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A METHOD OF PROCESS IDENTIFICATION OF CERTAIN LINEAR CONTROL SYSTEMS

by

E A Aronson

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ABSTRACT

Recent interest in adaptive control systems has led to considerable investigation of the problem of process identification; that is, the problem of describing the performance of an operating control system.

In the case of a linear-lumped control system with slowly changing performance characteristic, the process identification problem can be solved by evaluating, as functions of time, the coefficients of the polynomials which constitute the transfer function of the "undetermined" control system.

In this paper a method of estimating these coefficients is described and its advantages and limitations are discussed. The technique applies only to linear-lumped control systems whose zeros are known, and hence estimates only the poles of the system. In this technique the input and output of the undetermined system are fed into a set of fixed, known linear systems. These fixed systems may be designed so that they are isolated from the conditions which cause the coefficients (or parameters) of the undetermined system to change. The output of these fixed systems, along with the output of the undetermined system, is fed into a computer which estimates the unknown parameters. The method has three major advantages over other techniques in this area.

1) No extraneous inputs are introduced into the undetermined system.

2) The accuracy of the estimation does not depend on the time taken to evaluate the parameters.

3) Only the input and output of the undetermined system need be available for measurement. No derivative of these signals need be taken.

After a description of the method itself, a discussion is given concerning the accuracy of the estimation in the presence of noise. In particular, techniques are evolved to enable choice of the fixed systems to lessen any error in the estimation due to noise. Expressions are evolved for error estimation with gaussian noise and signals. Techniques are also discussed which further lessen the errors in the estimations once the fixed systems have been chosen.

Finally, a computer simulation of the estimation method is presented for a few simple cases. Conclusions drawn from the simulation indicate that the method does indeed measure the unknown parameter and yield good results for reasonable noise levels. The simulation also corroborates the error reduction techniques previously discussed.

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INTRODUCTION

In recent years there has been an increasing interest in the so-called "adaptive" control systems. In general, the term adaptive is applied to systems which are capable of maintaining proper performance over a wide range of environmental conditions. Almost all feedback control systems are somewhat adaptive in that they are not sensitive to small changes in their system parameters caused by a changing environment. However, the term adaptive is usually reserved for the more exotic systems in which a deliberate effort is made to overcome the effects of the environment on the system performance.

Many different techniques have been proposed for designing systems to be adaptive. One significant class of methods involves the continuing measurement of the system function, and on the basis of this information adjusts the compensation to bring the overall system performance within the design specifications.

In this paper a technique for measuring the system function of an operating system is proposed and its advantages and limitations are investigated. The technique is restricted to the analysis of linear-lumped systems, or systems which can reasonably be approximated by a linear-lumped system. The class of systems considered here is a very significant one.

The modification of the compensation in view of the information provided by this measurement is not discussed in this paper as the two parts of the problem are quite distinct.

PROCESS IDENTIFICATION OF LINEAR-LUMPED SYSTEMS

The problem of describing the performance of an operating control system, often called "process identification," [4]* has been studied for many years and from many points of view. Even restricting ourselves to linear-lumped time-invariant systems, we find that there are numerous functions which can be used to describe the system, and varied techniques for measuring the various functions and/or the parameters involved in them.

Probably the three most common ways of describing a linear-lumped, time-invariant system are the following:

1. Differential Equation.

$$\sum_{i=0}^{n} \alpha_i \frac{d^i y(t)}{dt^i} = \sum_{j=0}^{m} \beta_j \frac{d^j x(t)}{dt^j} \quad ; \quad \alpha_n, \beta_m \neq 0$$

where x and y represent, respectively, the input and the output of the system, and the α_i and β_j are parameters which completely characterize the system.

2. Transfer Function (frequency response).

$$H(s) = \frac{\sum_{j=0}^{m} \beta_j s^j}{\sum_{i=0}^{n} \alpha_i s^i} \quad ; \quad \alpha_n, \beta_m \neq 0$$

where the coefficients α_i and β_j again characterize the system, and are the same parameters that appear in the differential equation.

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*Square brackets denote references.

3. Impulse Response. For this restricted class of systems the output response of the system to an input of the form of an impulse can be written as

$$h(t) = \sum_{i=o}^{n} (K_{oi} e^{-c_{oi} t} + t K_{1i} e^{-c_{1i} t} + \dots + t^m K_{mi} e^{-c_{mi} t})$$

The K_i and c_i again completely characterize the system.

In theory, all of these descriptions are equivalent, since any one of them can be derived from any other. When it comes to practical matters, however, there is considerable difference in determining the system parameters from physical measurements. To illustrate the sort of difficulties that arise in the practical measurement of system parameters, three techniques are discussed in the examples below. These are not necessarily the best methods available, but are representative of the general approaches to the problem.

Example 1

The first method is based on the direct use of the differential equation describing the system. If the input and output, together with pertinent derivatives, are measured at a set of points, t_k, a set of linear simultaneous equations can be set up

$$\sum_{i=o}^{n} \alpha_i \frac{d^i j(t_k)}{dt^i} = \sum_{j=o}^{m} \beta_j \frac{d^j x(t_k)}{dt^j}$$

$$k = 0, 1, \dots, n + m + 1 \quad t_i \neq t_j \text{ if } i \neq j.$$

The solution of this set of equations will give the α_i and β_j which characterize the system.

The major disadvantage of this method is that it is generally very sensitive to any errors which might occur in the values of x, y, or their derivatives. This difficulty is compounded by the fact that the derivatives are not normally available, and when they are measured by differentiating the input and output signals, the noise may be amplified very seriously.

The advantages of this method are that it can use the input and output signals that occur naturally in the operation of the system, and the evaluation of the parameters requires only a finite, rather short, period of time.

Example 2

The second method involves a direct measurement of the frequency response. A very common method of characterizing a system is to measure its steady-state response to applied sinusoids of different frequencies. The coefficients of the transfer function can then be determined from the solution of a set of simultaneous linear equations; or, if the frequency response is measured as a continuous function of frequency, by certain techniques for approximating a curve by rational functions.

The major disadvantage of this method is the use of a generally extraneous input signal, the sinusoid. While this is often quite satisfactory for a piece of equipment in the laboratory, it can be quite disconcerting to use such a test signal on an aircraft in actual flight. Also, as the sinusoidal response must be measured after the transients have decayed, the time required to perform a reasonably complete frequency response test is likely to be excessive for a system whose characteristics can change moderately quickly, such as a rocket.

Example 3

The third method uses an autocorrelation technique. In a preceding paragraph we mentioned the general undesirability of introducing specific inputs to an operating control system. An input which is generally undesirable but which tends to minimize the undesirable effect on the output is white noise. This characteristic of white noise leads to an interesting method described by Truxal [4], which uses white noise as follows:

It is well known that if $\phi_x(\tau)$ represents the autocorrelation function of a stationary ergodic random process (Appendix B) which is the input to a linear-lumped time-invariant system whose weighting function is h(t), the cross correlation function of the input, x(t), and the output, y(t), is given by

$$\phi_{xy}(t) = \int_0^t h(\tau)\phi_x(t-\tau)\, d\tau$$

The time-invariant property of the system has been chosen since we assume the system parameters remain essentially constant over the period of time that it takes to compute the weighting function of the undetermined system with suitable accuracy.

Now, if the input is white noise,

$$\phi_x(t-T) = K\delta(t-\tau);$$

where δ is the Dirac delta function. Hence,

$$\phi_{xy}(t) = Kh(t)$$

Consequently, if we introduce white noise into the input of the undetermined system and perform the integration above, we may compute the weighting function of the system. Knowing the weighting function of the system, we have, at least in implicit form, all desirable information as to system operation.

Unfortunately, besides the existence of a generally undesirable output, this method involves an extremely difficult computation; namely, the computation of ϕ_{xy}. To compute ϕ_{xy} infinite delay is theoretically necessary, and hence considerable time delay may be necessary to yield reasonable accuracy. Consequently, we can evaluate the system weighting function only at discrete, perhaps widely spaced, times. Sufficiently accurate computation may therefore place serious restrictions on the maximum possible rate of change of the system parameters.

Another objection to this method is that the decomposition of the weighting function, given in some analog or digital form as an explicit time function, into useful information may be an extremely difficult procedure.

In view of the discussion of the above methods, one can list a number of desirable characteristics for a process identification method to be used in a system which is to adapt itself to significant changes in its operating environment.

1. <u>No extraneous inputs</u>. The technique should make use of the normal input and output of the system while it is in actual operation.

2. <u>Finite evaluation time</u>. If changes in the system parameters are to be detected, the evaluation time must be shorter than

the period of time in which significant changes can occur.

3. Reasonable accuracy.

In the next chapter a new method of process identification is proposed which attempts to meet these requirements. We call the process we desire to identify the "undetermined" system. Briefly, in this technique the input and output of the undetermined system are fed into a set of fixed, known linear systems. These fixed systems may easily be designed such that they are independent of the conditions which cause the parameters of the undetermined system to change. The outputs of these fixed systems, along with the output of the undetermined system are fed into a computer which generates a set of outputs which are estimates of the unknown parameters. A simplified block diagram of the process is shown in Fig. F 1.

After a basic discussion of the new method and the derivation of the equations involved in the computation, the major part of this paper is devoted to the problem of selection of the fixed systems in order to minimize the error in the estimated parameter values. Because of the complexity of the computation and its basic nonlinear nature, an exact solution for the optimum choice of fixed systems was not obtained; however, criteria for selecting the fixed systems to obtain "good" performance are presented. This proposed method was simulated on a computer, and the results are presented in the final chapter.

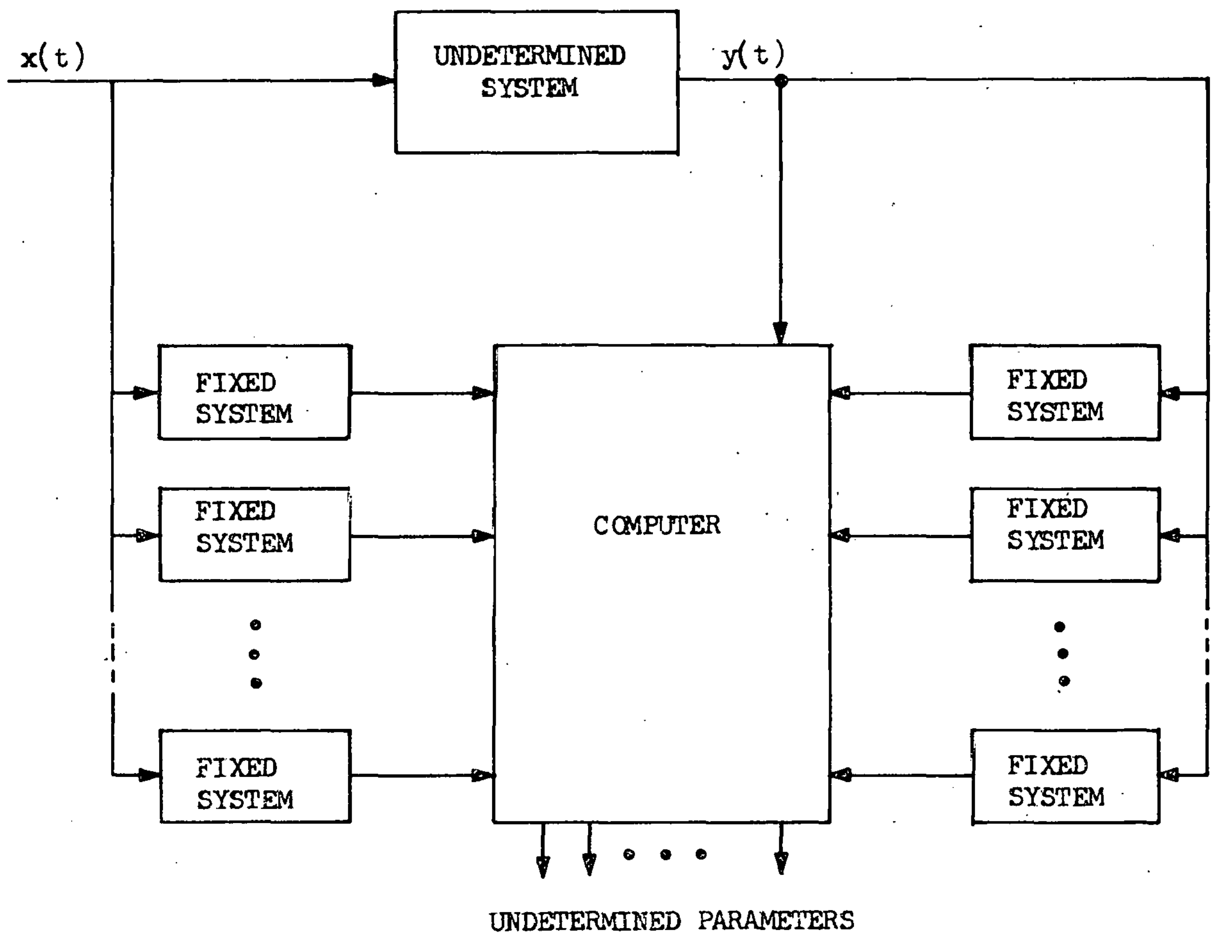

FIGURE F1. FLOW DIAGRAM OF PROPOSED SOLUTION

CHAPTER II

DESCRIPTION OF THE METHOD

2.1 Statement of the Problem

We assume that we have an undetermined linear-lumped, unconditionally stable, operating control system whose transfer function H(s) is represented by

$$H(s) = G(s)K(s) = G(s)\frac{1}{\sum_{i=0}^{n} c_i s^i} \quad ; \; c_n \neq 0 \qquad (2.1)$$

Since H(s) is a ratio of polynomials in s, the factoring of H(s) into G(s) and K(s) is always possible, and G(s) is a ratio of polynomials in s.

We restrict the discussion to those systems whose G(s) is known throughout the history of system operation, but that K(s) changes in a manner which can be suitably represented by varying at least some of the $\{c_i\}$ slowly with time in some unspecified fashion. In effect, we have stated that all of the zeros, if any, of H(s) are known, but at least some of the poles are not known.

It is our purpose here to develop a method for measuring the $\{c_i\}$ throughout the history of system operation, while the undetermined system is performing its control function. We shall call the $\{c_i\}$ the "undetermined parameters" of the undetermined system.

2.2 The Second Order System

Although the method presented in this paper is applicable to any control system which obeys the restrictions of section 2.1, we shall first describe the method as applied to a simple second order undetermined system. A discussion of the m^{th} order system will be found in section 2.3. A control system which is, or may be closely approximated by, a second order lumped-linear system is commonly encountered in practice; and hence a detailed analysis of the method as applied to such a system is of considerable importance. Except for section 2.3 we shall devote the remainder of this paper to the second order system described in this section.

We assume that the undetermined system contains no zeros and that the zero frequency gain remains constant at unity. The natural frequency, w_n, and the damping factor, ζ; however, do vary in some manner. For such a system, $G(s) = 1$, and the transfer function is given by

$$H(s) = \frac{1}{as^2 + bs + 1}$$

It is our purpose to measure a and b while the system is performing its control functions. Knowledge of the undetermined parameters a and b immediately yield the natural frequency and damping factor via the simple relations

$$w_n = 1/a^{\frac{1}{2}}$$

$$\zeta = bw_n/2$$

Consider two other systems whose transfer functions are given by

$$H_1(s) = \frac{1}{a_1 s^2 + b_1 s + 1}$$

$$a_1 \neq a_2, \quad b_1 \neq b_2$$

$$H_2(s) = \frac{1}{a_2 s^2 + b_2 s + 1}$$

where a_1, a_2, b_1, and b_2 are known and fixed. We shall call these constants the "fixed parameters" and the systems represented by H_1 and H_2 the "fixed systems." We stipulate that the fixed systems are unconditionally stable.

Let $x(t)$ be the input to the undetermined system and let $X(s)$ be the Laplace transform of this input. We define

$$Y(s) = H(s)X(s)$$

$$Y_1(s) = H_1(s)X(s)$$

$$Y_2(s) = H_2(s)X(s)$$

We also formally define the quantities a_1^*, a_2^*, b_1^*, and b_2^* by the relationships

$$a_1^* + a_1 = a_2^* + a_2 = a$$
$$b_1^* + b_1 = b_2^* + b_2 = b \qquad (2.2\text{-}1)$$

Consider the differences $Y_1 - Y$ and $Y_2 - Y$

$$Y_i - Y = H_iX - HX = (H_i - H)X \qquad i=1,\ 2$$

hence,

$$Y_i - Y = \left[\frac{1}{a_i s^2 + b_i s + 1} - \frac{1}{as^2 + bs + 1}\right] X$$

$$= \left[\frac{1}{a_i s^2 + b_i s + 1} - \frac{1}{(a_i^* + a_i)s^2 + (b_i^* + b_i) + 1}\right] X$$

$$= \left[\frac{a_i^* s^2 + b_i^* s}{(a_i s^2 + b_i s + 1)(as^2 + bs + 1)}\right] X$$

In terms of the transfer functions defined previously, we have,

$$Y_i - Y = a_i^* s^2 H_i HX + b_i^* s H_i HX$$

$$i=1,\ 2 \qquad (2.2\text{-}2)$$

$$= a_i^* s^2 H_i Y + b_i^* s H_i Y$$

From (2.2-1) we may express a_2^* and b_2^* in terms of a_1^* and b_1^*, vis.,

$$a_2^* = a_1^* + a_1 - a_2$$

$$b_2^* = b_1^* + b_1 - b_2$$

With these relationships we may rewrite, with some rearrangement, equations (2.2-2) as the two equations

$$Y_1 - Y = a_1^* s^2 H_1 Y + b_1^* s H_1 Y$$
$$Y_2 - Y - (a_1 - a_2) s^2 H_2 Y - (b_1 - b_2) s H_2 Y = a_1^* s^2 H_2 Y + b_1^* s H_2 Y \qquad (2.2\text{-}3)$$

It would now be advantageous to move to the time domain. Using the symbol L^{-1} to denote the inverse Laplace transform, we define the following Laplace transform pairs:

$$x(t) = L^{-1}[X(s)]$$
$$y(t) = L^{-1}[Y(s)]$$
$$y_i(t) = L^{-1}[Y_i(s)] \qquad i=1,2$$

that is, x is the input to the undetermined system, y is the output of the undetermined system, and y_i is the output of the i^{th} fixed system whose input is x.

Equations (2.2-3) contain terms of the form $s^j H_i Y$. We formally denote these expressions in the time domain as

$$z_i^{(j)}(t) = L^{-1}[s^j H_i Y] \qquad i=1,2;\ j-1,2$$

that is, $z_i^{(j)}$ is the output of the i^{th} fixed system whose transfer function is $s^j H_i$ and whose input is the output of the undetermined system. It is not without reason that the notation above implies differentiation. If the initial conditions on all time functions above were zero we could write

$$z_i^{(j)}(t) = \frac{d^j z_i(t)}{dt^j}$$

If the measurement of the undetermined parameters is begun when all systems are at rest; i.e., when all signals (except perhaps $x(t)$) and all of their derivatives are zero, zero initial conditions are assured. If the initial conditions are not all zero, the stability of the fixed and undetermined systems assures that the effect of non-zero initial conditions will become negligible within a relatively short time in the history of system operation. With these considerations in mind we take $z_i^{(j)}$ to be the j^{th} derivative of the function $z_i(t)$ for the remainder of the paper.

With this notation, equations (2.2-3) can be written in the time domain as

$$\begin{aligned} y_1 - y &= a_1^* z_1'' + b_1^* z_1' \\ y_2 - y - (a_1-a_2)z_2'' - (b_1-b_2)z_2' &= a_1^* z_2'' + b_1^* z_2' \end{aligned} \qquad (2.2\text{-}4)$$

Equations (2.2-4) represent two simultaneous linear algebraic equations in the two unknowns a_1^* and b_1^*. Once these unknowns have been found, the undetermined parameters a and b may be evaluated from equations (2.2-1).

Solution of (2.2-4) can be accomplished by Cramer's Rule, [1] yielding

$$\begin{aligned} a_1^* &= \frac{[y_1-y]z_2' - [y_2-y-(a_1-a_2)z_2''-(b_1-b_2)z_2']z_1'}{z_1''z_2' - z_2''z_1'} \\ b_1^* &= \frac{[y_2-y-(a_1-a_2)z_1''-(b_1-b_2)z_2']z_1'' - [y_1-y]z_2''}{z_1''z_2' - z_2''z_1'} \end{aligned} \qquad (2.2\text{-}5)$$

It should be noted that similar equations could have been written for the unknowns a_2^* and b_2^* by a parallel development, but that either pair will allow calculation of a and b.

A detailed discussion of the terms of (2.2-5) is now in order. The constants (a_1-a_2) and (b_1-b_2) are known and fixed. The function $y(t)$ is the output of the undetermined system. The functions $y_1(t)$ and $y_2(t)$ are the outputs of the two fixed systems whose transfer functions are, respectively, H_1 and H_2 and whose input is the input to the undetermined system. The functions z_1'', z_1', z_2'', and z_2' are the outputs of a set of fixed systems whose transfer functions are, respectively, s^2H_1, sH_1, s^2H_2, and sH_2, and whose input is $y(t)$, the output of the undetermined system. The signal flow is diagrammed in figure F2.2-1. We observe that all of these signals are explicitly available.

A question naturally arises as to the desirability of calculating the derivatives of $y(t)$, as the operations s^jH_iY seem to imply. Actually no differentiation as such takes place. The systems, s^jH_i, which appear to imply differentiation, never contain higher powers of s in the numerator than in the denominator. The important consequence of this fact is that the differentiation implied by s^jH_i can be accomplished internal to the operation H_i. A suggested analog circuit for H_i, j=1,2, is shown in figure F2.2-2. It is seen from this figure that the outputs, $z_i^{(j)}$, of the systems represented by s^jH_i are all explicitly available in the one circuit represented by H_i. It is to be observed that this circuit does not explicitly differentiate the output of the undetermined system and hence does not generally tend to amplify any noise which may accompany y(t). We may now replace the portion of F2.2-1 within the dashed lines by one circuit and represent the signal flow by figure F2.2-3.

