \right) + x \operatorname{erf}(x) - x \right]$$

The following brief table of values of $R(x)$ is of convenience in calculating the (normalized) temperature response of the opaque, isotropic solid, as outlined in Section 3.3.

$$R(x) = \sqrt{\pi} \left[\frac{1}{\sqrt{\pi}} e^{-x^2} - x \cdot \operatorname{erfc}(x) \right]$$

x	0	1	2	3	4	5	6	7	8	9
0.0	1.00000	0.98237	0.96495	0.94773	0.93070	0.91388	0.89725	0.88082	0.86460	0.84857
0.1	0.83274	0.81711	0.80167	0.78643	0.77139	0.75655	0.74190	0.72744	0.71318	0.69912
0.2	0.68524	0.67156	0.65807	0.64477	0.63166	0.61874	0.60601	0.59346	0.58110	0.56893
0.3	0.55694	0.54513	0.53350	0.52206	0.51079	0.49970	0.48879	0.47805	0.46749	0.45710
0.4	0.44688	0.43685	0.42696	0.41725	0.40771	0.39833	0.38911	0.38006	0.37117	0.36243
0.5	0.35385	0.34543	0.33717	0.32905	0.32109	0.31327	0.30561	0.29809	0.29071	0.28348
0.6	0.27638	0.26943	0.26262	0.25594	0.24940	0.24299	0.23671	0.23056	0.22454	0.21864
0.7	0.21287	0.20722	0.20169	0.19628	0.19099	0.18581	0.18075	0.17580	0.17096	0.16623
0.8	0.16160	0.15708	0.15267	0.14835	0.14414	0.14003	0.13601	0.13209	0.12826	0.12453
0.9	0.12088	0.11733	0.11386	0.11048	0.10718	0.10396	0.10083	0.09778	0.09480	0.09190

APPENDIX IV

Solution of the Composite Opaque Solid Model

Statement of the problem, from Section 3.5:

$$1) U_t(x,t) = \alpha_1 U_{xx}(x,t), \quad 0 \leq x < b, \quad t \geq 0$$

$$2) U_t(x,t) = \alpha_2 U_{xx}(x,t), \quad x > b, \quad t \geq 0$$

$$3) U(x,0) = 0, \quad x \geq 0$$

$$4) U(\infty, t) = 0, \quad t \geq 0$$

$$5) U_x(0,t) = - \frac{(1-R)H_0(t)}{k_1} = - \frac{H_0(t)}{k_1}, \quad t \geq 0$$

$$6) U(b-0, t) = U(b+0, t), \quad t \geq 0$$

$$7) k_1 U_x(b-0, t) = k_2 U_x(b+0, t), \quad t \geq 0.$$

Transforming by $\mathcal{L}_t\{\}$ and using (3):

$$1') s u(x,s) = \alpha_1 \frac{d^2 u(x,s)}{dx^2}, \quad 0 \leq x < b$$

$$2') s u(x,s) = \alpha_2 \frac{d^2 u(x,s)}{dx^2}, \quad x > b$$

$$4') u(\infty, s) = 0$$

$$5') \frac{du(0,s)}{dx} = - \frac{h_0(s)}{k_1}$$

$$6') u(b-0, s) = u(b+0, s)$$

$$7') k_1 \frac{du(b-0, s)}{dx} = k_2 \frac{du(b+0, s)}{dx}$$

Solving (1') and (2'):

$$8) u(x,s) = A e^{-\frac{x}{\sqrt{\alpha_1}} \sqrt{s}} + B e^{+\frac{x}{\sqrt{\alpha_1}} \sqrt{s}}, \quad 0 \leq x < b,$$

$$a) u(x,s) = C e^{-\frac{x}{\sqrt{\alpha_2}} \sqrt{s}} + D e^{+\frac{x}{\sqrt{\alpha_2}} \sqrt{s}}, \quad x > b.$$

Differentiating (8), and applying (5'):

$$-\frac{h_2(s)}{k_1} = -\sqrt{\frac{s}{\alpha_1}} A + \sqrt{\frac{s}{\alpha_1}} B$$

or

$$B = A - \sqrt{\frac{\alpha_1}{s}} \cdot \frac{h_2(s)}{k_1} = A - \frac{h_2(s)}{\sqrt{s}} \cdot \sqrt{\frac{k_1}{\mu_1}} \cdot \frac{1}{k_1} = A - \frac{h_2(s)}{\sqrt{\mu_1} \cdot \sqrt{s}}.$$

Applying (4') to (9), $D \equiv 0$.

Thus (8) and (9) become:

$$10) u(x,s) = A(e^{-\frac{x}{\sqrt{\alpha_1}\sqrt{s}} + e^{+\frac{x}{\sqrt{\alpha_1}\sqrt{s}}}) - \frac{h_2(s)}{\sqrt{\mu_1}} \cdot \frac{e^{+\frac{x}{\sqrt{\alpha_1}\sqrt{s}}}}{\sqrt{s}}, \quad 0 \leq x < b,$$

$$11) u(x,s) = C e^{-\frac{x}{\sqrt{\alpha_2}\sqrt{s}}}, \quad x > b.$$

The constants A and C are now evaluated by applying (6') and (7') to (10) and (11). Thus:

$$12) A(e^{-\frac{b}{\sqrt{\alpha_1}\sqrt{s}} + e^{+\frac{b}{\sqrt{\alpha_1}\sqrt{s}}}) - C e^{-\frac{b}{\sqrt{\alpha_2}\sqrt{s}}} = \frac{h_2(s)}{\sqrt{\mu_1}} \cdot \frac{e^{+\frac{b}{\sqrt{\alpha_1}\sqrt{s}}}}{\sqrt{s}},$$

and

$$k_1 \sqrt{\frac{s}{\alpha_1}} A(-e^{-\frac{b}{\sqrt{\alpha_1}\sqrt{s}} + e^{+\frac{b}{\sqrt{\alpha_1}\sqrt{s}}}) + k_2 \sqrt{\frac{s}{\alpha_2}} C e^{-\frac{b}{\sqrt{\alpha_2}\sqrt{s}}} = \frac{k_1 h_2(s)}{\sqrt{\mu_1}} \cdot \sqrt{\frac{s}{\alpha_1}} \cdot \frac{e^{+\frac{b}{\sqrt{\alpha_1}\sqrt{s}}}}{\sqrt{s}},$$

or, since $k_1/\sqrt{\alpha_1} = \sqrt{\mu_1}$,

$$13) A \sqrt{\mu_1}(-e^{-\frac{b}{\sqrt{\alpha_1}\sqrt{s}} + e^{+\frac{b}{\sqrt{\alpha_1}\sqrt{s}}}) + C \sqrt{\mu_2} e^{-\frac{b}{\sqrt{\alpha_2}\sqrt{s}}} = h_2(s) \frac{e^{+\frac{b}{\sqrt{\alpha_1}\sqrt{s}}}}{\sqrt{s}}.$$

Solving by determinants, let

$$\Delta = \begin{vmatrix} (e^{-\frac{b}{\sqrt{\alpha_1}\sqrt{s}} + e^{+\frac{b}{\sqrt{\alpha_1}\sqrt{s}}}) & -1 \\ \sqrt{\mu_1}(-e^{-\frac{b}{\sqrt{\alpha_1}\sqrt{s}} + e^{+\frac{b}{\sqrt{\alpha_1}\sqrt{s}}}) & \sqrt{\mu_2} \end{vmatrix} e^{-\frac{b}{\sqrt{\alpha_2}\sqrt{s}}}$$

or

$$\Delta = e^{-\frac{b}{\alpha_2} \sqrt{s}} \left[e^{-\frac{b}{\alpha_1} \sqrt{s}} (\sqrt{u_2} - \sqrt{u_1}) + e^{\frac{b}{\alpha_1} \sqrt{s}} (\sqrt{u_2} + \sqrt{u_1}) \right].$$

Then

$$\Delta \cdot A = h_a(s) \frac{e^{\frac{b}{\alpha_1} \sqrt{s}}}{\sqrt{s}} \begin{vmatrix} \frac{1}{\sqrt{u_1}} & -1 \\ 1 & \sqrt{u_2} \end{vmatrix} e^{-\frac{b}{\alpha_2} \sqrt{s}},$$

or

$$A = \frac{h_a(s) e^{\frac{b}{\alpha_1} \sqrt{s}} (1 + \sqrt{\frac{u_2}{u_1}})}{\sqrt{s} [e^{\frac{b}{\alpha_1} \sqrt{s}} (\sqrt{u_1} + \sqrt{u_2}) - e^{-\frac{b}{\alpha_1} \sqrt{s}} (\sqrt{u_1} - \sqrt{u_2})]} ;$$

while

$$\Delta \cdot C = \begin{vmatrix} (e^{-\frac{b}{\alpha_2} \sqrt{s}} + e^{\frac{b}{\alpha_2} \sqrt{s}}) & \frac{1}{\sqrt{u_1}} \\ \sqrt{u_1} (-e^{-\frac{b}{\alpha_2} \sqrt{s}} + e^{\frac{b}{\alpha_2} \sqrt{s}}) & 1 \end{vmatrix} h_a(s) \frac{e^{\frac{b}{\alpha_1} \sqrt{s}}}{\sqrt{s}}$$

or

$$C = \frac{2 h_a(s)}{\sqrt{s} e^{-\frac{b}{\alpha_2} \sqrt{s}} [e^{\frac{b}{\alpha_1} \sqrt{s}} (\sqrt{u_1} + \sqrt{u_2}) - e^{-\frac{b}{\alpha_1} \sqrt{s}} (\sqrt{u_1} - \sqrt{u_2})]}.$$

