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ELECTRON EFFECTS ON TRAPPED-
ELECTRON INSTABILITIES

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FINITE BETA AND RESONANT ELECTRON EFFECTS
ON TRAPPED-ELECTRON INSTABILITIES

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ABSTRACT

Finite β (plasma pressure/magnetic pressure) and resonant electron effects on the dissipative trapped-electron instabilities are analyzed. These modifications are shown to have a strong stabilizing influence on the finite ion gyroradius and electron temperature gradient driven modes, respectively. For relevant tokamak parameters complete stabilization may be possible. Stability criteria for radially local and nonlocal cases are presented.

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1. INTRODUCTION

Since experimental conditions in tokamaks are now very close to the collisionless or banana regime for electrons, there has been much recent interest in the trapped-electron instabilities [1-3]. In this regime, where the trapped-electron bounce frequency exceeds the effective collision frequency, drift-type waves can be driven unstable by the collisional detrapping of the electrons during a wave period. Analysis of these characteristically low-frequency ($\omega \ll \Omega_i$, $\Omega_i \equiv$ ion gyrofrequency), long-wavelength ($k\rho_i < 1$, $\rho_i \equiv$ ion gyroradius) modes has been carried out only in the electrostatic limit and by neglecting resonant electron effects. In this paper, we point out that since β , the ratio of plasma pressure to magnetic field pressure, falls in the range $m_e/m_i \ll \beta \ll 1$, in present generation experiments, it becomes important to consider electromagnetic modifications. In addition, we note that because electron-wave interactions are important even for parallel phase velocities as large as the electron thermal speed, resonant electron effects result in important modifications of the usual trapped-electron stability criterion in a sheared magnetic field.

Finite β effects are introduced into the analysis by inclusion of the parallel component of the magnetic vector potential, $A_{||}$. The perpendicular components, which give rise to compressional Alfvén waves, can be neglected provided $\beta \ll 1$. This condition is satisfied in tokamaks since β_p (ratio of plasma pressure to poloidal magnetic field pressure) must be less than the aspect ratio (ratio of major to minor radius, R/a) to

maintain equilibrium. Hence, we have $\beta \approx \beta_p (a/Rq)^2 \approx \epsilon/q^2 \ll 1$ where q is the safety factor, and $\epsilon \equiv a/R$ is the inverse aspect ratio.

In the radially local limit, the additional terms generated by $A_{||}$ lead to the familiar form of the dispersion relation [4] with the drift and shear Alfvén branches coupled by a term proportional to the ion gyroradius. The usual drift branch, of course, is modified by the trapped particle effects. It is found that the coupling between the two branches can be reasonably effective for typical tokamak parameters and leads to a possible complete stabilization of the universal (finite ion gyroradius) type dissipative modes [2]. The usual temperature gradient driven collisional mode is affected by finite β only indirectly, via the finite ion gyroradius term. As a result, significant finite β stabilization occurs only for the shorter wavelenghts ($k\rho_i \rightarrow 1$).

Analysis of the radial eigenmode problem results in equations that are found to be structurally similar to those previously obtained for finite β effects on ordinary drift waves [5]. However, some of the more important results occur in the $\beta \rightarrow 0$ limit in which the growth rate is derived including resonant electrons. The resonant electrons are shown to result in important modifications of both the temperature gradient and ion gyroradius terms, having stabilizing and destabilizing influences, respectively. If both the finite β and resonant electron effects are included in the calculation of the growth rate, one can see the indirect effect of finite β on the temperature gradient terms and also obtain an estimate of the critical β required to make the ion gyroradius terms stabilizing.

In this paper, we will present the basic theory and derive the eigenmode equations in Section 2 and the Appendix. In addition to retaining resonant electron and ion gyroradius effects in a sheared magnetic field, the derivation in the Appendix points out the equivalence of the usual Krook model of Section 2 to a BGK collision model. The radially local problem is analyzed in Section 3, and the radial structure with shear effects taken into account in Section 4. A brief summary of the results and a discussion of the conclusions reached are given in Section 5.

2. BASIC THEORY

The perturbed distribution function, f_j , can be obtained for an axisymmetric toroidal system by following the usual small gyroradius expansion procedure [6]. However, as noted in the preceding sections, we now must also include additional terms coming from the parallel component of the magnetic vector potential. In what follows, we will sketch how this is done [7]. Allowing for collisions with characteristic frequencies of the same order as the mode frequencies, the linearized Vlasov equation has the form

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - C_j + \underline{v} \cdot \underline{\nabla} + \Omega_j (\underline{v} \times \hat{n}) \cdot \frac{\partial}{\partial \underline{v}} \right) f_j \\ &= \frac{e_j}{M_j} \left(\underline{\nabla} \phi \cdot \frac{\partial}{\partial \underline{v}} + \frac{1}{c} (\underline{v} \times \underline{B}_1) \cdot \frac{\partial}{\partial \underline{v}} + \frac{1}{c} \frac{\partial}{\partial t} \underline{A} \cdot \frac{\partial}{\partial \underline{v}} \right) F_j^M \end{aligned} \quad (1)$$

where C_j is a collision operator, $\hat{n} = \underline{B}/B$, $\underline{E} = -\underline{\nabla} \phi - \frac{1}{c} \frac{\partial}{\partial t} \underline{A}$, $\underline{B}_1 = \underline{\nabla} \times \underline{A}$, and F_j^M is the Maxwellian equilibrium distribution function. The relevant ordering is $\omega/\Omega_j \sim \delta \ll 1$ with $\nabla_{||} \sim 1/L \sim \delta \nabla_{\perp}$ (L = characteristic scale length for equilibrium variations), so that Eq. (1), to lowest order, reduces to

$$\left(\underline{v}_{\perp} \cdot \underline{\nabla} - \Omega_j \frac{\partial}{\partial \zeta} \right) h_j = 0 \quad (2)$$

with $f_j = -F_j^M e_j \phi / T_j + h_j$ and ζ = perpendicular velocity azimuth angle. This result is further simplified by transforming to a coordinate system displaced by a gyro-orbit from a flux surface; i.e.,

$$\begin{aligned}\psi' &= \psi - \frac{1}{\Omega} \hat{n} \times \underline{v}_\perp \cdot \underline{\nabla} \psi \\ \beta' &= \beta - \frac{1}{\Omega} \hat{n} \times \underline{v}_\perp \cdot \underline{\nabla} \beta \\ \zeta' &= \zeta\end{aligned}\quad (3)$$

where $\underline{B} = \underline{\nabla} \psi \times \underline{\nabla} \beta$ and ψ and β label the field line. Equation (2) now reduces to $\partial h_j / \partial \zeta' = 0$, so that

$$h_j = h_j(E, s, \psi', \beta') \quad (4)$$

with E being the energy variable and s the position coordinate along the field line.

The specific form of h_j is calculated from a constraint equation obtained by averaging the first order form of Eq. (1) over the gyroperiod. After some straightforward but tedious algebra [7] one obtains

$$\left(i\omega + C_j - v_{||} \frac{\partial}{\partial s} \right) \hat{h}_j = i \frac{e_j}{T_j} F_j^M \left(\omega - \omega_{T_j}^* \right) \left\langle \left(\hat{\phi} - \frac{\hat{A}_{||} v_{||}}{c} \right) \exp \left(i \frac{\ell}{\Omega} \hat{n} \times \underline{v}_\perp \cdot \underline{\nabla} \beta \right) \right\rangle_{\zeta'} \quad (5)$$

where the angular brackets denote averages over ζ' , curvature drift terms have been neglected, and $\omega_{T_j}^* = \omega_j^* [1 + \eta_j (E/T - 3/2)]$, with $\eta_j \equiv d \ln T_j / d \ln n$ and $\omega_j^* =$ diamagnetic drift frequency for species j . Also, in arriving at Eq. (5) we have taken perturbation quantities to have an explicit dependence on one

of the field line labels (β) and the usual time dependence; e.g.,
 $\phi = \hat{\phi}(\psi, s) \exp[i(\ell\beta - \omega t)]$, and, in accordance with Eq. (4),
 $h = \hat{h}(\psi, s) \exp[i(\ell\beta - \omega t)]$. If we now take the ψ -dependence
to be $\exp[iS(\psi)]$ with $S(\psi)$ quickly varying in ψ , then the
 ζ -averaged term on the right side of Eq. (5) reduces to

$$\left(\hat{\phi}(s) - \hat{A}_{||}(s) \frac{v_{||}}{c} \right) \left\langle \exp \left[i \frac{1}{\Omega} \hat{n} \times \underline{v}_{\perp} \cdot \underline{\nabla}(\ell\beta + S) \right] \right\rangle_{\zeta},$$

$$= \left(\hat{\phi}(s) - \hat{A}_{||}(s) \frac{v_{||}}{c} \right) J_0 \left(\frac{k_{\perp} v_{\perp}}{\Omega} \right), \quad (6)$$

where $k_{\perp} \equiv |\underline{\nabla}(\ell\beta + S)|$, so that Eq. (5) becomes

$$\left(\omega - iC_j + iv_{||} \frac{\partial}{\partial s} \right) \hat{h}_j(E, s) = \frac{e_j}{T_j} F_j^M \left(\omega - \omega_{T_j}^* \right) \left(\hat{\phi}(s) - \hat{A}_{||}(s) \frac{v_{||}}{c} \right) J_0 \quad (7)$$

with the argument of the Bessel function indicated in Eq. (6).

