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NUMERICAL STUDY OF THE MHD SPECTRUM
IN TOKAMAKS WITH A
NONCIRCULAR CROSS SECTION

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NUMERICAL STUDY OF THE MHD SPECTRUM IN
TOKAMAKS WITH A NONCIRCULAR CROSS SECTION

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ABSTRACT. The behavior of a class of exact tokamak equilibria is investigated in the context of ideal magnetohydrodynamics. The equilibrium solution allows for a general plasma cross section shape, restricted to systems with essentially flat toroidal current profiles. The spectrum of the eigenmodes is evaluated by extremizing the Lagrangian using a Galerkin procedure. The different branches of the spectrum are appropriately represented by expanding the eigenfunctions in terms of the eigenfunctions of the corresponding straight limit. The perturbed vacuum field is expressed by a vector potential. The code is applied to various configurations with a fixed and with a free plasma boundary. The influence of ellipticity of the cross section and of the aspect ratio on the spectrum is discussed.

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1. INTRODUCTION

The magnetohydrodynamic (MHD) approximation for the description of a plasma confined in a toroidal configuration is very useful to understand present tokamak experiments and to design new ones. Here we describe a computer code which has been developed with this objective in mind. The behavior of a class of exact tokamak equilibria is investigated. We determine the spectrum of the linearized motion of the plasma around its equilibrium state. The method consists of extremizing the Lagrangian with a Galerkin procedure leading to a matrix eigenvalue problem. This indicates whether a configuration is stable or not and what the respective growth rates of the modes are. In addition, the spectrum and the eigenfunctions give details for understanding the stability behavior and may be useful for discussions of plasma heating. The eigenvalues and eigenfunctions of our code can be used as input data for a time dependent code or for calculations including resistivity. It is useful to treat analytical equilibria, since parameters can be varied freely thus allowing us to describe the equilibria in simple physical terms. The equilibrium solution is applicable to tokamak configurations like PLT, TFTR, PDX, JET and the belt pinch.

In an earlier stability calculation with this class of equilibria [1], the energy principle was used to find instabilities with the help of carefully chosen test functions. These test functions were similar to the eigenfunctions of the appropriate cylindrical equilibria - being accurate eigenfunctions up to first order in the inverse aspect ratio ϵ . The stability limit

was not fully determined. The purpose of this work is to obtain the full information contained in the linearized ideal MHD equations. We, therefore, extend the earlier code considerably.

The demands on the new code are much higher. The calculation must be performed in such a way that we get a well-posed matrix eigenvalue problem which can be solved with standard routines. This is complicated by the fact that the MHD spectrum splits into discrete and continuous branches with accumulation points for the eigenvalues (the growth rates) at infinity and at zero. To get a well posed matrix eigenvalue problem it is essential i) to introduce adapted coordinates, ii) to perform the calculation very accurately without using approximations, and iii) to choose appropriate expansion functions in the Galerkin procedure. Thereby, we make full use of the properties of the special class of our analytical equilibria. The equilibrium model enables a reasonable analytical mapping into suitable coordinates. This makes it possible to perform the integration with respect to the angle the short way around the torus algebraically minimizing the computer time requirement and obtaining good accuracy. Our method is complementary to a general equilibrium and stability code being developed at Princeton [2].

In section II the equilibrium solution is given and the mapping into a flux coordinate system is performed. The normal mode approach is described in section III and the expansion functions used in the Galerkin procedure are discussed.

Section IV shows a comparison of our approach with that used in another toroidal code. In section V the code is applied to various interesting cases.

2. EQUILIBRIUM AND COORDINATE SYSTEM

Static toroidal axisymmetric MHD equilibria can be derived from an equation for the flux function Ψ . A special choice for the pressure p and the poloidal current stream function $T = XB_\phi$ ($p = p_0 - p'\Psi$, $T^2 = R^2[B_0^2 + 2\delta p'/(1 + \alpha^2) \cdot \Psi]$) with $p_0, p', B_0, \delta, \alpha$ being constant leads to the very simple class of solutions [1].

$$\Psi = \left[Z^2(X^2 - \delta \cdot R^2) + \frac{\alpha^2}{4} (X^2 - R^2)^2 \right] \frac{p'}{2(1 + \alpha^2)} \quad (1)$$

inside the plasma. Here R is the radius of the magnetic axis and X, ϕ, Z , is the usual cylindrical coordinate system. The plasma-vacuum interface is a particular magnetic surface $\Psi = \Psi_b$. The plasma is surrounded by vacuum bounded by a conducting wall on an outer surface $\Psi = \Psi_w$. These equilibria have an almost constant volume current in the toroidal direction and also a poloidal current which produces a diamagnetic or paramagnetic current distribution. The plasma cross sections are roughly elliptic or D shaped. The ratio of half axis is given by $e = \alpha/(1 - \delta)^{1/2}$ with $\delta < 1$.

In order to perform the stability analysis with a high degree of accuracy it is useful to adapt a coordinate system in which a flux surface label ψ is one coordinate. We choose the ignorable

angle ϕ as the second coordinate and the third one in such a manner that the transformation between X, Z and $x^1 = \psi, x^3 = \theta$ can be performed analytically [1]. This is a non-orthogonal coordinate system, but the metric tensor is given by simple functions. The system is outlined in Fig. 1.

The new coordinates are introduced by the mapping

$$\psi = \left\{ \left[Z^2 (X^2 - \delta \cdot R^2) + \frac{\alpha^2}{4} (X^2 - R^2)^2 \right] \frac{1}{\alpha^2} \right\}^{1/2} \alpha \psi^{1/2}$$

$$\phi = \psi$$

$$\theta = \text{Arctg} \left(\frac{2}{\alpha} \frac{Z (X^2 - \delta \cdot R^2)^{1/2}}{X^2 - R^2} \right)$$

Introducing

$$r = \left[2\psi \cdot \cos\theta + R^2 \right]^{1/2} \quad \text{and} \quad \tilde{r} = \left[2\psi \cdot \cos\theta + R^2 (1 - \delta) \right]^{1/2} \quad (2)$$

we obtain for the covariant components of the metric tensor,

$$g_{11} = \frac{1}{\tilde{r}^2} \left[\frac{\tilde{r}^2}{r^2} \cos^2 \theta + \alpha^2 \sin^2 \theta + \frac{\alpha^2 \psi \sin^2 \theta}{\tilde{r}^2} \left(\frac{\psi \cos^2 \theta}{\tilde{r}^2} - 2 \cos \theta \right) \right],$$

$$g_{12} = g_{21} = 0 ,$$

$$g_{13} = g_{31} = \frac{\psi}{\tilde{r}^2} \left\{ \left(\alpha^2 - \frac{\tilde{r}^2}{r^2} \right) \sin \theta \cos \theta + \frac{\alpha^2 \psi \sin \theta}{\tilde{r}^2} \left[(\sin^2 \theta - \cos^2 \theta) - \frac{\psi \sin^2 \theta \cos \theta}{\tilde{r}^2} \right] \right\}$$

$$g_{22} = r^2 , \quad (3)$$

$$g_{23} = g_{32} = 0 ,$$

$$g_{33} = \frac{(\psi)^2}{\tilde{r}^2} \left[\frac{\tilde{r}^2}{r^2} \sin^2 \theta + \alpha^2 \cos^2 \theta + \frac{\alpha^2 \psi \sin^2 \theta}{\tilde{r}^2} \left(2 \cos \theta + \frac{\psi \sin^2 \theta}{\tilde{r}^2} \right) \right]$$

$$g^{1/2} = [\det(g_{ij})]^{1/2} = \alpha \psi / \tilde{r} \quad \text{is the Jacobian.}$$