Substituting in (10) :

$$u(x,s) = \frac{h_a(s)}{\sqrt{u_1} \cdot \sqrt{s}} \left\{ \frac{(1 + \sqrt{\frac{u_2}{u_1}}) (e^{-\frac{b-x}{\alpha_1} \sqrt{s}} + e^{+\frac{b+x}{\alpha_1} \sqrt{s}})}{e^{\frac{b}{\alpha_1} \sqrt{s}} (1 + \sqrt{\frac{u_2}{u_1}}) - e^{-\frac{b}{\alpha_1} \sqrt{s}} (1 - \sqrt{\frac{u_2}{u_1}})} \right. \\ \left. - \frac{(1 + \sqrt{\frac{u_2}{u_1}}) e^{+\frac{b+x}{\alpha_1} \sqrt{s}} - (1 - \sqrt{\frac{u_2}{u_1}}) e^{-\frac{b-x}{\alpha_1} \sqrt{s}}}{e^{\frac{b}{\alpha_1} \sqrt{s}} (1 + \sqrt{\frac{u_2}{u_1}}) - e^{-\frac{b}{\alpha_1} \sqrt{s}} (1 - \sqrt{\frac{u_2}{u_1}})} \right\}$$

or

$$14) u(x,s) = \frac{h_a(s)}{\sqrt{u_1} \cdot \sqrt{s}} \left\{ \frac{e^{+\frac{b-x}{\alpha_1} \sqrt{s}} (1 + \sqrt{\frac{u_2}{u_1}}) + e^{-\frac{b-x}{\alpha_1} \sqrt{s}} (1 - \sqrt{\frac{u_2}{u_1}})}{e^{\frac{b}{\alpha_1} \sqrt{s}} (1 + \sqrt{\frac{u_2}{u_1}}) - e^{-\frac{b}{\alpha_1} \sqrt{s}} (1 - \sqrt{\frac{u_2}{u_1}})} \right\}, 0 \leq x < b.$$

Substituting the expression for C in (11):

$$15) u(x,s) = \frac{h(s)}{\mu_1 \cdot \sqrt{s}} \left\{ \frac{2 e^{-\frac{x-b}{\mu_2} \sqrt{s}}}{e^{\frac{b}{\mu_1} \sqrt{s}} (1 + \sqrt{\frac{\mu_2}{\mu_1}}) - e^{-\frac{b}{\mu_1} \sqrt{s}} (1 - \sqrt{\frac{\mu_2}{\mu_1}})} \right\}, \quad x > b.$$

Now, define a dimensionless constant λ as

$$\lambda = \frac{1 - \sqrt{\frac{\mu_2}{\mu_1}}}{1 + \sqrt{\frac{\mu_2}{\mu_1}}}.$$

Note that since $(1 - 2\sqrt{\frac{\mu_2}{\mu_1}} + \frac{\mu_2}{\mu_1}) < (1 + 2\sqrt{\frac{\mu_2}{\mu_1}} + \frac{\mu_2}{\mu_1})$,
or $(1 - \sqrt{\frac{\mu_2}{\mu_1}})^2 < (1 + \sqrt{\frac{\mu_2}{\mu_1}})^2$, then it follows that
 $\lambda^2 < 1$.

Utilizing this constant, equation (14) may be re-written as:

$$\begin{aligned} u(x,s) &= \frac{h(s)}{\mu_1 \cdot \sqrt{s}} \left\{ \frac{e^{\frac{b-x}{\mu_2} \sqrt{s}} + \lambda e^{-\frac{b-x}{\mu_2} \sqrt{s}}}{e^{\frac{b}{\mu_1} \sqrt{s}} (1 - \lambda e^{-\frac{2b}{\mu_1} \sqrt{s}})} \right\} \\ &= \frac{h(s)}{\mu_1 \cdot \sqrt{s}} \left\{ \frac{e^{-\frac{x}{\mu_2} \sqrt{s}} + \lambda e^{-\frac{2b-x}{\mu_2} \sqrt{s}}}{1 - \lambda e^{-\frac{2b}{\mu_1} \sqrt{s}}} \right\}. \end{aligned}$$

The development to this point is very similar to that presented by R. V. Churchill in Operational Mathematics in Engineering, pp. 122-124, with slight changes in two of the boundary conditions. The next step in the development

follows from the observation that since $|\lambda| < 1$, and $\sqrt{s/\alpha_1} > 0$, then

$$[1 - \lambda e^{-\frac{2b}{\alpha_1}\sqrt{s}}]^{-1} = \sum_{n=0}^{\infty} \lambda^n e^{-\frac{2bn}{\alpha_1}\sqrt{s}}.$$

Now, the exponential terms here involve the square root of a complex number, s , so this formal expansion must be verified. This is best done by verification of the formal solution in $U(x,t)$, obtained from the inverse of the expression in $u(x,s)$. See R. V. Churchill, Phil. Mag., Ser. 7, Vol. 31, pp. 81-87 (1941). Here, this expansion will be verified by demonstrating that the solution in $U(x,t)$ reduces to that for the isotropic opaque solid when $\lambda = 0$ (i.e. when $\mu_1 = \mu_2$). Note that the above expansion is valid for $\lambda = 0$, under the usual definition, $0^0 \equiv 1$.

Equation (14) may now be written:

$$u(x,s) = \frac{h_0(s)}{\sqrt{\mu_1}\sqrt{s}} \left\{ \sum_{n=0}^{\infty} \lambda^n e^{-\frac{2nb+x}{\alpha_1}\sqrt{s}} + \sum_{n=0}^{\infty} \lambda^{n+1} e^{-\frac{2b(n+1)-x}{\alpha_1}\sqrt{s}} \right\}$$

or

$$(16) \quad u(x,s) = \frac{h_0(s)}{\sqrt{\mu_1}} \cdot \frac{1}{\sqrt{s}} \left\{ e^{-\frac{x}{\alpha_1}\sqrt{s}} + \sum_{n=1}^{\infty} \lambda^n \left[e^{-\frac{2nb+x}{\alpha_1}\sqrt{s}} + e^{-\frac{2nb-x}{\alpha_1}\sqrt{s}} \right] \right\},$$

$0 \leq x \leq b.$

Similarly, equation (15) becomes

$$\begin{aligned}
 u(x,s) &= \frac{h_a(s)}{\sqrt{\mu_1} \cdot \sqrt{s}} \left\{ \frac{2 e^{-\frac{x-b}{\sqrt{\mu_2}} \sqrt{s}} \cdot e^{-\frac{b}{\sqrt{\mu_1}} \sqrt{s}}}{(1 + \sqrt{\frac{\mu_2}{\mu_1}})(1 - \lambda e^{-\frac{2b}{\sqrt{\mu_1}} \sqrt{s}})} \right\} \\
 &= \frac{2 h_a(s)}{\sqrt{\mu_1}(1 + \sqrt{\frac{\mu_2}{\mu_1}})} \cdot \frac{1}{\sqrt{s}} \left\{ e^{-\left(\frac{x-b}{\sqrt{\mu_2}} + \frac{b}{\sqrt{\mu_1}}\right) \sqrt{s}} \sum_{n=0}^{\infty} \lambda^n e^{-\frac{2nb}{\sqrt{\mu_1}} \sqrt{s}} \right\}, \quad x > b.
 \end{aligned}$$

$$\text{But } \frac{2}{1 + \sqrt{\frac{\mu_2}{\mu_1}}} = \frac{(1 + \sqrt{\mu_2/\mu_1}) + (1 - \sqrt{\mu_2/\mu_1})}{1 + \sqrt{\mu_2/\mu_1}} = 1 + \lambda;$$

hence

$$17) \quad u(x,s) = \frac{h_a(s)}{\sqrt{\mu_1}} \cdot \frac{1}{\sqrt{s}} \left\{ (1 + \lambda) \sum_{n=0}^{\infty} \lambda^n e^{-\left[\frac{x-b}{\sqrt{\mu_2}} + \frac{(2n+1)b}{\sqrt{\mu_1}}\right] \sqrt{s}} \right\}, \quad x > b$$

Equations (16) and (17) now involve only the form $[(s)^{-1/2} e^{-k\sqrt{s}}]$ ($k \geq 0$), the inverse transform of which may be found immediately (R.V. Churchill, Operational Mathematics in Engineering, p. 299, No. 84) as

$$\frac{e^{-\frac{k^2}{4t}}}{\sqrt{\pi t}}.$$

Thus, using the convolution, the inverse transform of (16) becomes:

$$\begin{aligned}
 U(x,t) &= \frac{1}{\sqrt{\pi \mu_1}} \int_0^t \frac{H_a(t-\tau)}{\sqrt{\tau}} \left\{ \exp\left(-\frac{x^2}{4\mu_1 \tau}\right) + \sum_{n=1}^{\infty} \lambda^n \left[\exp\left(-\frac{[2nb+x]^2}{4\mu_1 \tau}\right) \right. \right. \\
 &\quad \left. \left. + \exp\left(-\frac{[2nb-x]^2}{4\mu_1 \tau}\right) \right] \right\} d\tau, \quad 0 \leq x < b,
 \end{aligned}$$

while that of (17) becomes:

$$U(x, t) = \frac{1}{\sqrt{\pi \mu_1}} \int_0^t \frac{H_a(t-\tau)}{\sqrt{\tau}} \left\{ (1+\lambda) \sum_{n=0}^{\infty} \lambda^n \exp\left(-\frac{\left[\frac{x-b}{\sqrt{4\alpha_1\tau}} + \frac{(2n+1)b}{\sqrt{4\alpha_1\tau}}\right]^2}{4\tau}\right) \right\} d\tau, \\ x > b.$$

These unwieldy expressions could be handled by machine methods, but for present purposes it is preferable to specialize the problem by considering an irradiance step function:

$$H_a(t) = 0, \quad t < 0 \\ = H_a, \quad t \geq 0.$$

Then $h_a(s) = H_a/s$, and the terms in (16) and (17) take the form $[s^{-3/2} e^{-R\sqrt{s}}]$. The inverse of this transform is

$$\mathcal{L}^{-1}\left\{\frac{e^{-R\sqrt{s}}}{s^{3/2}}\right\} = 2\sqrt{\frac{t}{\pi}} \exp\left(-\frac{R^2}{4t}\right) - R \operatorname{erfc}\left(\frac{R}{2\sqrt{t}}\right) \\ = \frac{2}{\sqrt{\pi}} \sqrt{t} R\left(\frac{R}{\sqrt{4t}}\right),$$

where $R(x) = \sqrt{\pi} \left[\frac{1}{\sqrt{\pi}} e^{-x^2} - x \operatorname{erfc}(x) \right]$, as defined in Section 3.3. (See Appendix III.)