In summation, equations (2.2-4) are two simultaneous linear algebraic equations in the two unknowns a_i^* and b_i^*. Once these unknowns have been found, the undetermined parameters, and hence the transfer function of the undetermined system, may be evaluated from equations (2.2-1). The signals x(t), y(t), $y_i(t)$ and $z_i^{(j)}(t)$ can all be easily made explicitly available for the computation (2.2-5).

It must be emphasized that the fixed systems are independent of the undetermined system, and thus may be insulated from the conditions which cause changes in the undetermined parameters. The fixed systems can be relatively simple networks built from simple electronic components. Since the fixed parameters are parameters of the fixed systems only, these parameters may be held constant.

The proposed method does indeed embody advantageous features mentioned in section 1.2. Only the input and output of the undetermined system is used as measurable information. No extraneous inputs are introduced into the undetermined system. The evaluation of (2.2-5) may be accomplished by continuous time analog devices or discrete time digital devices, or some combination of both. If a discrete time implementation of the method is employed, the accuracy of the measurement is not affected by the time interval between measurements.

Further advantages are available in using the method: No restriction has been made on the input to the undetermined system. Since the fixed systems can be designed so that they do not "load" the undetermined system, the independence of the control operation from the measurement is assured.

After the derivation of the measurement equations for the more general undetermined system, in the following section, some of the practical problems in the application of the method will be discussed.

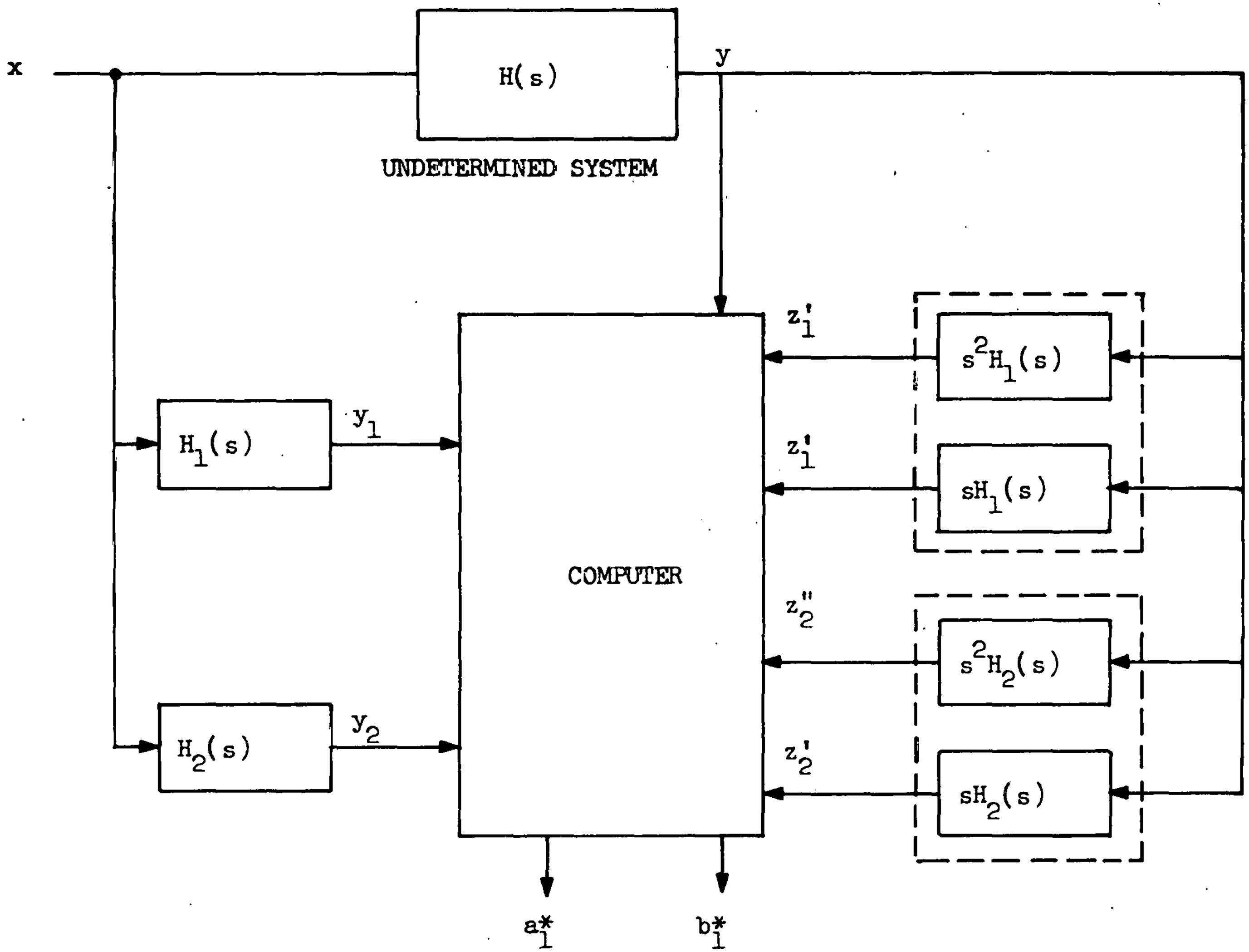

FIGURE F2.2-1 SECOND ORDER SOLUTION

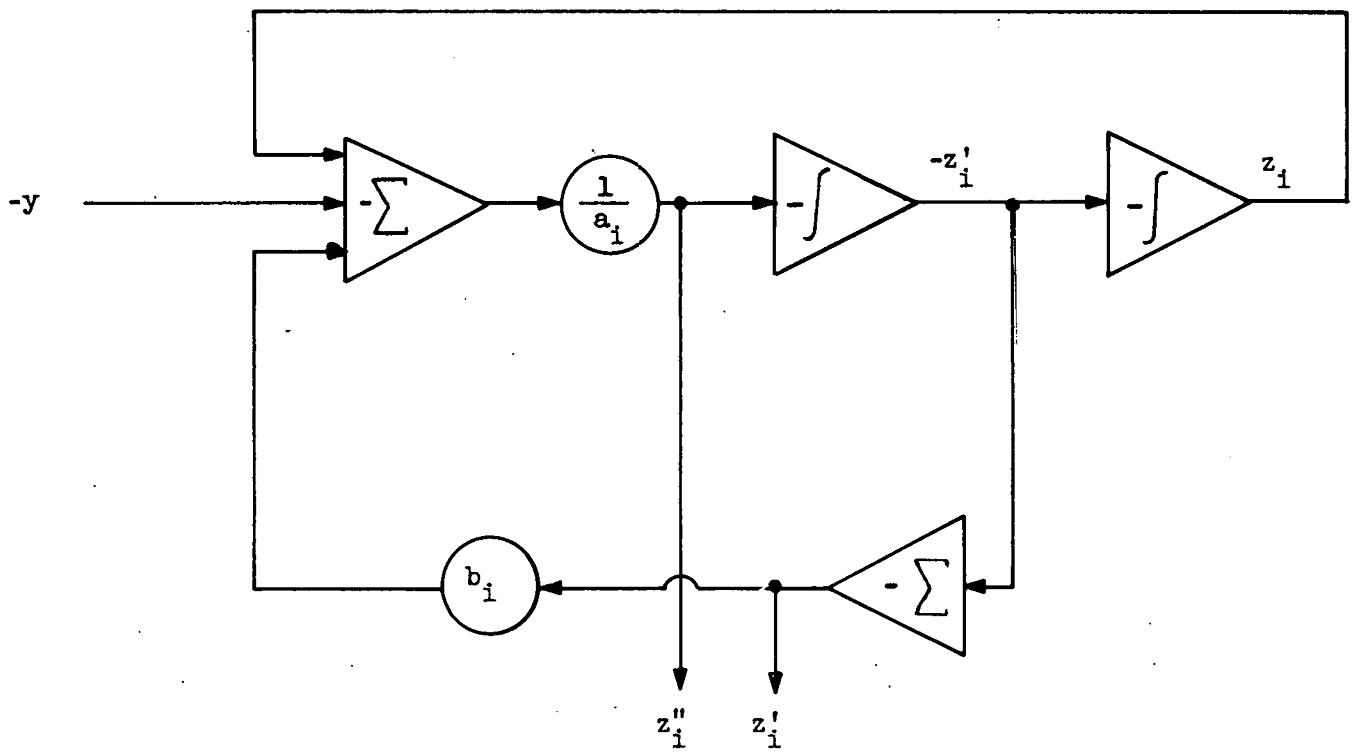

FIGURE F2.2-2 SUGGESTED ANALOG CIRCUIT FOR FIXED SYSTEM H_i

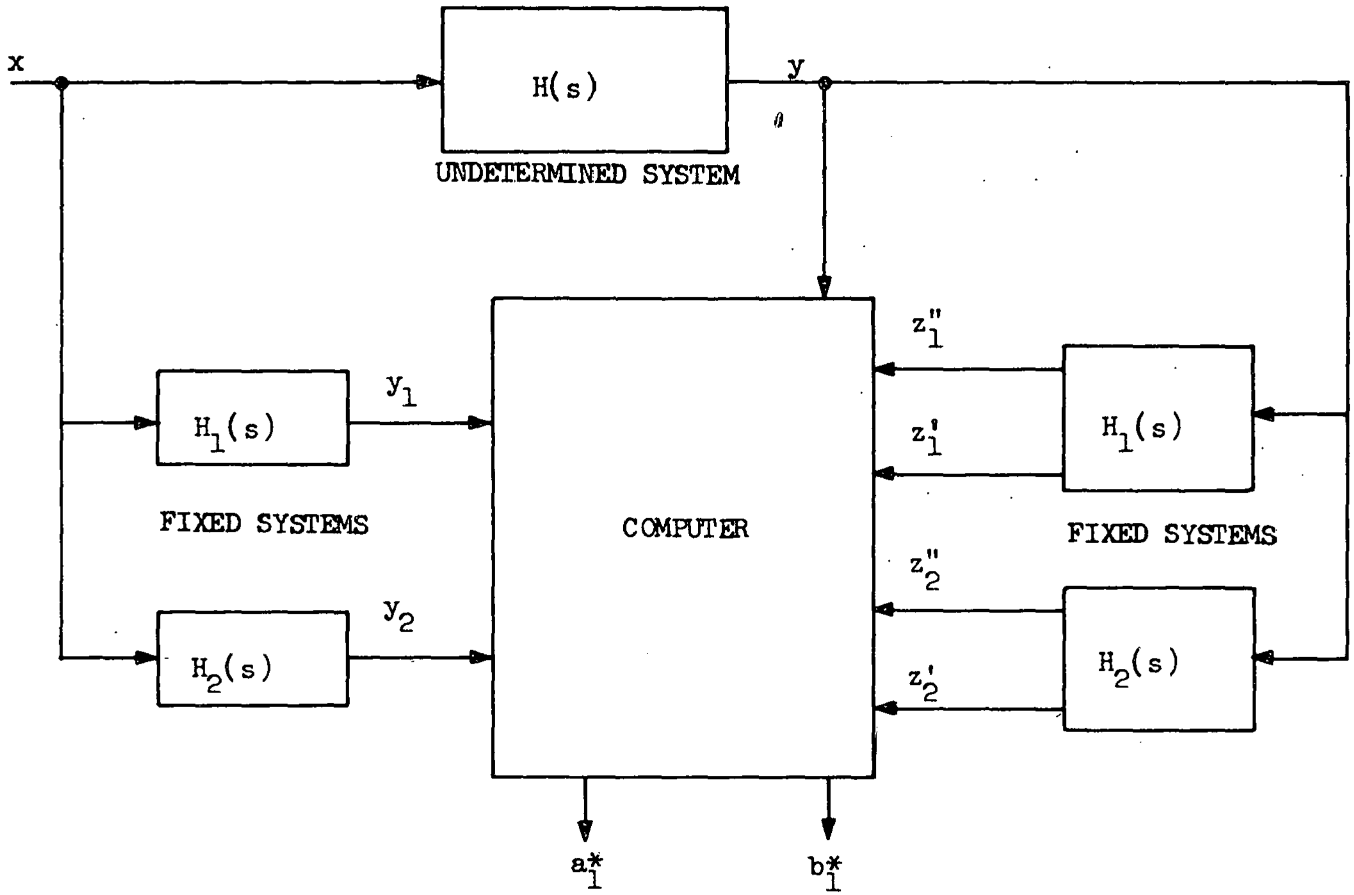

FIGURE F2.2-3 SIMPLIFIED SECOND ORDER SOLUTION

2.3 The m^{th} Order Undetermined System

In this section we shall present a more general development of the material of section 2.2. For simplicity, certain algebraic manipulations and discussion material which seems redundant to section 2.2 will not be presented.

By the m^{th} order undetermined system, we mean that the portion of the transfer function of the undetermined system which contains the undetermined parameters is of order m. With this definition we write the transfer function of the undetermined system as

$$H_o(s) = G(s)K_o(s) = G(s) \frac{1}{\sum_{j=o}^{m} c_{oj}s^j} \quad ; \qquad c_{om} \neq 0$$

We specify that the undetermined system be unconditionally stable. The only restrictions we impose on $G(s)$ is that it be linear, of finite order, and remain fixed throughout the history of operation of the undetermined system. The $\{c_{oj}\}$ are the undetermined parameters.

Let there be m+1 other unconditionally stable fixed systems with transfer functions

$$H_i(s) = G(s)K_i(s) = G(s) \frac{1}{\sum_{j=o}^{m} c_{ij}s^j} \quad ; \qquad i=1, 2, \ldots, m+1$$

The $\{c_{ij}\}$ are the fixed parameters. The only condition we impose upon them, except finiteness, is that for any chosen j, $c_{ij} = c_{kj}$

if and only if i=k.

As before, we define the parameter c^*_{ij} by the relation

$$c_{ij} + c^*_{ij} = c_{oj} \quad ; i=1, 2, \ldots, m+1; \quad j=0, 1, \ldots, m \quad (2.3\text{-}1)$$

We also define the output of the i^{th} system

$$Y_i(s) = H_i(s)X(s) = G(s)K_i(s)X(s) \qquad i=0,1,\ldots,m+1$$

where X(s) is the Laplace transform of the input, x(t), to the undetermined system.

We write the m+1 differences

$$Y_i - Y_o = H_iX - H_oX$$

$$= \frac{\sum_j c^*_{ij} s^j}{\left[\sum_j c_{ij} s^j\right]\left[\sum_j c_{oj} s^j\right]} G(s)X(s)$$

$$= \sum_j c^*_{ij} s^j K_i G K_o X \qquad \begin{matrix} i=1,2,\ldots,m+1 \\ j=0,1,\ldots m \end{matrix}$$

thus,

$$Y_i - Y_o = \sum_j c^*_{ij} s^j K_i Y_o \qquad (2.3\text{-}2)$$

For any fixed j, $c_{ij}+c^*_{ij} = c_{kj}+c^*_{kj}$, from equations (2.3-1). We may therefore express equations (2.3-2) in terms of any one set of c^*_{ij}. We may also assign any non-zero subscript to any of the transfer functions (H_i,K_i) of the fixed systems, provided no two

H_i or K_i are assigned the same subscript. Let us therefore express all the c^*_{ij} in terms of the set c^*_{1j}. We have, for each j,

$$c^*_{1j} = c_{1j} - c_{ij} + c^*_{ij} \quad ; \quad i=1, 2, \ldots, m+1.$$

Equations (2.2-2) now become, with a slight rearrangement

$$Y_i - Y_o - \sum_j (c_{1j}-c_{ij}) s^j K_i Y_o = \sum_j c^*_{1j} s^j K_i Y_o \qquad (2.3\text{-}3)$$

$$i=1, 2, \ldots, m+1$$

$$j=o, 1, \ldots, m$$

Using the definitions of section 2.2, we express (2.3-3) in the time domain as

$$y_i - y_o - \sum_j (c_{1j}-c_{ij}) z_i^{(j)} = \sum_j c^*_{1j} z_i^{(j)} \qquad (2.3\text{-}4)$$

$$i=1, 2, \ldots, m+1; \qquad j=o, 1, \ldots, m$$

Equations (2.3-4) are m+1 linear, simultaneous, algebraic equations in the m+1 unknowns c^*_{1j}. All time functions needed to generate equations (2.3-4) are easily available from the undetermined system and the fixed systems.

The only important point in this section, other than the extension to m+1 unknowns, is that $z_i^{(j)}$ is defined to be the inverse Laplace transform of $\{s^j K_i Y_o\}$; that is, the fixed systems whose input is the output of the undetermined system do not contain G(s) in their transfer functions. This point was not brought out in section 2.1 because G(s) was set to unity in that section. The m^{th} order case is diagrammed in figure F2.3.

It should again be observed that no actual differentiation of y(t) takes place, as seemingly implied by the operation $s^j K_i Y$, since the order of the denominator of $K_i(s)$ is always equal to or higher than j.

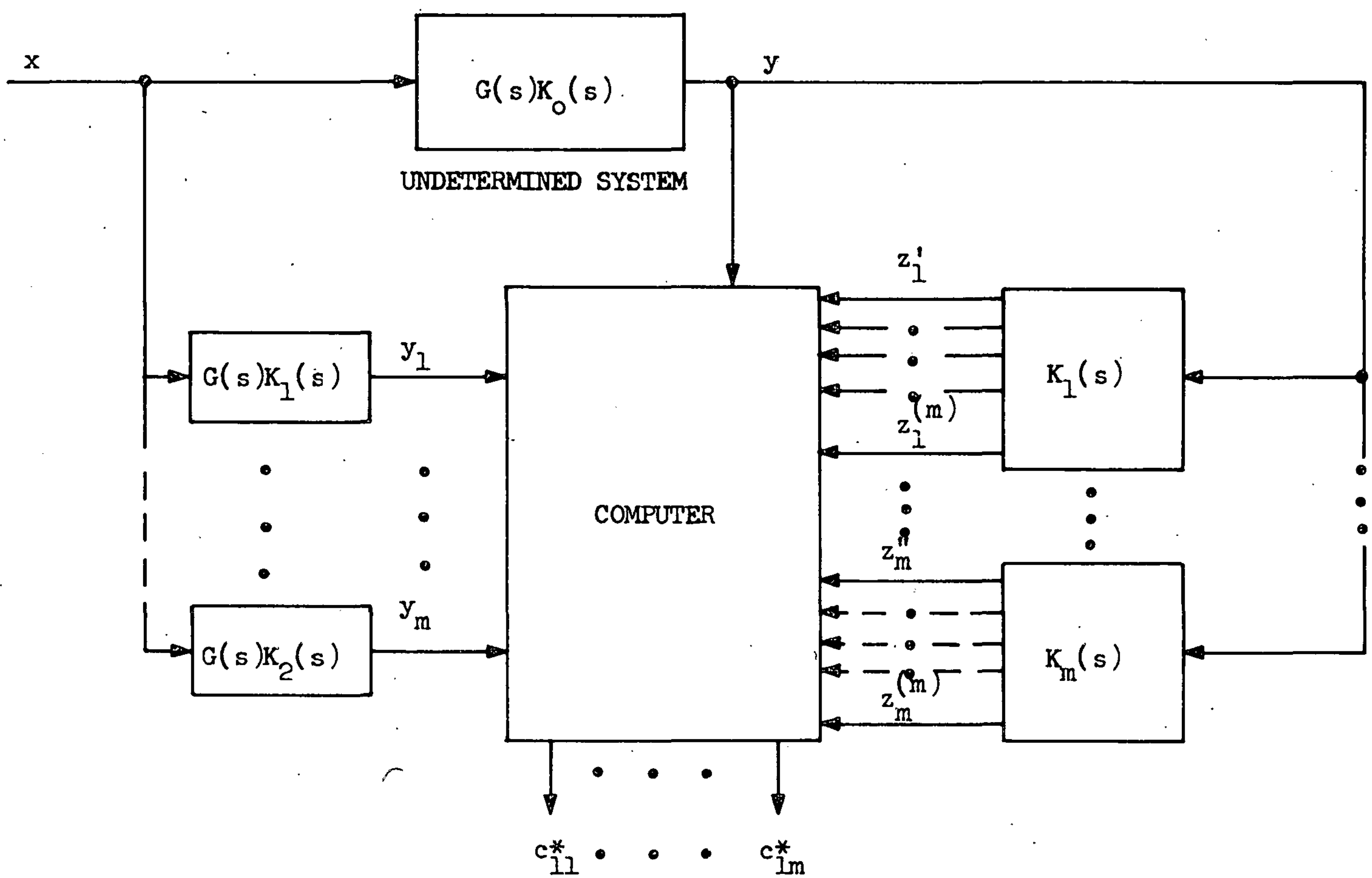

FIGURE F2.3 PROPOSED m^{th} ORDER SYSTEM

2.4 Difficulties with the Method

The major difficulty with the method is obviously the problem of accurate solution of equations (2.2-4). Accurate solution of a set of linear simultaneous equations is often a difficult task, especially if the order of the equations is high. Computational errors such as round-off and errors in signal measurement due to noise phenomena may present problems. These errors are especially severe if the signal level is small compared to the noise and computational levels.

The method does not preclude overdetermination and least squares solution, a process which usually tends to alleviate the accuracy problem. We shall not discuss this procedure here, but a brief discussion of this approach can be found in Appendix A.

The accuracy problem is particularly serious if the determinant of the coefficient matrix is singular, or almost singular. When the system is at rest, the value of this determinant will be identically zero, in the absence of noise or computational error, and the equations will become indeterminant. An indeterminancy will also result if the value of this determinant passes through zero (we have no reason to assume that this situation is not possible). The indeterminancy due to the rest condition is not surprising since no information about its performance is available from a system at rest.

In the presence of computational and/or noise errors it is evident that our results will be virtually meaningless when the value of the coefficient determinant is near zero. It therefore appears imperitive that any implementation of the method provide a procedure for alleviating this problem.

It should be noted that, while all the systems involved are assumed linear, and the basic problem of method accuracy is the solution of a set of linear algebraic equations, the computed values a_i^* and b_i^* (or c_{ij}^*) are <u>not</u> linear with respect to the various coefficients in the set of equations. In particular, the results of the computation depend in a nonlinear fashion on the input x(t), any noise that might be introduced into the system, and the set of fixed parameters chosen for the fixed systems. As will soon become apparent, this fact of nonlinearity considerably complicates the problem of optimizing the accuracy of this method of parameter measurement.