The contravariant components of the magnetic field and of the current density are

$$B^1 = 0 , \quad B^2 = \frac{1}{r^2} T , \quad B^3 = - \frac{\alpha p'}{1 + \alpha^2} \tilde{r} ,$$

$$J^1 = 0 , \quad J^2 = p' \left(1 - \frac{\delta \cdot R^2}{r^2 (1 + \alpha^2)} \right) , \quad J^3 = \frac{\alpha \cdot \delta \cdot R^2 (p')^2}{T (1 + \alpha^2)^2} \tilde{r}. \quad (4)$$

The toroidal current density is $J_\phi = p' \left[r - \delta \cdot R^2 / r (1 + \alpha^2) \right]$ and is flat for small inverse aspect ratio, i.e. $\psi_b / R^2 \ll 1$. The poloidal current J^3 is proportional to δ .

3. NORMAL MODE APPROACH

3.1. METHOD

This approach utilizes the variational principle for determining the magnetohydrodynamic spectrum associated with linearized perturbations around an equilibrium configuration. The normal modes of the system are determined by computing eigenvalues ω^2 and eigenvectors ξ that make the Lagrangian

$$L = \omega^2 K(\xi^*, \xi) - \delta W(\xi^*, \xi) \quad (5)$$

stationary with respect to variations of ξ . The displacement of a fluid element from its equilibrium position $\underline{r}$ is given by the real part $\text{Re}[\xi(\underline{r})\exp(-i\omega t)]$, and $\omega^2 K$ and δW are the kinetic and potential energy functionals.

$$K = 1/2 \int_{\text{plasma}} \rho |\xi|^2 d\tau \quad (6)$$

with ρ the plasma density.

$$\delta W = \delta W_p + \delta W_v$$

$$\delta W_p = 1/2 \int_{\text{plasma}} \left[|\underline{Q}|^2 + \xi^* \cdot (\underline{Q} \times \underline{j}) + (\nabla \cdot \xi^*) (\xi \cdot \nabla p) + \gamma p |\nabla \cdot \xi|^2 \right] d\tau, \quad (7)$$

where $\underline{Q} \equiv \nabla \times (\xi \times \underline{B})$ denotes the magnetic field perturbation in the plasma and γ is the ratio of specific heats.

$$\delta W_V = 1/2 \int_{\text{vacuum}} |\delta \underline{B}_V|^2 d\tau, \quad (8)$$

where $\underline{B}_V = \nabla \times \underline{a}$ is the perturbation of the magnetic field in vacuum expressed by a vector potential $\underline{a}$.

The Lagrangian is a function of ξ and first order derivatives of ξ with respect to $\underline{r}$, $\delta \xi / \delta \underline{r}_i$ ($i = 1, 2, 3$), alone. The corresponding Euler-Lagrange equations involve second order derivatives. Since, generally, integrations can be done more accurately than differentiations by computer, the variational technique is for our purpose more suitable than an iteration scheme for the differential equations.

The admissible variational functions are those for which the displacement has a finite kinetic energy norm. Maxwell's equations imply that the tangential electric field felt by an observer moving with the fluid is continuous at the plasma boundary and vanishes at the wall. This is expressed by the boundary conditions [3], that must be satisfied by the admissible functions,

$$\underline{n} \times \underline{a} = 0 \quad \text{at wall}, \quad (9)$$

$$\underline{n} \times \underline{a} = -(\underline{n} \cdot \underline{\xi}) \underline{B} \quad \text{at plasma boundary.}$$

Here $\underline{n}$ denotes the unit normal vector and the gauge is chosen so that the electric field is proportional to the vector potential $\underline{a}$.

We adopt the Galerkin method in which ξ is approximated by a linear superposition of M linearly independent expansion functions ϕ_v ,

$$\xi^{(M)} = \sum_{v=1}^M c_v \phi_v; \quad (10)$$

The c_v and ϕ_v are complex quantities. With this expansion, variation of Eq.(5) with respect to c_v^* , leads to the matrix eigenvalue problem

$$\sum_{v=1}^M \left\{ \omega^2 \langle \phi_v^* | K | \phi_v \rangle - \langle \phi_v^* | \delta W | \phi_v \rangle \right\} c_v = 0. \quad (11)$$

Following the arguments in Ref [4] we assume without proof that $\xi^{(M)}$ converges to a solution of Eq. (5). We know, however, that the lowest eigenvalue is always approached from above as the accuracy is increased. We cannot find instability for a stable system. The Euler equation resulting from the minimization of δW_v is $\nabla \times \nabla \times \underline{a} = 0$. An approximation for $\underline{a}$ results in more stable eigenfrequencies [3]. This fact is not strictly true for an approach to the δW_v using a scalar potential χ together with a Green's function technique for the perturbed magnetic field $\delta \underline{B}_v = \nabla \chi$.

3.2 EXPANSION FUNCTIONS FOR THE PLASMA DISPLACEMENT

In order to achieve a fast convergence in the set (10), we expand the eigenfunctions in terms of the exact analytical eigenfunctions of the appropriate straight equilibria. Therefore, we use a global expansion set for which the support of all ϕ_v is the entire plasma volume. As pointed out in [4] care must be taken to represent modes which are localized near singular magnetic surfaces. Because of axisymmetry we can separate the ϕ -dependence by an exponential $\exp(in\phi)$ with integer n . At present the algebra has been performed for $n \neq 0$. An extension for axisymmetric perturbations $n = 0$ is possible. The expansion in terms of straight system eigenfunctions leads to a Fourier series in θ . For the displacement vector ξ and for the vector potential a of the perturbed vacuum field we write

$$\xi(\psi, \phi, \theta) = P_0(u) \exp(in\phi) \sum_{m, v} \exp(im\theta) \xi_{m, v}(\psi), \quad (12)$$

$$a(\psi, \phi, \theta) = P_0(u) \exp(in\phi) \sum_{m, v} \exp(im\theta) a_{m, v}(\psi, \theta),$$

Choosing $P_0(u)$ (see Appendix A), to be a function in $u = 2\psi \cos\theta/R^2$, adapts the expansion functions to the toroidal case and improves, therefore, the convergence of the Fourier series [1]. In general, we can't expect a decoupling in m . We have to truncate the series in m for numerical treatment.