Thus equations (16) and (17) become:

$$18) \quad U^{\text{step}}(x, t) = \frac{2H_a}{\sqrt{\pi \mu_1}} \sqrt{t} \left\{ R\left(\frac{x}{\sqrt{4\alpha_1 t}}\right) + \sum_{n=1}^{\infty} \lambda^n \left[R\left(\frac{2nb+x}{\sqrt{4\alpha_1 t}}\right) \right. \right. \\ \left. \left. + R\left(\frac{2nb-x}{\sqrt{4\alpha_1 t}}\right) \right] \right\}, \quad 0 \leq x < b,$$

and

$$19) U^{step}(x,t) = \frac{2H_0}{\sqrt{\pi\mu_1}} \sqrt{t} (1+\lambda) \sum_{n=0}^{\infty} \lambda^n R\left(\frac{x-b}{\sqrt{4\alpha_2 t}} + \frac{(2n+1)b}{\sqrt{4\alpha_1 t}}\right), \quad x > b.$$

Note that if $\lambda = 0$ — in which case $\alpha_1 = \alpha_2 = \alpha$ — equations (18) and (19) both reduce to

$$U^{step}(x,t) \Big|_{\lambda=0} = \frac{2H_0}{\sqrt{\pi\mu}} \sqrt{t} R\left(\frac{x}{\sqrt{4\alpha t}}\right),$$

which is the correct expression for the isotropic opaque solid. Also, if $b \rightarrow \infty$, equation (18) reduces to

$$U^{step}(x,t) \Big|_{b \rightarrow \infty} = \frac{2H_0}{\sqrt{\pi\mu_1}} \sqrt{t} R\left(\frac{x}{\sqrt{4\alpha_1 t}}\right), \quad x \geq 0,$$

since $R(\infty) = 0$.

Consider now the case where $x = 0$. From (18)

$$20) U^{step}(0,t) = \frac{2H_0}{\sqrt{\pi\mu_1}} \sqrt{t} \left[1 + 2 \sum_{n=1}^{\infty} \lambda^n R\left(\frac{2nb}{\sqrt{4\alpha_1 t}}\right) \right].$$

As stated above

$$\sqrt{t} R\left(\frac{b}{\sqrt{4t}}\right) = \frac{\sqrt{\pi}}{2} \mathcal{L}^{-1} \left\{ \frac{e^{-b\sqrt{s}}}{s^{3/2}} \right\}.$$

But

$$\begin{aligned} \frac{d}{dt} \left[\sqrt{t} R\left(\frac{b}{\sqrt{4t}}\right) \right] &= \frac{\sqrt{\pi}}{2} \mathcal{L}^{-1} \left\{ s \cdot \frac{e^{-b\sqrt{s}}}{s^{3/2}} \right\} = \frac{\sqrt{\pi}}{2} \mathcal{L}^{-1} \left\{ \frac{e^{-b\sqrt{s}}}{\sqrt{s}} \right\} \\ &= \frac{\sqrt{\pi}}{2} \cdot \frac{e^{-\frac{b^2}{4t}}}{\sqrt{\pi t}}, \end{aligned}$$

$$\text{while } \frac{d}{d\sqrt{t}} \left[\sqrt{t} R\left(\frac{b}{\sqrt{4t}}\right) \right] = 2\sqrt{t} \frac{d}{dt} \left[\sqrt{t} R\left(\frac{b}{\sqrt{4t}}\right) \right] = e^{-\frac{b^2}{4t}}.$$

Hence,

$$21) \frac{dU^{\text{step}}(0,t)}{d\sqrt{t}} = \frac{2Ha}{\sqrt{\pi\mu_1}} \left[1 + 2 \sum_{n=1}^{\infty} \lambda^n e^{-\frac{n^2 b^2}{4t}} \right],$$

which is identical to equation 3-27 of Chapter III.

The two limiting cases there derived —

$$\frac{dU^{\text{step}}(0,0)}{d\sqrt{t}} = \frac{2Ha}{\sqrt{\pi\mu_1}}$$

and

$$\frac{dU^{\text{step}}(0,\infty)}{d\sqrt{t}} = \frac{2Ha}{\sqrt{\pi\mu_1}} \quad \text{—}$$

seem so eminently reasonable, that the formal verification of equations (18) and (19) was not made.

This, in the favorite text-book phrase, will be left as an exercise for the reader.

APPENDIX V

Solution of the Diathermanous Solid Model,
Double Exponential Absorption

Statement of the problem:

- 1) $U_t(x, t) = \alpha U_{xx}(x, t) + \frac{\gamma_1}{\nu} H_0(t) e^{-r_1 x}, 0 \leq x < b, t \geq 0$
- 2) $U_t(x, t) = \alpha U_{xx}(x, t) + \frac{\gamma_2}{\nu} H_0(t) e^{-r_1 b} e^{-r_2(x-b)}, x > b, t \geq 0$
- 3) $U(x, 0) = 0, x \geq 0$
- 4) $U(\infty, t) = 0, t \geq 0$
- 5) $U_x(0, t) = 0, t \geq 0$
- 6) $U(b-0, t) = U(b+0, t), t \geq 0$
- 7) $U_x(b-0, t) = U_x(b+0, t), t \geq 0$

Taking $\mathcal{L}_t\{ \}$, and using (2):

- 1') $s U(x, s) = \alpha \frac{d^2 U(x, s)}{dx^2} + \frac{\gamma_1}{\nu} h_0(s) e^{-r_1 x}, 0 \leq x < b$
- 2') $s U(x, s) = \alpha \frac{d^2 U(x, s)}{dx^2} + \frac{\gamma_2}{\nu} h_0(s) e^{-r_1 b} e^{-r_2(x-b)}, x > b$
- 4') $U(\infty, s) = 0$
- 5') $\frac{dU(0, s)}{dx} = 0$
- 6') $U(b-0, s) = U(b+0, s)$
- 7') $\frac{dU(b-0, s)}{dx} = \frac{dU(b+0, s)}{dx}$

The homogeneous solution of (1') is

$$U_H(x, s) = A e^{-\sqrt{\frac{s}{\alpha}} x} + B e^{+\sqrt{\frac{s}{\alpha}} x},$$

to which must be added a particular integral. Assume this has the form

$$u_p(x, s) = M e^{-r_1 x}.$$

Then $\frac{d^2 u_p(x, s)}{dx^2} = r_1^2 M e^{-r_1 x}$, and substituting in (1') :

$$s M e^{-r_1 x} = r_1^2 \alpha M e^{-r_1 x} + \frac{r_1}{\gamma} h_a(s) e^{-r_1 x}$$

or
$$M = \frac{r_1 h_a(s)}{\gamma(s - r_1^2 \alpha)}.$$

The complete solution of (1') is thus :

$$8) u(x, s) = A e^{-\sqrt{\frac{s}{\alpha}} x} + B e^{+\sqrt{\frac{s}{\alpha}} x} + \frac{r_1 h_a(s) e^{-r_1 x}}{\gamma(s - r_1^2 \alpha)}, \quad 0 \leq x \leq b.$$

For equation (2') the particular integral will have the form

$$u_p(x, s) = N e^{-r_2(x-b)};$$

hence

$$s N e^{-r_2(x-b)} = r_2^2 \alpha N e^{-r_2(x-b)} + \frac{r_2}{\gamma} h_a(s) e^{-r_1 b} e^{-r_2(x-b)}$$

and
$$N = \frac{r_2 h_a(s) e^{-r_1 b}}{\gamma(s - r_2^2 \alpha)}.$$

The complete solution of (2') is, then :

$$9) u(x, s) = C e^{-\sqrt{\frac{s}{\alpha}} x} + D e^{+\sqrt{\frac{s}{\alpha}} x} + \frac{r_2 h_a(s) e^{-r_1 b} e^{-r_2(x-b)}}{\gamma(s - r_2^2 \alpha)}, \quad x > b.$$

Differentiating (8) and applying (5') :

$$0 = -\sqrt{\frac{s}{\alpha}} A + \sqrt{\frac{s}{\alpha}} B - \frac{r_1^2 h_a(s)}{\gamma(s - r_1^2 \alpha)}$$

or

$$B = A + \sqrt{\frac{\alpha}{s}} \cdot \frac{r_1^2 h_a(s)}{\gamma(s - r_1^2 \alpha)}.$$

Substituting in (8):

$$10) u(x,s) = A(e^{-\frac{1}{2}\sqrt{s}x} + e^{+\frac{1}{2}\sqrt{s}x}) + \frac{r_1 h(s)}{\sqrt{s(s-r_1^2\alpha)}} (e^{-r_1 x} + \sqrt{\frac{\alpha}{s}} r_1 e^{+\frac{1}{2}\sqrt{s}x}),$$

$0 \leq x < b.$

Now, from (4') it follows that, in equation (9), $D \equiv 0$.

Thus

$$11) u(x,s) = C e^{-\frac{1}{2}\sqrt{s}x} + \frac{r_2 h(s) e^{-r_2 b} e^{-r_2(x-b)}}{\sqrt{s(s-r_2^2\alpha)}}, \quad x > b.$$

The two constants A and C are now to be evaluated by applying the continuity conditions (6') and (7'). From (6'):

$$12) A(e^{-\frac{1}{2}\sqrt{s}b} + e^{+\frac{1}{2}\sqrt{s}b}) - C e^{-\frac{1}{2}\sqrt{s}b}$$

$$= \frac{h(s)}{\sqrt{s}} \left[-\frac{r_1 e^{-r_1 b}}{s-r_1^2\alpha} - \sqrt{\frac{\alpha}{s}} \frac{r_1 e^{+\frac{1}{2}\sqrt{s}b}}{s-r_1^2\alpha} + \frac{r_2 e^{-r_2 b}}{s-r_2^2\alpha} \right]$$

and from (7'):

$$13) A(-e^{-\frac{1}{2}\sqrt{s}b} + e^{+\frac{1}{2}\sqrt{s}b}) + C e^{-\frac{1}{2}\sqrt{s}b}$$

$$= \frac{h(s)}{\sqrt{s}} \sqrt{\frac{\alpha}{s}} \left[\frac{r_1^2 e^{-r_1 b}}{s-r_1^2\alpha} - \frac{r_1^2 e^{+\frac{1}{2}\sqrt{s}b}}{s-r_1^2\alpha} - \frac{r_2^2 e^{-r_2 b}}{s-r_2^2\alpha} \right].$$