In calculating the perturbed distribution for electrons,
we can ignore electron gyroradius corrections and note that the
right side of Eq. (7) can be expressed as

$$- \frac{|e|}{T_e} F_e^M \left(1 - \frac{\omega_{Te}^*}{\omega} \right) \left[\left(\omega + iv_{||} \frac{\partial}{\partial s} - iv_{||} \frac{\partial}{\partial s} \right) \hat{\phi} - \frac{\omega}{c} v_{||} \hat{A}_{||} \right].$$

Hence, Eq. (7) for electrons can be rewritten in the form

$$\left(\omega - iC_e + iv_{||} \frac{\partial}{\partial s} \right) \left[\hat{h}_e + \frac{|e|}{T_e} F_e^M \left(1 - \frac{\omega_{Te}^*}{\omega} \right) \hat{\phi} \right]$$

$$= - \frac{|e|}{T_e} F_e^M \left(1 - \frac{\omega_{Te}^*}{\omega} \right) \left(\frac{\omega}{c} \hat{A}_{||} + i \frac{\partial}{\partial s} \hat{\phi} \right) v_{||}. \quad (8)$$

Scattering in energy due to collisions is ignored here so that the collision operator, C_e , acting on F_e^M and $v^2 F_e^M$ gives no contribution. Since the mode frequency of interest is much smaller than the average electron bounce frequency [$\omega \ll \omega_{be} \sim (v_e/\hat{\phi})(\partial\hat{\phi}/\partial s)$], we can operate on Eq. (3) with $\int^s ds'/v_{||}$ and obtain to lowest order in (ω/ω_{be}) ,

$$g_e = \frac{|e|}{T_e} F_e^M \left(1 - \frac{\omega_{Te}^*}{\omega}\right) \left(\hat{\phi} - \frac{i\omega}{c} \int^s ds' \hat{A}_{||}\right) + \tilde{g}_e \quad (9)$$

where $g_e \equiv \hat{h}_e + (|e|F_e^M/T_e)(1 - \omega_{Te}^*/\omega)\hat{\phi}$ and g_e is determined from the next order constraint as follows. First, we note that for trapped particles the appropriate boundary conditions for f_e and hence g_e are

$$g_+(s_0) = g_-(s_0) \text{ and } g_+(-s_0) = g_-(-s_0)$$

where $\pm s_0$ are turning points and the $\pm$ subscripts designate particles moving from left to right and vice versa. Operating on the first order terms in Eq. (8) with $\int_{-s_0}^{s_0} ds/v_{||}$ and summing over both directions ($\pm$) then yields the constraint

$$\int_{-s_0}^{s_0} \frac{ds}{v_{||}} (\omega - iC_e) g_e = 0. \quad (10)$$

Together with Eq. (9) this indicates that for trapped electrons

$$\tilde{g}_e = - \frac{|e|}{T_e} F_e^M \left(\frac{\omega - \omega_{Te}^*}{\omega + i\nu_{eff,e}} \right) \left(\hat{\phi} - \frac{i\omega}{c} \int^s ds' \hat{A}_{||} \right) \quad (11)$$

where the bar denotes orbit average and the term with the effective collision frequency $\nu_{\text{eff},e} \equiv \nu_e/\epsilon$, comes from using a Krook-type collision operator [1] for C_e . This is done because the $\beta \rightarrow 0$ limit is known to be relatively insensitive to the collision model [8]. In the Appendix the relationship between this Krook model and a BGK model is demonstrated. For circulating (untrapped) particles we operate on Eq. (8) with $\oint ds/v_{||}$ to obtain the constraint, $\oint (ds/v_{||})(\omega - iC_e)g_e = 0$. Again employing the result in Eq. (9) and noting in addition that $\hat{\phi}$ and $\hat{A}_{||}$ must be single-valued in s , we find for circulating electrons $\tilde{g}_e = 0$. However, it is important to note that untrapped electrons can give rise to a Landau-type resonance in first order in the ω/ω_{be} expansion. This contribution is obtained by assuming $\omega \sim k_{||}v_{||} \gg \nu_{\text{eff},e}$ and integrating Eq. (7) along the first-order characteristics of its left side to obtain (for $s \gg qR_0$)

$$\begin{aligned} \hat{h}_e^{(1)} &= \frac{|e|}{T_e} F_e^M (\omega - \omega_{Te}^*) \int_{-\infty}^s \frac{ds'}{v_{||}'} [\hat{\phi}(s') - \hat{A}_{||}(s') \frac{v_{||}'}{c}] \exp(-i\omega \int_s^{s'} \frac{ds''}{v_{||}''}) \\ &\approx \frac{|e|}{T_e} F_e^M \frac{\omega - \omega_{Te}^*}{\omega - k_{||}v_{||}} (\hat{\phi} - \hat{A}_{||} \frac{v_{||}}{c}) \end{aligned} \quad (12)$$

in which the distinction between local and bounce-averaged values is not retained. A rigorous justification of the untrapped particle contribution is given by Rosenbluth, et al. [9]. Collisional broadening of this resonance in the banana regimes and transition from banana to plateau regimes is considered by Hinton and Ross, and Tsang and Callen, respectively [10].

Equations (9), (11), and (12) determine the perturbed electron distribution function which can now be expressed as

$$f_e^T = \frac{|e|}{T_e} F_e^M \left[\hat{\phi} - \left(1 - \frac{\omega_{Te}^*}{\omega} \right) \frac{i\omega}{c} \int^S ds' \hat{A}_{||} - \left(\frac{\omega - \omega_{Te}^*}{\omega + i\nu_{eff,e}} \right) \left(\hat{\phi} - \frac{i\omega}{c} \int^S ds' \hat{A}_{||} \right) \right] \quad (13)$$

for trapped particles, and

$$f_e^U = \frac{|e|}{T_e} F_e^M \left[\hat{\phi} - \left(1 - \frac{\omega_{Te}^*}{\omega} \right) \frac{i\omega}{c} \int^S ds' \hat{A}_{||} - \frac{\omega - \omega_{Te}^*}{\omega - k_{||} v_e} \left(\hat{\phi} - \frac{v_{||}}{c} \hat{A}_{||} \right) \right] \quad (14)$$

for untrapped particles. Integrating over velocity space and retaining resonant electron corrections in the $k_{||} v_e \gg \omega$ limit then yields the electron density response

$$\frac{n_e}{n_0} = \frac{|e|}{T_e} \left\{ \frac{\omega_{Te}^*}{\omega} \hat{\phi} + \left(\hat{\phi} - \frac{i\omega}{c} \int^S ds' \hat{A}_{||} \right) \left[\left(1 - \frac{\omega_{Te}^*}{\omega} \right) (1 + iR_2) + i\eta_e \frac{\omega_{Te}^*}{\omega} R_1 \right] - (2\epsilon)^{1/2} \left\langle \frac{\omega - \omega_{Te}^*}{\omega + i\nu_{eff,e}} \right\rangle \left(\hat{\phi} - \frac{i\omega}{c} \int^S ds' \hat{A}_{||} \right) \right\} \quad (15)$$

with the angled brackets, $\langle \rangle$, denoting energy average and

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = \frac{\pi^{1/2} \omega}{|k_{||}| v_e} \begin{pmatrix} \frac{1}{2} \left[\left(1 - \frac{2\omega^2}{k_{||}^2 v_e^2} \right) \exp \left(\frac{-\omega^2}{k_{||}^2 v_e^2} \right) - \left(1 - \frac{\omega^2}{\epsilon k_{||}^2 v_e^2} \right) \exp \left(\frac{-\omega^2}{2\epsilon k_{||}^2 v_e^2} \right) \right] \\ \left[\exp \left(\frac{-\omega^2}{k_{||}^2 v_e^2} \right) - \exp \left(\frac{-\omega^2}{2\epsilon k_{||}^2 v_e^2} \right) \right] \end{pmatrix}.$$

In writing R_1 and R_2 we have not expanded the exponentials for two reasons: (i) even though $\omega^2/k_{||}^2 v_e^2 \ll 1$, $\omega^2/2\epsilon k_{||}^2 v_e^2$ need not be small compared to one; and, far more importantly, (ii) in the next section, in evaluating the contributions from R_1 and R_2 to the radial eigenmode analysis, the exponentials provide the effective cut-offs to the integrals of the perturbation theory [11]. Both R_1 and R_2 contain differences of two terms because the $v_{||}$ integration extends from 0 to $v_{||}/(2\epsilon)^{1/2}$ instead of from 0 to ∞ .