In order to get an expansion set $\xi_{m,v}(\psi)$, which is complete and orthogonal, special care is necessary to represent the different branches of the spectrum adequately. For a fast convergence and for completeness of the expansion functions it is essential to include the fundamental mode, i.e. the one with the least oscillations in the radial coordinate, for each branch.

In the sound modes, which have the lowest frequency of the spectrum, the fluid flows along the field lines. The shear-Alfvén waves have an almost divergence-free motion perpendicular to the magnetic field. The fast magnetosonic modes are due to this perpendicular compressibility. In order to decouple the branches and to get a diagonally dominant matrix - at least for small inverse aspect ratio ϵ - we expand the displacement vector ξ into the straight system eigenvectors for every branch. The contravariant components $(\xi_j^i)_{m,v}$ ($i = 1, 2, 3$; $j = K, F, S$) are constructed from the eigenfunctions of a cylindrical plasma with v modes. For the kink modes $(\xi_K^i)_{m,v}$ we use essentially Tayler's solution [5]. An expansion with respect to ellipticity of the plasma column shows that for the cylinder odd modes in θ are decoupled from even ones. Solving the lowest order of this perturbation calculation allows for a modification of Tayler's solution suitable for an elliptical cross section. The contravariant component in the ϕ direction, ξ_K^2 , is changed to make the displacement vector almost perpendicular to the magnetic field. Then ξ_K^3 is changed to make this perturbation divergence free again, at least in the cylindrical limit. The modifications of ξ_K^2 and ξ_K^3 are of order ϵ .

The dominant term in $Q_{\parallel} = Q \cdot B/B$ is $\nabla \cdot (\xi_{\perp}(B)^2) = r^2 \nabla \cdot (\xi_{\perp}/r^2)$.

This indicates that the shear-Alfvén modes are well described by displacements with $\nabla \cdot (\xi_{\perp}/r^2) = 0$. With the function $P_0(u)$ in Eq. (12) this condition can be fulfilled. This expansion set is characterized as

$$\xi_K^i = \begin{pmatrix} \xi_K^1 \\ \xi_K^2 \\ \xi_K^3 \end{pmatrix}_{m,v} \quad \text{with } \left(\xi_K \right)_{\parallel} \propto \xi_K \cdot B = O(\epsilon^2) \\ \text{and } \nabla \cdot \xi_K = O(\epsilon) .$$

The fast modes ξ_F are nearly orthogonal to the ξ_K , i.e. $\int g^{1/2} \xi_K \cdot \xi_F d\tau = 0$ for a cylinder. They have a divergence proportional to a complete set of Bessel functions with v nodes. Again we change the ϕ component to make the displacement vector orthogonal to B . This expansion set is characterized as

$$\xi_F^i = \begin{pmatrix} \xi_F^1 \\ \xi_F^2 \\ \xi_F^3 \end{pmatrix}_{m,v} \quad \text{with } \left(\xi_F \right)_{\parallel} \propto \xi_F \cdot B = O(\epsilon^2) \\ \text{and } \nabla \cdot \xi_F = O(1) .$$

The expansion functions for the sound modes are defined by the vector product of ξ_K and ξ_F . Thus ξ_S is almost parallel to B .

$$\xi_S = \xi_K \times \xi_F \quad \text{with } \left(\xi_S \right)_{\parallel} \propto \xi_S \cdot B = O(1) .$$

These expansion functions are square-integrable. Trying to decouple the discrete, global eigenfunctions from the continuous modes - at least in the limit of small inverse aspect ratio ϵ and small ellipticity - we must know that the

growth rates ω^2 for the fast modes scale like ϵ^{-2} , whereas the sound modes scale like ϵ^2 . The ratio of these growth rates ω_F^2/ω_S^2 is of the order ϵ^{-4} . With our algebraic calculation we can expect a decoupling only up to order ϵ^{-2} . It is therefore clear that we have some difficulties representing localized sound modes with a very small growth rate, especially for $m - nq \approx 0$. For $m - nq = O(1)$ and larger ϵ these modes are represented better. Because our equilibrium solution is symmetrical with respect to the X axis, the matrix elements for the real part and the imaginary part of the constants C_v in Eq.(10) don't couple. It is therefore sufficient to evaluate only the values of the real parts. This is equivalent to using the real part of the exponential functions $\exp(im\theta + in\phi)$. The functions $(\xi_j^i)_{m,v}$, $i = 1, 2, 3$; $j = K, F, S$, are listed in Appendix A.

3.3 TREATMENT OF THE VACUUM INTEGRAL

For our class of equilibria it is convenient to express the perturbed magnetic field in the vacuum region by a vector potential. Adopting the same coordinate system as in the plasma the two angle integrations in Eq.(8) can be performed algebraically. Doing the third integration in the radial direction numerically, we can perform the volume integral efficiently and evaluate δW_v very accurately. Since the vector potential has a unique, single valued solution, we avoid complications connected with the multivalueness of a scalar potential [6]. In comparison with a Green's function technique

we avoid problems coming from singular integrals.

The vacuum integral does not contribute to the kinetic energy matrix. We use the boundary condition

$\underline{n} \times \underline{a} = -(\underline{n} \cdot \underline{\xi}) \underline{B}$ to express the minimized vacuum contribution entirely in terms of the normal component of $\underline{\xi}$ at the interface. The functions $\underline{a}_{m,1}(\psi, \theta)$ in Eq.(12) are the appropriately modified straight system eigenfunctions, which are compatible with the boundary conditions. The $\underline{a}_{m,\mu}(\psi, \theta)$ for $\mu = 2, 3, \dots$, are chosen to be a complete system of functions vanishing at the interface and, of course, at the wall. In extremizing δW_v we express the free constants $D_{m,\nu}^i$ ($i = 1, 2, 3$) related to $\underline{a}_{m,\nu}(\psi, \theta)$ in terms of the fixed constant $C_{m,\nu}$ associated with $(\underline{n} \cdot \underline{\xi}_{m,\nu})$. This leads to a matrix inversion. Introducing a vector notation for the constants

$$\underline{y} = \begin{pmatrix} C_{m,1} \\ \vdots \\ D_{m,1}^1 \\ \vdots \\ D_{m,\mu}^3 \end{pmatrix}, \quad m = 1, 2, \dots, \text{Mod} (= \text{number of Fourier components}),$$

we can represent the vacuum integral in the form

$$\delta W_v = \underline{y}^+ \langle \bar{A} \rangle \underline{y}.$$

The C's are the constants introduced in Eq.(10). Extremizing the vacuum integral with respect to the free constants $D_{m,\nu}^i$

leads to an inhomogeneous system of linear equations for $D_{m,\nu}^i$, i.e. $\langle \bar{B} \rangle \underline{D} = \langle \bar{E} \rangle \underline{C}$. This we can solve for the D's in terms of the fixed constants C_v .

$$D_{m,\mu}^i = \sum_{v=1}^{\text{MOD}} (B^{-1} \cdot E)_{\lambda v} C_v \quad \begin{aligned} i &= 1, 2, 3 \\ \lambda &= (m-1) \cdot 3 + i \\ m &= 1, 2, \dots, \text{MOD} \\ \mu &= 1, 2, \dots, M_v \end{aligned}$$

We finally obtain a new matrix V

$$\delta W_v = \sum_{v,v'=1}^{\text{MOD}} V_{v,v'} C_v C_{v'}, \quad (13)$$

and add this contribution to the matrix δW_p in Eq (11).