Solving by determinants, let

$$\Delta = \begin{vmatrix} (e^{-\frac{1}{2}\sqrt{s}b} + e^{+\frac{1}{2}\sqrt{s}b}) & -1 \\ (-e^{-\frac{1}{2}\sqrt{s}b} + e^{+\frac{1}{2}\sqrt{s}b}) & +1 \end{vmatrix} e^{-\frac{1}{2}\sqrt{s}b}$$

$$= 2.$$

Then,

$$\begin{aligned}
 A &= \frac{h_0(s)}{2V} \sqrt{\frac{\alpha}{s}} \left[\left[\left(\frac{\sqrt{\frac{\alpha}{s}} r_1}{s-r_1^2 \alpha} + \frac{\sqrt{\frac{\alpha}{s}} r_2}{s-r_2^2 \alpha} \right) e^{-nb} - \frac{r_1^2 e^{\frac{b}{\sqrt{\alpha}} \sqrt{s}}}{s-r_1^2 \alpha} \right] - 1 \right] e^{-\frac{b}{\sqrt{\alpha}} \sqrt{s}} \\
 &\quad + \left[\left(\frac{r_1^2}{s-r_1^2 \alpha} - \frac{r_2^2}{s-r_2^2 \alpha} \right) e^{-nb} - \frac{r_1^2 e^{\frac{b}{\sqrt{\alpha}} \sqrt{s}}}{s-r_1^2 \alpha} \right] + 1 \right] e^{-\frac{b}{\sqrt{\alpha}} \sqrt{s}} \\
 &= \frac{h_0(s)}{2V} \sqrt{\frac{\alpha}{s}} e^{-\frac{b}{\sqrt{\alpha}} \sqrt{s}} \left\{ \left[\frac{r_1(r_1 - \sqrt{\frac{\alpha}{s}})}{s-r_1^2 \alpha} - \frac{r_2(r_2 - \sqrt{\frac{\alpha}{s}})}{s-r_2^2 \alpha} \right] e^{-nb} - \frac{2r_1^2 e^{\frac{b}{\sqrt{\alpha}} \sqrt{s}}}{s-r_1^2 \alpha} \right\} \\
 &= \frac{h_0(s)}{2V} \left\{ \left[-\frac{r_1(\sqrt{s} - r_1 \sqrt{\alpha})}{\sqrt{s}(s-r_1^2 \alpha)} + \frac{r_2(\sqrt{s} - r_2 \sqrt{\alpha})}{\sqrt{s}(s-r_2^2 \alpha)} \right] e^{-nb} e^{-\frac{b}{\sqrt{\alpha}} \sqrt{s}} - \frac{2\sqrt{\frac{\alpha}{s}} r_1^2}{s-r_1^2 \alpha} \right\};
 \end{aligned}$$

and

$$\begin{aligned}
 C &= \frac{h_0(s)}{2V} \sqrt{\frac{\alpha}{s}} \left| \begin{array}{cc} (e^{-\frac{b}{\sqrt{\alpha}} \sqrt{s}} + e^{\frac{b}{\sqrt{\alpha}} \sqrt{s}}) & \left[\left(\frac{\sqrt{\frac{\alpha}{s}} r_1}{s-r_1^2 \alpha} + \frac{\sqrt{\frac{\alpha}{s}} r_2}{s-r_2^2 \alpha} \right) e^{-nb} - \frac{r_1^2 e^{\frac{b}{\sqrt{\alpha}} \sqrt{s}}}{s-r_1^2 \alpha} \right] \\ (-e^{-\frac{b}{\sqrt{\alpha}} \sqrt{s}} + e^{\frac{b}{\sqrt{\alpha}} \sqrt{s}}) & \left[\left(\frac{r_1^2}{s-r_1^2 \alpha} - \frac{r_2^2}{s-r_2^2 \alpha} \right) e^{-nb} - \frac{r_1^2 e^{\frac{b}{\sqrt{\alpha}} \sqrt{s}}}{s-r_1^2 \alpha} \right] \end{array} \right| \\
 &= \frac{h_0(s)}{2V} \left\{ \left[\left(-\frac{r_1(\sqrt{s} - r_1 \sqrt{\alpha})}{\sqrt{s}(s-r_1^2 \alpha)} + \frac{r_2(\sqrt{s} - r_2 \sqrt{\alpha})}{\sqrt{s}(s-r_2^2 \alpha)} \right) e^{-nb} \right. \right. \\
 &\quad \left. \left. + \left(\frac{r_1(\sqrt{s} + r_1 \sqrt{\alpha})}{\sqrt{s}(s-r_1^2 \alpha)} - \frac{r_2(\sqrt{s} + r_2 \sqrt{\alpha})}{\sqrt{s}(s-r_2^2 \alpha)} \right) e^{\frac{b}{\sqrt{\alpha}} \sqrt{s}} \right] e^{-nb} - \frac{2\sqrt{\frac{\alpha}{s}} r_1^2}{s-r_1^2 \alpha} \right\}
 \end{aligned}$$

Substituting for A in (10):

$$\begin{aligned}
 u(x,s) &= \frac{h_0(s)}{2V} \left\{ \left[\frac{r_2(\sqrt{s} - r_2 \sqrt{\alpha})}{\sqrt{s}(s-r_2^2 \alpha)} - \frac{r_1(\sqrt{s} - r_1 \sqrt{\alpha})}{\sqrt{s}(s-r_1^2 \alpha)} \right] \left(e^{-\frac{b+x}{\sqrt{\alpha}} \sqrt{s}} + e^{-\frac{b-x}{\sqrt{\alpha}} \sqrt{s}} \right) e^{-nb} \right. \\
 &\quad \left. - 2\sqrt{\frac{\alpha}{s}} \left(\frac{r_1^2}{s-r_1^2 \alpha} \right) \left(e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}} + e^{\frac{x}{\sqrt{\alpha}} \sqrt{s}} \right) + \frac{2r_1 e^{-r_1 x}}{s-r_1^2 \alpha} + 2\sqrt{\frac{\alpha}{s}} \left(\frac{r_1^2}{s-r_1^2 \alpha} \right) e^{\frac{x}{\sqrt{\alpha}} \sqrt{s}} \right\}
 \end{aligned}$$

or

$$14) \quad u(x,s) = \frac{h_a(s)}{v} \left\{ \frac{r_1 e^{-r_1 x}}{s - r_1^2 \alpha} - \frac{r_1^2 \sqrt{\alpha} e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}}}{\sqrt{s}(s - r_1^2 \alpha)} \right. \\ \left. + \frac{e^{-r_1 b}}{2} \left(e^{-\frac{b+x}{\sqrt{\alpha}} \sqrt{s}} + e^{-\frac{b-x}{\sqrt{\alpha}} \sqrt{s}} \right) \left(\frac{r_2}{\sqrt{s}(\sqrt{s} + r_2 \sqrt{\alpha})} - \frac{r_1}{\sqrt{s}(\sqrt{s} + r_1 \sqrt{\alpha})} \right) \right\}$$

$$0 \leq x < b.$$

Similarly, substituting for c in (N):

$$15) \quad u(x,s) = \frac{h_a(s)}{v} \left\{ \frac{r_1 e^{-r_1 b} e^{-r_1(x-b)}}{s - r_1^2 \alpha} - \frac{r_1^2 \sqrt{\alpha} e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}}}{\sqrt{s}(s - r_1^2 \alpha)} \right. \\ \left. + \frac{e^{-r_1 b}}{2} \left[\left(\frac{r_2}{\sqrt{s}(\sqrt{s} + r_2 \sqrt{\alpha})} - \frac{r_1}{\sqrt{s}(\sqrt{s} + r_1 \sqrt{\alpha})} \right) e^{-\frac{x+b}{\sqrt{\alpha}} \sqrt{s}} \right. \right. \\ \left. \left. - \left(\frac{r_2}{\sqrt{s}(\sqrt{s} - r_2 \sqrt{\alpha})} - \frac{r_1}{\sqrt{s}(\sqrt{s} - r_1 \sqrt{\alpha})} \right) e^{-\frac{x-b}{\sqrt{\alpha}} \sqrt{s}} \right] \right\},$$

$$x > b.$$

Equations (14) and (15) are the complete solutions in $u(x,s)$. To approach the inverse transform, we turn immediately to the special case where $H_a(t)$ is a step function

$$H_a(t) = 0, \quad t < 0$$

$$= H_a, \quad t \geq 0.$$

Thus $h_a(s) = H_a/s$, and noting that

$$-\frac{r_1^2 \sqrt{\alpha} e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}}}{\sqrt{s}(s - r_1^2 \alpha)} = \frac{1}{2} \frac{r_1 e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}}}{\sqrt{s}(\sqrt{s} + r_1 \sqrt{\alpha})} - \frac{1}{2} \frac{r_1 e^{-\frac{x}{\sqrt{\alpha}} \sqrt{s}}}{\sqrt{s}(\sqrt{s} - r_1 \sqrt{\alpha})},$$

then (14) and (15) may be written as:

$$16) u(x,s) = \frac{Ha}{2\nu} \left\{ \frac{2r_1 e^{-r_1 x}}{s(s-r_1^2 a)} + \frac{r_1 e^{-\frac{x-b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}+r_1\sqrt{a})} - \frac{r_1 e^{-\frac{x-b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}-r_1\sqrt{a})} \right. \\ \left. - e^{-r_1 b} \frac{r_1 e^{-\frac{x+b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}+r_1\sqrt{a})} - e^{-r_1 b} \frac{r_1 e^{-\frac{x+b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}-r_1\sqrt{a})} \right. \\ \left. + e^{-r_1 b} \frac{r_2 e^{-\frac{x+b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}+r_2\sqrt{a})} + e^{-r_1 b} \frac{r_2 e^{-\frac{x+b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}-r_2\sqrt{a})} \right\} \\ 0 \leq x < b,$$

and

$$17) u(x,s) = \frac{Ha}{2\nu} \left\{ \frac{2r_2 e^{-r_2(x-b)}}{s(s-r_2^2 a)} + \frac{r_1 e^{-\frac{x-b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}+r_1\sqrt{a})} - \frac{r_1 e^{-\frac{x-b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}-r_1\sqrt{a})} \right. \\ \left. - e^{-r_1 b} \frac{r_1 e^{-\frac{x+b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}+r_1\sqrt{a})} + e^{-r_1 b} \frac{r_1 e^{-\frac{x+b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}-r_1\sqrt{a})} \right. \\ \left. + e^{-r_1 b} \frac{r_2 e^{-\frac{x+b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}+r_2\sqrt{a})} - e^{-r_1 b} \frac{r_2 e^{-\frac{x+b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}-r_2\sqrt{a})} \right\}, \\ x > b.$$

Now, we attack (16) term by term.

1st term:

$$\mathcal{L}^{-1} \left\{ \frac{2r_1 e^{-r_1 x}}{s(s-r_1^2 a)} \right\} = 2r_1 e^{-r_1 x} \mathcal{L}^{-1} \left\{ \frac{1}{s} \cdot \frac{1}{s-r_1^2 a} \right\} \\ = 2r_1 e^{-r_1 x} \int_0^+ e^{r_1^2 a \tau} d\tau \\ = \frac{2}{r_1 a} e^{-r_1 x} (e^{-r_1^2 a t} - 1)$$

The remaining terms all involve the form $\frac{e^{-\frac{x-b}{\sqrt{a}} \sqrt{s}}}{s\sqrt{s}(\sqrt{s}+a)}$.