In evaluating the perturbed ion distribution, we return to Eq. (7) and recall that the mode frequencies of interest exceed the average ion transit frequency [$\omega \gg \omega_{bi} \sim (v_i/\hat{\phi})(\partial\hat{\phi}/\partial s)$]. Since $\partial\hat{\phi}/\partial s$ and $\hat{A}_{||}\omega/c$ are treated as being of the same order in (ω/ω_{bi}) , the term $\hat{A}_{||}v_{||}/c$ appearing in Eq. (7) must be one order smaller than $\hat{\phi}$. Hence, order by order we find

$$\begin{pmatrix} \hat{h}_i^{(0)} \\ \hat{h}_i^{(1)} \\ \hat{h}_i^{(2)} \end{pmatrix} = \frac{|e|}{T_i} F_i^M \left(1 - \frac{\omega_{Ti}^*}{\omega} \right) J_0 \begin{pmatrix} \hat{\phi} \\ -i \frac{v_{||}}{\omega} \frac{\partial}{\partial s} \hat{\phi} - \frac{v_{||}}{c} \hat{A}_{||} \\ - \left(\frac{v_{||}}{\omega} \right)^2 \frac{\partial^2}{\partial s^2} \hat{\phi} - \frac{i v_{||}^2}{\omega c} \frac{\partial}{\partial s} \hat{A}_{||} \end{pmatrix} \quad (16)$$

where ion collisions are ignored. The corresponding expression for f_i carried to second order in (ω/ω_{bi}) and averaged over the velocity space phase angle ζ is

$$\begin{aligned} \langle f_i \rangle_{\zeta} = & - \frac{|e|}{T_i} F_i^M \left[\hat{\phi} - \left(1 - \frac{\omega_{Ti}^*}{\omega} \right) J_0^2 \hat{\phi} \right. \\ & \left. + \left(1 - \frac{\omega_{Ti}^*}{\omega} \right) \left(\frac{v_{||}^2}{\omega^2} \frac{\partial^2}{\partial s^2} \hat{\phi} - i \frac{v_{||}^2}{\omega c} \frac{\partial}{\partial s} \hat{A}_{||} \right) J_0^2 \right] \quad (17) \end{aligned}$$

Performing the velocity space integration gives the ion density response,

$$\begin{aligned} \frac{n_i}{n_0} = & - \frac{|e|}{T_i} \left[\left(1 - \Gamma_0 + \frac{\omega_i^*}{\omega} \Gamma_0 + \frac{\omega_i^*}{\omega} \eta_i b(\Gamma_1 - \Gamma_0) \right) \hat{\phi} \right. \\ & \left. + \left(\left[1 - \frac{\omega_i^*}{\omega} (1 + \eta_i) \right] \Gamma_0 - \frac{\eta_i \omega_i^*}{\omega} b(\Gamma_1 - \Gamma_0) \right) \left(\frac{v_{||}^2}{2\omega^2} \right) \left(\frac{\partial^2}{\partial s^2} \hat{\phi} - \frac{i\omega}{c} \frac{\partial}{\partial s} \hat{A}_{||} \right) \right] \quad (18) \end{aligned}$$

where $\Gamma_n \equiv I_n(b) \exp(-b)$, $I_n(b)$ are Bessel functions of order n , $b = k_{\perp}^2 \rho_i^2 / 2$, ρ_i (ion gyroradius) $= v_i / \Omega_i$, and $v_i = (2T_i / M_i)^{1/2}$. This result and the electron density response in Eq. (15) are then used in the quasineutrality condition, $\sum_j e_j n_j = 0$, to yield

$$\begin{aligned} & \left(\left(1 - \frac{\omega_e^*}{\omega}\right) (1 + iR_2) + i\eta_e \frac{\omega_e^*}{\omega} R_1 - (2\epsilon)^{1/2} \left\langle \frac{\omega - \omega_{Te}^*}{\omega + i\nu_{eff,e}} \right\rangle \right. \\ & \left. - \left(\frac{k_{\parallel} c_s}{\omega} \right)^2 \left\{ \left[1 + \frac{\omega_e^*}{\omega\tau} (1 + \eta_i) \right] \Gamma_0 - \frac{\omega_e^*}{\omega\tau} \eta_i b (\Gamma_0 - \Gamma_1) \right\} \right) \left(\phi - \frac{A_{\parallel} \omega}{k_{\parallel} c} \right) \\ & = - \left\{ \tau (1 - \Gamma_0) + \frac{\omega_e^*}{\omega} [1 - \Gamma_0 + \eta_i b (\Gamma_0 - \Gamma_1)] \right\} \end{aligned} \quad (19)$$

with $\tau \equiv T_e / T_i$ and $c_s \equiv (2T_e / M_i)^{1/2}$ is the sound speed. In arriving at Eq. (19), we have also assumed for simplicity that $\phi(s)$, $A_{\parallel}(s) \sim \exp(ik_{\parallel}s)$ where $|k_{\parallel}s_0| \ll 1$ with s_0 being a turning point for trapped particles. It now remains to relate ϕ and $A_{\parallel}$ through the parallel current response and Ampere's law.

The total parallel current response can be obtained by operating on Eq. (7) with $\sum_j e_j \int d^3v$. In terms of the perturbed distribution function, f_j , the resultant moment equation has the form

$$\begin{aligned}
 & - i\omega \sum_j e_j \int d^3v f_j + \frac{\partial}{\partial s} \sum_j e_j \int d^3v v_{||} f_j \\
 & = i\omega \sum_j \frac{e_j^2}{T_j} \phi \int d^3v F_j^M \left[1 - \left(1 - \frac{\omega_{Tj}^*}{\omega} \right) J_0^2 \right] \quad (20)
 \end{aligned}$$

where we have taken $\sum_j e_j \int d^3v C_j f_j = 0$ since collisions displace points in velocity while leaving the density in coordinate space unchanged. On the left side of Eq. (20), the first term vanishes from the quasineutrality condition and the second term is just $\partial j_{||} / \partial s$ with $j_{||}$ the total parallel current density. Performing the velocity space integration then gives

$$\frac{i}{\omega} \frac{\partial}{\partial s} j_{||} = - \frac{n_o e^2}{T_i} \phi \left[(1 - \Gamma_o) \left(1 - \frac{\omega_i^*}{\omega} \right) + \frac{\omega_i^*}{\omega} \eta_i b (\Gamma_1 - \Gamma_o) \right] \quad (21)$$

The magnetic vector potential is related to $j_{||}$ by Ampere's law, i.e.,

$$k_{\perp}^2 A_{||} = \frac{4\pi}{c} j_{||} \quad (22)$$

We can therefore eliminate $j_{||}$ and rewrite Eq. (21) as

$$\frac{k_{||} \omega}{c} k_{\perp}^2 A_{||} = \frac{\omega^2}{v_A^2} \left[\frac{1 - \Gamma_o}{b} \left(1 - \frac{\omega_i^*}{\omega} \right) - \frac{\omega_i^*}{\omega} \eta_i (\Gamma_o - \Gamma_1) \right] k_{\perp}^2 \phi \quad (23)$$

where the Alfvén velocity is $v_A = B/(4\pi n_0 M_i)^{1/2}$ and the s -dependence of $A_{||}$, as before, is assumed to vary as $\exp(ik_{||} s)$. Equations (19) and (23) comprise the basic equations to be solved. An alternative derivation in which the limit $b < 1$, which includes resonant electron effects in a sheared magnetic field, is presented in the Appendix. In treating the radial eigenmode problem, the factor $k_{\perp}^2$ will be replaced by a differential operator of the form $(k_y^2 - \partial^2/\partial x^2)$ and only the longer wavelength modes ($b < 1$) will be analyzed. For the radially local problem analyzed in the next section this factor, of course, cancels out of the equation.