The functions $a_{m,v}(\psi, \phi, \theta)$ are shown in Appendix B.

3.4 EVALUATION OF MATRICES

After having developed our method for evaluating the normal modes and after discussing the expansion functions we briefly outline the evaluation of the matrices. With $Q^i = \epsilon^{ipq} \partial_p (\epsilon_{qmn} \xi_B^{m,n})$, ϵ^{ipq} and ϵ_{ipq} denoting the contravariant and covariant components of the antisymmetric ϵ tensor and $\partial_p = \partial/\partial \chi^p$, we get in the new coordinate system from Eqs. (6)-(8).

$$K = \frac{1}{2} \sum_{m_1, m_2=M_0}^{M_1} \int_0^{\psi_b} d\psi \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta \, g^{1/2} (g_{ij} \xi_1^i \xi_2^j)_p,$$

$$\begin{aligned}
 \delta W_P = & \frac{1}{2} \sum_{m_1, m_2 = M_0}^{M_1} \int_0^{\psi_b} d\psi \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta g^{1/2} \left[g_{ij} Q_1^i Q_2^j \right. \\
 & + \xi_1^i \epsilon_{ikl} Q_2^k J^l + g^{-1/2} \partial_i (g^{1/2} \xi_1^i) \xi_2^j \partial_j p \\
 & \left. + \frac{5}{3} p (\partial_i (g^{1/2} \xi_1^i)) (\partial_j (g^{1/2} \xi_2^j)) / g \right]. \\
 \\
 \delta W_V = & \frac{1}{2} \sum_{m_1, m_2 = M_0}^{M_1} \int_{\psi_b}^{\psi_w} d\psi \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta g^{1/2} \left(g_{ij} \epsilon^{ikl} \partial_k a_{l(m_1)} \right. \\
 & \left. \epsilon^{jk'l'} \partial_{k'} a_{l'(m_2)} \right).
 \end{aligned}$$

Indices occurring as subscripts and superscripts indicate summation from 1 to 3. The algebra connected with the evaluation of the integrands and the performing of the ϕ, θ integrations are done on a computer with the REDUCE system [7]. The algebraic calculation is described in detail in Ref.[8]. The remaining integration with respect to ψ is performed numerically using Gaussian quadratures. The solving of the matrix eigenvalue problem is done without any difficulty by using standard library routines.

The size of the matrices δW and K is given by $S = 3 \cdot \text{MOD} \cdot M$ with MOD being the number of Fourier components in θ and M the number of radial expansion functions. The size of the matrices A,B,V for the vacuum calculation is

$S_A = (4 + 3 \cdot (M_V - 1)) \cdot \text{MOD} - 2\delta_{\text{mo}}$, $S_B = 3 \cdot M_V \cdot \text{MOD} - 2\delta_{\text{mo}}$,
 $S_V = \text{MOD} \cdot \text{MOD}$ with M_V being the number of expansion functions
 in vacuum. If the mode $m = 0$ is included we have to lower the
 size by 2. Typical numbers, which give a good convergence are
 $\text{MOD} = 5$ to 11 , $M = 4$ to 7 , and $M_V = 2$ or 3 , leading to
 $S = 110$, $S_A = 50$.

3.5 RELATION TO OTHER TOROIDAL CODES

Stability calculations for axisymmetric systems are very
 difficult and complicated and can be performed in general only
 using a computer. Therefore, the comparison of the different
 codes is a useful check. Here we compare our approach with that
 of Sykes and Wesson [9] and Bateman and Schneider [10] We
 show the main differences between the two methods in the
 Table 1. In the applications we refer in some cases to the
 results of Ref. [9] .

TABLE 1. COMPARISON WITH OTHER (TOROIDAL) MHD CODES

Code	Kerner (PPPL)	Wesson/Sykes; Bateman/Schneider
Equilibrium	Analytic Special class	Numerical
Coordinates	Flux coordinates Poloidal angle with B-lines not straight	Cylindrical coordinates
Mapping	Analytic	
Expansion	Global	
Functions		
Polarizations of ξ	Similar to straight system Approximate decoupling of branches of spectrum	Cylindrical coordinates
Vacuum	Vector potential a Galerkin procedure	Scalar potential Finite difference
Method for Evaluation of Growth rates	Matrix eigenvalues Extremizing the Lagrangian; Galerkin procedure	Integration in time of the differential equations with explicit difference scheme
Integrations	$d\phi$, $d\theta$ analytical $d\psi$ numerical	
Comments	Efficient code for full spectrum; good convergence for global modes	General code for lowest eigenvalue; possible extension to nonlinear effects

5. APPLICATION

In this section we apply the code to configurations which are of practical interest. We distinguish between two cases where the conducting wall is placed directly at the plasma boundary (fixed boundary) or away from the plasma (free boundary). In the fixed boundary case the boundary condition is $\underline{n} \cdot \underline{\xi} = 0$ at the plasma boundary and there is no vacuum contribution. These modes are called internal modes. The growth rates are normalized to the poloidal Alfvén speed at the magnetic axis.

$$\omega_N^2 = \omega^2 \rho_0 \psi_b^2 (1 + \alpha^2) / (2 \cdot R^2 (1 - \delta) p_0) \quad (14)$$

The discussion is done for a constant density ρ_0 .