A partial fraction expansion gives

$$\frac{e^{-r\sqrt{s}}}{s\sqrt{s}(\sqrt{s}+a)} = \frac{e^{-r\sqrt{s}}}{a^2\sqrt{s}(\sqrt{s}+a)} - \frac{e^{-r\sqrt{s}}}{a^2s} + \frac{e^{-r\sqrt{s}}}{a s^{3/2}}.$$

Thus, for the 2nd term:

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{r_1 e^{-\frac{x}{\sqrt{s}}\sqrt{s}}}{s\sqrt{s}(\sqrt{s}+r_1\sqrt{a})} \right\} &= \frac{e^{-r_1 x} e^{r_1^2 a t}}{r_1 \alpha} \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t} \right) - \frac{\operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} \right)}{r_1 \alpha} \\ &\quad + \frac{2}{\alpha} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{x^2}{4\alpha t}} - \frac{x}{\alpha} \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} \right). \end{aligned}$$

3rd term:

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{r_1 e^{-\frac{x}{\sqrt{s}}\sqrt{s}}}{s\sqrt{s}(\sqrt{s}-r_1\sqrt{a})} \right\} &= \frac{e^{-r_1 x} e^{r_1^2 a t}}{r_1 \alpha} \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} - r_1 \sqrt{\alpha t} \right) - \frac{\operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} \right)}{r_1 \alpha} \\ &\quad - \frac{2}{\alpha} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{x^2}{4\alpha t}} + \frac{x}{\alpha} \operatorname{erfc} \left(\frac{x}{2\sqrt{\alpha t}} \right). \end{aligned}$$

4th term:

$$\begin{aligned} \mathcal{L}^{-1} \left\{ e^{-r_1 b} \frac{r_1 e^{-\frac{b+x}{\sqrt{s}}\sqrt{s}}}{s\sqrt{s}(\sqrt{s}+r_1\sqrt{a})} \right\} &= \frac{e^{-r_1 b}}{r_1 \alpha} e^{r_1(b+x)} e^{r_1^2 a t} \operatorname{erfc} \left(\frac{b+x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t} \right) \\ &\quad - \frac{e^{-r_1 b}}{r_1 \alpha} \operatorname{erfc} \left(\frac{b+x}{2\sqrt{\alpha t}} \right) + \frac{2e^{-r_1 b}}{\alpha} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(b+x)^2}{4\alpha t}} \\ &\quad - \frac{e^{-r_1 b} (b+x)}{\alpha} \operatorname{erfc} \left(\frac{b+x}{2\sqrt{\alpha t}} \right). \end{aligned}$$

5th term:

$$\begin{aligned} \mathcal{L}^{-1} \left\{ e^{-r_1 b} \frac{r_1 e^{-\frac{b-x}{\sqrt{s}}\sqrt{s}}}{s\sqrt{s}(\sqrt{s}+r_1\sqrt{a})} \right\} &= \frac{e^{-r_1 b}}{r_1 \alpha} e^{r_1(b-x)} e^{r_1^2 a t} \operatorname{erfc} \left(\frac{b-x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t} \right) \\ &\quad - \frac{e^{-r_1 b}}{r_1 \alpha} \operatorname{erfc} \left(\frac{b-x}{2\sqrt{\alpha t}} \right) + \frac{2e^{-r_1 b}}{\alpha} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(b-x)^2}{4\alpha t}} \\ &\quad - \frac{e^{-r_1 b} (b-x)}{\alpha} \operatorname{erfc} \left(\frac{b-x}{2\sqrt{\alpha t}} \right). \end{aligned}$$

The 6th and 7th terms are obvious from the 4th and 5th above. The inverse of (16) can now be written out as:

$$\begin{aligned}
 U(x,t) = \frac{Ha}{2\sqrt{\alpha}} \bigg\{ & \frac{2}{r_1} e^{-r_1 x} e^{r_1^2 \alpha t} - \frac{2}{r_1} e^{-r_1 x} + \frac{e^{r_1 x} e^{r_1^2 \alpha t}}{r_1} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t}\right) \\
 & - \frac{1}{r_1} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) + 4\sqrt{\frac{\alpha t}{\pi}} e^{-\frac{x^2}{4\alpha t}} - 2x \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \\
 & - \frac{e^{-r_1 x} e^{r_1^2 \alpha t}}{r_1} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} - r_1 \sqrt{\alpha t}\right) + \frac{1}{r_1} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \\
 & - \frac{e^{-r_1 b} e^{r_1(b-x)} e^{r_1^2 \alpha t}}{r_1} \operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t}\right) \\
 & + \frac{e^{-r_1 b}}{r_1} \operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}}\right) - 2e^{-r_1 b} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(b+x)^2}{4\alpha t}} \\
 & + e^{-r_1 b} (b+x) \operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}}\right) + \frac{e^{-r_1 b}}{r_1} \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}}\right) \\
 & - \frac{e^{-r_1 b} e^{r_1(b-x)} e^{r_1^2 \alpha t}}{r_1} \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t}\right) \\
 & - 2e^{-r_1 b} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(b-x)^2}{4\alpha t}} + e^{-r_1 b} (b-x) \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}}\right) \\
 & + \frac{e^{-r_2 b} e^{r_2(b+x)} e^{r_2^2 \alpha t}}{r_2} \operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}} + r_2 \sqrt{\alpha t}\right) - \frac{e^{-r_2 b}}{r_2} \operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}}\right) \\
 & + 2e^{-r_2 b} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(b+x)^2}{4\alpha t}} - e^{-r_2 b} (b+x) \operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}}\right) \\
 & + \frac{e^{-r_2 b} e^{r_2(b-x)} e^{r_2^2 \alpha t}}{r_2} \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}} + r_2 \sqrt{\alpha t}\right) - \frac{e^{-r_2 b}}{r_2} \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}}\right) \\
 & + 2e^{-r_2 b} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(b-x)^2}{4\alpha t}} - e^{-r_2 b} (b-x) \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}}\right) \bigg\} \\
 & \quad 0 \leq x < b.
 \end{aligned}$$

This can be cleared up extensively to give:

$$\begin{aligned}
 18) \quad U(x,t) = & \frac{H_0}{k} \left\{ 2\sqrt{\frac{\alpha t}{\pi}} e^{-\frac{x^2}{4\alpha t}} - x \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \frac{e^{-n x}}{r_1} \right. \\
 & + \frac{e^{n_1 x t}}{2r_1} \left[e^{r_1 x} \operatorname{erfc}\left(r_1 \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) + e^{-n x} \operatorname{erfc}\left(r_1 \sqrt{\alpha t} - \frac{x}{2\sqrt{\alpha t}}\right) \right] \\
 & + \frac{e^{-n b}}{2} \left[\left(\operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}}\right) + \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}}\right) \right) \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right. \\
 & \quad - \frac{e^{r_1(b+x)} e^{n_1 x t}}{r_1} \operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t}\right) \\
 & \quad + \frac{e^{r_2(b+x)} e^{n_2 x t}}{r_2} \operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}} + r_2 \sqrt{\alpha t}\right) \\
 & \quad - \frac{e^{r_1(b-x)} e^{n_1 x t}}{r_1} \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t}\right) \\
 & \quad \left. \left. + \frac{e^{r_2(b-x)} e^{n_2 x t}}{r_2} \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}} + r_2 \sqrt{\alpha t}\right) \right] \right\} \\
 & \quad 0 \leq x < b,
 \end{aligned}$$

where the identity

$$\operatorname{erfc}(x) \equiv 2 - \operatorname{erfc}(-x)$$

has been employed.

The inverse of (17) may now be written out by inspection as:

$$\begin{aligned}
 U(x,t) = & \frac{H_0}{2\sqrt{\alpha}} \left\{ 2 \frac{e^{-n b} e^{-r_2(x-b)}}{r_2} e^{-n_2 x t} - \frac{2 e^{-n b} e^{-r_2(x-b)}}{r_2} - \frac{1}{r_1} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \right. \\
 & + \frac{e^{n x} e^{n_1 x t}}{r_1} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t}\right) + 4\sqrt{\frac{\alpha t}{\pi}} e^{-\frac{x^2}{4\alpha t}} \\
 & - 2x \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \frac{e^{-n x} e^{n_1 x t}}{r_1} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}} - r_1 \sqrt{\alpha t}\right) \\
 & + \frac{1}{r_1} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \frac{e^{-n b} e^{n(r_1 b)} e^{n_1 x t}}{r_1} \operatorname{erfc}\left(\frac{x+b}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t}\right) \\
 & \quad \quad \quad (\text{cont})
 \end{aligned}$$

$$\begin{aligned}
& \dots + \frac{e^{-r_1 b}}{r_1} \operatorname{erfc}\left(\frac{x+b}{2\sqrt{\alpha t}}\right) - 2e^{-r_1 b} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(x+b)^2}{4\alpha t}} \\
& + e^{-r_1 b} (x+b) \operatorname{erfc}\left(\frac{x+b}{2\sqrt{\alpha t}}\right) - \frac{e^{-r_1 b}}{r_1} \operatorname{erfc}\left(\frac{x+b}{2\sqrt{\alpha t}}\right) \\
& + \frac{e^{-r_1 b} e^{-r_2(x+b)} e^{r_2^2 \alpha t}}{r_2} \operatorname{erfc}\left(\frac{x+b}{2\sqrt{\alpha t}} + r_2 \sqrt{\alpha t}\right) \\
& + 2e^{-r_1 b} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(x+b)^2}{4\alpha t}} - e^{-r_1 b} (x+b) \operatorname{erfc}\left(\frac{x+b}{2\sqrt{\alpha t}}\right) \\
& + \frac{e^{-r_1 b} e^{-r_1(x-b)} e^{r_1^2 \alpha t}}{r_1} \operatorname{erfc}\left(\frac{x-b}{2\sqrt{\alpha t}} - r_1 \sqrt{\alpha t}\right) - \frac{e^{-r_1 b}}{r_1} \operatorname{erfc}\left(\frac{x-b}{2\sqrt{\alpha t}}\right) \\
& - 2e^{-r_1 b} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(x-b)^2}{4\alpha t}} + e^{-r_1 b} (x-b) \operatorname{erfc}\left(\frac{x-b}{2\sqrt{\alpha t}}\right) \\
& - \frac{e^{-r_1 b} e^{-r_2(x+b)} e^{r_2^2 \alpha t}}{r_2} \operatorname{erfc}\left(\frac{x-b}{2\sqrt{\alpha t}} - r_2 \sqrt{\alpha t}\right) + \frac{e^{-r_1 b}}{r_2} \operatorname{erfc}\left(\frac{x-b}{2\sqrt{\alpha t}}\right) \\
& + 2e^{-r_1 b} \sqrt{\frac{\alpha t}{\pi}} e^{-\frac{(x-b)^2}{4\alpha t}} - e^{-r_1 b} (x-b) \operatorname{erfc}\left(\frac{x-b}{2\sqrt{\alpha t}}\right) \Big\} \\
& \quad x > b.
\end{aligned}$$