The remainder of this paper will be devoted to the determinant zero-crossing problem and accuracy of parameter measurement in the presence of noise. In order to keep the complexity of the problem to a reasonable level, we shall restrict ourselves to the second order system; and hence to the accuracy of the computation described in equations (2.2-5).

CHAPTER III

CHOICE OF THE FIXED PARAMETERS

3.1 General Remarks

In this chapter we shall be concerned with choosing the fixed parameters so as to minimize errors in the computation (2.2-5) which are caused by noise. Under certain simplifying assumptions, techniques will be developed which enable us to make a judicious choice of the fixed parameters.

The first assumption is that no error is caused by any device used to perform the computation. In the final analysis, factors such as size, cost, etc. would determine the computational accuracy in terms of hardware and computer sophistication.

The second assumption is that the fixed systems generate no noise in themselves, and are unaffected by the environmental changes which affect the undertermined system. Since the fixed systems H_1 and H_2 can easily be isolated from the undetermined system, they may be designed to relatively severe specifications.

Thirdly, we assume that the input to the undetermined system is also the input to the fixed systems which generate Y_1 and Y_2. Thus any noise that might be superimposed on the original input merely becomes part of the actual input and produces no error in the measurement. It is of course possible that the input to the undetermined system must be fed through a cable and/or transducer to provide input to the fixed systems. In this event, we assume that any noise caused by this transmission is negligible.

With these restrictions on the noise we must assume that any and all noise is generated within or at the output of the undetermined system. Since a "perfect" computer and "perfect" (in the sense of zero noise generation) fixed systems have been assumed, it is evident that all noise must pass through the fixed systems which generate the $z_i^{(j)}$. The order of the fixed systems is specified by the undetermined system. Consequently, once the input to the undetermined system, the undetermined system itself, and the noise have been specified, only the choice of the fixed parameters is available to attempt to reduce any error in the measurement.

The importance of the fixed parameters with respect to error reduction may easily be seen from the fact that if the two fixed systems are chosen to be identical to each other, solution of (2.2-4) is not even possible. It is then at least intuitively evident that with specified input, noise, and undetermined system, there must exist sets of fixed parameters which allow less error in the measurement than other sets of fixed parameters under the same set of given conditions.

In order to make a suitable choice of the fixed parameters we must have some knowledge of the input to the undetermined system, $x(t)$, and the noise. In general, we may not specify the input and noise as known functions of time, but it must be assumed that we have at least some statistical knowledge of these functions. Indeed, if the input and/or noise were known functions of time it is conceivable that another, more generally accurate, method could be devised to measure the undetermined parameters.

The computer and fixed systems have been eliminated as noise generators in the sense that if any noise appears on their outputs it is wholly attributable to noise on their inputs. The input x(t) is not subject to noise for reasons discussed above. Consequently, any noise in the overall system must be that noise which appears at the output of the undetermined system. We shall therefore assume that the output of the undetermined system appears as y(t)+e(t), where e(t) is the noise and y(t) is the noiseless output of the undetermined system due to the input x(t).

In the techniques of error reduction discussed below it is often necessary to specify the undetermined parameters. We are not begging the question by specifying what we wish to find, but are merely pointing out that the results found from error reduction computation are themselves functions of the undetermined parameters. In general, if a best (in the sense of error reduction) set of fixed parameters is chosen, this set is best only for a certain set of undetermined parameters. Fortunately, this set should be a "good" set for relatively small variations in the fixed parameters. If the measurement is being used to control the undetermined parameters, no difficulties arise if the control is adequate, since the undetermined parameters will always be near some specified operating point.

In the remainder of this chapter we shall discuss the statistical description of the signals and noise and present methods for choosing the fixed parameters which tend to minimize the error in the measurement of the undetermined parameters.

3.2 Statistical Description of the Input and Noise

In the analysis below we shall represent $x(t)$ and $e(t)$ as two sample functions of two independent stationary ergodic gaussian random processes with zero mean and known power spectral density. The above properties have been chosen because such processes are often reasonable approximations to the actual phenonema, and are mathematically tractable. Besides assuming that $e(t)$ is independent of $x(t)$, we assume that $e(t)$ is independent of the overall system configuration; that is, $e(t)$ is in no way dependent on H_1 or H_2, nor on the undetermined system, nor on any device used to solve (2.2-4). The ergodic property states that averages across the ensemble; i.e., "across the process," may be replaced with averages across time of a sample function. The stationary assumption assures that the statistical properties of the processes will be independent of time. A gaussian process has been chosen because such a process is "highly random" and is capable of relatively simple analysis. Further discussion of these concepts may be found in Appendix B.

The control designer often has little information about what functions his system will be subject to in the field beyond the frequency content of the input and of the noise. When the power spectral density (PSD) of the input and noise are specified it is not necessarily implied that these representations apply exactly to these functions. What is implied is that this representation is the best

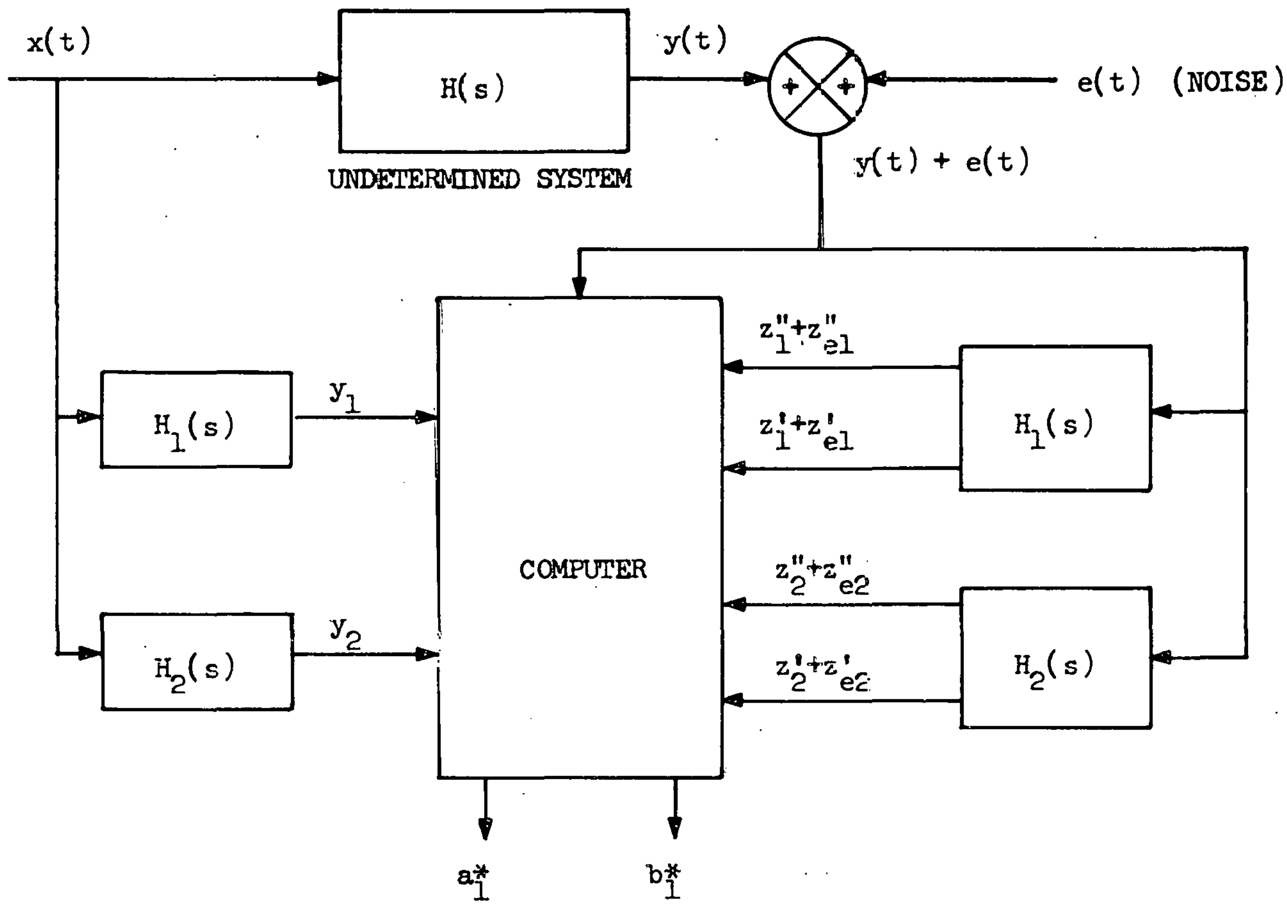

FIGURE F3.1 PLACEMENT OF THE NOISE

available. Also implied is that this representation defines the frequency region over which we desire the measurement of the undetermined parameters to be more accurate with respect to the choice of the fixed parameters. For instance, if we cut off these spectra at some frequency, we only imply that we are not particularly concerned with (say) the high frequency portion of the spectra.

We have chosen $x(t)$ and $e(t)$ to be sample functions of random processes. It therefore follows that the outputs of the undetermined and fixed systems are also sample functions of random processes, related to their inputs by linear transformations. Due to the statistical nature of the signals and noise, it is evident that some sort of averaging process must be used to evaluate, and hence attempt to reduce, the errors in the measurement of the undetermined parameters. In particular, we shall use the mean square error, or a suitable approximation to the mean square error, as a measure of the error in the estimation of the undetermined parameters.

3.3 Some Notation

In the following material all formulas for integration over a specturm will be given the limits $(0, \infty)$ or $(-\infty, \infty)$. This convention does not exclude the possibility of finite limits if the spectrum in question is non-zero over only a finite range of frequencies.

It is advantageous at this point to introduce some additional notation. We define

$$D = z_1'' z_2' - z_2'' z_1'$$

$$Na = [y_1 - y] z_2' - [y_2 - y - (a_1 - a_2) z_2'' - (b_1 - b_2) z_2'] z_1' \qquad (3.3\text{-}1)$$

$$Nb = [y_2 - y - (a_1 - a_2) z_2'' - (b_1 - b_2) z_2'] z_1'' - [y_1 - y] z_2''$$

We have suppressed the argument (t) in (3.3-1) and shall continue to do so when no confusion would result. It should be noted now, however, that since the y_i and the $z_i^{(j)}$ are functions of time, D, Na, and Nb are functions of time.

The function D may be recognized as the denominator and the functions Na and Nb as the numerators of the expressions in equations (2.2-5); therefore we may write

$$a^* = Na/D$$
$$(3.3\text{-}2)$$
$$b^* = Nb/D$$

The subscripts on a* and b* have been suppressed for brevity. Since a* and b* each differ from the undetermined parameters only by an additive constant (a_1 and b_1, respectively), a* and b* are functions of time only if the undetermined parameters are functions of time.

Since the undetermined and fixed systems are linear, the principle of superposition applies; that is, if the input to any of these devices is composed of signal and additive noise, the output may be represented simply as the sum of the output of the system with signal only as the input plus the output of the system with noise only as the input. This fact leads naturally to the following notation:

We shall use the previous notation to refer to the signal portion of the variables. The noise portion will be subscripted with the symbol "e." The subscript "m" will refer to the variables actually measured by the parameter estimation system; i.e., the sum of the noise and signal. We thus write

$$y_m(t) = y(t) + y_e(t) = y(t) + e(t)$$

$$z_{mi}^{(j)}(t) = z_i^{(j)}(t) + z_{ei}^{(j)}(t)$$

Noting that y_1 and y_2 include no noise terms, we write

$$y_{mi} = y_i$$

Although Na, Nb, and D are not generated by linear operations, it is possible to separate the signal and noise portions of these variables by collecting terms. We have

$$D_m(t) = D(t) + D_e(t)$$

$$Na_m(t) = Na(t) + Na_e(t)$$

$$Nb_m(t) = Nb(t) + Nb_e(t)$$

If, for brevity, we write

$$u = y_1 - y$$

$$v = y_2 - y - (a_1-a_2)z_2'' - (b_1-b_2)z_2'$$

$$u_e = -e$$

$$v_e = -e \; -(a_1-a_2)z''_{e2} \; -(b_1-b_2)z'_{e2}$$

Na_m and Nb_m can be written

$$Na_m = (u+u_e)(z'_2+z'_{e2}) - (v+v_e)(z'_1+z'_{e1})$$

$$Nb_m = (v+v_e)(z''_1+z''_{e1}) - (u+u_e)(z''_2+z''_{e2})$$

Thus,

$$Na_e = Na_m-Na = uz'_{e2}+u_e z'_2-vz'_{e1}-v_e z'_1+u_e z'_{e2}-v_e z'_{e1}$$

$$Nb_e = Nb_m-Nb = vz''_{e1}+v_e z''_1-uz''_{e2}-u_e z''_2+v_e z''_{e1}-u_e z''_{e2} \qquad (3.3\text{-}3)$$

$$D_e = D_m - D = z''_1 z'_{e2}+z'_2 z''_{e1}-z'_1 z''_{e2}-z''_2 z'_{e1}+z''_{e1}z'_{e2}-z''_{e2}z'_{e1}$$

The parameter measurement system computes the following variables

$$a^*_m(t) = Na_m(t)/D_m(t) = a^* + a^*_e(t)$$

$$b^*_m(t) = Nb_m(t)/D_m(t) = b^* + b^*_e(t)$$

Simple algebra yields

$$a^*_e = \frac{Na_e - a^*D_e}{D_m}$$

$$(3.3\text{-}4)$$

$$b^*_e = \frac{Nb_e - b^*D_e}{D_m}$$

The quantities $a_e^*(t)$ and $b_e^*(t)$ are the errors in the measurement of the undetermined parameters caused by noise. It should be observed that a_e^* and b_e^* are somewhat complicated non-linear functions of the undetermined parameters, the fixed parameters, the input to the undetermined system, and the noise. It is the purpose of Chapters III and IV to investigate techniques for reducing these errors in some sense.

3.4 The Possibility of an Exact Solution

Initial reasoning would suggest that a possible approach toward reducing the error in a* and b* would be to find the fixed parameters, with x and e given and a and b given and fixed, such that the magnitude of the quantity

$$E_e = q_a \overline{a_e^{*2}} + q_b \overline{b_e^{*2}} \qquad (3.4\text{-}1)$$

be minimized; that is, the weighted sum of the mean square errors be minimized. The positive numbers q_a and q_b are arbitrary weights, chosen such that $q_a + q_b = 1$. Unfortunately, such an approach is not feasible, for reasons stated in the remainder of this section. In the absence of an exact solution, it appears that an approximate solution must be used. Such an approximate solution is discussed in the next section.

The most serious difficulty in evaluating E_e is that there appears to be nothing inherent in the form of a_e^* and b_e^* (3.3-4) that precludes the possibility of D_m passing through zero in such a fashion that a_e^* and b_e^* have non-integrable singularities across time. Indeed, it is possible, and highly probable, that when D_m is zero, either $Na_e - a^*D_e$ or $Nb_e - b^*D_e$, the numerators of a_e^* and b_e^*, will not be zero. Since integration is not possible, averaging is not possible, at least in the sense that the mean square value of a_e^* and b_e^* will in general be non-finite.

It may also be observed that the quantities $Na_e - a^*D_e$, $Nb_e - b^*D_e$, and D_m are all statistically mutually dependent. No general practical theory exists for generating the probability density function of the ratio of two statistically dependent functions, and hence the probability density function of a_e^* and b_e^*, whether or not the ratio exhibits singularities.

If the singularities did not exist, it would be possible to expand D_m about some point, say $\overline{D_m}$, and write

$$\overline{a_e^{*2}} = \left\{ (Na_e - a^*D_e) \sum_{n=0}^{\infty} (-1)^n \left[\frac{D_m - \bar{D}_m}{\bar{D}_m} \right]^n \right\}^2 \qquad (3.4\text{-}2)$$

a similar expression holds for b_e^*. With x and e chosen to be the Gaussian processes described in section 3.2, it is possible to evaluate expressions of the form

$$\overline{(Na_e - a^*D_e)^2 (D_m - D_m)^n}$$

and hence evaluate $\overline{a_e^{*2}}$.

For large n, about 3 or 4, the terms of (3.4-2) become exceedingly complicated; and thus, if the convergence of the series is not rapid, the tediousness of the computation would render the series essentially useless.

It is possible to remove the singularities of a_e^* and b_e^* by imposing a constraint on D_m. Consider the following constraint.

Let

$$a^* = Na_m/D_{cm} \text{ and } b^* = Nb_m/D_{cm}$$

where, for some positive V,

$$D_{cm} = D_m \; ; \; |D_m| > V$$

$$D_{cm} = -V \; ; \; 0 \geq D_m \geq -V$$

$$D_{cm} = V \; ; \; 0 \leq D_m \leq V$$

Although such a constraint is quite arbitrary, it will be ahown in section 4.1 that this constraint is very useful if V is judiciously chosen. The imposition of this constraint is important to the parameter measurement method described in this paper and it will be more fully discussed in section 4.1.

We have removed the singularities in a_e^* and b_e^*, but the discontinuous probability density function of D_{cm} now makes the evaluation of $\overline{a_e^{*2}}$ and $\overline{b_e^{*2}}$ even more difficult. The imposition of a constraint which allows the probability density function of D_{cm} to be continuous and yields a value of zero with zero probability would be virtually impossible to implement.

In view of the severe complications discussed above, it seems that some sort of approximation scheme must be used to determine the H_1 and H_2 which reduce the error in a^*_m and b^*_m.

Incidently, it may be considered that an advantage may be gained if a^*_2 (b^*_2) were computed in lieu of a^*_1 (b^*_1). It can be shown that by virtue of the fact that $a^*_{m1} - a^*_{m2}$ ($b^*_{m1} - b^*_{m2}$) is constant, independent of signal and noise, the error in a^*_{m2} (b^*_{m2}) is identical to the error in a^*_{m1} (b^*_{m1}). Hence, there is no advantage in choosing one set of fixed parameters over another set (all fixed parameters having been chosen,) to use as the reference values.

3.5 An Approximate Solution

A first order approximation which tends to reduce the errors in the undetermined parameter measurements in the presence of noise is to choose the fixed parameters such that the quantity

$$R_e = \frac{q_a \sqrt{\overline{(Na_e - a^*D_e)^2}} + q_b \sqrt{\overline{(Nb_e - b^*D_e)^2}}}{\sqrt{\overline{D_m^2}}} \qquad (3.5\text{-}1)$$

is minimized, where q_a and q_b are weights chosen such that $q_a + q_b = 1$.

In minimizing R_e, it is implied that since a^*_e and b^*_e are both the ratio of two functions, these errors will tend to be small if their respective numerators are small with respect to their denominator. Although minimization of R_e does not necessarily yield the best choice of the fixed parameters, in the sense of mean square error reduction, it does offer a useful guide towards fixed parameter choice.

Little error is introduced in the evaluation of R_e if the constraint described in sections 3.4 and 4.1 is ignored, even though the constraint may be used in an actual implementation of the parameter measurement method. The constraint is only useful if V^2 is small compared to $\overline{D_m^2}$, and hence the replacement of D_{cm} with D_m makes little difference in the evaluation of R_e.

We shall now evaluate R_e where x(t) and e(t) are each sample functions of two independent stationary ergodic gaussian random processes with zero mean and known PSD. In order to perform this evaluation, we shall first show how the mean value of the product of the outputs of two linear-lumped filters with the same stationary ergodic process as input may be evaluated.

Let g(t) be a sample function of a random process described above. Also, let g be the input to two linear-lumped time invariant filters whose transfer functions are $K_1(s)$ and $K_2(s)$. Let $r_1(t)$ and $r_2(t)$ be the respective outputs. We may write (Property 4, Appendix B)

$$\overline{r_1 r_2} = \frac{1}{2} \int_{-\infty}^{\infty} G_{r_1 r_2}(w)\, dw$$

where $G_{r_1 r_2}(w)$ is the CSD of $r_1(t)$ and $r_2(t)$. With the aid of properties 5 and 6, the CSD of r_1 and r_2 can be written as

$$G_{r_1 r_2}(w) = K_1(-jw)K_2(jw)G_g(w) \ ; \ j^2 = -1$$

Since $G_{r_1r_2}$ exhibits odd symmetry of its imaginary part and even symmetry of its real part, we can write

$$\overline{r_1r_2} = \int_0^\infty \text{Real part of } \{K_1(-jw)K_2(jw)G_g(w)\}dw \qquad (3.5\text{-}2)$$

Using the relation (3.5-2), R_e, as given in (3.5-1) can be evaluated. In the remainder of this section we shall describe the evaluation of $\overline{D_m^2}$ in some detail. The evaluation of the numerators of R_c can be found in Appendix C.