5.1 FIXED PLASMA BOUNDARY

Because of the very small growth rates of the unstable modes - especially for $n = 1$ - the stability behavior in toroidal plasmas is different from that of the corresponding straight plasmas, even if the toroidal effects are small. The Mercier criterion [11] being valid for perturbations localized on a singular magnetic surface $\psi = \psi_s$, allows us to evaluate the stability boundary for our equilibrium easily. The stability boundary found with our code agrees well with that of Mercier's criterion as is shown in Fig. 2, $q(0)$ is the value of the safety factor at the magnetic axis. Decreasing the aspect ratio from 10 to 2.8 increases the shear considerably in our equilibrium model. The ratio of q at the surface to that at the magnetic axis is 1.04 for $\epsilon^{-1} = 10$ and 1.74 for $\epsilon^{-1} = 2.8$, if $\delta = 0$. The values of q , necessary for stability, are mostly

restricted by interchange modes localized near the magnetic axis. Thus we have plotted Mercier's criterion only near the axis in Fig. 2. It is interesting that in the vicinity of these marginal localized modes there are global modes with relative large growth rates. These are the dangerous modes in practice. For $\alpha = 1$ there is a deviation from the interchange criterium. The case $\alpha = 1$ describes a circular cross section only for small ϵ . With increasing ϵ the cross sections become D shaped [1]. The stabilization for $\alpha = 1$, found with our code, is due to the fact that there is no unstable mode with wave numbers $n = 1$ $m_0 = 1$, where m_0 corresponds to the dominant Fourier component in θ . An expansion of δW with respect to ϵ , done by Frieman et al. [12], shows marginal stability in second order $\delta W^{(2)}$. The decision about stability is then determined by the fourth order $\delta W^{(4)}$. This indicates that the internal kink mode has a very small growth rate, if it is unstable at all. Our results are compatible with this. The unstable modes have wave numbers $n = \ell + 1$ and $m_0 = \ell$, $\ell = 1, 2, 3, \dots$ with a value for the critical q , $q_{crit} \approx m_0/n = \ell/(\ell + 1)$. For $\epsilon = 0.1$ the most unstable mode has $n = 6$, $m_0 = 5$ with $q_{crit} = 0.85$. The normalized growth rate ω_N is $\omega_N = 1.4 \cdot 10^{-2}$ being a factor of 50-100 smaller than a strong MHD instability $\omega_N = 1$. For small ellipticity there are unstable internal kink modes ($n = 1$, $m_0 = 1$). The normalized growth rate for the case of $\alpha = b/a = 2$ is $\omega_N = 3.4 \cdot 10^{-2}$. For a higher ellipticity there occur instable modes with $n = 1$, $m_0 = 2$. (for $\epsilon = 0.1$ with $\alpha > 3.3$) and $n = 1$, $m_0 = 3$ (for $\epsilon = 0.1$ with $\alpha > 5$).

For increasing inverse aspect ratio ϵ the results are qualitatively the same with the stability boundary slightly shifted to higher q values. In Figs. 3a and 3b the spectrum is plotted versus the ellipticity of the plasma cross section for an aspect ratio $\epsilon^{-1} = 10$ and $\epsilon^{-1} = 2.8$ with q on axis equal to 1. The wave numbers are $n = 1$ and $m = -2, -1, \dots, 4$. Six radial expansion functions are used for each Fourier component. In the large aspect ratio case $\epsilon^{-1} = 10$, Fig. 3a, the different branches are separated. The continuous Alfvén spectrum is well represented. With increasing ellipticity more unstable internal kink modes occur. Some of the sound modes are not accurately calculated, especially for $m = 1$, because of a coupling of order ϵ^2 with the fast modes. In the (almost) circular cross section, $\alpha = 1$, the fast modes are degenerated. This degeneration vanishes with ellipticity. For a small aspect ratio $\epsilon^{-1} = 2.8$, Fig. 3b, the growth rates of the fast modes are decreased by a factor ϵ^2 , whereas the ones of the sound modes are increased by ϵ^{-2} . Therefore, the different branches are less separated. Note that the continuous shear-Alfvén modes are still well described. For $\alpha = 1$ the fast modes are no longer degenerated, since the cross section is D shaped. The configuration with $\alpha = 1.75$ is a good description for the JET device. The results of Sykes and Wesson [9] for internal modes apply to an equilibrium with a circular cross section and a radially decreasing current profile, which is approximated in our equilibrium solution with $\alpha = 1$ and $\delta = 0$. Though the two equilibria differ slightly it is interesting to see how well both calculations agree:

i) There is no unstable internal mode with a toroidal wave number $n = 1$,

ii) The most unstable mode corresponds to a $n = 3$, $m = 2$ mode.

Fig. 4 shows this unstable mode ($n = 3$, $m = 2$) together with the $n = 2$, $m = 1$ mode.

5.2 FREE PLASMA BOUNDARY

The first example is a large aspect ratio case, $\epsilon^{-1} = 20$. If the wall is sufficiently far away from the plasma, i.e., $\Lambda = r_W/r_p \geq 2$, the toroidal effects became negligible for small ϵ . Thus, we are able to compare our results with those obtained by the straight system code of Chance et al, [13]. We consider a plasma with elliptical cross section $\alpha = b/a = 2$ and $\delta = 0$, so that from Eq. (4) the poloidal current vanishes, with a value of the safety factor at axis $q(0) = 1.5$ and $\Lambda = r_W/r_p = 2$. The value of the plasma beta is $\beta_T = 0.0055$ and the poloidal beta $\beta_p = 1$. The toroidal wave number n is equal to 1. For the straight elliptical plasmas, the even and the odd modes of the Fourier series in θ decouple, whereas in the toroidal case they don't. With our code we obtain a good convergence using 5 Fourier components $m = 0, 1, 2, 3, 4$ and 6 global radial expansion functions leading to a 90×90 matrix. To get the same accuracy in the straight code we have to use 9 finite elements in the radial direction. The size of the matrices δW and K is then 81 for the even ($m = 0, 2, 4$) and 58 for the odd ($m = 1, 3$) modes. The spectrum is listed in Table 2 to facilitate comparing of details.

TABLE 2. NORMALIZED EIGENVALUES ω^2 FOR TOROIDAL AND CYLINDRICAL CODE

NO.	TOROIDAL ODD + EVEN	CYLINDRICAL		DESCRIPTION
		ODD	EVEN	
11	$1.08 \cdot 10^5$	$1.19 \cdot 10^5$		Fast magnetosonic
12	$1.02 \cdot 10^5$		$1.37 \cdot 10^5$	
13	$9.65 \cdot 10^4$		$1.09 \cdot 10^5$	
14	$8.47 \cdot 10^4$		$9.42 \cdot 10^4$	
15	$7.82 \cdot 10^4$	$9.10 \cdot 10^4$		
16	$6.35 \cdot 10^4$		$7.16 \cdot 10^4$	
17	$6.20 \cdot 10^4$	$6.53 \cdot 10^4$		
18	$5.68 \cdot 10^4$		$6.11 \cdot 10^4$	
19	$5.08 \cdot 10^4$	$5.78 \cdot 10^4$		
20	$4.91 \cdot 10^4$		$5.43 \cdot 10^4$	
21	$3.66 \cdot 10^4$		$3.57 \cdot 10^4$	
22	$3.12 \cdot 10^4$	$3.15 \cdot 10^4$		
23	$2.96 \cdot 10^4$		$3.12 \cdot 10^4$	
24	$2.70 \cdot 10^4$	$2.82 \cdot 10^4$		
25	$1.91 \cdot 10^4$		$2.07 \cdot 10^4$	
26	$1.84 \cdot 10^4$		$1.85 \cdot 10^4$	
27	$1.35 \cdot 10^4$	$1.37 \cdot 10^4$		
28	$9.73 \cdot 10^3$		$1.00 \cdot 10^4$	
29	$6.60 \cdot 10^3$	$6.63 \cdot 10^3$		
30	$3.18 \cdot 10^3$		$3.21 \cdot 10^3$	
31	9.39		7.43	m=4 kink
32	6.90		6.74	m=4 Alfvén
33	6.84			
34	8.80			
35	6.70		6.67	