Again, considerable canceling and regrouping can be done to give:

$$\begin{aligned}
19) \quad U(x,t) &= \frac{H\alpha}{k} \left\{ 2\sqrt{\frac{\alpha t}{\pi}} e^{-\frac{x^2}{4\alpha t}} - x \operatorname{erfc}(x) - \frac{e^{-(r_1-r_2)b} e^{-r_2 x}}{r_2} - \left(\frac{1}{r_1} - \frac{1}{r_2}\right) e^{-r_1 b} \right. \\
& + \frac{e^{r_1^2 \alpha t}}{2r_1} \left[e^{r_1 x} \operatorname{erfc}\left(r_1 \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}\right) + e^{-r_1 x} \operatorname{erfc}\left(r_1 \sqrt{\alpha t} - \frac{x}{2\sqrt{\alpha t}}\right) \right] \\
& + \frac{e^{-r_1 b}}{2} \left[\left(\operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}}\right) + \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}}\right) \right) \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \right. \\
& \quad \left. - \frac{e^{r_1(b+x)} e^{r_1^2 \alpha t}}{r_1} \operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t}\right) \right] \\
& \quad \quad \quad (\text{can't})
\end{aligned}$$

$$\begin{aligned}
& \dots + \frac{e^{r_2(b+x)} e^{r_2^2 \alpha t}}{r_2} \operatorname{erfc}\left(\frac{b+x}{2\sqrt{\alpha t}} + r_2 \sqrt{\alpha t}\right) \\
& - \frac{e^{r_1(b-x)} e^{r_1^2 \alpha t}}{r_1} \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}} + r_1 \sqrt{\alpha t}\right) \\
& + \frac{e^{r_2(b-x)} e^{r_2^2 \alpha t}}{r_2} \operatorname{erfc}\left(\frac{b-x}{2\sqrt{\alpha t}} + r_2 \sqrt{\alpha t}\right) \Bigg\}, \\
& x > b.
\end{aligned}$$

Now, we define the dimensionless variables

$$\xi = r_1 x, \quad \lambda_1 = r_1 b,$$

$$\theta = r_1^2 \alpha t, \quad \lambda_2 = r_1 / r_2,$$

$$\text{and } \Psi(\xi, \theta) = \frac{k r_1 U(\xi / r_1, \theta / r_1^2 \alpha)}{H_a},$$

whereupon (18) and (19) may immediately be re-written as:

$$\begin{aligned}
20) \quad \Psi(\xi, \theta) &= \frac{2}{\sqrt{\pi}} \sqrt{\theta} e^{-\frac{\xi^2}{4\theta}} - \xi \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\theta}}\right) - e^{-\xi} \\
&+ \frac{e^{\theta}}{2} \left[e^{\xi} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\xi}{2\sqrt{\theta}}\right) + e^{-\xi} \operatorname{erfc}\left(\sqrt{\theta} - \frac{\xi}{2\sqrt{\theta}}\right) \right] \\
&+ \frac{e^{-\lambda_1}}{2} \left[\left(\operatorname{erfc}\left(\frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) + \operatorname{erfc}\left(\frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) \right) (1 - \lambda_2) \right. \\
&\quad \left. - e^{\lambda_1 + \xi} e^{\theta} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) + \lambda_2 e^{\frac{\lambda_1 + \xi}{\lambda_2}} e^{\frac{\theta}{\lambda_2^2}} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) \right. \\
&\quad \left. - e^{\lambda_1 - \xi} e^{\theta} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) + \lambda_2 e^{\frac{\lambda_1 - \xi}{\lambda_2}} e^{\frac{\theta}{\lambda_2^2}} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) \right] \\
&0 \leq \xi < \lambda_1
\end{aligned}$$

and

$$\begin{aligned}
21) \Psi(\xi, \theta) = & \frac{2}{\sqrt{\pi}} e^{-\frac{\xi^2}{2\theta}} - \xi \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\theta}}\right) - e^{-\lambda_1} \left[\lambda_2 \left(e^{\frac{\lambda_1 - \xi}{\lambda_2}} - 1 \right) + 1 \right] \\
& + \frac{e^{\theta}}{2} \left[e^{\xi} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\xi}{2\sqrt{\theta}}\right) + e^{-\xi} \operatorname{erfc}\left(\sqrt{\theta} - \frac{\xi}{2\sqrt{\theta}}\right) \right] \\
& + \frac{e^{-\lambda_1}}{2} \left[\left(\operatorname{erfc}\left(\frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) + \operatorname{erfc}\left(\frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) \right) (1 - \lambda_2) \right. \\
& - e^{\lambda_1 \xi} e^{\theta} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) + \lambda_2 e^{\frac{\lambda_1 + \xi}{\lambda_2}} e^{\frac{\theta}{\lambda_2}} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) \\
& \left. - e^{\lambda_1 - \xi} e^{\theta} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) + \lambda_2 e^{\frac{\lambda_1 - \xi}{\lambda_2}} e^{\frac{\theta}{\lambda_2}} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) \right], \\
& \xi > \lambda_1.
\end{aligned}$$

Equations (20) and (21), the complete normalized solutions for this model, are seen to be identical to equations (4-22) and (4-23) of Chapter IV. For further discussion of these relations, see p. 77 et seq.

The next task is to derive the expressions for $\partial U(x,t)/\partial t$. Noting that

$$\frac{\partial U(x,t)}{\partial t} = \mathcal{L}^{-1} \{ s u(x,s) \},$$

it follows from (16) and (17) that for the step-function form of $H_a(t)$:

$$\begin{aligned}
\frac{\partial U(x,t)}{\partial t} = & \frac{H_a}{2\sqrt{\pi}} \mathcal{L}^{-1} \left\{ \frac{2r_1 e^{-r_1 x}}{s - r_1 \sqrt{a}} + \frac{r_1 e^{-\frac{r_1}{\sqrt{a}} \sqrt{s}}}{\sqrt{s}(\sqrt{s} + r_1 \sqrt{a})} - \frac{r_1 e^{-\frac{r_1}{\sqrt{a}} \sqrt{s}}}{\sqrt{s}(\sqrt{s} - r_1 \sqrt{a})} \right. \\
& + e^{-nb} \left[-\frac{r_1 e^{-\frac{r_1}{\sqrt{a}} \sqrt{s}}}{\sqrt{s}(\sqrt{s} + r_1 \sqrt{a})} - \frac{r_1 e^{-\frac{r_1}{\sqrt{a}} \sqrt{s}}}{\sqrt{s}(\sqrt{s} + r_1 \sqrt{a})} \right. \\
& \left. \left. + \frac{r_2 e^{-\frac{r_2}{\sqrt{a}} \sqrt{s}}}{\sqrt{s}(\sqrt{s} + r_2 \sqrt{a})} + \frac{r_2 e^{-\frac{r_2}{\sqrt{a}} \sqrt{s}}}{\sqrt{s}(\sqrt{s} + r_2 \sqrt{a})} \right] \right\}, \quad 0 \leq x < b;
\end{aligned}$$

hence ,

$$\begin{aligned} \frac{\partial U(x,t)}{\partial t} = \frac{H_0}{2\nu} & \left\{ 2r_1 e^{-r_1 x} e^{r_1^2 x t} + r_1 e^{r_1 x} e^{r_1^2 x t} \operatorname{erfc}\left(r_1 \sqrt{at} + \frac{x}{2\sqrt{at}}\right) \right. \\ & - 2r_1 e^{-r_1 x} e^{r_1^2 x t} + r_1 e^{-r_1 x} e^{r_1^2 x t} \operatorname{erfc}\left(r_1 \sqrt{at} - \frac{x}{2\sqrt{at}}\right) \\ & + e^{-r_1 b} \left[-r_1 e^{+r_1(b+x)} e^{r_1^2 x t} \operatorname{erfc}\left(r_1 \sqrt{at} + \frac{b+x}{2\sqrt{at}}\right) \right. \\ & \quad - r_1 e^{+r_1(b-x)} e^{r_1^2 x t} \operatorname{erfc}\left(r_1 \sqrt{at} + \frac{b-x}{2\sqrt{at}}\right) \\ & \quad + r_2 e^{+r_2(b+x)} e^{r_2^2 x t} \operatorname{erfc}\left(r_2 \sqrt{at} + \frac{b+x}{2\sqrt{at}}\right) \\ & \quad \left. \left. + r_2 e^{+r_2(b-x)} e^{r_2^2 x t} \operatorname{erfc}\left(r_2 \sqrt{at} + \frac{b-x}{2\sqrt{at}}\right) \right] \right\} \end{aligned}$$

or

$$\begin{aligned} 22) \frac{\partial U(x,t)}{\partial t} = \frac{H_0}{2\nu} & \left\{ r_1 e^{+r_1 x t} \left[e^{r_1 x} \operatorname{erfc}\left(r_1 \sqrt{at} + \frac{x}{2\sqrt{at}}\right) + e^{-r_1 x} \operatorname{erfc}\left(r_1 \sqrt{at} - \frac{x}{2\sqrt{at}}\right) \right] \right. \\ & - e^{-r_1 b} \left[r_1 e^{r_1(b+x)} e^{r_1^2 x t} \operatorname{erfc}\left(r_1 \sqrt{at} + \frac{b+x}{2\sqrt{at}}\right) \right. \\ & \quad + r_1 e^{r_1(b-x)} e^{r_1^2 x t} \operatorname{erfc}\left(r_1 \sqrt{at} + \frac{b-x}{2\sqrt{at}}\right) \\ & \quad - r_2 e^{r_2(b+x)} e^{r_2^2 x t} \operatorname{erfc}\left(r_2 \sqrt{at} + \frac{b+x}{2\sqrt{at}}\right) \\ & \quad \left. \left. - r_2 e^{r_2(b-x)} e^{r_2^2 x t} \operatorname{erfc}\left(r_2 \sqrt{at} + \frac{b-x}{2\sqrt{at}}\right) \right] \right\}, \\ & \quad 0 \leq x < b; \end{aligned}$$

and ,

$$\begin{aligned}
23) \frac{\partial U(x,t)}{\partial x} = \frac{H_0}{2v} & \left\{ 2r_2 e^{-r_1 b} e^{-r_2(x-b)} e^{r_2^2 x t} + r_1 e^{r_1 x} e^{r_1^2 x t} \operatorname{erfc}\left(r_1 \sqrt{x t} + \frac{x}{2\sqrt{x t}}\right) \right. \\
& - r_1 e^{-r_1 x} e^{r_1^2 x t} \operatorname{erfc}\left(-r_1 \sqrt{x t} + \frac{x}{2\sqrt{x t}}\right) \\
& - e^{-r_1 b} \left[r_1 e^{r_1(x+b)} e^{r_1^2 x t} \operatorname{erfc}\left(r_1 \sqrt{x t} + \frac{x+b}{2\sqrt{x t}}\right) \right. \\
& - r_1 e^{-r_1(x-b)} e^{r_1^2 x t} \operatorname{erfc}\left(-r_1 \sqrt{x t} + \frac{x-b}{2\sqrt{x t}}\right) \\
& - r_2 e^{r_2(x+b)} e^{r_2^2 x t} \operatorname{erfc}\left(r_2 \sqrt{x t} + \frac{x+b}{2\sqrt{x t}}\right) \\
& \left. \left. + r_2 e^{-r_2(x-b)} e^{r_2^2 x t} \operatorname{erfc}\left(-r_2 \sqrt{x t} + \frac{x-b}{2\sqrt{x t}}\right) \right] \right\}, \\
& x > b.
\end{aligned}$$