We have defined

$$D_m = z''_{m1}z'_{m2} - z''_{m2}z'_{m1}$$

Thus,

$$D_m^2 = z''^2_{m1}z'^2_{m2} + z''^2_{m2}z'^2_{m1} - 2z''_{m1}z''_{m2}z'_{m1}z'_{m2}$$

Since the average of a sum is the sum of the averages,

$$\overline{D_m^2} = \overline{z''^2_{m1}z'^2_{m2}} + \overline{z''^2_{m2}z'^2_{m1}} - \overline{2z''_{m1}z''_{m2}z'_{m1}z'_{m2}} \qquad (3.5\text{-}3)$$

Since x is gaussian, y is gaussian (Theorem 1). Since e is gaussian, y+e is gaussian (Theorem 2). Since y+e is the gaussian input to the filters which generate the functions $z_i^{(j)}(t)$, these functions are jointly gaussian (Theorem 3). Hence, from (B-1),

$$\overline{D_m^2} = (\overline{z''_{m1} z''_{m1}})\,(\overline{z'_{m2} z'_{m2}}) + 2(\overline{z''_{m1} z'_{m2}})^2$$

$$+ (\overline{z''_{m2} z''_{m2}})\,(\overline{z'_{m1} z'_{m1}}) + 2(\overline{z''_{m2} z'_{m1}})^2$$

$$-2\,(\overline{z''_{m1} z''_{m2}})\,(\overline{z'_{m1} z'_{m2}}) -2\,(\overline{z''_{m1} z'_{m1}})\,(\overline{z''_{m2} z'_{m2}}) \qquad (3.5\text{-}4)$$

$$-2\,(z''_{m1} z'_{m2})\,(z''_{m2} z'_{m1})$$

In order to evaluate the means in (3.5-4) we must evaluate expressions of the form

$$G_{z_{mi}^{(p)} z_{mk}^{(n)}} \;;\quad i, k = 1, 2 \text{ and } p, n = 1, 2$$

For convenience, we define

$$M_i(w) = |H_i(jw)|^2 = H_i(jw)H_i(-jw) \qquad (3.5\text{-}5)$$

With the PSD of x given as G_x, the PSD of y is

$$G_y = MG_x$$

With the PSD of e given as G_e, we define

$$G = G_{y+e} = MG_x + G_e \qquad (3.5\text{-}6)$$

We now write

$$G_{z_{mk}^{(n)}} = (-1)^n j^{2n} w^{2n} M_k G$$

and therefore

$$G_{z_{mi}^{(p)} z_{mk}^{(n)}} = (-1)^p j^{p+n} w^{p+n} H_i(-jw) H_k(jw) G$$

$$= (-1)^p (jw)^{p+n} M_i M_k G \; \{[a_i a_k w^4 + (b_i b_k - a_i - a_k) w^2 + 1]$$

$$+ jw[(a_i b_k - a_k b_i) w^2 + b_i - b_k]\}$$

We now define

$$R_{ik} = [a_i a_k w^4 + (b_i b_k - a_i - a_k) w^2 + 1] M_i M_k$$

$$I_{ik} = [(a_i b_k - a_k b_i) w^2 + b_i - b_k] M_i M_k \qquad (3.5\text{-}7)$$

A few properties of R and I are apparent:

1) $R_{ik} = R_{ki} = R \qquad (i \neq k)$

2) $R_{ii} = M_i$

3) $I_{ik} = -I_{ki}$

4) $I_{ii} = I_{kk} = 0$

5) R and I are even functions of w

With the new notation the required CSD is given by

$$G_{z_{mi}^{(p)} z_{mk}^{(n)}} = (-1)^p (jw)^{p+n} (R_{ik} + jwI_{ik}) G \qquad (3.5\text{-}8)$$

Using the stated properties of R and I, evaluating (3.5-8) for the indicated i, k, p, and n, and using these values in (3.5-4) yields

$$D_m^2 = \int_0^\infty Gw^4 M_1 dw \int_0^\infty Gw^2 M_2 dw + \int_0^\infty Gw^4 M_2 dw \int_0^\infty Gw^2 M_1 dw$$

$$+6 \left[\int_0^\infty Gw^4 I_{12} dw \right]^2 -2 \int_0^\infty Gw^4 R dw \int_0^\infty Gw^2 R dw \tag{3.5-9}$$

where R and I are given by (3.5-7), G is given by (3.5-6), and the M_i are given by (3.5-5).

The expressions for the mean square values of Na_e-a^*D_e and Nb_e-b^*D_e can be derived by essentially the same methods. Expressions for these averages are found in Appendix C. If G_x and G_e are each the ratio of polynomials, as is often the case, all the integrals in (3.5-9) and Appendix C are closed-form integrable via partial fraction expansion. The expansion can be extremely complicated, however, and the use of a high speed digital computer to perform a numerical integration may be usually more expeditious.

It is thus possible to express R_e as the ratio of functions involving a_1, a_2, b_1, and b_2, these functions also depending on the undetermined system H(s), and the PSD assumed for the input x(t) and the noise e(t), with the aforementioned restrictions of x and e. The usual straightforward minimization of R_e involves the differentiation of R_e with respect to a_1, a_2, b_1, and b_2, setting these expressions equal to zero, and solving simultaneously for a_1, a_2, b_1, and b_2. This is an extremely complicated procedure, however, and the minimumization of R_e appears to be more easily accomplished using a four dimensional search for a_1, a_2, b_1, and b_2 (assuming the availability of a high speed digital computer).

3.6 Choice of the Fixed Parameters with Unspecified Noise

It is possible that the control system designer will have knowledge of the input to the undetermined system, but will not have sufficient information available to adequately describe the noise. In this situation it is still possible to make a generally judicious choice of the fixed parameters.

Consider the error terms

$$a_e^* = \frac{Na_e - a^*D_e}{D + D_e}$$
$$b_e^* = \frac{Nb_e - b^*D_e}{D + D_e} \qquad (3.6\text{-}1)$$

Without information about the noise it is not possible to compute averages of Na_e, Nb_e, and D_e. It is possible, however, to compute averages of D, since the input to the undetermined system is assumed to be specified.

Examination of (3.6-1) shows that a_e^* and b_e^* will tend to be lessened if D is increased. It therefore seems reasonable, initially, that choosing the fixed parameters such that the mean square value of D is maximized would tend to decrease the mean square values of a_e^* and b_e^*.

Unfortunately, simply maximizing $\overline{D^2}$ does not appear to be a valid method. The quantity $\overline{D^2}$ may be increased by merely increasing the mean square values of the $z_i^{(n)}$. Examination of (3.3-3) indicates that increasing the mean square values of the $z_i^{(n)}$ will also tend to increase Na_e, Nb_e, and D_e. It is therefore not evident that this scheme would yield minimization, or even lessening, of $\overline{a_e^*}$ and $\overline{b_e^*}$.

Also, merely increasing the $z_i^{(n)}$ essentially amounts to multiplying the equations (2.1-4) through by a constant. In solving a set of simultaneous equations, no accuracy improvement can be achieved by this operation.

We may also consider the following argument against merely increasing the mean square value of the $z_i^{(n)}$: It is evident that such an increase can be accomplished by setting b_1 and b_2 to zero; that is, by designing the fixed systems to have zero damping. Zero-damped fixed systems violate the unconditionally stable stipulation mentioned in Chapter II. Also, undamped systems exhibit large outputs in only a relatively small range of frequencies, and lessening of the errors would be effected only over this small band, if at all. It would be generally more advantageous to lessen the errors over a broad input frequency range.

It therefore appears desirable to maximize $\overline{D^2}$ while keeping the $z_i^{(n)}$ somewhat small. This operation can be accomplished by "normalizing" D with respect to the $z_i^{(n)}$ in some fashion. We choose here to maximize $\overline{D^2}$ with respect to the mean square value of some function of the $z_i^{(n)}$. A useful normalizing function is

$$D_N[z_i^{(n)}] = \sqrt{(z_1''^2 + z_1'^2)\,(z_2''^2 + z_2'^2)} \qquad (3.6\text{-}2)$$

We therefore shall choose the fixed parameters such that the ratio

$$\frac{\overline{D^2}}{\overline{D_N^2}} = \frac{\overline{(z_1''z_2' - z_2''z_1')^2}}{\overline{(z_1''^2+z_1'^2)\,(z_2''^2+z_2'^2)}} \qquad (3.6\text{-}3)$$

is maximized.

The function (3.7-2) was chosen because it can be shown that

$$D_N^2 \geq D^2$$

for all (finite) $z_i^{(n)}$, and consequently (3.6-3) is maximized if $\overline{D^2}/\overline{D_N^2}$ is unity. Evaluation of (3.6-3) for given input, undetermined parameters, and fixed parameters will always yield a number between zero and one; and hence, an easily interpreted index of the degree of maximization is available.

Using the results of section 3.5 it is possible to compute the mean square values of D and D_N.

$$\overline{D^2} = \int_0^\infty G_y w^4 M_1 dw \int_0^\infty G_y w^2 M_2 dw + \int_0^\infty G_y w^4 M_2 dw \int_0^\infty G_y w^2 M_1 dw$$

$$-2 \int_0^\infty G_y w^4 R dw \int_0^\infty G_y w^2 R dw + 6 \left[\int_0^\infty G_y w^4 I_{12} dw \right]^2 \qquad (3.6\text{-}4)$$

$$\overline{D_N^2} = \left[\int_o^{\infty} G_y w^4 M_1 dw + \int_o^{\infty} G_y w^2 M_1 dw\right]\left[\int_o^{\infty} G_y w^4 M_2 dw + \int_o^{\infty} G_y w^2 M_2 dw\right]$$

$$+ 2\left[\int_o^{\infty} G_y w^4 R dw\right]^2 + 2\left[\int_o^{\infty} G_y w^2 R dw\right]^2 + 4\left[\int_o^{\infty} G_y w^4 I_{12} dw\right]^2$$

If we consider the solution of (2.2-4) to be the intersection of two vectors, the maximization of (3.6-3) implies that the vectors are mutually orthogonal. In general, the errors in the solution of simultaneous linear equations tends to be relatively insensitive to errors in the coefficients of the equations if the vectors represented by the equations are mutually orthogonal.

In terms of the theory of the solution of simultaneous linear equations, we have specified in this section that the fixed parameters be chosen such that the equations (2.2-4) be generally "well-conditioned;" that is, that the solutions of (2.2-4) be relatively insensitive to errors in the coefficients of the unknowns. If the solutions tend to be very sensitive to these errors, the equations are said to be "ill-conditioned."

A common way of evaluating the condition of a set of simultaneous linear equations is as follows: [1]

Consider the set of equation

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad & \vdots \\ a_{n1}x_1 + \dots a_{nn}x_n &= b_n \end{aligned}$$

The set is transformed by dividing the ith equation by

$$\sqrt{\sum_{j=1}^{n} a_{ij}^{2}}$$

and evaluating the magnitude of the coefficient determinant of the transformed equations. If this magnitude is near unity, the equations are well-contitioned. If this magnitude is small compared to unity, the equations are ill-conditioned.

From the discussion above it is evident the values of the fixed parameters which maximize (3.6-3) will tend to keep (2.2-4) well-conditioned across time.

CHAPTER IV

ADDITIONAL METHODS OF REDUCING THE ERRORS

4.1 Constraining the Measurement

It was mentioned in section 2.4 that in order to avoid grotesque errors in the measurement of the undetermined parameters when $D_m(t)$ is near zero, it is necessary to constrain the measurement in some fashion. Even in the absence of generally serious noise, little accuracy can be obtained when D_m is small. Indeed, if the undetermined system is at rest, no computation is possible and any results from the measurements can be attributed wholly to noise.

If the parameter measurement is to be used simply to yield a time history of the undetermined parameters, it is only necessary to provide a time history of D_m so that the parameter measurement results may be properly assessed in any region of time.

If the parameter measurement is to be used to provide a signal to control some device, it appears generally judicious either to delete the control function or to constrain the measurement to a reasonable level when D_m is near zero.

Whatever the application of the measurement, a simple and effective approach to this problem is to constrain the measurement whenever

$$|D_m| \leq V$$

where V is some preassigned positive number. In preassigning V we do not preclude that V be a function of time, or any other variable which may be useful for this purpose.

If the measurement is to be used to provide a signal to control some device and it is not feasible to remove the control when the magnitude of D_m is less than V, a reasonable constraint may be applied as follows:

If at time t_1, $|D_m|$ becomes less than V, and remains less than V until time t_2, we constrain the measurement by setting

$$a_m^*(t) = a_m^*(t_1)$$

$$b_m^*(t) = b_m^*(t_1) \qquad (4.1\text{-}1)$$

for all t such that

$$t_1 \leq t \leq t_2$$

Application of (4.1-1) tends to assure the absence of large errors in the measurement of the undetermined parameters when D_m is relatively small.

With knowledge of the noise and the input to the undetermined system, it is a simple matter to choose a reasonable value of V. We note that when

$$D_m^2 \leq \overline{D_e^2}$$

during the measurement, it is highly probable that the value of D_m is composed almost wholly of noise. Therefore, a judicious choice of V appears to be

$$V = \sqrt{\overline{D_e^2}}$$

With the techniques of section 3.5, it is easy to compute the value of $\overline{D_e^2}$; vis.,

$$\begin{aligned}
\overline{D_e^2} &= \int_o^\infty G_y w^4 M_1 dw \int_o^\infty G_e w^2 M_2 dw + \int_o^\infty G_y w^4 M_2 dw \int_o^\infty G_e w^2 M_1 dw \\
&+ \int_o^\infty G_y w^2 M_1 dw \int_o^\infty G_e w^4 M_2 dw + \int_o^\infty G_y w^2 M_2 dw \int_o^\infty G_e w^4 M_1 dw \\
&-2 \int_o^\infty G_y w^4 R dw \int_o^\infty G_e w^2 R dw -2 \int_o^\infty G_y w^2 R dw \int_o^\infty G_e w^4 R dw \\
&+ 4 \int_o^\infty G_y w^4 I_{12} dw \int_o^\infty G_e w^4 I_{12} dw \qquad (4.1\text{-}2) \\
&+ \int_o^\infty G_e w^4 M_1 dw \int_o^\infty G_e w^2 M_2 dw + \int_o^\infty G_e w^4 M_2 dw \int_o^\infty G_e w^2 M_1 dw \\
&-2 \int_o^\infty G_e w^4 R dw \int_o^\infty G_e w^2 R dw + 6 \left[\int_o^\infty G_e w^4 I_{12} dw \right]^2
\end{aligned}$$

It is conceivable that one is not particularly concerned with the errors themselves in a_m^* and b_m^*, but finds it generally more desirable to prevent the application of the constraint over as much time as possible without using an excessively small value of V. If this is the case, excellent results can be obtained by choosing the fixed parameters such that the ratio

$$\overline{D_e^2} \;/\; \overline{D^2}$$

is minimized. Use of this criterion has the effect of generally keeping a_e^* and b_e^* at reasonable levels and assures that the magnitude of D_m will be less than V as infrequently as possible.

If the noise is not known, V may still be estimated by choosing

$$V = p \sqrt{\overline{D^2}}$$

where p is some positive number less than one.

4.2 Filtering the Computed Parameter Values

In addition to choosing a "good" set of fixed parameters further reduction in the errors in the measurements can usually be obtained by passing the computed values of $a_m^*(t)$ and $b_m^*(t)$ through low pass filters. Since we have specified that a* and b* are slowly varying functions of time, it is evident that any relatively high frequency content in a_m^* and b_m^* must be due to noise. If a_m^* and b_m^* were passed through low pass filters much of the high frequency noise would be reduced.

Since the power spectra of the error functions a_e^* and b_e^* cannot easily be determined, the optimum low pass filter cannot be easily designed. However, almost any low pass filter will tend to reduce the errors. When this method of parameter measurement is applied to an actual system, an appropriate low pass filter can be selected and the cutoff frequency determined on the basis of experiment. In the simulation study described in Chapter V, such low pass filters were found to be generally effective in reducing the noise content of a_m^* and b_m^*.

CHAPTER V

A SIMULATION OF THE METHOD

5.1 Introduction

A simulation of the proposed method of parameter evaluation was performed using a digital computer.* The results of the simulation are given in this chapter. The difficulties entailed in generating stationary Gaussian random variables with known PSD by the digital computer precluded using the exact inputs and noise described in Chapter III in the simulation. As an approximation to these signals, a set of sinusoids were used and the fixed and undetermined systems were assumed to be in the steady state. These sinusoidal signals could be interpreted as random functions with rather specialized frequency spectra. The particular sinusoids used are described in Section 5-2.

Although sinusoids can hardly be described as general signals, it was felt that insight into the behavior and accuracy of the method could be obtained with these signals. In particular, the strong dependence of the accuracy of the method, in the presence of noise, upon the fixed parameter could be clearly shown. It must be remembered that knowledge of the sinusoidal nature of the inputs and noise was not used *per se* in the simulation.

The simulation is intended to give an indication of the operation and noise sensitivity of the method for a few simple cases. In particular, the aims of this simulation study were threefold:

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*Courtesy Sandia Corporation, Albuquerque, New Mexico

1. To indicate that the method does indeed measure the parameters of an operating (second order) linear-lumped control system.
2. To show the sensitivity of the method to noise and the choice of the fixed parameters.
3. To offer evidence that use of the techniques discussed in Chapter III and Chapter IV does reduce the errors in the measurements.

5.2 Input and Noise Signals

The input signal was chosen to be

$$x(t) = \sum_{n=1}^{10} X(n) \sin (0.2nt)$$

and the noise

$$e(t) = \sum_{n=1}^{10} \sin (0.2nt)$$

The X(n) were determined by the shape and level of the PSD assumed for the input signal. In all cases, the noise spectrum was flat and E was determined by the chosen noise level.

An advantage to using periodic noise and input, and assuming that the fixed and undetermined systems were in the steady state, is that the results repeat periodically. It thus becomes unnecessary to use excessive machine time. From the functions above, it may be seen that the measurements repeat every 10 π seconds of simulation time.

In the two undetermined systems examined, the natural frequency of each undetermined system was chosen to be one rad./sec.; consequently, the input and noise spanned the major operating frequency range of the undetermined system.

5.3 Simulation Procedure

Briefly, the procedure consists of two computational phases. An undetermined system, noise level and input spectrum were chosen. With given noise, input, and undetermined system, the quantity R_e (equation 3.5-1) was computed for different sets of fixed parameters. Based on the value of R_e, two sets of fixed parameters were chosen, a "good" set and a "poor" set. The "good" set would presumably allow less error in the measurement than the "poor" set. Under the given condition, simulations were then run using each set of the chosen fixed parameters. Additional simulations were run with the same undetermined system and input function, but with different noise levels, for both sets of fixed parameters. In all, four sets of undetermined parameters and input spectra were examined.

Details of the procedure are as follows:

1. The natural frequency and damping factor of a second order linear-lumped "undetermined system" were chosen. The natural frequency of the two undetermined systems examined here was taken to be 1.0 rad./sec.

2. An input spectrum was chosen, the frequencies themselves being taken as described in section 5.2. The noise spectrum was taken to be flat for all cases.

3. The noise to signal ratio was chosen at the relatively high value of 0.1. The noise to signal ratio is defined as

$$L = \sqrt{\overline{e^2} / \overline{y^2}} \qquad (5.3\text{-}1)$$

where e is the noise in the measurement system at the output of the undetermined system and y is the (noiseless) output of the undetermines system. The quantity L was defined with respect to the <u>output</u> of the undetermined system so that errors in the measurements at comparable noise to signal levels could be compared for different undetermined systems.

4. For different sets of fixed parameters the quantity

$$R_e = \frac{\sqrt{\overline{(a^*D_e - Na_e)^2}} + \sqrt{\overline{(b^*D_e - Nb_e)^2}}}{\sqrt{\overline{D_m^2}}}$$

was computed. R_e is described in detail in section 3.5. In that section it was pointed out that values of the fixed parameters which tend to minimize R_e tend to reduce the measurement errors.

5. Two sets of fixed parameters were chosen such that one set yielded a relatively large value of R_e and the other set yielded a relatively small value of R_e. It is expected that the "good" set (small R_e) will give measurements which are less

sensitive to noise than the "poor" set (large R_e). No attempt was made to find a best set. It was felt that values of R_e which were less than the noise to signal ratio (equation 5.3-1) should give tolerable errors.

The "good" and "poor" sets were also chosen such that the parameters of the reference fixed system (a_1, b_1) were the same for both sets. This restriction assured a basis for comparison between sets since, in the absence of noise, both sets should yield the same measurements.

6. A simulation was run for each set of chosen fixed parameters. Additional simulations were then run with other noise to signal ratios for both sets.

7. A constraint (section 4.1) was used in the simulation. The constraint was imposed as follows:

If $|D_m(t)| \leq V$, when $t_1 \leq t \leq t_2$; then the measurement was constrained by setting

$$a_m^*(t) = a_m^*(t_1)$$

$$b_m^*(t) = b_m^*(t_1) \; ; \quad t_1 \leq t \leq t_2$$

For each noise to signal ratio and set of fixed parameters, the quantity

$$d = \sqrt{\overline{D_e^2}} \, ,$$

was computed. In all but one pair of simulations (per set of given conditions) V was set equal to d; where d, of course, depended on the noise to signal ratio used in the particular simulation. At the high noise to signal ratio of 0.1, a pair of simulations was run with V = d and also with V = d/4. The latter choice of V was included because the measurements tended to be constrained for considerable time at the high noise to signal ratio, and consequently averages across time of the errors tended to be severly affected by the values that the errors had when the constraint was imposed. Lowering the value of V tended to remove this effect.

8. Measurements were made every $\pi/240$ seconds; i.e., 2400 measurements of a_m^* and b_m^* were made over 10π seconds. For each simulation, the mean and rms values of a_e^* and b_e^* were computed.

9. The percent of time that the measurements were not constrained (P) was also computed.

10. During each simulation a_m^* and b_m^* were passed through low pass filters with transfer functions $1/(1+s)$ to examine the filtered output as discussed in section 4.2. The rms values of the filtered a_e^* and b_e^* were computed.

In all, forty simulations were run. Two undetermined systems were examined, each with two input spectra. For easy reference each simulation has been assigned a code. The code consists of a letter to indicate the undetermined system and input spectrum, followed by a number to indicate the set of fixed parameters used, followed by another number to indicate the noise to signal ratio and value of V used.

The undetermined systems and input spectra are listed in Table 1.

TABLE 1

	w	ζ	a	b	input
A--	1.0	0.3	1.0	0.6	$X(n) = 1.0$
B--	1.0	0.3	1.0	0.6	$X(n) = 1.112/(1+.04n^2)$
C--	1.0	1.0	1.0	2.0	$X(n) = 1.0$
D--	1.0	1.0	1.0	2.0	$X(n) = 1.112/(1+.04n^2)$

The 1.112 appears in the numerator of the input for sets B and D so that the rms value of the input to these sets will be equal to the rms value of the input to sets A and C.

The sets of fixed parameters were chosen at noise to signals of 0.1 (L=0.1). The choices are indicated in Table 2.

TABLE 2. (L = 0.1)

	w_1	ζ_1	w_2	ζ_2	a*	b*	R_e
A1-	0.9	0.1	1.2	0.1	-.2346	.3778	.0604
A2-	0.9	0.1	1.2	1.4			.1889
B1-	1.2	1.1	0.4	0.7	.1667	-1.233	.0985
B2-	1.2	1.1	0.8	0.2			.1321
C1-	0.8	1.2	1.2	1.2	-.5625	-1.300	.2077
C2-	0.8	1.2	1.2	0.2			.3752
D1-	0.7	0.7	1.1	0.7	-1.0508	0.0	.2419
D2-	0.7	0.7	1.1	0.2			.2607

The noise to signal ratios and values of V used are indicated in Table 3.