TABLE 2. NORMALIZED EIGENVALUES ω^2 FOR TOROIDAL AND CYLINDRICAL CODE

NO.	TOROIDAL		CYLINDRICAL		DESCRIPTION
	ODD	EVEN	ODD	EVEN	
36	6.36			6.25	
37	5.3				sound
38	4.7				sound
39	3.3			3.0	m=0 kink
40	2.9			2.67	m=0 Alfvén
41	2.8				
42	2.7				
50	2.5				
53	2.2			2.66	
43	2.70		2.69		m=3 Alfvén
44	2.68				
45	2.66				
46	2.63		2.63		
48	2.23		2.25		
49	2.10		1.90		m=3 kink
52	1.7		1.6		m=1 kink
60	0.24		0.22		m=1 Alfvén
62	0.24				
63	0.23				
64	0.22		0.20		
65	0.21		0.14		
66	0.19			0.19	m=2 Alfvén
67	0.19				
	0.18				
	0.17			0.19	

TABLE 2. NORMALIZED EIGENVALUES ω^2 FOR TOROIDAL AND CYLINDRICAL CODE

NO.	TOROIDAL ODD + EVEN	CYLINDRICAL		DESCRIPTION
		ODD	EVEN	
70	0.14		0.14	
71	0.13			
72	0.10		$2.7 \cdot 10^{-2}$	sound
73	0.07			
74	0.06			
75	$1.7 \cdot 10^{-2}$	1.10^{-2}		
76	1.10^{-2}			
77	1.10^{-2}			
			$1.7 \cdot 10^{-3}$	
89	7.10^{-4}	5.10^{-4}		
90	-0.49		-0.51	m=2 kink

Note that the eigenvalues range over 10 orders of magnitude. There is a good agreement for all the discrete modes, especially for the unstable kink mode. The difference between the stable $m = 4$ kink modes (eigenvalue no. 31) may come from the missing coupled $m = 6$ mode. For the fast magnetosonic waves, the fundamental modes agree very well. Usually, the toroidal code gives more accurate results in this branch, because the global functions can better represent the eigenfunctions with several nodes. Therefore, the eigenvalues are smaller in the toroidal case. In the continuous branches the Alfvén waves agree well, too. Some of the sound modes, e.g. eigenvalues no. 37 and 38, are off due to a coupling of order ϵ^2 with fast modes. It is worth noting that for a case with circular plasma cross section the agreement between the two codes is even better. Next we consider configurations with aspect ratio $\epsilon^{-1} = 10$ and a free plasma boundary $\Lambda = 2$. For an (almost) circular cross section, $\alpha = 1$ and $\delta = 0$, we obtain instabilities for $m_0 - 1 \leq nq_s \leq m_0$ with $n = 1$ and $m_0 = 1, 2, 3, 4, \dots, 7$. We have no difficulties in finding instabilities which corresponds to $m_0 = 7, 8$ modes. For elliptical cross sections the unstable regions are broadened. These results are very similar to these of a cylindrical plasma [14] the only difference being that even and odd modes are coupled. We now examine the influence of toroidicity on these modes. For an aspect ratio $\epsilon^{-1} = 6$ these modes are still unstable. In Fig. 5 the growth rates of the unstable modes with $n = 1$ are plotted for values of $\alpha = 1$ and 4. The points with $q_s = 1, 2, 3, \dots$, being marginal for $\alpha = 1$ and $\epsilon \rightarrow 0$, are now

considerably unstable. These calculations are performed using 9 and 11 Fourier components and 4 global expansion functions in the radial direction. The convergence of the eigenvalues becomes worse for high m values, especially for modes with $m = 6$ and 7. This is probably due to an overestimation of the stabilizing vacuum contribution δW_v . As noted earlier, the lowest eigenvalue is always approached from above. With decreasing aspect ratio our choice of the wall shape leads to a restriction of the possible distance of the wall from the plasma. The parameter Λ has to be chosen so that $2\Lambda\psi_p/R^2$ is less than unity. For $\epsilon^{-1} = 4$ we consider configurations with $\Lambda = 1.5$. The results are shown in Fig. 6 for values of $\alpha = 1$ and 4 and $n = 1$. The calculation is done with 11 Fourier components and 4 global expansion functions. The modes with different m values are strongly coupled and no points with marginal stability occur for $q_s < 6$. The small growth rates of the modes with $5 < q_s < 6$ and the stabilization for $q_s > 6$ is probably due to an overestimation of the vacuum integral. The slow convergence implies the need for 11 Fourier components. The comparison with the results of Sykes and Wesson [9] for the JET configuration is difficult because of the different distance of the wall from the plasma. For the value of $\epsilon^{-1} = 2,8$ the parameter $\Lambda = r_w/r_p$ is restricted to values less than 1.3. This implies that the wall is very close to the plasma, especially on the outer side of the plasma. Therefore, our results are more optimistic with respect to stability than theirs. Because of the strong coupling of the different modes the wall stabilization might be different

for a very fat tokamak from that of a straight system. The influence of the wall distance on the stability behavior - especially for fat tokamaks, is being examined in connection with the general Princeton code and will be reported later elsewhere. This discussion shows that for the equilibrium solution treated here increasing the aspect ratio does not stabilize unstable free boundary modes, at least for values of the safety factor at the plasma surface less than six. It appears that a conducting wall has a stronger stabilizing influence than in a straight system.

SUMMARY

A method for computing the MHD spectrum for a class of analytical tokamak equilibria is presented. The spectrum is obtained by extremizing the Lagrangian of the system with a Galerkin procedure. A well-posed matrix eigenvalue problem is obtained by using global expansion functions derived from the appropriate straight system eigenfunctions. Since no typical tokamak approximation is used, all parameters in the equilibrium solution can be varied freely. Therefore, we overcome the limitations of most of the tokamak stability calculations done so far, which use expansion techniques or surface current models.

The application of the code in a large aspect ratio, free boundary case yields a good agreement with a complementary straight system code [13]. Increasing the inverse aspect ratio in the equilibrium model treated does not provide a stabilization of free boundary modes, for values of the safety factor at the plasma surface less than six.

For a fixed plasma boundary the stability limit agrees well with Mercier's criterion. Furthermore, the results obtained for a fat tokamak, $\epsilon^{-1} = 2.8$, agree with those of a code which integrates the time dependent differential equations [9]. The application of our code to various configurations gives us confidence in the accuracy of our method. Therefore, this code will provide a good check for the general MHD code being developed at Princeton.