Now, note that

$$\lim_{x \rightarrow -\infty} \operatorname{erfc}(x) = 2,$$

and $\lim_{x \rightarrow \infty} \operatorname{erfc}(x) = 0.$

Equations (22) and (23) may now be evaluated in the limit as t approaches zero from above. Thus:

$$\begin{aligned}
24) \frac{\partial U(x, 0+)}{\partial t} &= \frac{H_0}{2v} \{ r_1 [e^{-r_1 x} \cdot 2] \} \\
&= \frac{H_0 r_1}{v} e^{-r_1 x}, \quad 0 \leq x < b,
\end{aligned}$$

and

$$\begin{aligned}
25) \frac{\partial U(x, 0+)}{\partial t} &= \frac{H_0}{2v} \{ 2r_2 e^{-r_1 b} e^{-r_2(x-b)} \cdot 1 \} \\
&= \frac{H_0 r_2}{v} e^{-r_1 b} e^{-r_2(x-b)}, \quad x > b.
\end{aligned}$$

These two equations are identical to (4-24) of Chapter IV.

It is interesting to note the following rather startling situation. By employing again the fact that $\text{erfc}(x) = 2 - \text{erfc}(-x)$, equation (23) may be re-written as:

$$\begin{aligned}
 \frac{\partial U(x,t)}{\partial t} &= \frac{H_0}{2V} \left\{ 2r_2 e^{-r_1 b} e^{r_2(b-x)} e^{r_1^2 x t} + r_1 e^{r_1 x} e^{r_1^2 x t} \text{erfc}(r_1 \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}) \right. \\
 &\quad - 2r_1 e^{-r_1 x} e^{r_1^2 x t} + r_1 e^{-r_1 x} e^{r_1^2 x t} \text{erfc}(r_1 \sqrt{\alpha t} - \frac{x}{2\sqrt{\alpha t}}) \\
 &\quad - e^{-r_1 b} \left[r_1 e^{r_1(b+x)} e^{r_1^2 x t} \text{erfc}(r_1 \sqrt{\alpha t} + \frac{b+x}{2\sqrt{\alpha t}}) \right. \\
 &\quad + r_1 e^{r_1(b-x)} e^{r_1^2 x t} \text{erfc}(r_1 \sqrt{\alpha t} + \frac{b-x}{2\sqrt{\alpha t}}) \\
 &\quad - r_2 e^{r_2(b+x)} e^{r_2^2 x t} \text{erfc}(r_2 \sqrt{\alpha t} + \frac{b+x}{2\sqrt{\alpha t}}) \\
 &\quad \left. - r_2 e^{r_2(b-x)} e^{r_2^2 x t} \text{erfc}(r_2 \sqrt{\alpha t} + \frac{b-x}{2\sqrt{\alpha t}}) \right] \\
 &\quad \left. + 2r_1 e^{-r_1 b} e^{r_1(b-x)} e^{r_1^2 x t} - 2r_2 e^{-r_1 b} e^{r_2(b-x)} e^{r_2^2 x t} \right\} \\
 &= \frac{H_0}{2V} \left\{ r_1 e^{r_1^2 x t} \left[e^{r_1 x} \text{erfc}(r_1 \sqrt{\alpha t} + \frac{x}{2\sqrt{\alpha t}}) + e^{-r_1 x} \text{erfc}(r_1 \sqrt{\alpha t} - \frac{x}{2\sqrt{\alpha t}}) \right] \right. \\
 &\quad - e^{-r_1 b} \left[r_1 e^{r_1(b+x)} e^{r_1^2 x t} \text{erfc}(r_1 \sqrt{\alpha t} + \frac{b+x}{2\sqrt{\alpha t}}) \right. \\
 &\quad + r_1 e^{r_1(b-x)} e^{r_1^2 x t} \text{erfc}(r_1 \sqrt{\alpha t} + \frac{b-x}{2\sqrt{\alpha t}}) \\
 &\quad - r_2 e^{r_2(b+x)} e^{r_2^2 x t} \text{erfc}(r_2 \sqrt{\alpha t} + \frac{b+x}{2\sqrt{\alpha t}}) \\
 &\quad \left. - r_2 e^{r_2(b-x)} e^{r_2^2 x t} \text{erfc}(r_2 \sqrt{\alpha t} + \frac{b-x}{2\sqrt{\alpha t}}) \right] \Bigg\}, \\
 &\quad x > b,
 \end{aligned}$$

which is identical to equation (22)! Thus the form of $\partial U(x,t)/\partial t$ is the same for the

two regions, $0 \leq x < b$, and $x > b$, although the limiting values (as $t \rightarrow 0$ from above) are different. Further, reference to equations (24) and (25) above shows that

$$\lim_{x \rightarrow b^+} \frac{\partial U(x, 0^+)}{\partial t} \neq \lim_{x \rightarrow b^-} \frac{\partial U(x, 0^+)}{\partial t}.$$

In fact, it can be shown by substitution of $x = b$ in either (18) or (19) and direct differentiation w.r.t. t , that

$$\begin{aligned} \frac{\partial U(b, 0^+)}{\partial t} &= \frac{1}{2} \left[\lim_{x \rightarrow b^-} \frac{\partial U(x, 0^+)}{\partial t} + \lim_{x \rightarrow b^+} \frac{\partial U(x, 0^+)}{\partial t} \right] \\ &= \frac{H_0 e^{-\gamma b}}{\gamma} \left(\frac{\gamma_1 + \gamma_2}{2} \right). \end{aligned}$$

Note that this expression cannot be derived by substitution of $x = b$ in (22) (or the equivalent (23)), since one is faced with the indeterminate form $\operatorname{erfc}(\frac{0}{0})$.

One important task remains. The solution of a partial differential equation by the method of the Laplace transform must always be justified by demonstrating that the solution satisfies the original d.e. and the boundary values. In the present case, with the above-demonstrated discontinuity in the first partial w.r.t. time, this is particularly important. This "brute force" verification is actually considerably more lengthy and involved than the establishing of the solution itself, which is usually the case. Hence, here, it will only be sketched in outline. The complete

verification is on file in this laboratory.

It is convenient to revert now to the normalized solution Ψ in ξ and θ . Note first that

$$\begin{aligned}\frac{\partial U}{\partial t} &= \frac{\partial U}{\partial \theta} \frac{d\theta}{dt} = \frac{d\theta}{dt} \cdot \frac{Ha}{kr_1} \frac{\partial \Psi}{\partial \theta} \\ &= \frac{r_1^2 k}{\nu} \cdot \frac{Ha}{kr_1} \frac{\partial \Psi}{\partial \theta} \\ &= \frac{r_1 Ha}{\nu} \frac{\partial \Psi}{\partial \theta},\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 U}{\partial x^2} &= \frac{\partial}{\partial x} \left[\frac{\partial U}{\partial \xi} \cdot \frac{d\xi}{dx} \right] = \frac{\partial^2 U}{\partial \xi^2} \left(\frac{d\xi}{dx} \right)^2 = r_1^2 \cdot \frac{Ha}{kr_1} \frac{\partial^2 \Psi}{\partial \xi^2} \\ &= \frac{r_1 Ha}{k} \frac{\partial^2 \Psi}{\partial \xi^2}.\end{aligned}$$

The original d.e. and boundary conditions may now be written as:

$$26) \frac{\partial \Psi(\xi, \theta)}{\partial \theta} = \frac{\partial^2 \Psi(\xi, \theta)}{\partial \xi^2} + e^{-\xi}, \quad 0 \leq \xi < \lambda_1$$

$$27) \frac{\partial \Psi(\xi, \theta)}{\partial \theta} = \frac{\partial^2 \Psi(\xi, \theta)}{\partial \xi^2} + \frac{e^{-\lambda_1} \cdot e^{-\frac{\xi-\lambda_1}{\lambda_2}}}{\lambda_2}, \quad \xi > \lambda_1$$

$$28) \Psi(\xi, 0) = 0, \quad \xi \geq 0$$

$$29) \Psi(\infty, \theta) = 0, \quad \theta \geq 0$$

$$30) \frac{\partial \Psi(0, \theta)}{\partial \xi} = 0, \quad \theta \geq 0$$

$$31) \Psi(\lambda_1 - 0, \theta) = \Psi(\lambda_1 + 0, \theta), \quad \theta \geq 0$$

$$32) \frac{\partial \Psi(\lambda_1 - 0, \theta)}{\partial \xi} = \frac{\partial \Psi(\lambda_1 + 0, \theta)}{\partial \xi}, \quad \theta \geq 0$$

That the form of Ψ given by (20) and (21) satisfies the last two continuity conditions was demonstrated in Chapter IV. We may then turn to (26) above and note that (20) becomes

$$\Psi(\xi, 0) = -e^{-\xi} + \frac{1}{2} [e^{-\xi} \cdot 2] = 0, \quad 0 \leq \xi < \lambda_1,$$

while (21) gives:

$$\begin{aligned} \Psi(\xi, 0) &= -e^{-\lambda_1} [\lambda_2 (e^{\frac{\lambda_1 - \xi}{\lambda_2}} - 1) + 1] + e^{\xi} + e^{-\lambda_1} (1 - \lambda_2) \\ &\quad - e^{\xi} + e^{-\lambda_1} \lambda_2 e^{\frac{\lambda_1 - \xi}{\lambda_2}} \\ &= 0, \quad \xi > \lambda_1. \end{aligned}$$

Hence (28) is satisfied. The verification of (29) is a trifle more difficult, since one must first demonstrate that $\lim_{x \rightarrow \infty} x \operatorname{erfc}(x) = 0$, and also $\lim_{x \rightarrow \infty} e^x \operatorname{erfc}(x) = 0$.