TABLE 3

	L	V
--1	0.1	d
--2	0.1	d/4
--3	0.05	d
--4	0.025	d
--5	0.005	d

The results of the simulations are in the next section.

5.4 Results of the Simulations

Figures F5.4-1 through F5.4-10 are plots of the errors in a^*_m and b^*_m versus time for case A. A smooth curve connects each plotted point, which is the average of 48 computed points. These plots are constructed so that an easy comparison can be made between cases A1 and A2. Error versus time was computed for the other cases also, but the results of these simulations will not be presented in as much detail as the results of case A.

It can readily be observed that the errors are considerably smaller for the "good" set of fixed parameters (A1) than for the "poor" set (A2), at all noise to signal ratios. Of the four cases examined, case A offers the most striking evidence of the usefulness of the criterion that R_e be minimized.

At the high noise to signal ratio of 0.1 it can be seen that the measurements are constrained more often with A2 than with A1; hence, A1 is actually measuring the undetermined parameters over more time than A2.

Figures F5.4-11 through F5.4-18 are plots of the rms and average values of the unfiltered errors versus noise to signal ratio and the rms values of the filtered errors versus noise to signal ratio.

Examination of these plots of the rms errors for cases A, B and C show that at almost every noise to signal ratio the rms errors are less for the "good" set of fixed parameters than for the "poor" set. The decrease in rms errors is particularly evident at high noise to signal ratios.

In case D, the reducing of rms error is found in the measurement of a_m^* but not in the b_m^*. It may be seen, however, that the rms errors in b_m^* are essentially equal for both D1 and D2.

The rms errors in b_m^* are generally greater than the rms errors in a_m^* in all cases. This situation could have been changed by weighting R_e such that the b_e^* portion of the evaluation of R_e contributed more than the a_e^* portion.

No generalization appears possible concerning the mean value of the errors beyond the fact that these errors tend to be less for small noise to signal ratios, as might be expected.

The plots indicate the filtering reduces the rms error in all cases examined. Improvement is especially notable in the high noise to signal ratio regions.

Table 4 is a compilation of the mean and rms errors in tabular form. The quantity P, which represents the percent of time that the measurement is not constrained, is also listed. It may be seen that P is greater for the "good" sets of fixed parameters than the "poor" sets for all cases considered.

5.5 Conclusions from the Simulations

1. The errors are lessened if L, the noise to signal ratio, is lessened. This intuitively obvious conclusion makes it evident that at the limiting condition of L = 0, no error exists; and hence, the system does indeed measure the parameters of the undetermined system.

2. In general, the errors are less for smaller R_e. In particular, the rms values of the errors are reduced with smaller R_e. This phenomena is especially evident for conditions A, B, and C.

3. It was mentioned in section 5.3 that values of R_e which are less than the associated value of L should yield relatively small errors. It is noted that for conditions C and D, the chosen values of R_e are consistently greater than the associated values of L. This phenomena is due to the fact that the undetermined systems used in A and B amplify the signal in a frequency range (about 1.0 rad/sec) while conditions C and D do not amplify the signal. It was therefore possible to choose the fixed parameters in cases A and B to take advantage of this amplification,

while in cases C and D no such advantage appeared. It therefore appears that the method can be used to greater advantage with a lightly damped undetermined system and essentially flat noise and input spectra than a highly damped undetermined system. The relatively low errors in case D seem to contradict this statement, but it appears that in this case the phasing of the noise and signal has been a significant contribution in error reduction. If the noise and signal had been random, the phasing could not contribute as heavily to error reduction.

4. It appears that for values of L greater than .05, the errors are high enough to render the method not feasible, at least for the conditions used in this simulation.

5. Filtering in general makes a significant contribution in reducing the errors, especially for the cases where L is large. Unfortunately, filtering alone cannot reduce the mean errors, which are generally not zero. It is evident that if a filter with a lower cut-off frequency had been used, a further reduction of rms error would have occurred.

6. The method exhibits highly non-linear properties; that is, conclusions drawn from one set of conditions cannot be extended to another set of conditions related to the original set by a simple magnification or attenuation of the frequency scale.

TABLE 4

		Unfiltered				Filtered			
	R_e	$\sqrt{\overline{a_e^{*2}}}$	$\sqrt{\overline{b_e^{*2}}}$	$\overline{a_e^{*}}$	$\overline{b_e^{*}}$	$\sqrt{\overline{a_e^{*2}}}$	$\sqrt{\overline{b_e^{*2}}}$	d	P
A11	.0604	.2022	.1298	.0318	.0568	.1608	.0988	2.3047	79.6
A12	.0604	.2145	.2633	-.0243	.0817	.1154	.1726	2.3047	98.8
A13	.0301	.0904	.1236	-.0086	.0416	.0558	.0850	1.2301	95.4
A14	.0150	.0475	.0612	-.0048	.0204	.0255	.0416	.6369	98.8
A15	.0030	.0077	.0117	-.004	.0041	.0055	.0084	.1320	99.8
A21	.1889	.2723	.2733	-.1261	-.0164	.1983	.2036	.9083	42.5
A22	.1889	.5010	.7173	.0128	-.0633	.2540	.3949	.9083	88.0
A23	.0883	.1755	.1626	.0111	-.0212	.1360	.1174	.4654	58.3
A24	.0428	.1214	.1762	.0123	-.0385	-.0691	.1018	.2357	85.9
A25	.0084	.0749	.0817	.0051	-.0136	.0192	.0326	.0480	98.1
B11	.0985	.2934	.2212	-.0535	-.0499	.2330	.1758	.1379	79.6
B12	.0985	.2582	.3677	-.0400	-.0298	.1948	.2823	.1379	100.0
B13	.0488	.1376	.1494	-.0354	-.0123	.1075	.1146	.0650	95.6
B14	.0243	.0648	.0760	-.0124	-.0090	.0499	.0575	.0316	100.0
B15	.0062	.0140	.0165	-.0030	-.0043	.0105	.0118	.0048	100.0
B21	.1321	.1999	.2193	-.0313	-.0811	.1577	.1615	.8965	57.5
B22	.1321	.5970	.3912	-.0303	-.0354	.2117	.1710	.8965	98.8
B23	.0642	.2191	.1467	.0151	-.0093	.1170	.0924	.4380	89.9
B24	.0317	.1667	.0866	-.0019	-.0150	.0697	.0399	.2167	96.9
B25	.0433	.0646	.0315	-.006	-.0071	.0176	.0118	.0063	99.8

TABLE 4, Cont'd.

	R_e	$\sqrt{\overline{a_e^{*2}}}$	$\sqrt{\overline{b_e^{*2}}}$	$\overline{a_e^*}$	$\overline{b_e^*}$	$\sqrt{\overline{a_e^{*2}}}$	$\sqrt{\overline{b_e^{*2}}}$	d	P
C11	.2077	.5602	.5529	.3344	.2425	.4693	.3870	.0117	33.0
C12	.2077	.7699	.7503	.0371	-.0104	.2531	.2588	.0117	59.5
C13	.1037	.3054	.3188	.0734	.0458	.1487	.1433	.0056	41.5
C14	.0522	.2603	.2824	.0363	.0013	.0750	.0910	.0028	63.8
C15	.0107	.1322	.1394	.0001	.0049	.0192	.0327	.0006	94.1
C21	.3752	.5804	.4795	-.1006	.2624	.4905	.4280	.1160	26.8
C22	.3752	1.0527	1.4938	.1435	.3896	.7928	1.1938	.1160	41.1
C23	.1935	.4273	.3725	.0706	.1035	.3069	.3110	.0562	33.2
C24	.0983	.3330	.3101	.0178	.0745	.1677	.2367	.0277	42.3
C25	.0202	.0985	.1284	-.0043	.0238	.0268	.0856	.0056	84.3
D11	.2419	.2403	.1431	-.2061	-.1156	.2391	.1314	.0659	38.1
D12	.2419	.2033	.3173	-.0574	.0291	.1638	.2229	.0659	94.0
D13	.1204	.1031	.1003	-.0187	.0089	.0865	.0640	.0312	77.3
D14	.0599	.0536	.0719	-.0182	.0044	.0449	.0481	.0152	95.5
D15	.0121	.0133	.0163	-.0071	-.0005	.0114	.0089	.0030	100.0
D21	.2965	.5497	.1975	.3840	.0859	.5302	.1710	.1950	35.7
D22	.2965	.3563	.3012	-.0253	.0289	.1774	.2010	.1950	95.7
D23	.1512	.1337	.0900	-.0050	-.0057	.0814	.0590	.0938	70.8
D24	.0760	.0857	.0680	-.0083	.0047	.0473	.0436	.0458	96.3
D25	.0155	.0214	.0142	.0047	.0105	.0105	.0084	.0091	99.5

