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Numerical Techniques for Studying
Repetitively Pulsed
Reactor Time-Dependent Behavior

by

G. C. Hopkins

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NUMERICAL TECHNIQUES FOR STUDYING
REPETITIVELY PULSED REACTOR TIME-DEPENDENT BEHAVIOR

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ABSTRACT

Computationally efficient numerical techniques are developed for studying the time-dependent behavior of repetitively pulsed reactors. Reactivity perturbations synchronous with the external pulse are also considered. Three types of reactivity are considered: step change, sinusoidal, and thermal feedback. Noniterative techniques, based upon Laplace transforms and residues, are used to solve the first two types, and continuous analytic continuation is used to solve the third.

I. INTRODUCTION

In a repetitively pulsed reactor system excited by some form of accelerator and synchronously pulsed reactivity, a problem arises in determining the neutronic response in a computationally efficient manner. Computationally efficient here means a mathematical method that involves as few parameters as possible, is easily programmable, requires a relatively short computation time, and retains a maximum degree of accuracy with a minimum amount of approximation.

The system chosen for analysis is a small, fast assembly, fueled with ^{239}Pu , and repetitively pulsed by an accelerator. The possibility of a reactivity perturbation, synchronous with the externally induced neutron pulse, is also considered.

The pulse characteristics are shown in Fig. 1. The duration of the source pulse is for a period t_p , with the total cycle time T . Synchronous with this pulse, the background reactivity k_2 is perturbed to k_1 , both k_1 and k_2 being subcritical.

The problem is to obtain the neutronic response over the entire cycle, $t = 0 \rightarrow t = T$, after it has become quasi-stabilized. This means that the value of the neutron response at any point in time in any cycle is equal to the response at the same point in any other cycle.

Fig. 1. Relation of input source pulse, reactivity pulse, and neutron output.

Once the quasi-stabilized initial value of the response, at $t = 0$, is known, the response over the remainder of the cycle can be calculated by a variety of numerical methods. In fact, the problem can be solved by assuming an arbitrary initial value, using some method to calculate the entire cycle, taking the end point as the initial point for the following cycle, and then repeating the process until quasi-stabilization occurs, but that method would fall outside the definition of computationally efficient.

Therefore, the methods to be examined will be used to calculate, in a noniterative manner, the initial values of the pertinent variables after quasi-stabilization has occurred for all forms of reactivity except that associated with thermal feedback. For that case, however, a convergence acceleration technique will be employed to greatly reduce the computation time.

The starting point for all the methods is the point reactor kinetic equations, with one delayed group being assumed.

$$\frac{dN(t)}{dt} = \frac{\rho(t) - \beta}{\Lambda} N(t) + \lambda C(t) + S(t), \quad (1)$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} N(t) - \lambda C(t). \quad (2)$$

These kinetic equations are the sole basis for the methods developed. No other parameters, such as fission rate or precursor production rates between pulses, are needed, nor are any intermediate mathematical operations on variables contained in the equations necessary. Simplicity is one of the primary characteristics of this development.

Many of the difficulties encountered in the numerical part of the solution are indirectly a result of the very short generation time Λ considered: 5.0×10^{-9} sec. Other parameters typical of the reference system are a pulse width t_p of 5.0×10^{-6} sec, a cycle duration T of 5.0×10^{-3} sec, and a peak source strength of 10^{17} neutrons/sec.

Previous studies of this problem have involved approximating the precursor density as a constant,¹ computationally long asymptotic approaches to the quasi-stabilized state,² and considering the power pulse as a delta function.³ None of these restrictions are contained in the present formulations.

II. STEP CHANGE IN REACTIVITY

Three forms of reactivity will be considered, the first of which is analogous to the reactivity shown in Fig. 1, namely, a step change in reactivity.

$$\rho(t) = (k_1 - 1)/k_1 \quad t = 0 \rightarrow t = t_p, \quad (3)$$

$$\rho(t) = (k_2 - 1)/k_2 \quad t = t_p \rightarrow t = T. \quad (4)$$

The method used for this case is the Laplace transform-residue method, the detailed development of which is contained in the appendix. Basically, the kinetics equations are Laplace transformed, the coefficient matrix is formed, and, after some algebraic manipulation, the resulting equation system is inversely transformed back to the time domain by using residues.