Thus

$$\begin{aligned} \lim_{x \rightarrow \infty} x \operatorname{erfc}(x) &= \lim_{x \rightarrow \infty} \frac{\frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\lambda^2} d\lambda}{x} = \frac{0}{0} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{2}{\sqrt{\pi}} e^{-x^2}}{1} = 0, \end{aligned}$$

$$\begin{aligned} \text{and } \lim_{x \rightarrow \infty} e^x \operatorname{erfc}(x) &= \lim_{x \rightarrow \infty} \frac{\frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-\lambda^2} d\lambda}{e^{-x}} = \frac{0}{0} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{2}{\sqrt{\pi}} e^{-x^2}}{-e^{-x}} = 0. \end{aligned}$$

Then, from (21)

$$\Psi(\infty, 0) = -e^{-\lambda_1} (1 - \lambda_2) + e^{-\lambda_1} (1 - \lambda_2) = 0,$$

and (29) is satisfied.

Next we must find $\frac{\partial \Psi(\xi, 0)}{\partial \xi}$. This is a tedious process, at best, and only the final result will be given here.

$$\begin{aligned} \frac{\partial \Psi(\xi, 0)}{\partial \xi} &= e^{-\xi} - \operatorname{erfc}\left(\frac{\xi}{2\sqrt{\theta}}\right) + \frac{\xi}{2} \left[e^{\xi} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\xi}{2\sqrt{\theta}}\right) - e^{-\xi} \operatorname{erfc}\left(\sqrt{\theta} - \frac{\xi}{2\sqrt{\theta}}\right) \right] \\ &\quad + \frac{e^{-\lambda_1}}{2} \left[-e^{\lambda_1 \xi} e^{\xi} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) + e^{\frac{\lambda_1 + \xi}{\lambda_2}} e^{\frac{\xi}{\lambda_2}} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) \right. \\ &\quad \left. + e^{\lambda_1 - \xi} e^{\xi} \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) - e^{\frac{\lambda_1 - \xi}{\lambda_2}} e^{\frac{\xi}{\lambda_2}} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) \right], \\ &\quad 0 \leq \xi < \lambda_1, \end{aligned}$$

hence

$$\begin{aligned} \frac{\partial^2 \Psi(0, \theta)}{\partial \xi^2} &= 1 - 1 + \frac{e^0}{2} \left[\operatorname{erfc}(\sqrt{\theta}) - \operatorname{erfc}(\sqrt{\theta}) \right] \\ &\quad + \frac{e^{-\lambda_1}}{2} \left[-e^{\lambda_1} e^0 \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1}{2\sqrt{\theta}}\right) + e^{\frac{\lambda_1}{\lambda_2}} e^{\frac{\lambda_1}{\lambda_2}} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1}{2\sqrt{\theta}}\right) \right. \\ &\quad \left. + e^{\lambda_1} e^0 \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1}{2\sqrt{\theta}}\right) - e^{\frac{\lambda_1}{\lambda_2}} e^{\frac{\lambda_1}{\lambda_2}} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1}{2\sqrt{\theta}}\right) \right] \\ &= 0, \end{aligned}$$

and (30) is satisfied.

Finally, the second partial w.r.t. ξ of Ψ is found to be

$$\begin{aligned} \frac{\partial^2 \Psi(\xi, \theta)}{\partial \xi^2} &= -e^{-\xi} + \left\{ \frac{e^0}{2} \left[e^0 \operatorname{erfc}\left(\sqrt{\theta} + \frac{\xi}{2\sqrt{\theta}}\right) + e^{-\xi} \operatorname{erfc}\left(\sqrt{\theta} - \frac{\xi}{2\sqrt{\theta}}\right) \right] \right. \\ &\quad + \frac{e^{-\lambda_1}}{2} \left[-e^{\lambda_1 \xi} e^0 \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) + \frac{e^{\frac{\lambda_1 + \xi}{\lambda_2}} e^{\frac{\xi}{\lambda_2}}}{\lambda_2} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1 + \xi}{2\sqrt{\theta}}\right) \right. \\ &\quad \left. \left. - e^{\lambda_1 - \xi} e^0 \operatorname{erfc}\left(\sqrt{\theta} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) + \frac{e^{\frac{\lambda_1 - \xi}{\lambda_2}} e^{\frac{\xi}{\lambda_2}}}{\lambda_2} \operatorname{erfc}\left(\frac{\sqrt{\theta}}{\lambda_2} + \frac{\lambda_1 - \xi}{2\sqrt{\theta}}\right) \right] \right\}, \end{aligned}$$

$$\theta \geq 0, 0 \leq \xi < \lambda_1,$$

and

$$\frac{\partial^2 \Psi(\xi, \theta)}{\partial \xi^2} = -\frac{e^{-\lambda_1} e^{-\frac{\xi - \lambda_1}{\lambda_2}}}{\lambda_2} + \left\{ G(\xi, \theta) \right\}, \quad \theta \geq 0, \xi > \lambda_1,$$

where $G(\xi, \theta)$ is the expression in brackets in the expression above for $0 \leq \xi < \lambda_1$. Now, inspection of equation (22) or the equivalent (23) reveals that

$$\frac{\partial^2 \Psi(\xi, \theta)}{\partial \xi^2} = G(\xi, \theta), \quad \xi \geq 0, \theta \geq 0,$$

where $G(\xi, \theta)$ is as defined above. Thus, substituting

in (26) and (27)

$$G(\xi, \theta) = -e^{-\xi} + G(\xi, \theta) + e^{-\xi}, \quad 0 \leq \xi < \lambda_1,$$

or $0 = 0,$

and

$$G(\xi, \theta) = -\frac{e^{-\lambda_1} e^{-\frac{\xi-\lambda_1}{\lambda_2}}}{\lambda_2} + G(\xi, \theta) + \frac{e^{-\lambda_1} e^{-\frac{\xi-\lambda_1}{\lambda_2}}}{\lambda_2}, \quad \xi > \lambda_1,$$

or $0 = 0.$

Q. E. D. !

It might be noted in passing that an even more interesting problem is that of the diathermanous composite solid with double exponential absorption, where

$$\left. \begin{array}{l} \tau = \tau_1 \\ \nu = \nu_1 \\ k = k_1 \end{array} \right\} 0 \leq x < b \quad \text{and} \quad \left. \begin{array}{l} \tau = \tau_2 \\ \nu = \nu_2 \\ k = k_2 \end{array} \right\} x > b.$$

This has so far resisted a few not overly vigorous attempts at solution. There is also the rather unpleasantly messy but otherwise straightforward problem of triple exponential absorption. However, as noted in Chapter IX, the first problem which must be next attacked is that of an opaque thermoelement imbedded in a diathermanous material.

Siste viator !

ACKNOWLEDGEMENTS*

The author would first like to express his profound gratitude for the lively interest and continual encouragement of Dr. H. E. Pearse, Department of Surgery, and Dr. William F. Bale, Department of Radiation Biology, in the prosecution of this study. The time-worn phrase, "Without their assistance this paper could never have been written," has become so trite as to be almost devoid of meaning; in this case, however, it is quite literally true. For their support, the author is deeply grateful.

Of equal value has been the direct assistance and moral support of the writer's many associates in the Flash Burn Section of The University of Rochester Atomic Energy Project; to each of these associates, past and present, the author acknowledges his indebtedness. In particular, mention should be made of the early proposal of Dr. H. D. Kingsley, formerly Chief of this section, that a study of temperature response of radiated skin should be undertaken. It is hoped that this paper, coming some nine years after this proposal, will indicate to him that we are at least beginning to implement his suggestion.

The researches of Dr. J. R. Hinshaw, present Chief of this section, have not only paved the way for this study, but also have served as models of precise and thorough experimentation in the field of radiant energy burns. Indeed, the entire problem of the prediction of radiant energy burns, of which this study is a small part, is based almost entirely upon the work of Dr. Hinshaw. It is a pleasure to acknowledge the advice, assistance, and encouragement of this surgeon and scholar.

*This paper is based on work performed under contract with the United States Atomic Energy Commission at The University of Rochester Atomic Energy Project, Rochester, New York.

It would be a grave error to omit the name of Dr. George Mixter, Jr., from this listing of those who have assisted in this study. Formerly associated with this section, Dr. Mixter pioneered in the application of analytical procedures to the extensive experimental results he obtained. It was largely through the close but too brief association with this ebullient and astounding intellect that the writer was first attracted to the interdisciplinary field of biophysics.

The immediate co-workers of the writer, in the physics laboratory of the Flash Burn Section, have contributed extensively to this study. Dr. R. M. Blakney, formerly in charge of the laboratory, and L. J. Krolak were largely responsible for the initial development of the carbon arc source employed for all the experimental work reported herein. Mr. Krolak also performed the careful spectrophotometric measurements necessary for the proper interpretation of the experimental results. The author's present associates, J. A. Basso and Mrs. Marilyn Aldrich Guck, have been of constant and invaluable assistance throughout the entire program. Much of the specialized equipment used in this study was constructed by Mr. Basso, who also is responsible for the calibration of the carbon arc image furnace used as the radiation source. Mrs. Guck assisted in all the experimental work, and also carried out the initial data reduction. Mr. Basso prepared the figures in this paper, and with Mrs. Guck proofread the final copy. Truly it is impossible to acknowledge adequately their aid except to say once more that without it this paper could never have been written.

It is pleasant also to acknowledge the invaluable help of Mrs. Cornelia Winslow, who labored long and conscientiously in translating the manuscript into readable typescript and prepared the final product. The author is particularly grateful for her finding and correcting many errors and inconsistencies in the original copy.

Finally, it is a pleasant but impossible task to attempt to acknowledge properly the help of my wife. Certainly, her patience has made it possible for me to devote so much time to this work. Even more important, her interest and enthusiasm have made it all the more interesting to me. Most important, her love has made it all eminently worthwhile.