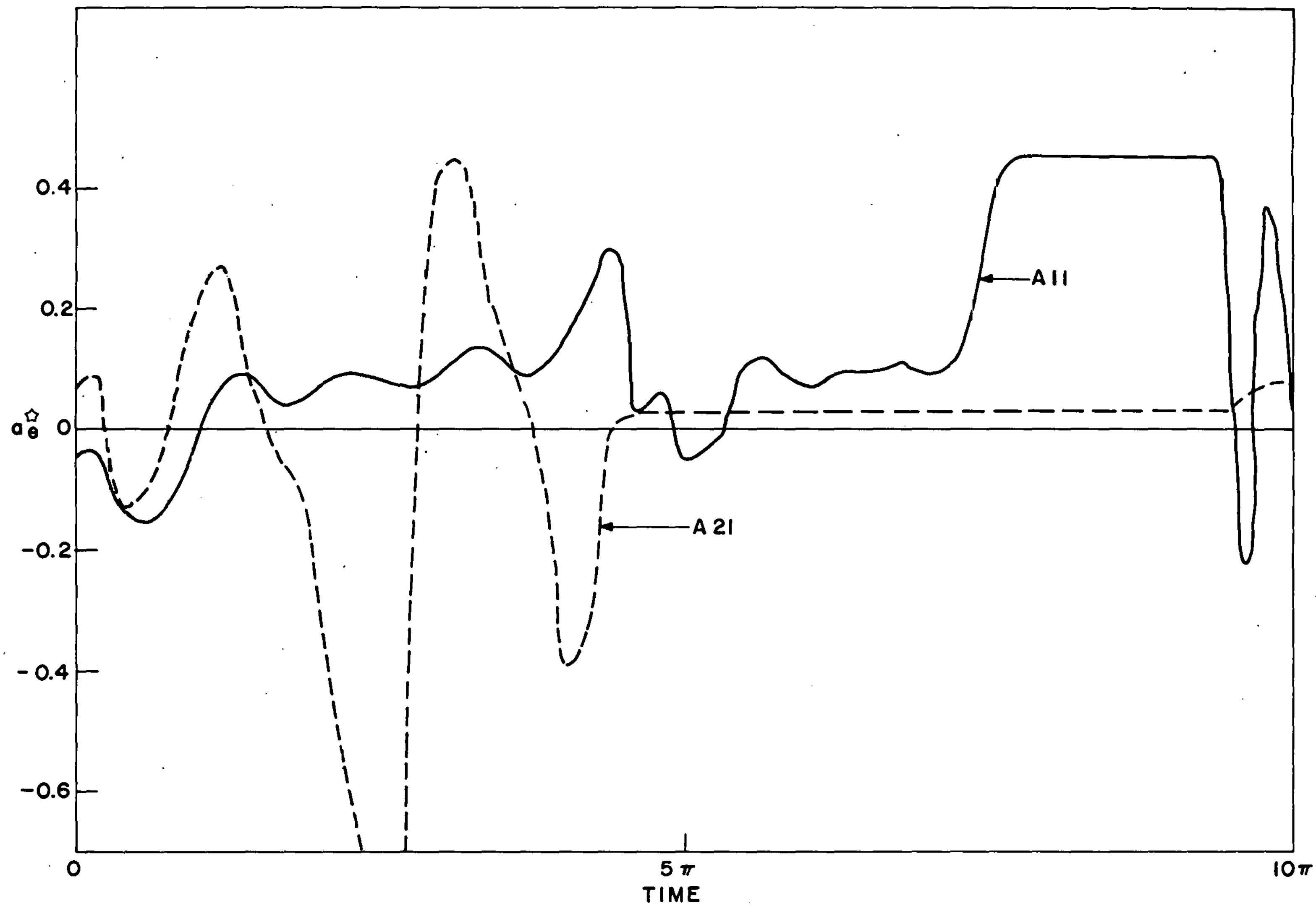

FIGURE F5.4-1, a_c^* vs TIME FOR CONDITIONS A11 AND A21

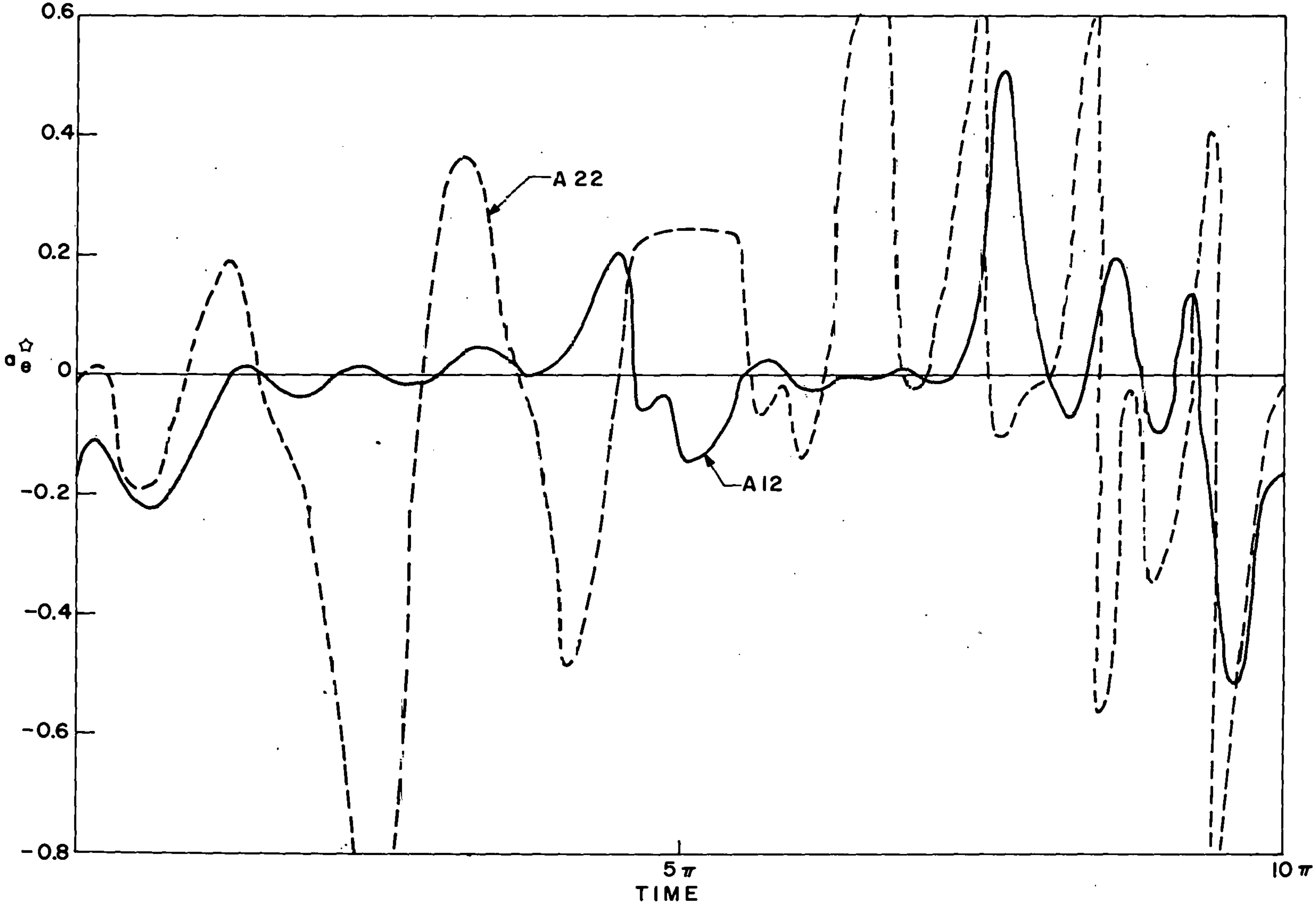

FIGURE F5.4-2, a_e^* vs TIME FOR CONDITIONS A12 AND A22

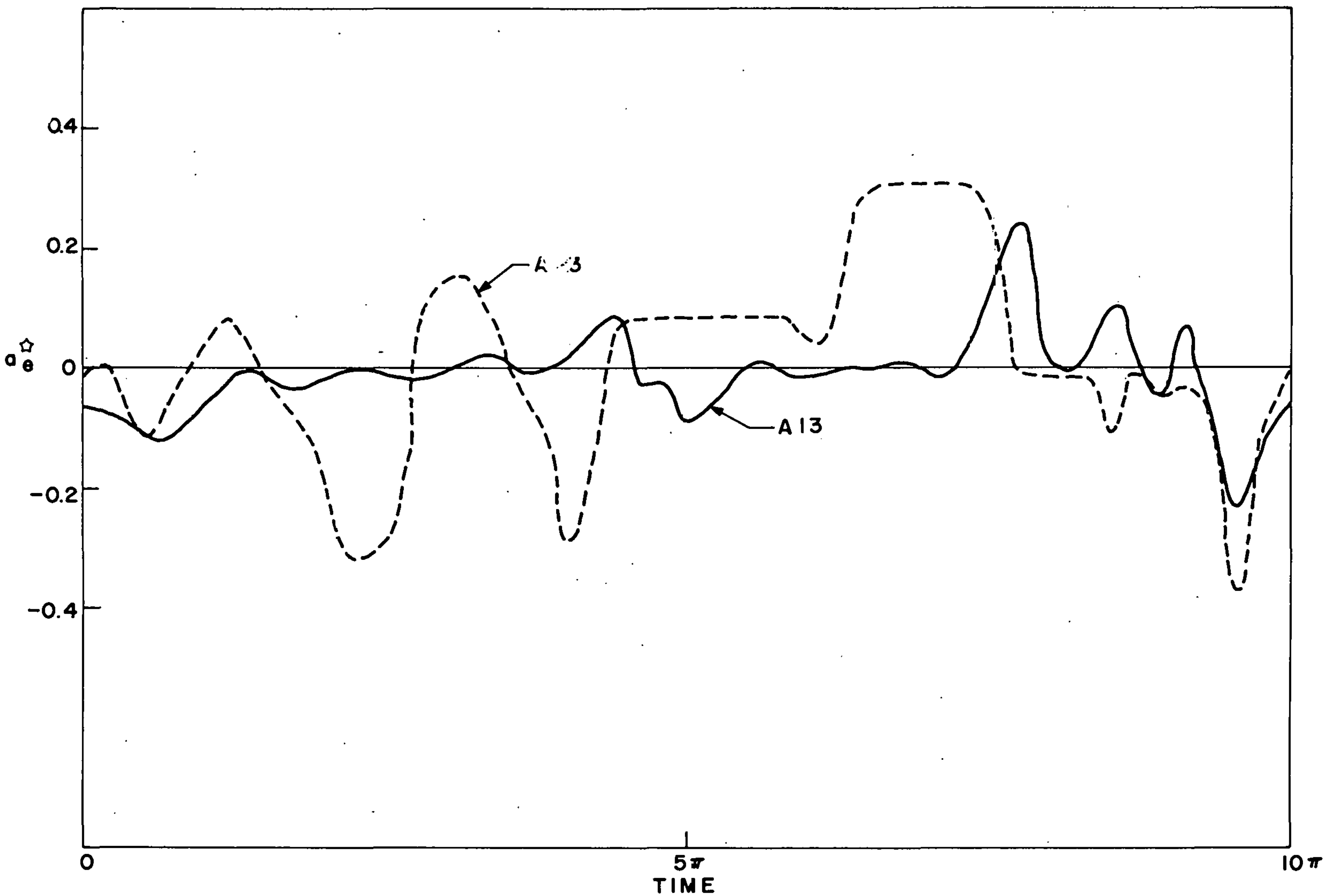

FIGURE F5.4-3, a_e^* vs TIME FOR CONDITIONS A13 AND A23

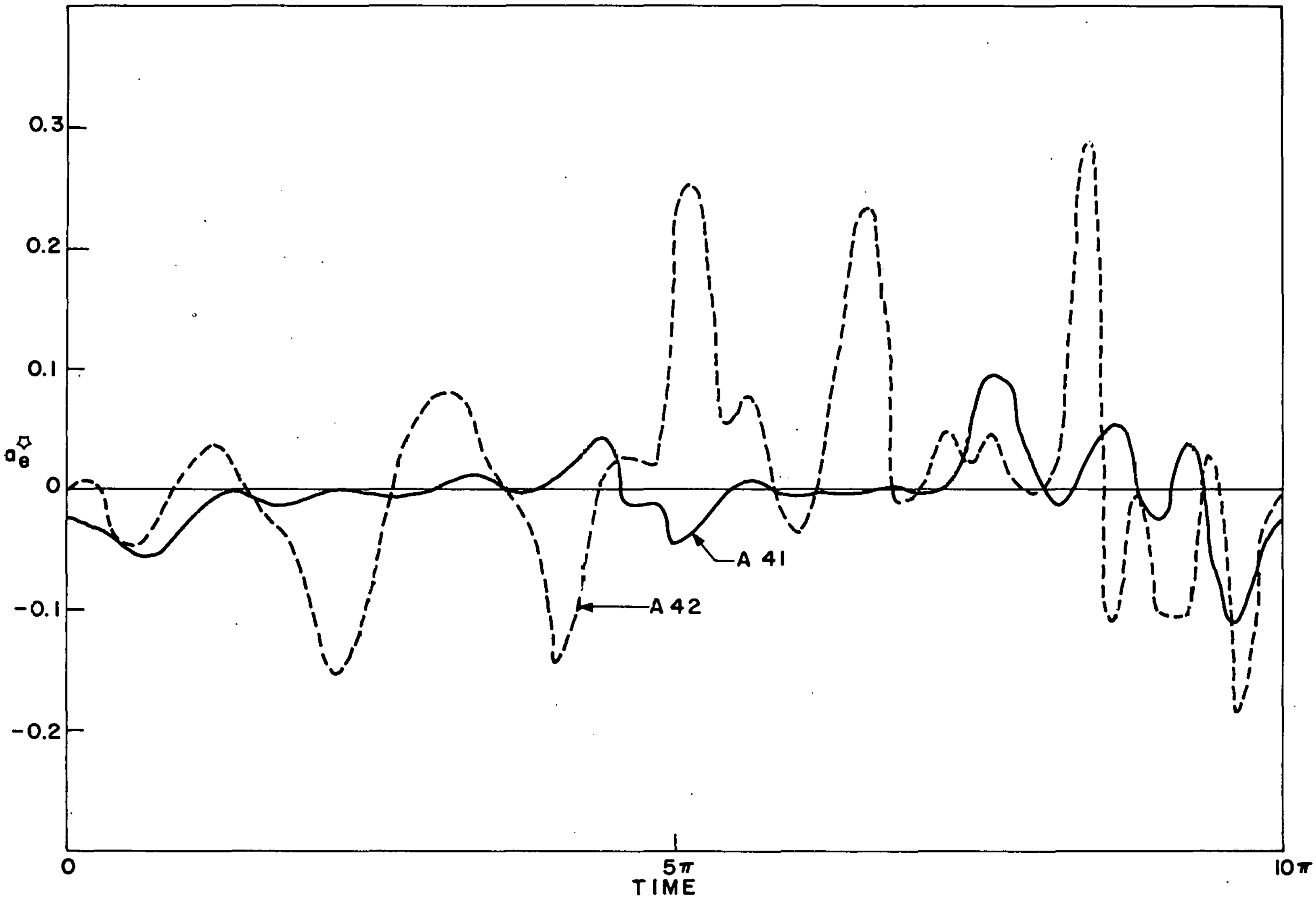

FIGURE F5.4-4, a_e^* vs TIME FOR CONDITIONS A14 AND A24

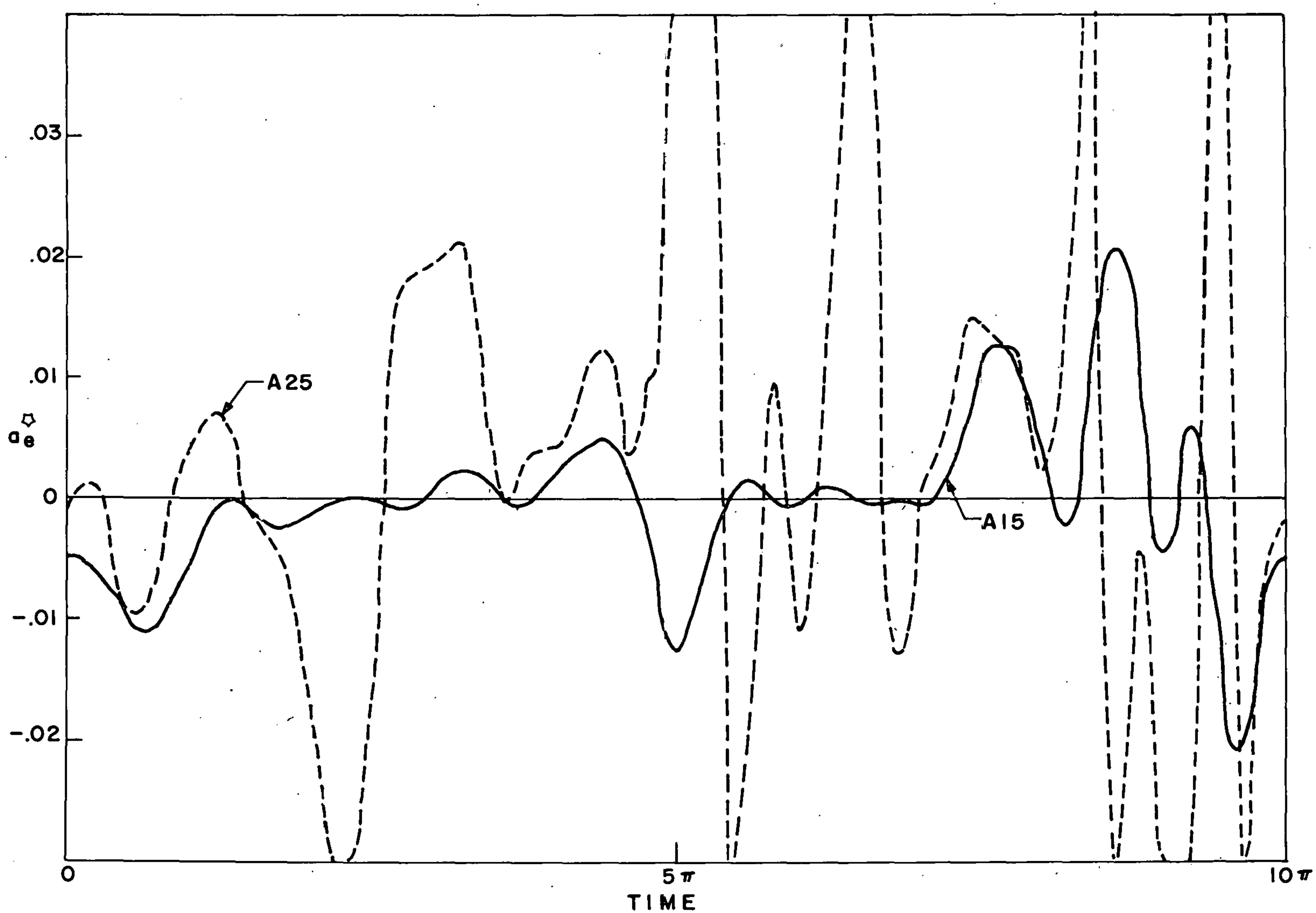

FIGURE F5.4-5, a_e^* vs TIME FOR CONDITIONS A15 AND A25

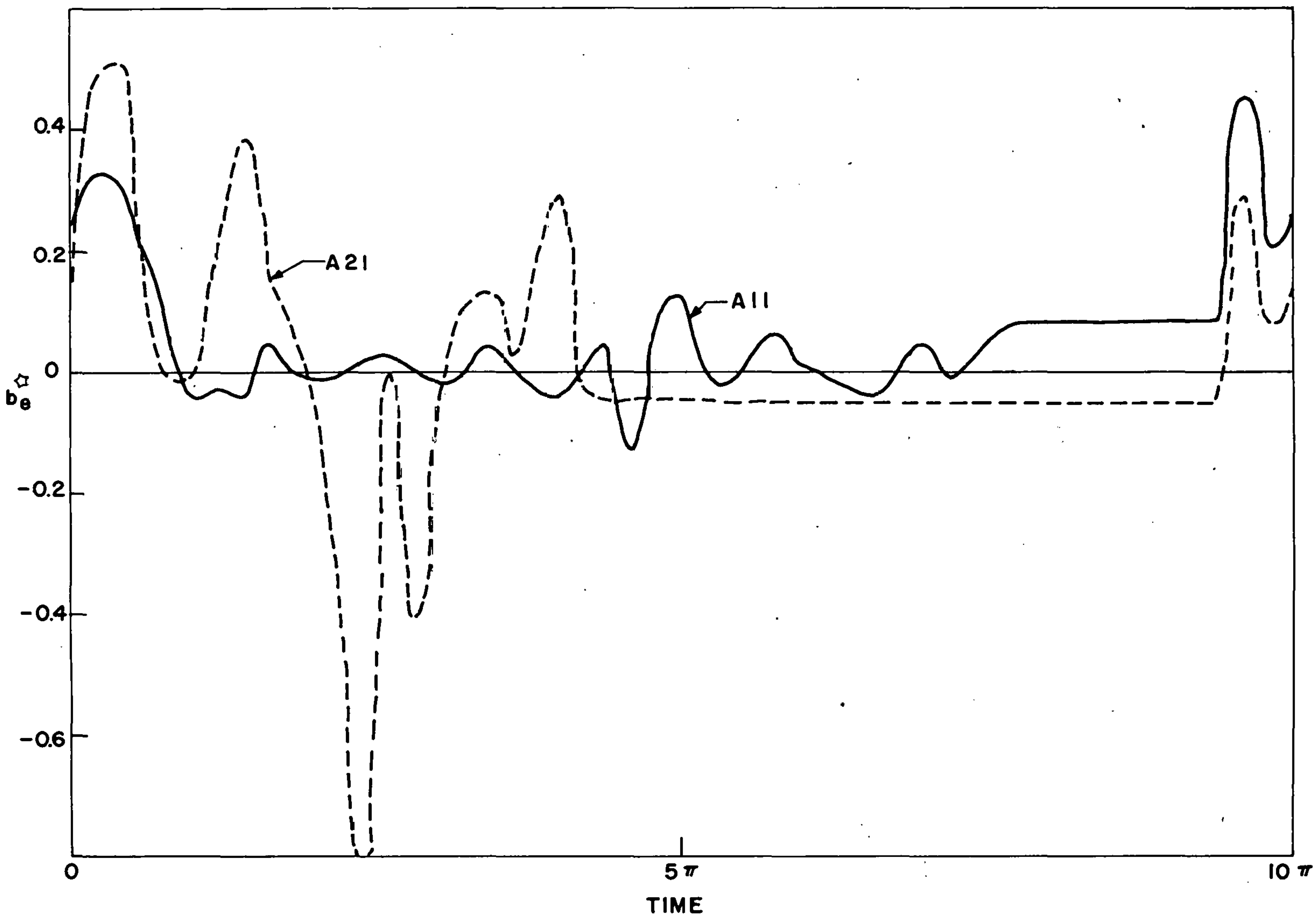

FIGURE F5.4-6, b_e^* vs TIME FOR CONDITIONS A11 AND A21

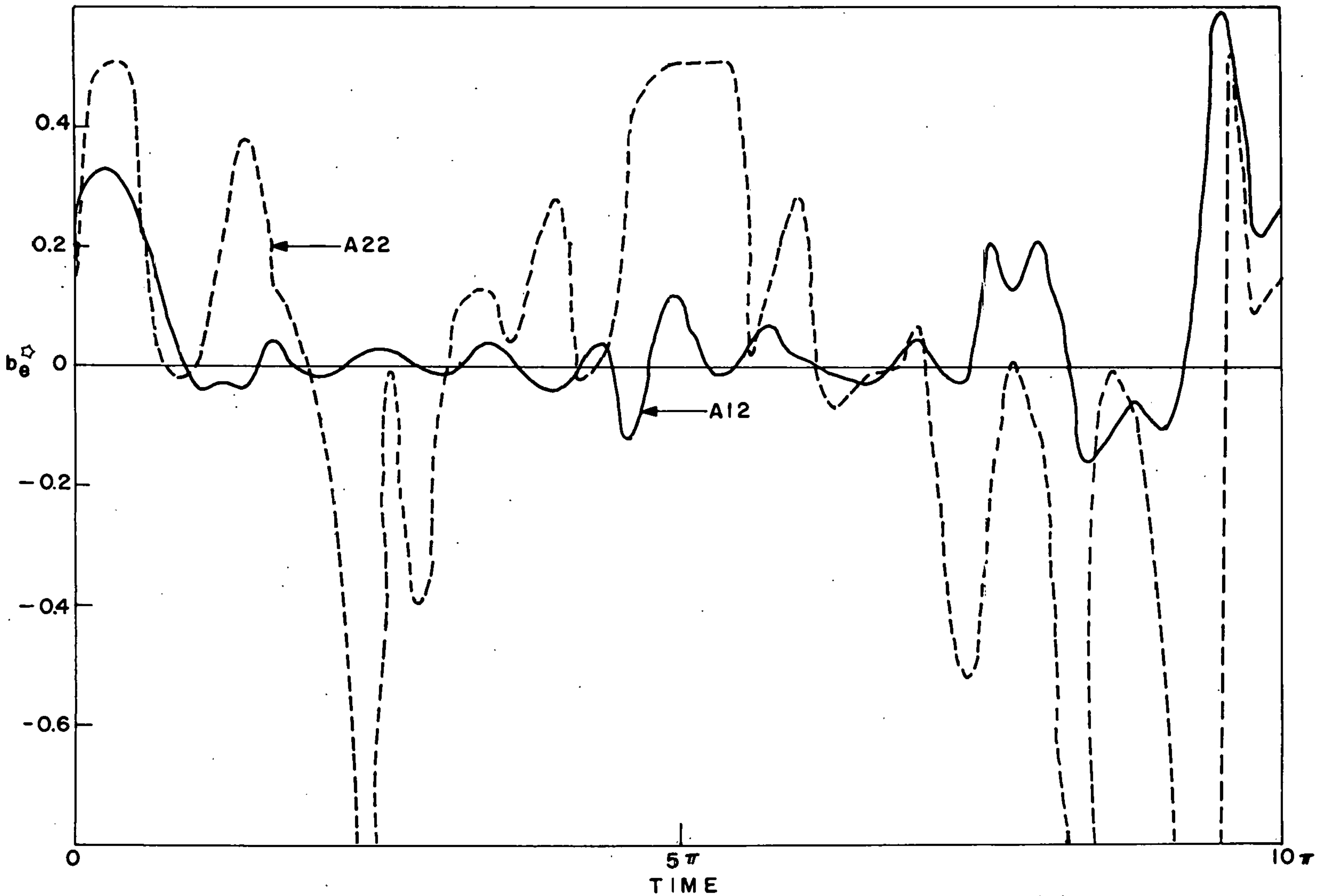

FIGURE F5.4-7, b_e^* vs TIME FOR CONDITIONS A12 AND A22

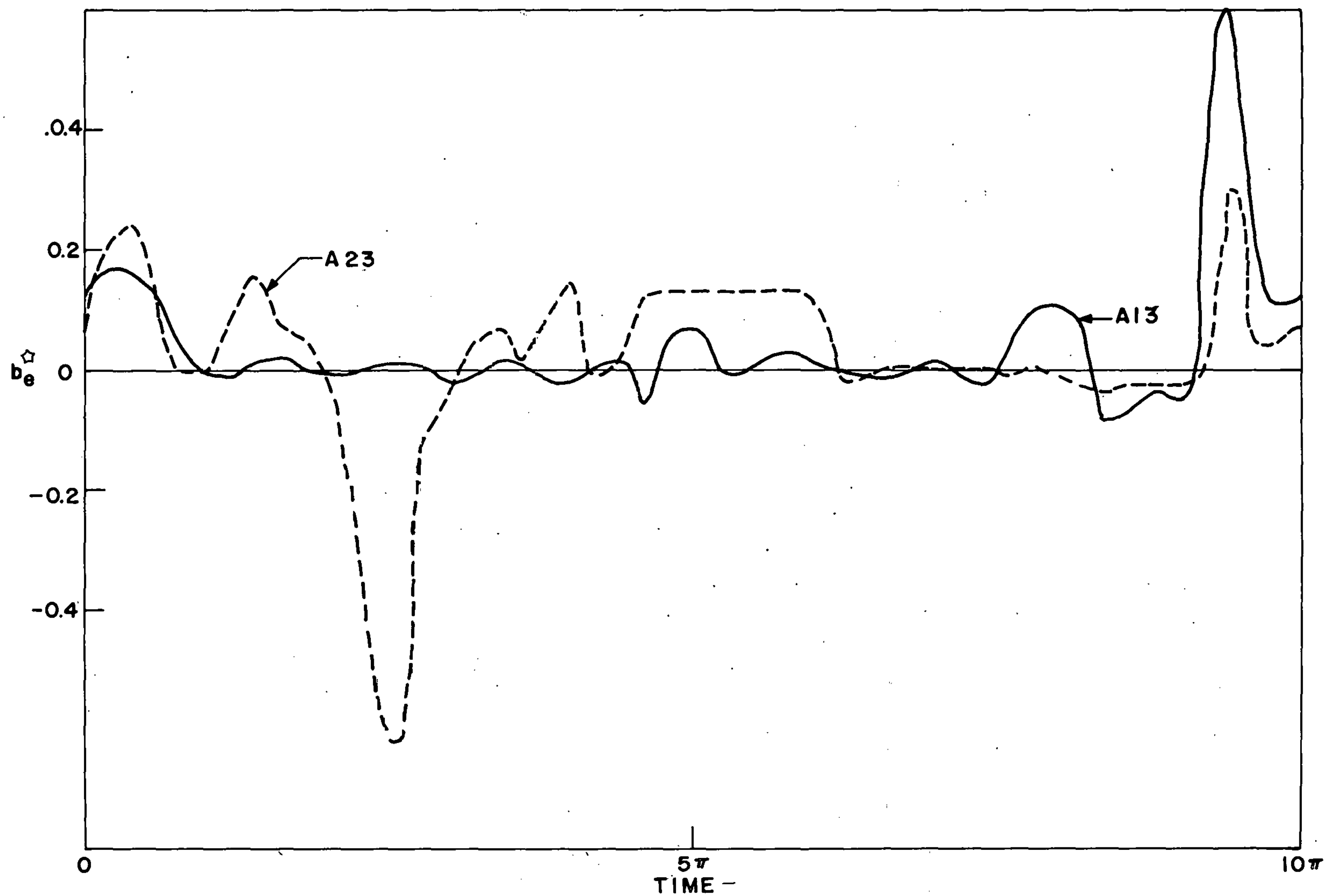

FIGURE F5.4-8, b_e^* vs TIME FOR CONDITIONS A13 AND A23

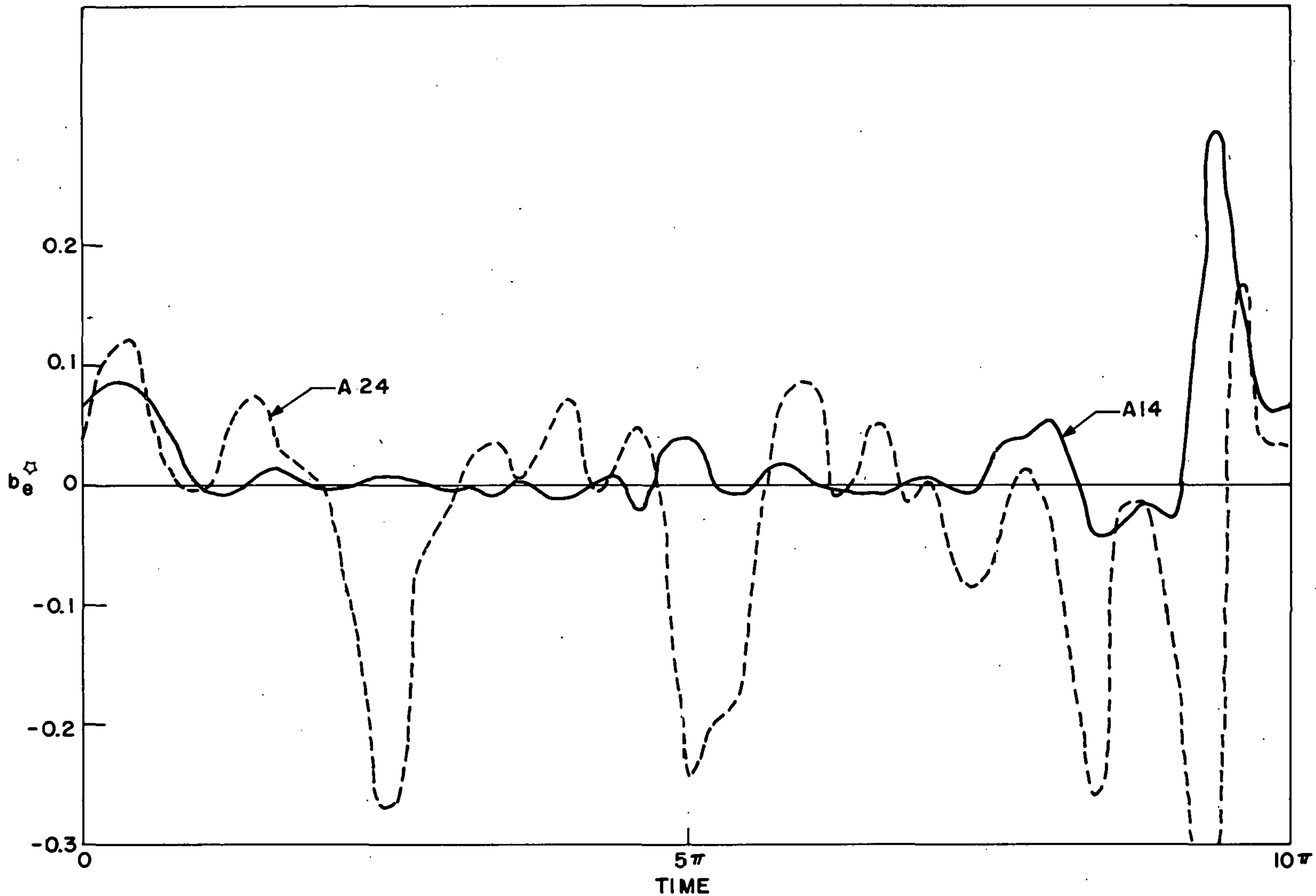

FIGURE F5.4-9, b_e^* vs TIME FOR CONDITIONS A14 AND A24

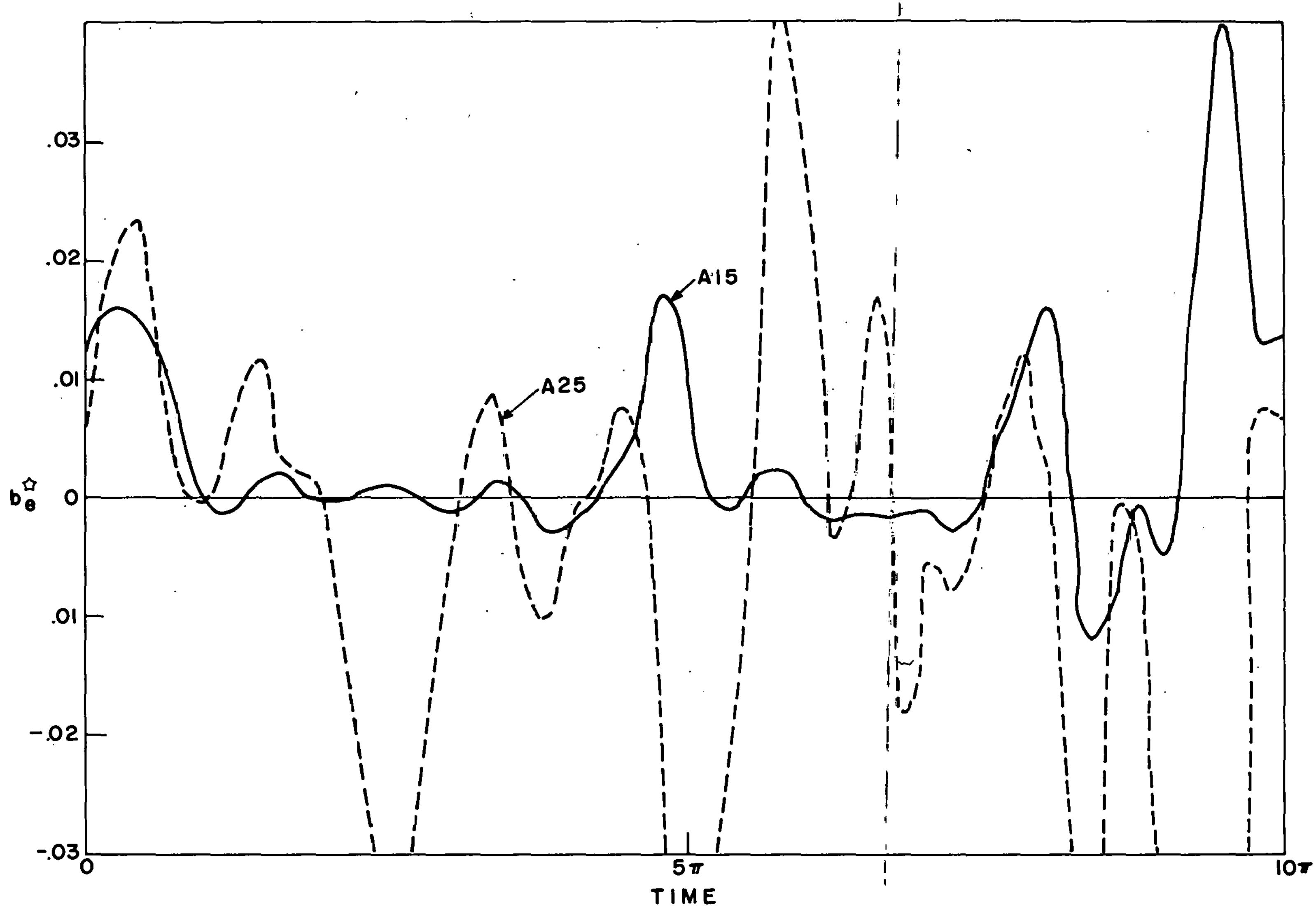

FIGURE F5.4-10, b_e^* vs TIME FOR CONDITIONS A15 AND A25

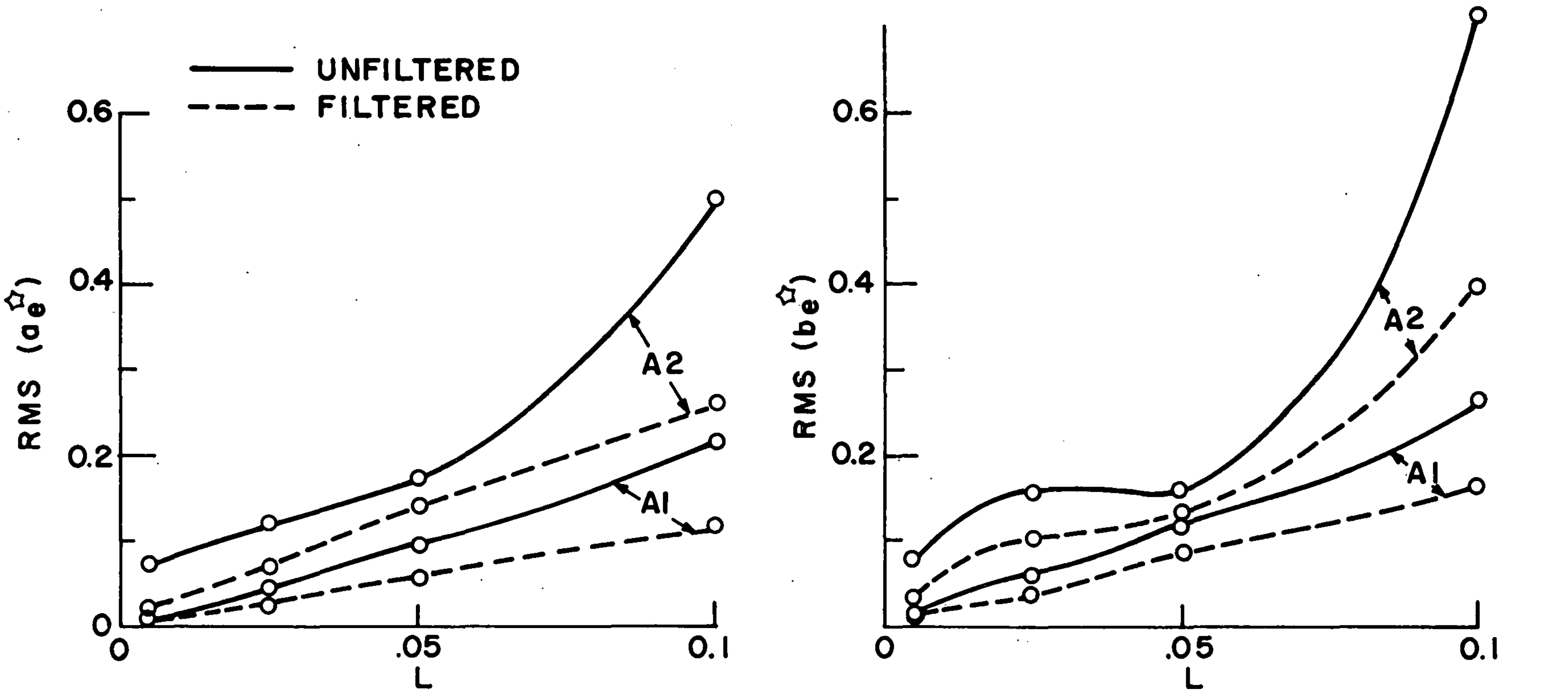

FIGURE F5.4-11, RMS ERRORS vs NOISE TO SIGNAL RATIO FOR CONDITION A

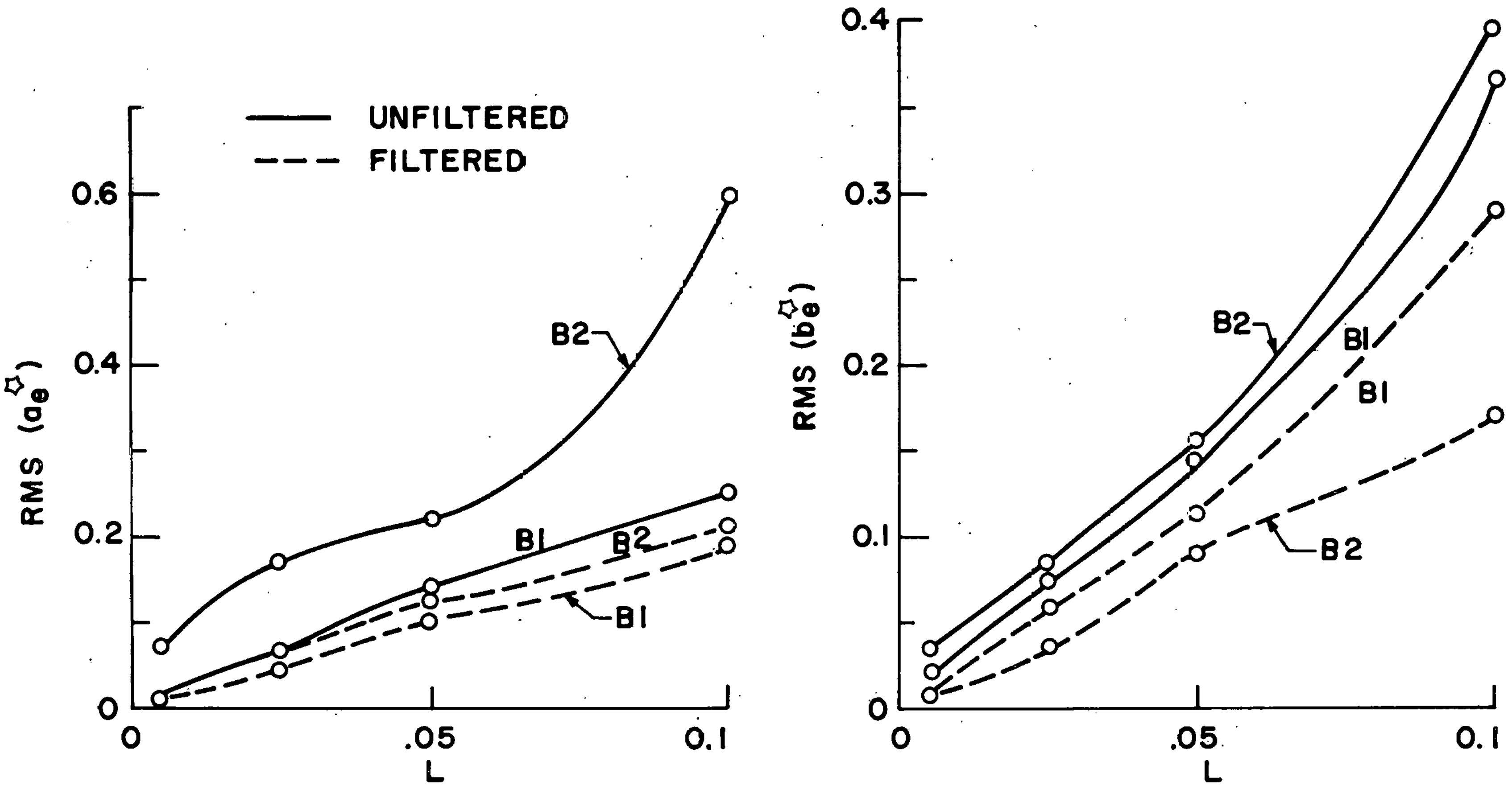

FIGURE F5.4-12, RMS ERRORS vs NOISE TO SIGNAL RATIO FOR CONDITION B

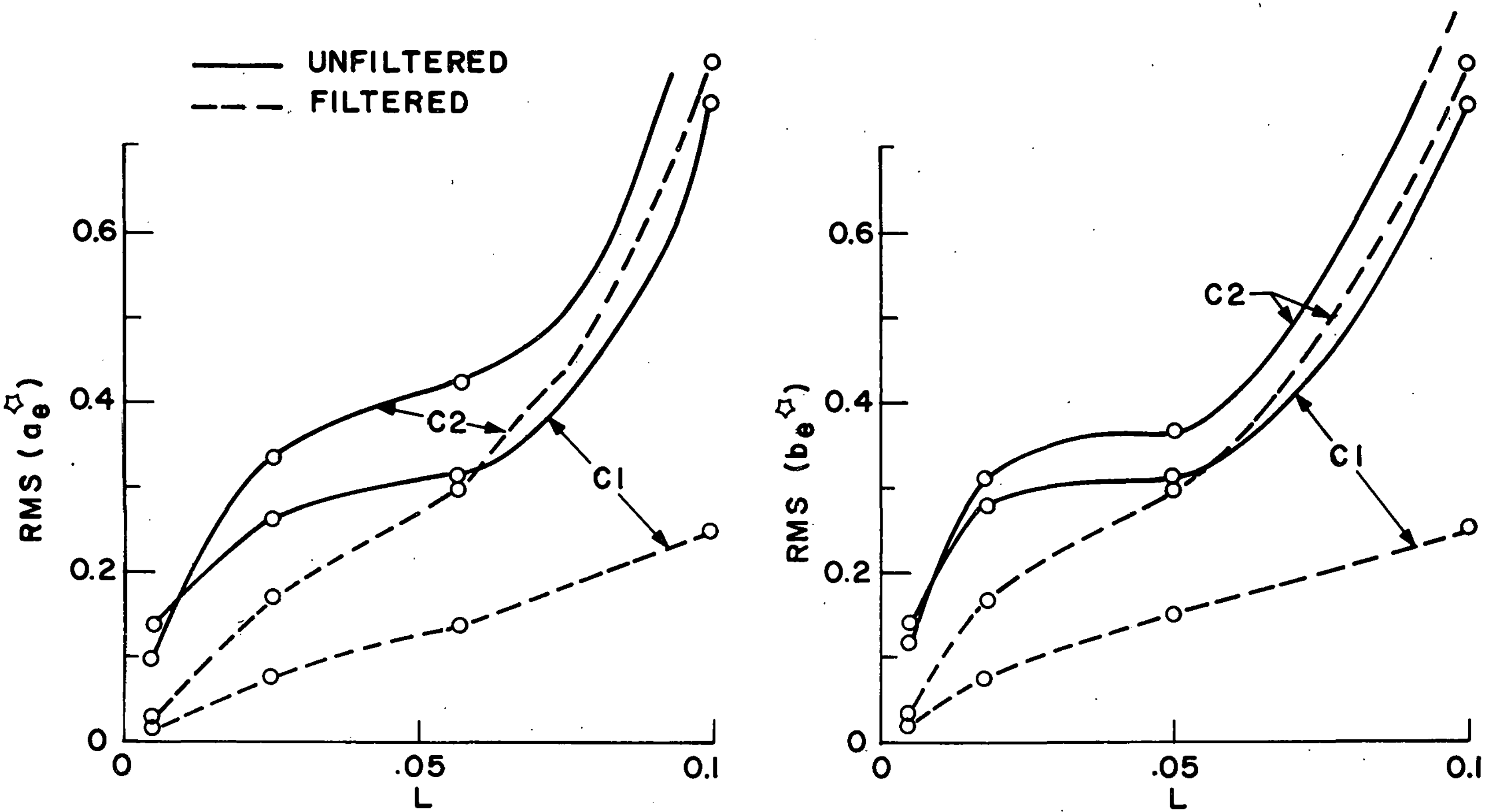

FIGURE F5.4-13, RMS ERRORS vs NOISE TO SIGNAL RATIO FOR CONDITION C

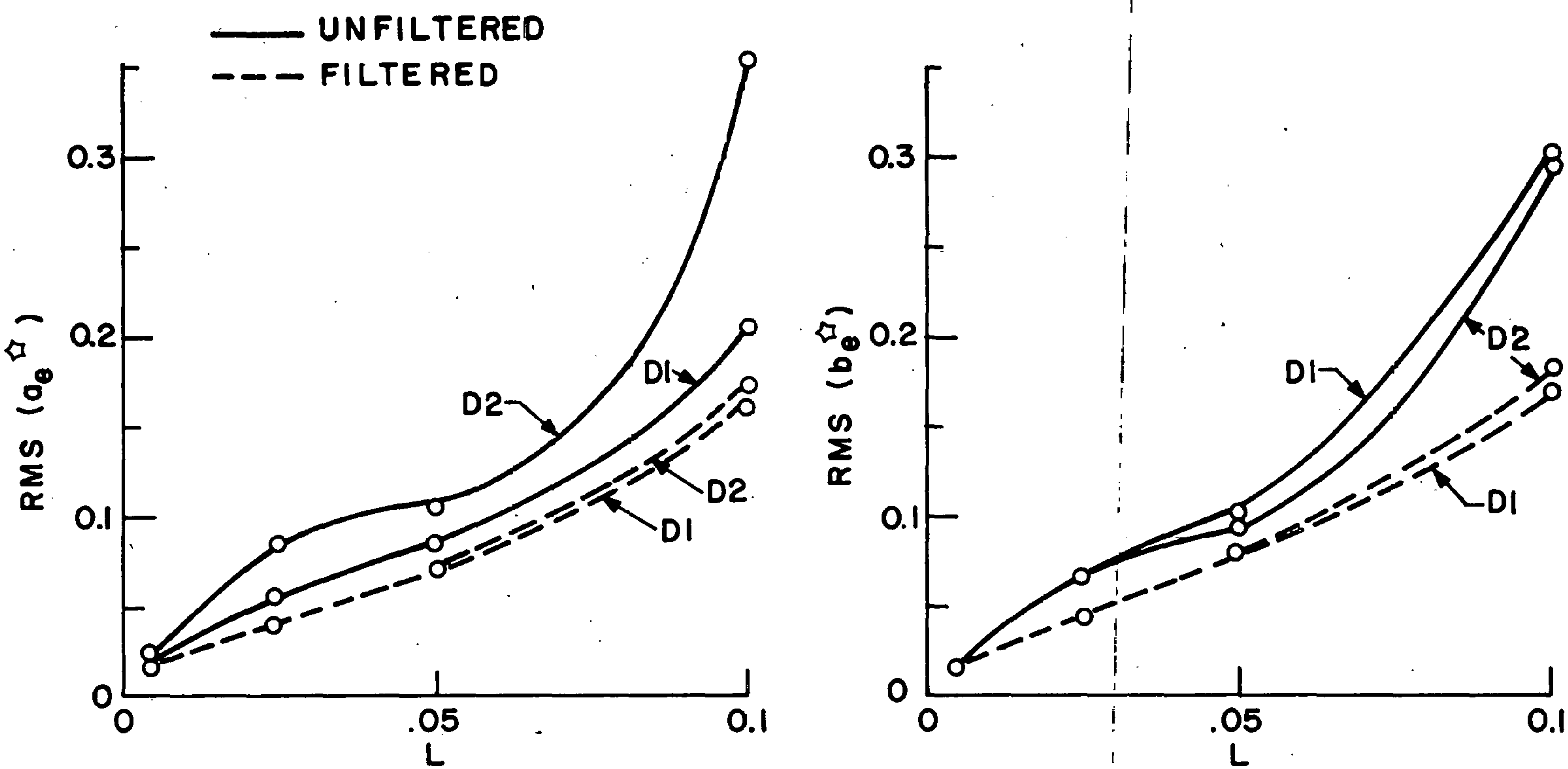

FIGURE F5.4-14, RMS ERRORS vs NOISE TO SIGNAL RATIO FOR CONDITION D

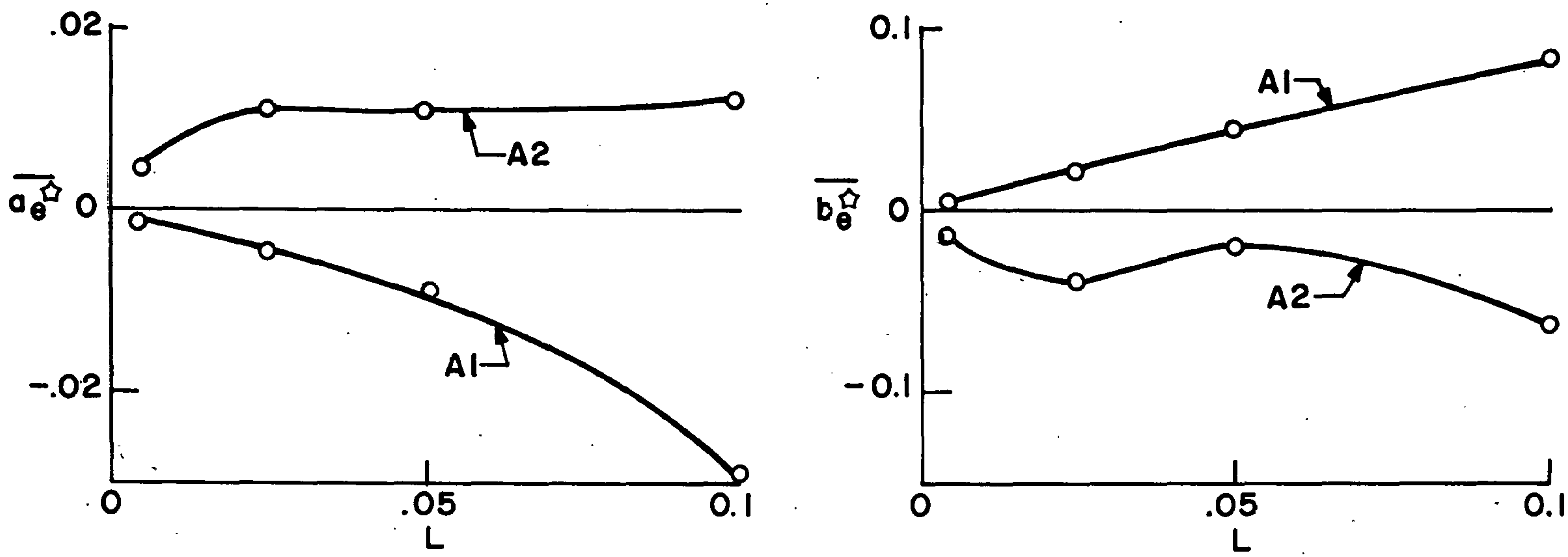

FIGURE F5.4-15, MEAN ERRORS vs NOISE TO SIGNAL RATIO FOR CONDITION A

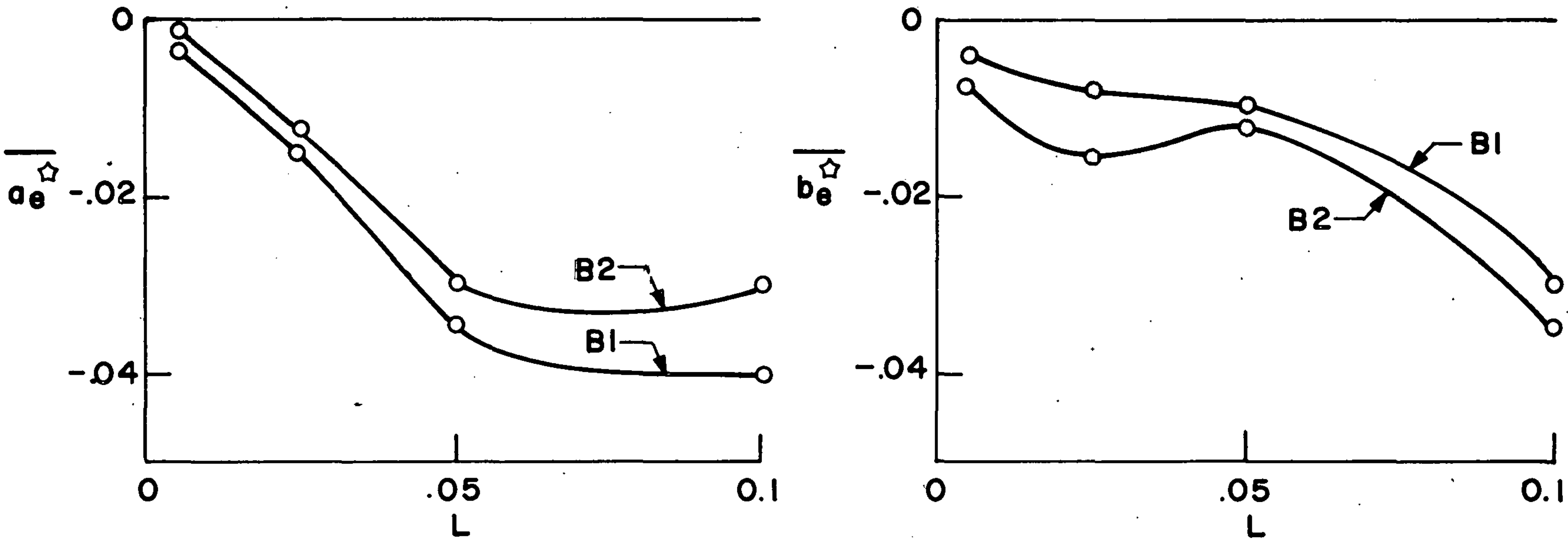

FIGURE F5.4-16, MEAN ERRORS vs NOISE TO SIGNAL RATIO FOR CONDITION B

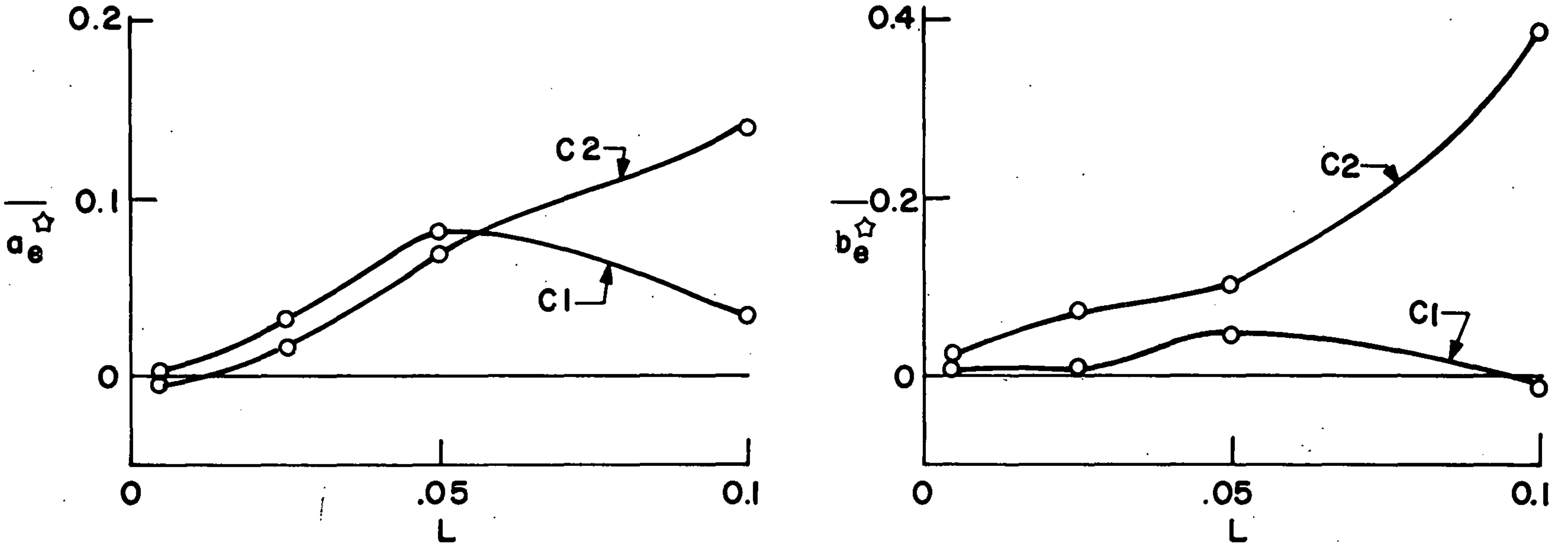

FIGURE F5.4-17, MEAN ERRORS vs NOISE TO SIGNAL RATIO FOR CONDITION C

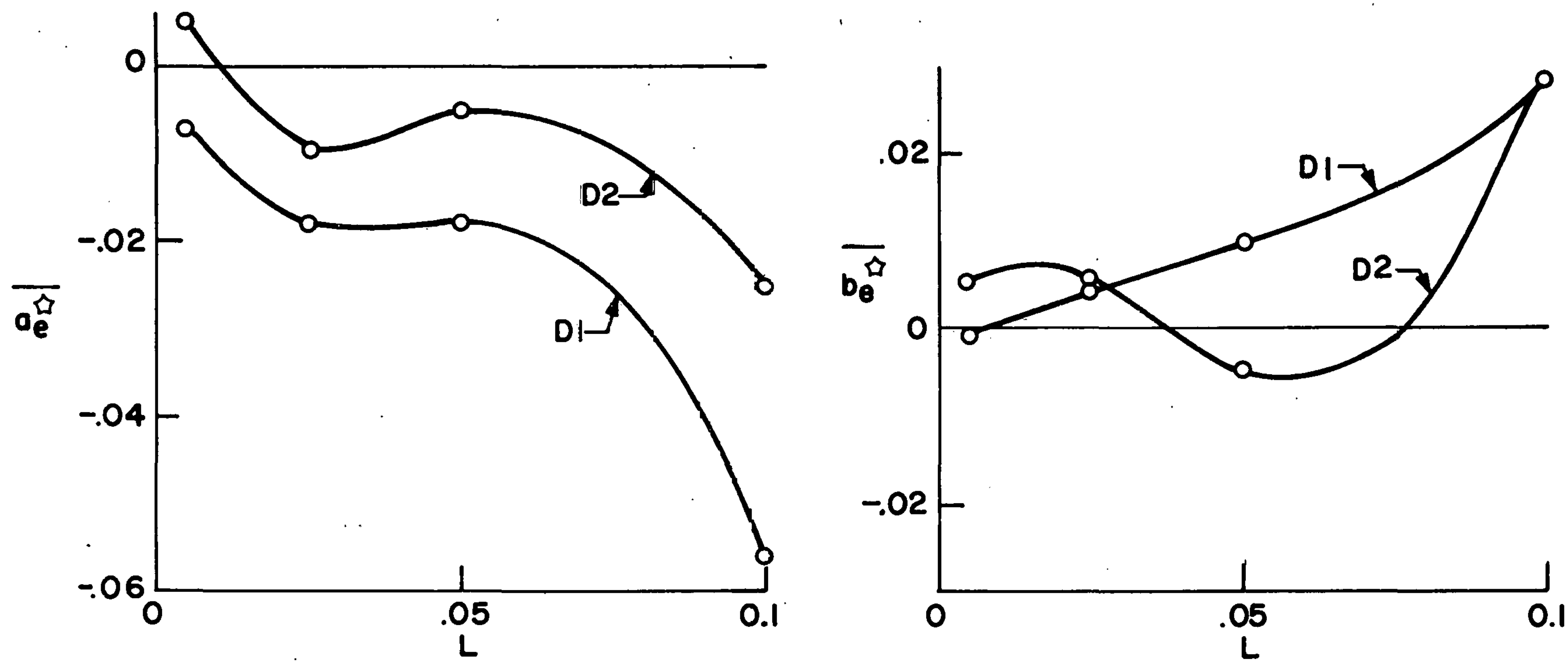

FIGURE F5.4-18, MEAN ERRORS vs NOISE TO SIGNAL RATIO FOR CONDITION D

APPENDIX A

OVERDETERMINATION AND LEAST SQUARES [5]

It is generally true that when the coefficients of a set of linear equations are subject to random errors, "overdetermination" is useful in reducing the errors in the solutions.

Let there be n equations in the m unknowns $\{x_j, j=1, \ldots, m\}$ vis.

$$\begin{array}{ccccc} a_{11}x_1 & + \cdots + & a_{1m}x_m & = & c_1 \\ \vdots & & \vdots & & \vdots \\ a_{n1}x_1 & + \cdots + & a_{nm}x_m & = & c_n \end{array}$$

If $n > m$, the set is said to be "overdetermined." The generally most useful and practical way to solve the set is in the "least squares" sense; that is, defining

$$R_i = \sum_{j=1}^{m} (a_{ij}x_j - c_i)$$

to find the unknowns such that

$$S = \sum_{i=1}^{n} R_i^2$$

is minimized.

The minimization is accomplished by differentiating S with respect to each of the unknowns and setting each of the partial derivatives to zero, yielding the m equations

$$\frac{\partial S}{\partial x_j} = \sum_{i=1}^{n} a_{ij}R_j = 0 \qquad j=1,\ldots,m \qquad \text{(A-1)}$$

Equations (A-1) represent m simultaneous linear equations which may be solved for the m unknowns.

This approach may be used to solve (2.1-4) as follows:

Let there be m fixed systems

H_i; $i=1,\ldots,m$; $m > 2$

We write the following two equations in the two unknowns a_1^* and b_1^*

$$a_1^* \sum_{i=1}^{m}(z_i'')^2+b_1^* \sum_{i=1}^{m} z_i''z_i' = \sum_{i=1}^{m}(y_i-y-(a_1-a_i)z_i''-(b_1-b_i)z_i')z_i''$$

$$\text{(A-2)}$$

$$a_1^* \sum_{i=1}^{m}z_i''z_i'+b_1^* \sum_{i=1}^{m}(z_i)^2 = \sum_{i=1}^{m}(y_i-y-(a_1-a_i)z_i''-(b_1-b_i)z_i')z_i'$$

The solutions of (A-2) are generally less sensitive to errors in the y_i and $z_i^{(n)}$ than the solutions of (2.1-4).

APPENDIX B

STATISTICAL THEORY

To illuminate the material of Chapters III and IV a brief description of certain statistical relations useful in this paper is given in this appendix. All relationships are given without proof. A more extensive treatment can be found in many references. [2, 3] The material here is essentially from Lanning and Battin. [3]

Property 1

If x and y are random variables of two statistically independent random processes, then

$$E[xy] = E[x]E[y]$$

where E[z] denotes the expected value (average value) of the variable [z].

In the body of the paper extensive use is made of the notion of a sample function of a stationary ergodic random process. Such a process is defined as follows:

Definition 1

A random process is ergodic (or possesses the ergodic property) if averages across the ensemble of random variables describing the process may be replaced with averages taken across time from a sample function (representitive random variable) of the process

Definition 2