Appendix A - Expansion Functions for ξ

The straight system eigenfunctions must be modified to be suitable as expansion functions in the toroidal case. The evaluation of the perturbed magnetic fields shows that there are additional terms of order unity, instead of order ϵ as desired for a good approach. These terms are due to the derivation of the toroidal field $B^2 \propto 1/r^2$ and the Jacobian $g^{1/2} \propto 1/\tilde{r}$, with r and $\tilde{r}$ given in Eq. (2). We introduce a function P_0 and solve a simple differential equation for achieving that these contributions in lowest order vanish. The solution of this equation is proportional to $r^2 \tilde{r}$. The polynomial $P_0(u) = \sum_{v=0}^3 d_v u^v$ with $u = \psi \cdot \cos\theta / R^2$, used in [1], is an approximation for

$$P_0 = r^2 \cdot \tilde{r} / R^3 (1 - \delta)^{1/2} . \quad (A1)$$

The code allows for both versions. The constants d_v in the polynomial are given by

$$d_0 = 1 , \quad d_1 = [2 + 1/(1-\delta)]/2 , \quad d_2 = (3 - d_1) / 4(1-\delta) , \quad (A2)$$

$$d_3 = (d_1 - 2) / 8(1-\delta)^2 .$$

The part of the displacement vector, depending on the radial coordinate ψ , $\xi(\psi)$ in Eq. (12), is expanded in terms of the set

$$\xi^i(\psi) = \sum_{m=M_0}^{M_1} \sum_{v=1}^M \left\{ c^1_{m,v} \left(\xi_K^i \right)_{m,v} + c^2_{m,v} \left(\xi_F^i \right)_{m,v} + c^3_{m,v} \left(\xi_S^i \right)_{m,v} \right\},$$

$i = 1, 2, 3$, representing the kink (and shear-Alfvén) modes, the fast magnetosonic and the sound modes. We introduce the auxiliary functions for $m \neq 0$

$$f^1_{m,v}(\psi) = R \cdot n \left(\frac{1 + \alpha^2}{2} \frac{1}{m^2 \beta_v^2 - 1} \right)^{1/2} \left[\frac{m \beta_v - 1}{2m} J_{m+1}(Y_v) + \frac{m \beta_v + 1}{2m} J_{m-1}(Y_v) \right]$$

$$f^2_{m,v}(\psi) = n \frac{1 + \alpha^2}{2} J_m(Y_v) / m \cdot R$$

$$f^3_{m,v}(\psi) = R \cdot n \left(\frac{1 + \alpha^2}{2} \frac{1}{m^2 \beta_v^2 - 1} \right)^{1/2} \left[\frac{m \beta_v - 1}{2m} J_{m+1}(Y_v) - \frac{m \beta_v + 1}{2m} J_{m-1}(Y_v) \right]$$

where J_m are Bessel functions of order m with the argument

$$Y_v = \left\{ [1 + \alpha^2] / 2 (m^2 \beta_v^2 - 1) \right\}^{1/2} n \cdot \psi / R^2. \quad \beta_v \text{ are constants}$$

determined by the boundary conditions at the plasma boundary. For

$m = 0$ the Bessel functions depend on the argument

$$\tilde{Y}_v = \left\{ [1 + \alpha^2] / 2 \cdot (\beta_v^2 - 1) \right\}^{1/2} n \psi / R^2. \quad \text{Note that we consider here}$$

only the case $n \neq 0$.

i) fixed boundary case

$$\left. \begin{aligned} \left(\xi_K^1 \right)_{m,v} &= f^1_{m,v}(Y_v) \\ \left(\xi_K^2 \right)_{m,v} &= G(\psi) \cdot \xi^3_{m,v}(Y_v) \\ \left(\xi_K^3 \right)_{m,v} &= \frac{n(f^2_{m,v}(Y_v) - G(\psi) f^3_{m,v}(Y_v)/\psi)}{m + n G(\psi)} \\ &\quad + f^3_{m,v}(Y_v)/\psi \end{aligned} \right\} \quad \text{for } m \neq 0 \quad (A3)$$

$$\left. \begin{aligned} (\xi_K^1)_{m,v} &= R \cdot J_0(\tilde{Y}_v) \cdot \psi \\ (\xi_K^2)_{m,v} &= G(\psi) \cdot \xi_{m,v}^3(\tilde{Y}_v) \\ (\xi_K^3)_{m,v} &= -R \cdot \beta_v J_1(\tilde{Y}_v) / \psi \end{aligned} \right\} \text{for } m = 0 \quad (\text{A4})$$

The function $G(\psi) = (\psi/R^2)^2 (1 - \delta + \alpha^2)/2q(1 - \delta)\alpha O(\epsilon^2)$ represents the value of g_{33} averaged over the angle θ and q the safety factor. Neglecting the higher order term g_{13} we achieve $\xi_K \cdot B = O(\epsilon^2) \ll 1$. The constants β_v are determined by the v^{th} zero of $f_{m,v}^1$ at the plasma boundary $\psi = \psi_b$. The fundamental mode has a ψ -component vanishing at the boundary and a θ -component with one node at $\psi < \psi_b$.

The expansion functions for the magnesonic modes are given in terms of

$$\left. \begin{aligned} (\xi_F^1)_{m,v} &= f_{m,v}^3(Y_v) \\ (\xi_F^2)_{m,v} &= G(\psi) f_{m,v}^1(Y_v) / \psi \\ (\xi_F^3)_{m,v} &= f_{m,v}^1(Y_v) / \psi \end{aligned} \right\} \text{for } m \neq 0 \quad (\text{A5})$$

$$\left. \begin{aligned} (\xi_F^1)_{m,v} &= R \cdot \beta_v \cdot J_1(\tilde{Y}_v) \\ (\xi_F^2)_{m,v} &= R \cdot G(\psi) \cdot J_0(\tilde{Y}_v) \\ (\xi_F^3)_{m,v} &= R \cdot J_0(\tilde{Y}_v) \end{aligned} \right\} \text{for } m = 0 \quad (\text{A6})$$

The constants β_v are determined by the v^{th} zero of $f_{m,v}^3$ at the plasma boundary. The fundamental mode has a ψ -component vanishing at the boundary and a θ -component without a node.

Finally, the sound modes are represented in terms of

$$\left(\xi_s^1\right)_{m,v} = 0$$

$$\left(\xi_s^2\right)_{m,v} = \alpha \cdot \psi \left[\left(\xi_K^3\right)_{m,v} \left(\xi_F^1\right)_{m,v} - \left(\xi_K^1\right)_{m,v} \left(\xi_F^3\right)_{m,v} \right] / H(\psi)$$

$$\left(\xi_s^3\right)_{m,v} = -G^2(\psi) \left[\left(\xi_K^3\right)_{m,v} \left(\xi_F^1\right)_{m,v} - \left(\xi_K^1\right)_{m,v} \left(\xi_F^3\right)_{m,v} \right] \cdot H(\psi) / \alpha \cdot \psi^3$$

with $H(\psi) = R^3(1-\delta)^{1/2}(1+d_2 2(\psi/R^2)^2)$ and d_2 defined in Eq. (A2).

ii) free boundary case

For the expansion function of the kink modes we add to the set in (A3) and (A4) one more function $\left(\xi_K^i\right)_{m,0}$ (for $i = 1, 2, 3$) not vanishing at the plasma boundary. This is done by choosing a value of the constant β_0 (with $1/|m| < \beta_0 < \beta_1$) near the value $2/|m|$.