Expressions for the neutron and precursor densities as functions of time appear in the form

$$N(t) = \sum_{i=1}^3 f(N_0, C_0, R_i) e^{R_i t}, \quad (5)$$

$$C(t) = \sum_{i=1}^3 g(N_0, C_0, R_i) e^{R_i t}, \quad (6)$$

where N_0 and C_0 are the initial values of the variables $N(t)$ and $C(t)$, respectively, in the interval $0 \rightarrow t$, and the R_i are the roots obtained from the coefficient matrix.

At this point, one of the advantages of this method can be seen. Some numerical techniques, such as finite difference, introduce an approximation for the derivatives. As a result, the equations equivalent to Eqs. (5) and (6) advance the solution over many small time steps, the values of the variables at point i being functions of the same variables at point $i - 1$. As more accuracy is desired, the time steps must be made smaller, which increases the computing time. This is necessary even when the reactivity is constant.

With the Laplace-residue technique, Eqs. (5) and (6) are valid for any value of t inserted into them, as long as the reactivity remains constant. There is no dependency upon variables at the previous step, but rather, only upon the initial values.

For the form of reactivity being considered from Eqs. (3) and (4), only two time steps are required, which lead to two relationships resulting from Eq. (5).

$$N(t_p) = f(N_0) \quad , \quad (7)$$

$$N(T) = g[N(t_p)] \quad . \quad (8)$$

A third equation, $N(T) = N_0$, relates the equality of the variables at the beginning and end of the cycle, which exists only after quasi-stabilization has occurred. With three equations in three unknowns, the initial values of the variables are uniquely determined noniteratively. Once N_0 and C_0 have been found, $N(t)$ can be calculated for any value of t during the cycle from Eq. (5).

III. SINUSOIDAL REACTIVITY

The second form of reactivity, more realistic than the first, is a sinusoidal variation over the first part of the cycle from $t = 0 \rightarrow t = t_p$. The method used to solve this problem is actually applicable to any *a priori* known form of reactivity, either linear or nonlinear.

The reactivity expression for the entire cycle is

$$\rho(t) = \rho_0 + \rho_1 \sin\left(\frac{t}{t_p} \pi\right) \quad t = 0 \rightarrow t = t_p, \quad (9)$$

$$\rho(t) = \rho_0 \quad t = t_p \rightarrow t = T. \quad (10)$$

Once again, the Laplace method can be used. For this case, however, instead of only two time intervals, there are 1001, as shown in Eqs. (11) and (12).

$$N(t_{i+1}) = f[N(t_i)] \quad t = 0 \rightarrow t = t_p, \quad (11)$$

$$N(T) = g[N(t_p)] \quad t = t_p \rightarrow t = T. \quad (12)$$

Since the reactivity does not change after $t = t_p$, there is only one time interval from t_p to T . However, from $t = 0 \rightarrow t = t_p$, the reactivity is approximated as a constant over each of 1000 intervals. As seen in Eq. (11), the value of the variables at the end of each interval is taken as the initial value for the next interval, using the relationship of Eq. (5). As before, during quasi-stabilization, $N(T) = N_0$. This results in 1002 equations in 1002 unknowns.

Many numerical techniques in computer subroutines for solving simultaneous equations cannot be applied in this case, however, because the 2004 by 2004 array that results, for the variables $N(t)$ and $C(t)$, is too large.

However, an examination of the coefficient matrix reveals that no row contains more than three elements and that every diagonal element is a "1." This makes the problem ideally suited for Gaussian elimination, the solution again being obtained noniteratively.

IV. THERMAL FEEDBACK REACTIVITY

A. Analytic Continuation

The final form of reactivity considered is the nonlinear case of thermal feedback. Although thermal feedback is of little significance in the system being considered, one computational method for treating the general mathematical problem is presented.

The thermal feedback mechanism, assumed to exist over the entire cycle, is

$$\rho(t) = \rho_0 + \alpha [T_f(t) - T_f^0] \quad t = 0 \rightarrow t = T, \quad (13)$$

$$C_f \frac{dT_f(t)}{dt} = HN(t) - h [T_f(t) - T_c(t)], \quad (14)$$

$$C_c \frac{dT_c(t)}{dt} = h [T_f(t) - T_c(t)] - C_c \dot{\omega} [T_{c,o}(t) - T_{c,i}], \quad (15)$$

where

- α = temperature coefficient,
- T_f = temperature of the fuel,
- T_c = temperature of the coolant,
- C_f = heat capacity of the fuel,
- C_c = heat capacity of the coolant,
- h = heat transfer coefficient from fuel to coolant,
- H = a conversion factor from neutron level to power level,
- $\dot{\omega}$ = coolant flow rate,
- $T_{c,o}$ = temperature of the coolant at the outlet,
- $T_{c,i}$ = temperature of the coolant at the inlet, assumed to be a constant.