3. RADIALLY LOCAL ANALYSIS

In the radially local limit, Eq. (23) gives a simple relation between ϕ and $A_{||}$ which can be substituted in Eq. (19) to yield the following dispersion relation,

$$\begin{aligned} & \left((1 + iR_2) \left(1 - \frac{\omega_*}{\omega} \right) + i\eta_e \frac{\omega_*}{\omega} R_1 - (2\epsilon)^{1/2} \left\langle \frac{\omega - \omega_{Te}^*}{\omega + i v_{eff}} \right\rangle \right. \\ & - \left(\frac{k_{||} c_s}{\omega} \right)^2 \left\{ \left[1 + \frac{\omega_*}{\omega \tau} (1 + \eta_i) \right] \Gamma_0 - \frac{\omega_*}{\omega \tau} \eta_i b (\Gamma_0 - \Gamma_1) \right\} \left[1 \right. \\ & \left. - \frac{1}{b} \left(\frac{\omega}{k_{||} v_A} \right)^2 \left[(1 - \Gamma_0) \left(1 + \frac{\omega_*}{\omega \tau} \right) + b \frac{\omega_*}{\omega \tau} \eta_i (\Gamma_0 - \Gamma_1) \right] \right\} \\ & = -\tau \left[(1 - \Gamma_0) \left(1 + \frac{\omega_*}{\omega \tau} \right) + b \frac{\omega_*}{\omega \tau} \eta_i (\Gamma_0 - \Gamma_1) \right] \end{aligned} \quad (24)$$

where $c_s = (2T_e/M_i)^{1/2}$ is the speed of sound and $\omega_* \equiv \omega_e^*$. In Eq. (24) the drift and Alfvén branches are coupled via the finite ion gyroradius term of the right-hand side. From the form of Eq. (24), the ratio of the finite β term over the coupling term is seen to scale as $[(\omega/k_{||} v_A)^2 / \tau b] \equiv \kappa$. For $b \ll 1$, $\omega \approx \omega_* \propto b^{1/2}$ giving $\kappa \propto b^0$ while for $b \gg 1$, $\omega \propto b^0$ giving $\kappa \propto b^{-1}$. Consequently, the finite β modification of the drift branch is slightly more effective for $b \lesssim 1$. As a result, in what follows we will consider only the $b < 1$ limit. However, we emphasize that because the b -dependence of the coefficients of the coupling and finite β

terms are so simply related, the results are expected to be at least qualitatively correct for $b \sim 1$.

Retaining the resonant electron terms from Eq. (24) in the $b < 1$ limit, the dispersion relation may be written as

$$b = \left\{ \frac{\omega[\omega\tau + \omega_*(1+\eta_i)]}{\tau k_{||}^2 v_A^2} - 1 \right\} \left\{ \frac{(\omega - \omega_*)[1 + (2\varepsilon)^{1/2} A_2 + iR_2] - \eta_e \omega_* [(2\varepsilon)^{1/2} A_1 - iR_1]}{\tau\omega + \omega_*(1+\eta_i)} - \frac{k_{||}^2 v_i^2}{2\omega^2} \right\} \quad (25)$$

with

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = -2\pi^{-1/2} \int_0^\infty dx x^{1/2} \frac{\exp(-x)}{1 + iux^{-3/2}} \begin{pmatrix} x - \frac{3}{2} \\ 1 \end{pmatrix}, \quad (26)$$

where $u \equiv v_e/\varepsilon\omega$.

In the drift limit with $\omega = \omega_* + \Delta\omega$, $|\Delta\omega/\omega_*| < 1$, and $\omega_*^2(\tau + 1 + \eta_i) \neq \tau k_{||}^2 v_A^2$, Eq. (25) gives the growth rate $\gamma \equiv \text{Im } \omega$, in the form

$$\frac{\gamma}{\omega_*} = (\tau + 1 + \eta_i) \left\{ b \left[1 - \frac{(\tau + 1 + \eta_i) \omega_*^2}{\tau k_{||}^2 v_A^2} \right]^{-1} - \frac{k_{||}^2 v_i^2}{2 \omega_*^2} \right\} [(2\varepsilon)^{1/2} \text{Im} A_2 + R_2] \\ + \eta_e [(2\varepsilon)^{1/2} \text{Im} A_1 - R_1] \quad (27)$$

provided $|(2\varepsilon)^{1/2} A_2|$, $|R_2| \ll 1$. We note here that A_1 , A_2 , R_1 , and R_2 are evaluated at $\omega = \omega_*$, and any product involving any two of these four quantities is neglected.

In some limits, simple analytic estimates of A_1 and A_2 ,

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \approx \begin{cases} \frac{i 4 \varepsilon \omega}{\pi^{1/2} v_e} \begin{pmatrix} 3/2 \\ 1 \end{pmatrix} & \frac{v_e}{\varepsilon \omega} \gg 1 \\ \frac{i 3 v_e}{\pi^{1/2} \varepsilon \omega} \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \ln \left(\frac{\varepsilon \omega}{v_e} \right) & \frac{v_e}{\varepsilon \omega} \ll 1, \end{cases}$$

and R_1 and R_2 ,

$$\begin{pmatrix} R_2 \\ R_1 \end{pmatrix} \approx \frac{\pi^{1/2} \omega^3}{2 \varepsilon |k_{||}|^3 v_e^3} \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} \quad \frac{\omega^2}{2 \varepsilon k_{||}^2 v_e^2} \ll 1,$$

may be employed to simplify Eq. (25) and/or Eq. (27). The most important features, however, are that R_1 , R_2 , and $\text{Im} A_2$ are always positive, while $\text{Im} A_1 > 0$ for $v_e/\omega \gtrsim 1$ and $\text{Im} A_1 < 0$ otherwise.

As noted earlier, at very long wavelengths ($b \ll 1$) the finite β as well as finite ion gyroradius modifications are negligible, so that for typical $\eta_e \sim 1$, the basic mode remains the usual temperature gradient driven instability [1] except for $2\epsilon < (\omega/k_{\parallel} v_e)^2 < 1$ when R_1 takes on a maximum value $\sim \omega/|k_{\parallel}| v_e$. However, at shorter wavelengths ($b \lesssim 1$), where the universal type instability [2] contributes, the finite β factor introduces a significant stabilizing influence. This occurs when the denominator of b in Eq. (27) changes sign; i.e.,

$$\left(\frac{\omega_*}{k_{\parallel} v_A}\right)^2 (\tau + 1 + \eta_i) > 1. \quad (28)$$

To estimate the minimum value of $\beta_i = 8\pi N_o T_i / B_o^2$ for this effect to be operative, we take $(k_{\parallel})_{\min} \sim 1/R_q$ and note that $v_A = v_i / \beta_i^{1/2}$. The condition thus becomes

$$\beta_i > \frac{2}{\tau + 1 + \eta_i} \left(\frac{L_n}{R_q}\right)^2 \frac{1}{b} \quad (29)$$

where L_n is the density scale length. This implies that the finite β effect renders the previously destabilizing finite ion gyroradius contributions [2] stabilizing for values of β_i typically exceeding a few percent. The preceding, of course, represents a rough estimate since we have not explicitly calculated the mode structure along the field line with finite β terms included.

For completeness we present the results of solving Eq. (25) for the growth rates in the Alfvén and drift-Alfvén limits. In the

Alfvén limit the lowest order frequency ω_0 is assumed to satisfy $\omega_0 [\omega_0 \tau + \omega_* (1 + \eta_i)] = \tau k_{\parallel}^2 v_A^2$ and $\omega_0 \neq \omega_*$ so that writing $\omega = \omega_0 + \Delta\omega$, Eq. (25) can be solved to obtain

$$\frac{\gamma}{\omega_0} \approx \frac{b [\omega_0 \tau + \omega_* (1 + \eta_i)]^2}{2\omega_0 \tau + \omega_* (1 + \eta_i)} \left\{ \frac{(2\epsilon)^{1/2} \text{Im} A_2 + R_2}{\omega_* - \omega_0} + \frac{\eta_e \omega_* [(2\epsilon)^{1/2} \text{Im} A_1 - R_1]}{(\omega_* - \omega_0)^2} \right\}$$

where $(\omega_0 - \omega_*) / [\omega_0 \tau + \omega_* (1 + \eta_i)]$ is assumed to be the dominant drift contribution to Eq. (25) in this Alfvén limit and where A_1 , A_2 , R_1 , and R_2 are to be evaluated at $\omega = \omega_0$. For $\omega_* < \omega_0 \leq v_e / \epsilon$ only the temperature gradient driven term becomes stabilizing; however, unlike the drift limit, $\gamma \rightarrow 0$ as $b \rightarrow 0$ (weak coupling).