For the expansion functions of the fast modes we have to change the argument y_v of the functions in Eq. (A3) and (A4) in such a way that the β_v are determined by the v^{th} zero of $f_{m,v}^1$. For the $m = 0$ Fourier components the zero's of $J_0(\tilde{y}_v)$ have to be evaluated. The fundamental fast mode has a θ -component vanishing at the boundary $\psi = \psi_b$ and a ψ component without a node.

Appendix B - Expansions Functions for $\underline{a}$

The vector potential $\underline{a}$ has a unique, single valued solution with the boundary conditions of Eq.(9). In order to obtain a complete set we must add to the unique straight system solution a set of functions vanishing at the plasma boundary and, of course, at the wall. For the ψ, θ dependent part (see Eq.12) we expand the covariant components of the vector $\underline{a}(\psi, \theta)$ $(a_i)_{m,v}(\psi, \theta)$, $i = 1, 2, 3$. For $v = 1$ we have the modified cylindrical solution

$$(a_1)_{m,1} = \alpha \left[-CA \cdot \phi_1 + T(\psi_b)/(r^2 \cdot \tilde{r}) \Gamma_1 \right] \quad (B1)$$

$$(a_2)_{m,1} = \alpha \cdot CA \cdot \phi_2$$

$$(a_3)_{m,1} = \alpha \left[-CA \cdot \phi_3 + T(\psi_b)/(r^2 \cdot \tilde{r}) \Gamma_3 \right]$$

with the constant $CA = \alpha \cdot p'/(1 + \alpha^2)$. The functions $r(\psi, \theta)$ and $\tilde{r}(\psi, \theta)$ of Eq.(2) have to be introduced to adapt the straight system solution to the boundary conditions (9) in the toroidal geometry. The functions in (B1) are given by

$$\phi_1 = \left(\frac{k_1}{k_2^b} - \frac{g_1 k_3^b}{k_2^b g_3^b} \right) \frac{\psi_b \cdot \xi_b^1}{R^2 \cdot C} \quad (B2)$$

$$\Gamma_1 = \frac{g_1}{g_3^b} \xi_b^1$$

$$\phi_2 = \frac{k_2}{k_2^b} \psi_b \xi_b^1$$

$$\phi_3 = \left(\frac{k_3}{k_2} - \frac{g_3 k_3^b}{k_2 g_3^b} \right) \frac{\psi}{R^2 \cdot C} \psi_b \xi_b^1 \quad (B2)$$

$$\Gamma_3 = \frac{g_3}{g_3^b} \cdot \psi \cdot \xi_b^1$$

The subscript b denotes the value of the functions at the plasma boundary, ξ_b^1 the normal component of the displacement vector.

$C = (1 + \alpha^{2-\delta/2\alpha^2})^{1/2}$. The functions k_1, k_2, k_3, g_1, g_3 depend on $y = \left[(1 + \alpha^2)/2 \right]^{1/2} n\psi/R^2$ and are defined as follows

$$\begin{aligned} k_1 &= K_{m-1}(y) - CW I_{m-1}(y) - m \frac{(K_{m-1}^w - CW I_{m-1}^w) I_m(y)}{I_m'^w} \\ k_2 &= K_m(y) + CW I_m(y) \\ k_3 &= K_{m-1}(y) - CW I_{m-1}(y) - \frac{(K_{m-1}^w - CW I_{m-1}^w) I_m'(y)}{I_m'^w} \\ g_1 &= \frac{K_m(y)}{y} - \frac{K_m'^w I_m(y)}{I_m'^w y} \\ g_3 &= \frac{1}{m} \left(K_m'(y) - \frac{K_m'^w}{I_m'^w} I_m'(y) \right) \end{aligned} \quad (B3)$$

with $m \neq 0$ and $n \neq 0$, $CW = -K_m^w/I_m^w$. The index w denotes the value of the function at the wall. I_m, K_m are modified Bessel functions, primes denote derivatives with respect to the argument.

For $m = 0$ we have

$$\Gamma_1 = 0, \quad \phi_3 = 0,$$

and in Eq. (B2) we have to insert the functions

$$k_1 = K_1(y) + \frac{K_O^W}{I_O^W} I_O(y)$$

$$k_2 = K_O(y) - \frac{K_O^W}{I_O^W} I_O(y)$$

$$k_3 = 0$$

$$g_1 = 0$$

$$g_3 = K_1(y) - K_1^W / I_1^W I_1(y)$$

The expansion functions for $v = 2, 3 \dots$ are given by the complete set of functions vanishing at the plasma boundary and at the wall

$$(f)_{m,\mu}(\psi) = J_{m,\mu}(z_\mu) - J_{m,\mu}(z_\mu^b) \cdot Y_{m,\mu}(z_\mu) / Y_{m,\mu}(z_\mu^b)$$

with $\mu = v - 1$ and $z = (1 + \alpha^2/2)^{1/2} \beta_v n\psi/R^2$. The values of the constants β_v are determined by the μ^{th} zeros of $f_{m,\mu}$ at the wall $\psi = \psi_w$. We then get for $m \neq 0$

$$\phi_1 = 0 = \phi_3, \quad \phi_2, \quad \Gamma_1, \quad \Gamma_3 = f_{m,\mu} \quad (\text{B4})$$

and for $m = 0$

$$\Gamma_1 = \phi_3 = 0, \quad \phi_1, \quad \phi_2, \quad \Gamma_3 = f_{m,\mu} \quad (\text{B5})$$

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Fig. 1. Coordinate System

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Fig. 2. Stability boundary for internal modes; $\delta = 0$

(a) 762013

(b) 762012

Fig. 3. Spectrum for internal modes with $q(0) = 1$, $\delta = 0$

a) $\epsilon^{-1} = 10$

b) $\epsilon^{-1} = 2.8$

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Fig. 4. Unstable internal modes for $\epsilon^{-1}=2.8$, $\alpha=1$, $\delta=0$.

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Fig. 5. Unstable free boundary modes for $\epsilon^{-1}=6$, $\Lambda=2$, $\delta=0$, dots and connecting solid curve represent $\alpha=1$ circles and connecting dashed curve represent $\alpha=4$.

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Fig. 6. Unstable free boundary modes for $\epsilon^{-1}=4$, $\Lambda=1.5$, $\delta=0$, dots and connecting solid curve represent $\alpha=1$, circles and connecting dashed curve represent $\alpha=4$.