A random process is stationary if its statistical properties are invariant with translations in time.

The Power Spectral Density and the Cross Spectral Density

Let $g(t)$ be a sample function of a random process. The auto-correlation function of $g(t)$ is given as

Definition 3

$$\phi_g(t_1,t_2) = E[g(t_1)g(t_2)]$$

In particular, if the random process is stationary and ergodic, the autocorrelation function is a function of the difference t_2-t_1. With $\tau = t_2-t_1$, $\phi_g(\tau)$ can be expressed as

Definition 4

$$\phi_g(\tau) = \overline{g(t)g(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(t)g(t+\tau)\,dt$$

Definition 5

If $g(t)$ and $y(t)$ are sample functions of two stationary ergodic random processes, the cross-correlation function is defined as

$$\phi_{gy}(\tau) = E[g(t)y(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g(t)y(t+\tau)\,dt$$

Definition 6

The power spectral density (PSD) is the Fourier transform of the autocorrelation function, vis.,

$$G_g(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_g(\tau) \exp(-jw\tau)\, d\tau \; ; \qquad j^2 = -1$$

Definition 7

The cross spectral density (CSD) is the Fourier transform of the cross correlation function

$$G_{gy}(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi_{gy}(\tau) \exp(-jw\tau)\, d\tau$$

The PSD and CSD exhibit the following important properties:

Property 2 The PSD is always real and non-negative. Also, the PSD is an even function of w.

$$G_g(w) \geq 0 \qquad \text{and} \qquad G_g(w) = G_g(-w)$$

for all w.

Property 3 If g(t) is a sample function of a random process, the mean square value of g(t) is given by

$$\overline{g(t)^2} = \frac{1}{2} \int_{-\infty}^{\infty} G_g(w) dw = \int_{0}^{\infty} G_g(w)\, dw$$

Property 4 The mean value of the product of two functions g(t) and y(t) is given by

$$\overline{g(t)y(t)} = \frac{1}{2} \int_{-\infty}^{\infty} G_{gy}(w)\, dw$$

In particular, if G_{gy} is an even function of w,

$$\overline{g(t)y(t)} = \int_0^\infty G_{gy}(w)\, dw$$

Property 5 If $G_g(w)$ is the PSD of the input to a linear, time invariant system whose transfer function is K(s), the PSD of the output (y) is

$$G_y(w) = K(jw)K(-jw)G_g(w) = |K(jw)|^2 G_g(w)$$

Property 6 If G_g is the PSD of the input to a linear, time invariant system whose transfer function is K(s), the CSD of the input and output is

$$G_{gy}(w) = K(jw)G_g(w)$$

Property 7 If g(t) and y(t) are sample functions of two independent random processes with PSD $G_g(w)$ and $G_y(w)$ respectively, the PSD of the sum of g and y is

$$G_{(g+y)}(w) = G_g(w) + G_y(w)$$

Gaussian Random Variables

Definition 8

A random variable (g) is gaussian if its probability density function is given as

$$p(g) = \frac{1}{\sqrt{2\pi}\, u} \exp\left[- \frac{(g-m)^2}{2u^2} \right]$$

where $m = E[g]$ and $u^2 = E[g^2]$.

In particular, if g(t) is a sample function of a stationary ergodic gaussian random process with zero mean, the probability density function of the function is described by

$$p(g) = \frac{1}{\sqrt{2\pi}\, u} \exp\left[- \frac{g(t)^2}{2u^2} \right]$$

where $u^2 = E[g(t)^2]$.

Theorem 1

If y(t) is a sample function of the output of a linear system whose input is a sample function of a gaussian process, y(t) is also a sample function of a gaussian process; or more briefly, y(t) is gaussian.

Theorem 2

If g(t) and y(t) are gaussian random variables, then the sum of g and y is a gaussian random variable.

Theorem 3

Let g(t) be the gaussian input to n linear filters, and let $\{y_i;\ i=1,\ldots,n\}$ be the outputs. The functions $y_i(t)$ possess a joint gaussian distribution; that is,

$$p(y_1,\ldots,y_n) = \frac{1}{(2n)^{n/2}\sqrt{|M|}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} m_{ki}^{-1} y_i y_k\right]$$

The quantity $|M|$ is the determinant of the so-called moment matrix

$$|M| = \begin{Vmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & & \vdots \\ m_{n1} & \cdots & m_{nn} \end{Vmatrix}$$

where the m_{ik} are given by

$$m_{ik} = E[y_i y_k]$$

The m_{ki}^{-1} are the elements of the inverse of the moment matrix.

Theorem 4

If the functions y_i possess a joint gaussian distribution, then

$$E\left[y_1^{r1},\ldots,y_n^{rn}\right] = \frac{r1!\ \ldots\ rn!}{2^q\ q!} \sum (m_{i_1k_1} \cdots m_{i_qk_q})$$

$$2q = \sum_j rj\ ; \qquad rj \text{ integers}$$

where the summation represents all permutations and combinations of the products such that r1 of the indices $(i_1k_1\ldots i_qk_q)$ are unity, r2 of them are two, etc. The formula above obviously becomes very unwieldy for even small q.