For the drift-Alfvén limit $\omega_0 = \omega_*$ and $\omega_*^2 (\tau + 1 + \eta_i) = \tau k_{\parallel}^2 v_A^2$. Employing $\omega = \omega_0 + \Delta\omega$ in Eq. (25) results in

$$\frac{\gamma}{\omega_0} \approx [(2\tau + 1 + \eta_i)b]^{1/2} [(2\epsilon)^{1/2} \text{Im} A_2 + R_2] + \frac{1}{2} \eta_e [(2\epsilon)^{1/2} \text{Im} A_1 - R_1]$$

for $4b / (2\tau + 1 + \eta_i) \gg (k_{\parallel}^2 v_i^2 / \omega_*^2) + |\eta_e [(2\epsilon)^{1/2} A_1 - iR_1] / (\tau + 1 + \eta_i)|$.

This result is qualitatively the same as the low β limit of Eq. (27) when $b \sim 1$ and $k_{\parallel}^2 v_i^2 / 2\omega_*^2 \ll 1$, and is not valid in the $b \rightarrow 0$ limit.

4. RADIAL EIGENMODE ANALYSIS

Turning now to the radial eigenmode problem we again consider the long wavelength regime ($b < 1$). Replacing $k_{\perp}^2$ with $k_y^2 - \partial^2/\partial x^2$ and $k_{\parallel}$ with $k_y x/L_s$ ($L_s \equiv$ shear length) in Eqs. (19) and (23) and retaining resonant electron corrections for the circulating electrons (see Appendix) results in

$$\frac{\partial^2}{\partial x^2} \phi - [\lambda + k_y^2 - \mu^2 x^2 + i\sigma(x)] \phi = - [\lambda - \mu^2 x^2 + i\sigma(x)] \frac{v}{\alpha} \frac{1}{x} A_{\parallel} \quad (30)$$

and

$$\left(\frac{\partial^2}{\partial x^2} - k_y^2 \right) A_{\parallel} = \frac{\alpha}{x} \left(\frac{\partial^2}{\partial x^2} - k_y^2 \right) \phi \quad (31)$$

with

$$\lambda \equiv \frac{2}{\rho_i} \left(\frac{(\omega - \omega_*) [1 + (2\varepsilon)^{1/2} A_2] - \eta_e \omega_* (2\varepsilon)^{1/2} A_1}{\omega \tau + \omega_* (1 + \eta_i)} \right)$$

$$\sigma = \sigma(x) \equiv \frac{2 [(\omega - \omega_*) R_2 + \eta_e \omega_* R_1]}{\rho_i^2 [\omega \tau + \omega_* (1 + \eta_i)]}$$

$$\mu \equiv \frac{k_y v_{Ti}}{L_s \omega \rho_i}, \quad \frac{v}{\alpha} \equiv \frac{\omega L_s}{c k_y}, \quad \tau = \frac{T_e}{T_i},$$

and

$$v = \frac{\omega[\omega\tau + \omega_*(1 + \eta_i)]L_s^2}{\tau k_y^2 v_A^2} = \frac{\beta'_i \omega[\omega\tau + \omega_*(1 + \eta_i)]L_s^2}{\tau k_y^2 v_i^2}$$

In terms of toroidal variables $\partial^2/\partial x^2$ and k_y correspond respectively to $\partial^2/\partial r^2$ and $\ell q/r$ with ℓ the toroidal mode number.

Equations (30) and (31) have the same structure as the equations encountered in the analysis of finite β effects on collisionless drift waves [5]. In Ref. 5, the resonant electrons are treated via a perturbation technique. Subsequent work in the $\beta \rightarrow 0$ limit. [11] has substantiated the validity of this perturbative approach, which we also employ.

Employing the techniques developed in Ref. 5, the eigenvalue equation for the radial mode believed to be the least stable (the lowest, even mode) can be determined. Retaining the additional terms due to both the electron and ion temperature gradients and the new resonant electron terms which result because the trapped electrons cannot resonate with the wave, leads to the rather complicated eigenvalue equation

$$\begin{aligned} & \lambda + i\mu - \mu^2 v + \frac{2}{3} \left(\sigma - \frac{1}{2}\right) k^2 \left[1 + 2^{(\sigma+1)/2} \left(\frac{2\sigma+1}{\sigma+2}\right) {}_2F_1\left(1 + \frac{\sigma}{2}; \frac{1}{2} - \frac{\sigma}{2}; \frac{\sigma}{2} + 2; \frac{1}{2}\right) \right] \\ & + \frac{4i(i\mu)^{1/2} \omega |L_s| (i\delta)^{\sigma/2}}{\rho_i^2 [\omega\tau + \omega_*(1 + \eta_i)] |k| v_e \Gamma(\sigma - \frac{1}{2})} \left\{ [\omega - \omega_*(1 - \frac{1}{2} \eta_e)] (i\delta)^{-1/2} \left[K_{\sigma-1} \left(2(i\delta)^{1/2} \right) \right. \right. \\ & \left. \left. - (2\epsilon)^{\frac{1-\sigma}{2}} K_{\sigma-1} \left(2\left(\frac{i\delta}{2\epsilon}\right)^{1/2} \right) \right] - \eta_e \omega_* \left[K_{\sigma-2} \left(2(i\delta)^{1/2} \right) - (2\epsilon)^{-\sigma/2} K_{\sigma-2} \left(2\left(\frac{i\delta}{2\epsilon}\right)^{1/2} \right) \right] \right\} = 0, \end{aligned}$$

where $\omega \approx \omega_*$, $\sigma \equiv 1 + i\mu\nu$, $\delta \equiv m|L_s|/2 ML_n$. In Eq. (32),

$K_\nu(z) = \int_0^\infty dt \cosh(\nu t) \exp[(-z) \cosh(t)]$ with $\text{Re}(z) > 0$ is a modified Bessel function, ${}_2F_1(a, b; c; z)$ is a hypergeometric function, and $\Gamma(z)$ is a gamma function.

Writing $\omega = \omega_* + \Delta\omega$ and assuming $\delta \ll 2\epsilon$ and $|\mu\nu \ln \delta| \ll 1$; i.e.,

$$\frac{m|L_s|}{ML_n} \ll 4\epsilon \quad \text{and} \quad \beta_i(\tau + 1 + \eta_i) \frac{|L_s|}{2L_n} \ln \frac{2ML_n}{m|L_s|} \ll 1,$$

results in a considerable simplification of Eq. (32); namely,

$$\begin{aligned} \frac{\Delta\omega}{\omega_*} [1 + (2\epsilon)^{1/2} A_2] - \eta_e (2c)^{1/2} A_1 - \frac{\beta_i(\tau + 1 + \eta_i)^2}{2\tau} + \frac{i(\tau + 1 + \eta_i)L_n}{\tau|L_s|} \\ + (\tau + 1 + \eta_i)b \left(1 + i(1.2)\beta_i(\tau + 1 + \eta_i) \frac{|L_s|}{L_n} \right) \\ - \frac{(1-i)}{2} \left(\frac{m|L_s|}{ML_n} \right)^{1/2} \left[\left(\frac{\Delta\omega}{\omega_*} + \frac{1}{2} \eta_e \right) \left(\ln \frac{1}{2\epsilon} \right) \left(1 - \frac{i\beta_i(\tau+1+\eta_i)|L_s|}{4L_n} \ln \frac{2ML_n}{m|L_s|} \right) \right] = 0 \end{aligned} \quad (33)$$

where the shear stabilizing term $i(\tau + 1 + \eta_i)L_n/\tau|L_s|$ is a result of ion inertia in a sheared field.

In order to determine the importance of retaining the resonant electron terms, we first consider the $\beta_i \rightarrow 0$ limit. Taking advantage of the small parameters, the growth rate in this

$\beta_i \rightarrow 0$ limit is found from Eq. (33) to be

$$\begin{aligned} \frac{\gamma}{\omega_*} = & (\tau + 1 + \eta_i) b \left[(2\epsilon)^{1/2} \text{Im}A_2 + \frac{1}{2} \left(\frac{m|L_s|}{ML_n} \right)^{1/2} \ln \left(\frac{1}{2\epsilon} \right) \right] \\ & + \eta_e \left[(2\epsilon)^{1/2} \text{Im}A_1 - \frac{1}{4} \left(\frac{m|L_s|}{ML_n} \right)^{1/2} \ln \left(\frac{1}{2\epsilon} \right) \right] - \frac{(\tau + 1 + \eta_i)L_n}{\tau|L_s|} . \quad (34) \end{aligned}$$

This result can, of course, be found directly from Eq. (30) with the right hand side set equal to zero. Consequently, the removal of the trapped electrons from the resonant electron contribution results in the usual $\ln(ML_n/m|L_s|)$ being replaced by $\ln(1/2\epsilon)$. As a result, the resonant electron contributions to the growth rate are about the same magnitude as the usual dissipative trapped electron terms (A_1 and A_2) for typical tokamak parameters. Recalling that $\text{Im}A_2 > 0$ always and $\text{Im}A_1 > 0$ for $v_e/\epsilon\omega \gtrsim 1$, Eq. (34) shows that, typically, the resonant electron contributions are stabilizing in the temperature gradient driven term and destabilizing in the finite ion gyroradius term.