For this form of nonlinear reactivity, the method of solution becomes, of necessity, iterative. One iterative technique will be examined and compared with two others.

The technique used for this problem is continuous analytic continuation,⁴ the algorithm for which is

$$N(t_{i+1}) = \sum_{k=0}^K \frac{1}{k!} N^{(k)}(t_i) (t_{i+1} - t_i)^k, \quad (16)$$

$$(t_{i+1} - t_i) = f[\epsilon, N(t_i), N^{(K+1)}(t_i)]. \quad (17)$$

An order K Taylor expansion of the time-dependent variables is made about the point t_i . $N^{(k)}$ is the kth derivative of $N(t_i)$. As seen from Eq. (16), this is a one-step method because at each time step solutions are obtained from the values of the dependent variables and their derivatives at the previous step only.

The size of the time step itself, shown in Eq. (17), is expanded or contracted automatically, depending upon the current behavior of the variables. In addition, the time-step size is an explicit function of an input degree-of-accuracy parameter, ϵ . This is one of the distinguishing features of this method not found in many other numerical techniques. The method has also been further optimized in that the order K resulting in the least computation time is periodically checked and reset, if necessary.

For this reactivity case, five variables are calculated as functions of time: $N(t)$, $C(t)$, $\rho(t)$, $T_f(t)$, and $T_c(t)$. Since the problem is nonlinear, the method is iterative, with the functional values at the end of a cycle not being equal to the values at the beginning of the cycle. Rather, the end-point values, at $t = T$, serve as the initial values for the following cycle, and the process is repeated until quasi-stabilization occurs.

Although this method is faster than two other techniques that will be discussed, it is still desirable to accelerate convergence to quasi-stabilization, if possible. This can be done by using spline functions.⁵ Spline functions can be used for many purposes, including extrapolation, more accurately than many other curve-fitting techniques.

For this problem, the solution over the entire cycle at quasi-stabilization is equivalent to knowing only the initial values of the variables at quasi-stabilization. Therefore, the basic analytic

continuation technique is used to generate 20 successive initial values. These initial values are then fed into a spline extrapolation subroutine, which extrapolates out, in time, a factor of five and returns a new set of initial values. This process is then repeated until all variables have become quasi-stabilized.

B. Comparison with Other Techniques

Two other numerical techniques were examined to form a comparison with continuous analytic continuation: Runge-Kutta and a modified version of the Laplace method discussed previously. Both were iterative techniques, used in conjunction with spline functions, but neither allowed an accuracy input parameter nor variable time steps.

The modified Laplace method treated the cycle as a series of intervals over each one of which the reactivity was assumed to be a constant as calculated from the previous interval using Eq. (13). Since feedback effects, for this particular system with arbitrarily chosen representative parameters, were almost insignificant, the reactivity changed little within each of the two intervals from $t = 0 \rightarrow t = t_p$ and $t = t_p \rightarrow t = T$. This, in turn, resulted in the modified Laplace method possessing both stability and accuracy for very few time steps, which led to a considerably shorter computation time than that of the continuous analytic continuation method. However, in a problem where feedback effects are significant, the Laplace technique would require many more time steps and, therefore, more computing time. The computing time for analytic continuation, meanwhile, would remain essentially unchanged, since the neutron time behavior, not the reactivity behavior, is the predominant factor in determining time-step size. Also, for complicated problems, the characteristic identity of the differential equations is not lost in the translation to computer programming for analytic continuation, whereas it is lost for the Laplace technique, making it more difficult to debug.

The third reactivity case was also solved by using the Runge-Kutta method to form a comparison with the continuous analytic continuation (Taylor series) method. All computer programming, except for the algorithms, was identical. Several parametric studies were made varying the generation time,

reactivity magnitudes, and source pulse duration time. For these studies, the ratio of central processor times, of Runge-Kutta to analytic continuation, varied from 1.9 to 2.4. This is due partly to the fact that analytic continuation contains fewer multiplications and additions than Runge-Kutta for the same accuracy. Also, Runge-Kutta does not permit variable time steps. Furthermore, because of similar mathematical characteristics, these results should hold for most other neutron kinetics problems.