Fortunately, we are only concerned with two simple results of the formula above; namely,

$$E[y_1 y_2 y_3 y_4] = E[y_1 y_2]E[y_3 y_4] + E[y_1 y_3]E[y_2 y_4] + E[y_1 y_4]E[y_2 y_3]$$

and (B-1)

$$E[y_1^{r1} \ldots\ldots y_n^{rn}] = 0 \quad \text{if} \quad \sum_j rj \text{ is odd.}$$

APPENDIX C

EVALUATION OF $\overline{(Na_e-a^*D_e)^2}$ AND $\overline{(Nb_e-b^*D_e)^2}$

The expressions for the mean square values of $Na_e-a^*D_e$ and $Nb_e-b^*D_e$ can be evaluated by the methods of section 3.5. These expressions are necessary in the evaluation of R_e (equation 3.5-1). The signal and noise are assumed to be sample functions of two independent, ergodic, stationary Gaussian random processes with zero mean and known PSD. Let the PSD of x(t) be G_x and the PSD of e(t) be G_e.

We define

$$G_y = G_x M + G_e \tag{C-1}$$

where

$$M_i(w) = |H_i(jw)|^2 = H_i(jw)H_i(-jw) \quad ; \quad j^2 = -1 \tag{C-2}$$

The expressions for Na_c, Nb_e, and D_e are given in equation (3.3-3). Expand $(Na_e-a^*D_e)^2$ and $(Nb_e-b^*D_e)^2$ and average these quantities. Note the following properties of the expansion and averaging process:

1) Each term of the expansion contains four functions.
2) The average value of any term containing two noise functions and two signal functions is the product of the average of the noise functions and the average of the signal functions, by virtue of the independence of signal and noise (Theorem 1).

3) The average value of any term containing three noise functions and one signal function is zero, because the average value of each individual signal (and noise) term is zero.

4) The average value of any term containing four noise functions can be evaluated (B-1) in the same fashion that (3.5-4) was derived from (3.5-3).

If we define, for i = 1, 2; k = 1, 2;

$$R = \left[a_i a_k w^4 + (b_i b_k - a_i - a_k) w^2 + 1\right] M_i M_k \qquad (i \neq k)$$

$$R_i = \left[a a_i w^4 + (b b_i - a - a_i) w^2 + 1\right] M M_i$$

$$I_{ik} = -I_{ki} = \left[(a_i b_k - b_i a_k) w^2 + b_i - b_k\right] M_i M_k \qquad (i \neq k) \qquad \text{(C-3)}$$

$$Q'_{ik} = w^2 \left[bR + (aw^2 - 1) I_{ik}\right] M_i M_k \qquad (i \neq k)$$

$$Q''_{ik} = w^2 \left[R(aw^2 - 1) + w^2 b I_{ik}\right] M_i M_k \qquad (i \neq k)$$

we may write the desired averages as

$$\overline{z_i'' z_k''} = \int_0^\infty w^4 G_y R \, dw \qquad (i \neq k) \qquad (1) \qquad \text{(C-4)}$$

$$\overline{z_i'' z_i''} = \int_0^\infty w^4 G_y M_i \, dw \qquad (2)$$

$$\overline{z_i'z_k'} = \int_0^\infty w^2 G_y R dw \qquad (i \neq k) \qquad (3)$$

$$\overline{z_i'z_i'} = \int_0^\infty w^2 G_y M_i dw \qquad (4)$$

$$\overline{z_i''z_k'} = \int_0^\infty w^4 G_y I_{ik} dw \qquad (i \neq k) \qquad (5)$$

$$\overline{z_i''z_i'} = 0 \qquad (6)$$

$$\overline{yy_i} = \int_0^\infty G_x R_i dw \qquad (7)$$

$$\overline{y_i y_k} = \int_0^\infty R G_x dw \qquad (i \neq k) \qquad (8)$$

$$\overline{y_i y_i} = \int_0^\infty M_i G_x dw \qquad (9)$$

$$\overline{yz_i'} = \int_0^\infty G_y w^2 b_i M_i dw \qquad (10)$$

$$\overline{yz_i''} = \int_0^\infty G_y w^2 (aw^2 - 1) M_i dw \qquad (11)$$

$$\overline{y_k z_i'} = \int_0^\infty G_y Q_{ik}' dw \qquad (i \neq k) \qquad (12)$$

$$\overline{y_k z_i''} = \int_0^\infty G_y Q_{ik}'' dw \qquad (i \neq k) \qquad (13)$$

$$\overline{y_i z_i'} = \int_0^\infty G_y w^2 b M_i \, dw \qquad (14) \qquad \text{(C-4 Cont.)}$$

$$\overline{y_i z_i''} = \int_0^\infty G_y w^2 (aw^2 - 1) M_i \, dw \qquad (15)$$

$$\overline{e z_{ei}'} = \int_0^\infty G_e b_i w^2 M_i \, dw \qquad (16)$$

$$\overline{e z_{ei}''} = \int_0^\infty G_e w^2 (a_i w^2 - 1) M_i \, dw \qquad (17)$$

$$\overline{z_{ei}'' z_{ek}''} = \int_0^\infty G_e w^4 R \, dw \qquad (i \neq k) \qquad (18)$$

$$\overline{z_{ei}'' z_{ei}''} = \int_0^\infty G_e w^4 M_i \, dw \qquad (19)$$

$$\overline{z_{ei}' z_{ek}'} = \int_0^\infty G_e w^2 R \, dw \qquad (i \neq k) \qquad (20)$$

$$\overline{z_{ei}' z_{ei}'} = \int_0^\infty G_e w^2 M_i \, dw \qquad (21)$$

$$\overline{z_{ei}'' z_{ek}'} = \int_0^\infty G_e w^4 I_{ik} \, dw \qquad (i \neq k) \qquad (22)$$

$$\overline{z_{ei}'' z_{ei}'} = 0 \qquad (23)$$

If G_x and G_e are each the ratio of polynomials (as is often the case) all the integrals above are closed-form integrable via partial fraction expansion. The partial fraction expansion can be quite complicated, however, and the use of a high-speed digital computer to perform a numerical integration may be generally more practical. The minimization of R_e (equation (3.5-1)) appears to be most easily found by a four dimensional search for a_1, a_2, b_1, and b_2.

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