The reason that the resonant electron contributions are not substantially reduced, as one might expect by looking at the $\omega^2/2\epsilon k_{||}^2 v_e^2 \ll 1$ limit of R_1 and R_2 , is that the resonant electron contributions are important all the way up to $\omega^2/k_{||}^2 v_e^2 \sim 1$. Noting that it has been verified in detail in Ref. 11 that the effective cut-off for the integrals of the perturbation theory are determined by the exponentials in R_1 and R_2 , it is clear that the leading terms

for R_1 and R_2 must also depend on the difference of the logarithms of the arguments of the exponentials in R_1 and R_2 ; namely, $\ln(1/2\epsilon)$.

Returning to the finite β_i expression given in Eq. (33), we treat the β_i , $L_n/|L_s|$, A_1 , A_2 , and resonant electron terms as small corrections, so that products of small terms may be neglected. The resulting expression for the growth rate with the finite β_i effects included may be written as

$$\begin{aligned} \frac{\gamma}{\omega_*} = & (\tau + 1 + \eta_i) \left(b - \frac{\beta_i (\tau + 1 + \eta_i)}{2\tau} \right) \left[(2\epsilon)^{1/2} \text{Im}A_2 + \frac{1}{2} \left(\frac{m|L_s|}{ML_n} \right)^{1/2} \ln\left(\frac{1}{2\epsilon}\right) \right. \\ & \left. - (1.2)\beta_i (\tau + 1 + \eta_i) \frac{|L_s|}{L_n} \right] - \frac{(\tau + 1 + \eta_i)L_n}{\tau|L_s|} \\ & + \eta_e \left[(2\epsilon)^{1/2} \text{Im}A_1 - \frac{1}{4} \left(\frac{m|L_s|}{ML_n} \right)^{1/2} \ln\left(\frac{1}{2\epsilon}\right) \right] . \end{aligned} \quad (35)$$

From Eq. (35) for $\eta_e > 0$, we arrive at the following conclusions. The electron temperature gradient driven terms are unaffected by finite β ; however, as previously discussed, the resonant electron corrections are a strong stabilizing influence. The finite β correction to the ion inertia or shear term completely stabilizes the finite ion gyroradius term for $b < \beta_i (\tau + 1 + \eta_i)/2\tau$. For smaller wavelengths the finite β modification of the ion gyroradius coupling has a strong stabilizing influence on the ion gyroradius terms for $(1.2)\beta_i (\tau + 1 + \eta_i) |L_s|/L_n + [(2\epsilon)^{1/2} \text{Im}A_2 + \frac{1}{2} (m|L_s|/ML_n)^{1/2} \ln(1/2\epsilon)]$. From the $\beta_i \rightarrow 0$ limit, we expect

$$\left[(2\varepsilon)^{1/2} \text{Im} A_2 + \frac{1}{2} (m|L_s|/ML_n)^{1/2} \ln(1/2\varepsilon) \right] \leq L_n/\tau b|L_s|$$
 for stability, so that this last condition can be roughly restated to say that the finite β modifications of the coupling have a strong influence for $(1.2)\beta_i(\tau + 1 + \eta_i)\tau bL_s^2/L_n^2 \rightarrow 1$. Unfortunately, the perturbation theory treatment [5] is strictly valid only for $\beta_i(\tau + 1 + \eta_i)\tau bL_s^2/L_n^2 < 1$ in order that the shift in the eigenvalue be small compared to the spacing between lowest order eigenvalues. However, similar difficulties arise due to the treatment of the resonant electrons via perturbation theory, and a detailed treatment [11] has verified that the perturbative treatment is far better than the small shift in eigenvalue argument indicates. Consequently, this last result is expected to be at least qualitatively correct. It is also interesting to note that for $qR \sim L_s$ this last result corresponds to the crude estimate obtained from the local theory and given in Eq. (29).

5. CONCLUSIONS

In this paper we have calculated the effects of finite β on the dissipative trapped-electron instabilities. The basic eigenmode equations were derived in Section 2 and analyzed for the radially local case in Section 3. Here, it was found that the coupling between the drift and shear Alfvén branches can be quite effective in changing the finite-ion gyroradius contribution from a destabilizing [2] to a stabilizing one. For typical tokamak parameters $L_n/L_s \sim \epsilon L_n/qR \sim 1/30$, this can occur at values of β_i of a few percent which is certainly a relevant regime.

In dealing with the radial eigenmode problem in Section 4, we first pointed out the importance of including the resonant electrons and then determined the stabilizing influence of the finite β modifications for an isolated rational surface; that is, for the distance between adjacent rational surfaces, $(l\partial q/\partial r)^{-1}$, much less than, for example, the radial localization width,

$\rho_i (\tau L_s/L_n)^{1/2}$ for $\omega \approx \omega_*$. There are five principle results.

(I) The resonant electrons can stabilize the electron temperature gradient driven term for $(m|L_s|/ML_n)^{1/2} \ln(1/2\epsilon) > 4(2\epsilon)^{1/2} |\text{Im}A_1|$.

(II) The resonant electrons result in a further destabilizing influence on the ion gyroradius driven term, $(2\epsilon)^{1/2} \text{Im}A_2 +$

$(2\epsilon)^{1/2} \text{Im}A_2 + \frac{1}{2} (m|L_s|/ML_n)^{1/2} \ln(1/2\epsilon)$. (III) The finite β

modification of the ion gyroradius coupling has a strong stabilizing influence for $(1.2)\beta_i(\tau + 1 + \eta_i)|L_s|/L_n \rightarrow [(2\varepsilon)^{1/2} \text{Im}A_2 + \frac{1}{2} (m|L_s|/ML_n)^{1/2} \ln(1/2\varepsilon)]$, or roughly speaking, as $\beta_i(\tau + 1 + \eta_i)\tau b L_s^2/L_n^2 \rightarrow 1$. (IV) Finite β effects result in no new temperature gradient driven terms and therefore can only affect η_e terms indirectly via (III) for $b \rightarrow 1$; that is, for shorter poloidal wavenumbers. (V) The finite β correction to the ion inertia or shear term stabilizes the finite ion gyroradius term for $b < \beta_i(\tau + 1 + \eta_i)/2\tau$.

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APPENDIX

In this Appendix an alternative derivation of the finite β Eqs. (30) and (31) is presented which includes resonant electron corrections. Working in a toroidal geometry with circular, concentric flux surfaces, the magnetic field is written as $\underline{B}_0 = B_T \hat{\underline{\zeta}} + B_p \hat{\underline{\theta}} = (B_0 R_0 / R) \{ \hat{\underline{\zeta}} + [\epsilon / q(r)] \hat{\underline{\theta}} \}$ with $\epsilon = r / R_0$, $R = R_0 + r \cos \theta$, θ and ζ the poloidal and toroidal angle variables, and r the small radial variable measured from the center of the concentric circles located at R_0 . Suppressing species subscripts and neglecting neoclassical corrections, the unperturbed distribution function f_0 is taken to be

$$f_0 = N(p_\zeta) \left[\frac{M}{2\pi T(p_\zeta)} \right]^{3/2} \exp \left[\frac{-Mv^2}{2T(p_\zeta)} \right],$$

where $p_\zeta = MRv_\zeta + (Ze/c)\psi$ is the canonical angular momentum and $\psi = \psi(r)$ is the poloidal flux function ($RB_p = -\partial\psi/\partial r$).

A linearized BGK model [12] is assumed for the trapped electrons whereby electrons are untrapped at a rate ν_{eff} and replenished at the same rate but with a Maxwellian distribution. The collision model for the circulating electrons is then found by requiring the number of trapped (circulating) electrons lost equal the number of circulating (trapped) electrons gained. The collisional effects on the ion distribution function are neglected. Consequently, for drifts small compared to the thermal speeds the v_ζ dependence of N and T may be neglected in all terms except $\nabla_v f_0$, and the linearized kinetic equations are of the form

$$\begin{aligned} \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{Ze}{Mc} \underline{v} \times \underline{B}_O \cdot \nabla_v f + \frac{Ze}{M} \left(\underline{E} + \frac{1}{c} \underline{v} \times \underline{B} \right) \cdot \nabla_v f_O \\ = -v(f - f_O S) + G f_O, \end{aligned} \quad (A1)$$

where for the trapped electrons ($Z = -1$) $G = 0$, $v = v_{eff}$, and

$$S = \frac{n}{N} + \left(\frac{v^2}{v_e^2} - \frac{3}{2} \right) \frac{\delta T}{T_e}$$

with

$$\begin{aligned} N &= \int d^3v f_O, \quad \frac{3}{2} N T_e = \int d^3v \frac{1}{2} m v^2 f_O, \\ n &= \int d^3v f, \quad \frac{3}{2} N \delta T = T_e \int d^3v \left(\frac{v^2}{v_e^2} - \frac{3}{2} \right) f; \end{aligned}$$

while for circulating electrons ($Z = -1$) $v = 0$ and

$$G = \frac{\int_T d^3v v_{eff} (f - f_O S)}{\int_C d^3v f_O};$$

and for the ions ($Z = +1$) $v = 0$ and $G = 0$. The subscripts T and C on the integrals denote that the integration is over the trapped and circulating electrons, respectively; absence of a subscript denotes that the integration is over all velocity space.