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APPENDIX

LAPLACE TRANSFORM-RESIDUE TECHNIQUE

The Laplace transform-residue technique can be used over a broad time interval with *a priori* constant reactivity or successively over many smaller intervals, in each of which the reactivity is assumed constant. For either case, the method is valid only for a constant reactivity. A constant source will also be assumed, although any source that is an analytic function could be treated equally well. Zero subscripts refer to the initial values of the variables at the beginning of the applicable interval.

The method begins with the point-reactor kinetics equations, assuming one delayed group.

$$\frac{dN(t)}{dt} = \frac{\rho - \beta}{\Lambda} N(t) + \lambda C(t) + \bar{S} \quad , \quad (A-1)$$

$$\frac{dC(t)}{dt} = \frac{\beta}{\Lambda} N(t) - \lambda C(t) \quad , \quad (A-2)$$

where

- $N(t)$ = time-dependent neutron level,
- $C(t)$ = time-dependent precursor level,
- ρ = reactivity,
- β = total delayed neutron fraction,
- λ = precursor decay constant,
- Λ = prompt neutron generation time,
- $\bar{S}$ = source strength.

The Laplace transform of these equations is formed by

$$s\bar{N}(s) - N_0 = \frac{\rho - \beta}{\Lambda} \bar{N}(s) + \lambda \bar{C}(s) + \frac{\bar{S}}{s} \quad , \quad (A-3)$$

$$s\bar{C}(s) - C_0 = \frac{\beta}{\Lambda} \bar{N}(s) - \lambda \bar{C}(s) \quad , \quad (A-4)$$

where s is the Laplace independent variable and barred functions of s are the Laplace transforms of functions of t .

Equations (A-3) and (A-4) are now recast into matrix form.

$$\underline{MN} = \underline{S} \quad , \quad (A-5)$$

where

$$\underline{N} = \begin{pmatrix} \bar{N}(s) \\ \bar{C}(s) \end{pmatrix} \quad , \quad (A-6)$$

$$\underline{S} = \begin{pmatrix} N_0 + \frac{\bar{S}}{s} \\ C_0 \end{pmatrix} \quad , \quad (A-7)$$

and

$$\underline{M} = \begin{pmatrix} s - \frac{\rho - \beta}{\Lambda} & -\lambda \\ -\frac{\beta}{\Lambda} & s + \lambda \end{pmatrix} \quad . \quad (A-8)$$

The solution to Eq. (A-5) is

$$\underline{N} = M^{-1} \underline{S}, \quad (A-9)$$

which results in

$$\bar{N}(s) = \frac{s[(s + \lambda)N_o + \lambda C_o] + (s + \lambda)\bar{S}}{D}, \quad (A-10)$$

and

$$\bar{C}(s) = \frac{s\left[\frac{\beta}{\Lambda}N_o + \left(s - \frac{\rho - \beta}{\Lambda}\right)C_o\right] + \frac{\beta}{\Lambda}\bar{S}}{D}, \quad (A-11)$$

where

$$D = s\left[\left(s - \frac{\rho - \beta}{\Lambda}\right)(s + \lambda) - \frac{\lambda\beta}{\Lambda}\right]. \quad (A-12)$$

Equation (A-12) is now set equal to zero, and the three roots of s , R_1 , R_2 , and R_3 are solved. The time-dependent solutions for $N(t)$ and $C(t)$ are formed by the sum of the residues for Eqs. (A-10) and (A-11), respectively, obtained by using the roots

$$N(t) = \sum_{i=1}^3 \frac{(N_o R_i + \bar{S})(R_i + \lambda) + R_i \lambda C_o}{\prod_{\substack{j=1 \\ j \neq i}}^3 (R_i - R_j)} e^{R_i t} \quad (A-13)$$

and

$$C(t) = \sum_{i=1}^3 \frac{\frac{\beta}{\Lambda}(R_i N_o + \bar{S}) + \left(R_i - \frac{\rho - \beta}{\Lambda}\right)C_o}{\prod_{\substack{j=1 \\ j \neq i}}^3 (R_i - R_j)} e^{R_i t}. \quad (A-14)$$

As can be seen, these solutions are valid for any value of t inserted into Eqs. (A-13) and (A-14), provided that the roots R_i remain unchanged. This, in turn, is seen to be the case, in Eq. (A-12), as long as the reactivity remains constant. When the reactivity does change, the values of $N(t)$ and $C(t)$ at the point of change become N_o and C_o , respectively, for the next interval. Then, a new set of roots is obtained from Eq. (A-12), and Eqs. (A-13) and (A-14) are solved again by using the new N_o , C_o , and R_i .