Forming the trajectory integral solution of (A1) with $\nabla_v f_O$ inserted, employing the source free Maxwell equations to eliminate E_r, E_θ , and B_ζ , integrating by parts, and seeking solutions of the form $\exp(-i\omega t - i l \zeta)$ ($l \neq 0$) results in

$$\begin{aligned}
 f = & \frac{izef_0}{\ell T} \left(RE_\zeta + \int_{-\infty}^0 d\tau \exp [-i(\omega + i\nu)\tau - i\ell(\zeta' - \zeta)] \{ i(\omega - \omega_d') R' (E_\zeta' \right. \\
 & + \frac{v_r'}{c} B_\theta' - \frac{v_\theta'}{c} B_r') - \nu (R' E_\zeta' + \frac{i\ell T}{ze} S') - \frac{i\ell T}{ze} G' - \frac{\omega \omega_d'^2 R'^2}{\ell^2 c} (B_r' \sin \theta' \\
 & + B_\theta' \cos \theta') + \frac{i\omega_d' R'}{\ell^2} \cos \theta' \frac{\partial}{\partial r'} (R' E_\zeta') - \frac{i\omega_d' R'}{\ell^2 r'} \sin \theta' \frac{\partial}{\partial \theta'} (R' E_\zeta') \\
 & \left. - \frac{i\omega_d' R'}{\ell^2 cr'} (v_\theta' \cos \theta' + v_r' \sin \theta') \left[\frac{\partial}{\partial r'} (r' R' B_r') + \frac{\partial}{\partial \theta'} (R' B_\theta') \right] \right\} \right), \quad (A2)
 \end{aligned}$$

where trajectory variables and functions of trajectory variables are denoted by primes, and

$$\omega_d \equiv \frac{q\ell cT}{zeB_0 R_0} \left[1 + n \left(\frac{Mv^2}{2T} - \frac{3}{2} \right) \right] \frac{\partial}{\partial r} \ln N. \quad (A3)$$

Employing a θ -dependence which is the product of a fast dependence, $\exp(-im\theta)$, and a slow, even, periodic dependence [i.e., $\cos(\theta/2)$], where the number of trapped electrons is zero at $\theta = \pm\pi$, various terms in (A2) can be estimated in the vicinity of the m, ℓ rational surface, where $q(r = r_{m\ell}) = m/\ell$. For q/ℓ , $|\ell^{-2} R \partial / \partial r|$, and $1/\ell^2$ small compared to one, the only E_ζ and B_r terms that need be retained under the integral in (A2) are $(\omega - \omega_d) [E_\zeta - (v_\theta/c) B_r]$ and the collisional terms. Furthermore, no B_θ terms need be retained if, in addition, $\beta_i \omega / \Omega_i \ll k^2 \rho_i^2$ ($k = m/r$) and displacement currents are neglected. This last result follows by noting that, for phase velocities intermediate between the electron and ion thermal speeds, the current from the $\underline{E} \times \underline{B}_0$ drift of the electrons is

approximately canceled by the current due to the electron pressure gradient, while for the ions no such cancellation of the $\underline{E} \times \underline{B}_0$ drift occurs. Therefore, to lowest order the perpendicular current is on the order of that due to the $\underline{E} \times \underline{B}_0$ drift of the ions. Consequently, using the $\exp(-im\theta)$ dependence, only

$$f = \frac{iZeRf_0}{\ell T} \{ RE_\zeta + i \int_{-\infty}^0 d\tau [(\omega - \omega'_d) (E'_\zeta - \frac{v'_\theta}{c} B'_r) - v'(\frac{\ell T}{Ze} S' - iE'_\zeta) - \frac{\ell T}{ZeR} G'] \exp [-i(\omega + iv)\tau - i\ell(\zeta' - \zeta) + im(\theta' - \theta)] \} \quad (A4)$$

need be retained from (A2).

Employing the trajectory variables ($dr'/d\tau = v'_r$, $d\theta'/d\tau = v'_\theta/r$, $d\zeta'/d\tau = v'_\zeta/R' \approx v'_\zeta/R_0$, $v'_r \approx v_\perp \cos \chi'$, $v'_\theta \approx (\epsilon'/q')v'_\parallel - v_\perp \sin \chi'$, $v'_\zeta \approx v'_\parallel + (\epsilon'/q')v_\perp \sin \chi'$, and $\chi = [(\int_0^\tau d\tau' \Omega') - \xi]$, expanding $m - \ell q(r)$ about the $r = r_{m\ell}$ rational surface, and defining

$$k = \frac{m}{r}, \quad \frac{1}{L_s} = \frac{\epsilon}{q} \frac{\partial q}{\partial r}, \quad k_\parallel = \frac{m - \ell q}{qR_0} = -\frac{k}{L_s} (r - r_{m\ell}) \quad (A5)$$

gives

$$\begin{aligned} m(\theta' - \theta) - \ell(\zeta' - \zeta) &\approx \int_0^\tau d\tau' \left[\frac{mv'_\theta}{r'} - \frac{\ell v'_\zeta}{R_0} \right] \approx \int_0^\tau d\tau' \left[\frac{m - q'\ell}{q'R_0} v'_\parallel - \frac{kv_\perp}{\Omega} [\cos \chi - \cos \xi] \right] \\ &\approx \int_0^\tau d\tau' \left[k'_\parallel v'_\parallel - \frac{kv_\perp}{\Omega} [\cos \chi - \cos \xi] \right], \end{aligned} \quad (A6)$$

where ξ , $v_{\perp}$, and $v_{\parallel}$ are the values of the cylindrical velocity variable at $\tau = 0$.

For the ions ($v = 0 = G$), the times of interest are short compared to the bounce and circulation times so local values ($r' = r$, $\theta' = \theta$, $v'_{\parallel} = v_{\parallel}$, $\Omega' = \Omega$, $\omega'_d = \omega_d$, $v'_\theta = (\epsilon/q)v_{\parallel} - v_{\perp} \sin \chi$, $\chi = \Omega\tau - \xi$, $B'_r = B_r$) may be employed except for E'_z where the leading gyroradius correction is desired. As a result, $E'_z = E_z + (r' - r) \partial E_z / \partial r + (1/2)(r' - r)^2 \partial^2 E_z / \partial r^2$ with $r' - r = (v_{\perp}/\Omega) [\sin(\Omega\tau - \xi) + \sin \xi]$ is employed. This limit is the usual limit of slab geometry and results in

$$\int d^3 v f_i = - \frac{ieNR}{\ell T_e} \left\{ \frac{\omega_*}{\omega} E_z + \frac{\omega\tau + \omega_*(1+\eta_i)}{\omega} \left[\left(\frac{k_{\parallel}^2 v_i^2}{2\omega^2} - b \right) E_z - \frac{k_{\parallel}^2 v_i^2 \epsilon}{2\omega c q} B_r + \frac{1}{2} \rho_i^2 \frac{\partial^2 E_z}{\partial r^2} \right] \right\},$$

$$\int d^3 v v_{\parallel} f = \frac{\omega}{k_{\parallel}} \int d^3 v f - \frac{1}{k_{\parallel}} \int d^3 v (\omega - k_{\parallel} v_{\parallel}) f,$$

$$\int d^3 v (\omega - k_{\parallel} v_{\parallel}) f_i = - \frac{ieNR}{\ell T_e} \left\{ \omega_* E_z - [\omega\tau + \omega_*(1+\eta_i)] \left[b E_z - \frac{1}{2} \rho_i^2 \frac{\partial^2 E_z}{\partial r^2} \right] \right\}, \quad (A7)$$

where $v_i^2 = 2T_e/M_i$, $\tau = T_e/T_i$, $b = k^2 \rho_i^2/2$, $\rho_i = v_i/\Omega_i$, $\omega_* = -(k c T_e / e B_0) \partial \ln N / \partial r \equiv \tau k v_i \rho_i / 2 L_n$.

For the electrons, gyroradius corrections are negligible so (A6) becomes $m(\theta' - \theta) - \ell(\zeta' - \zeta) \approx k_{\parallel} \int_0^{\tau} d\tau' v'_{\parallel}$. For the circulating

electrons ($v = 0$) $k_{||} \int_0^{\tau} d\tau' v_{||} \approx k_{||} v_{||} \tau$ is assumed and the slow θ' periodicity of E'_{ζ} , B'_r , G' in (A4) is employed to rewrite the integral from $-\infty$ to 0 as an integral over a single period equal to a circulation time the short way, $\tau_c \sim 2\pi q R_o / v_{||}$. For $|\omega - k_{||} v_{||}| \tau_c \ll 1$ the distribution function for the circulating electrons, f_c , is found to be

$$f_c = - \frac{ieRf_{oe}}{\ell T_e} \left[E_{\zeta} - \frac{\omega - \omega_{Te}^*}{\omega - k_{||} v_{||}} (E_{\zeta} - \frac{v_{||} \epsilon}{cq} B_r) - \frac{(\ell T_e / eR)}{\omega - k_{||} v_{||}} G \right],$$

where $\omega_{Te}^* = \omega_* \{1 + \eta_e [(v^2/v_e^2) - 3/2]\}$. Performing the velocity integrations for $|\omega/k_{||}| \ll v_e$, retaining resonant electron terms, and noting that in this limit the G and B_r terms may be neglected in $\int_c d^3 v f$ and $\int_c d^3 v (\omega - k_{||} v_{||}) f$, respectively, yields

$$\begin{aligned} \int_c d^3 v f_c = & - \frac{ieRN}{\ell T_e} \{ [1 - (2\epsilon)^{1/2}] [E_{\zeta} - \frac{(\omega - \omega_*) \epsilon}{k_{||} cq} B_r] + \\ & + i [(1 - \frac{\omega_*}{\omega}) R_2 + \eta_e \frac{\omega_*}{\omega} R_1] [E_{\zeta} - \frac{\omega \epsilon}{k_{||} cq} B_r] \}, \end{aligned} \quad (A8)$$

$$\int_c d^3 v (\omega - k_{||} v_{||}) f_c = - \frac{ieRN}{\ell T_e} [1 - (2\epsilon)^{1/2}] \omega_* E_{\zeta} + i \int_T d^3 v v_{eff} (f_T - f_{oe} S),$$

where R_1 and R_2 are defined in Eq. (16). Both R_1 and R_2 contain differences of two terms because the $v_{||}$ integration is from 0 to $v_{||}/(2\epsilon)^{1/2}$ instead of from 0 to ∞ .

For the trapped electrons ($G = 0$ and $v = v_{\text{eff}}$) $v_{\parallel}' \approx qR_0 d\theta'/d\tau$ gives $k_{\parallel} \int_0^{\tau} d\tau' v_{\parallel}' \approx k_{\parallel} qR_0 (\theta' - \theta)$. Again employing periodicity, here of the bounce motion which has a bounce frequency $\omega_b \sim (\epsilon)^{1/2} v_{\perp}/qR_0$, the trapped electron distribution function f_T is found to be

$$f_T = - \frac{i e R f_{oe}}{\ell T_e} \left\{ E_{\zeta} - \frac{\omega - \omega_{Te}^*}{\omega + i v_{\text{eff}}} E_{\zeta} - \frac{i v_{\text{eff}}}{\omega + i v_{\text{eff}}} \left[E_{\zeta} + \frac{\ell T_e n}{i e R N} + \frac{\ell \delta T}{i e R} \left(\frac{v_{\perp}^2}{v_e^2} - \frac{3}{2} \right) \right] \right\}$$

for $|k_{\parallel} q R_0| \ll 1$ and $|\omega| \ll \omega_b$. To lowest order in $\epsilon^{1/2}$, f_c may be employed to obtain

$$\frac{\ell T_e n}{i e R N} + E_{\zeta} \approx \frac{(\omega - \omega_{Te}^*) \epsilon}{k_{\parallel} c q} B_r \quad \text{and} \quad \frac{\ell \delta T}{i e R} \approx - \frac{\omega_{Te}^* \eta_e \epsilon}{k_{\parallel} c q} B_r \quad (\text{A9})$$

so that f_T may be taken to be

$$f_T = - \frac{i e R f_{oe}}{\ell T_e} \left\{ E_{\zeta} - \frac{\omega - \omega_{Te}^*}{\omega + i v_{\text{eff}}} \left[E_{\zeta} + \frac{i v_{\text{eff}} \epsilon}{k_{\parallel} c q} B_r \right] \right\} \quad (\text{A10})$$

Denoting a trapped particle average by $\langle \rangle$, $\langle A \rangle = \left[\int_T d^3 v f_{oe} A \right] \div [N(2\epsilon)^{1/2}]$, and carrying out the velocity space integrations, the results can be written as

$$\int_T d^3 v f_T = - \frac{i e R N (2\epsilon)^{1/2}}{\ell T_e} \left\{ E_{\zeta} - \frac{(\omega - \omega_{Te}^*) \epsilon}{k_{\parallel} c q} B_r - \left\langle \frac{\omega - \omega_{Te}^*}{\omega + i v_{\text{eff}}} \right\rangle \left(E_{\zeta} + \frac{\omega \epsilon}{k_{\parallel} c q} B_r \right) \right\}$$

$$\int_T d^3 v (\omega - k_{\parallel} v_{\parallel}) f_T \approx \omega \int_T d^3 v f_T \quad (\text{A11})$$

From quasi-neutrality ($\int_T d^3v f_T + \int_C d^3v f_C = \int d^3v f_i$) and the parallel component of Ampere's law [using (A9) to rewrite S in (A8)] the two coupled second-order differential Eqs. (30) and (31) are obtained when the identifications $E_\zeta \approx -R_O^{-1} \partial \phi / \partial \zeta = (i\ell/R_O) \phi$ and $B_r \approx r^{-1} \partial A_\parallel / \partial \theta = -ik A_\parallel$ are employed.

It is also interesting to note that using ϕ and $A_\parallel$ and combining (A9) and S results in the trapped-electron collision operator,

$$-v_{\text{eff}}(f_T - f_{oe} S) = -v_{\text{eff}} \left\{ f - f_{oe} \frac{e}{T_e} \left[\phi + \frac{\omega - \omega_{Te}^*}{k_\parallel c} A_\parallel \right] \right\} = -v_{\text{eff}} \tilde{g}_e,$$

employed in the drift kinetic approach, Eqs. (9) - (11). Furthermore, $\delta T \equiv 0$ may be employed in the electrostatic limit.

REFERENCES

- * Now at Department of Physics and Astronomy, University of Maryland, College Park, Maryland, U.S.A.
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- [1] KADOMTSEV, B. B., POGUTSE, O. P., Dokl. Akad. Nauk SSSR 186, (1969) 553 [Sov. Phys.-Dokl. 14, (1969) 470].
- [2] LIU, C. S., ROSENBLUTH, M. N., TANG, W. M., Princeton University MATT Report No. 1125, March, 1975.
- [3] HORTON, W., Jr., ROSS, D. W., TANG, W. M., BERK, H. L., FRIEMAN, E. A., LAQUEY, R. E., LOVELACE, R. V., MAHAJAN, S. M., ROSENBLUTH, M. N., RUTHERFORD, P. H., Princeton University MATT Report No. 1075, Nov., 1974 and Plasma Physics and Controlled Nuclear Fusion Research (Proc. 5th Int. Conf., Tokyo, 1974) IAEA-CN-33/A14-3.
- [4] KADOMTSEV, B. B., Plasma Turbulence, Academic Press, N.Y., (1965) 83.
- [5] CATTO, P. J., EL-NADI, A. M., LIU, C. S., ROSENBLUTH, M. N., Nucl. Fusion 14, (1974) 405.
- [6] See, for example, P. H. Rutherford and E. A. Frieman, Phys. Fluids 11, 569 (1968).
- [7] A detailed finite beta derivation of f_j has been carried out by B. Newberger and P. H. Rutherford (private communication).

- [8] CALLEN, J. D., BEASLEY, C. O., FISCHER, S. K., HICKS, N. R., and SEELY, J. F., Bull. Am. Phys. Soc. 19, 863 (1974).
- [9] ROSENBLUTH, M. N., ROSS, D. W., and KOSTOMAROV, D. P., Nucl. Fusion 12, 3 (1972).
- [10] HINTON, F. L., and ROSS, D. W.; and TSANG, K. T. and CALLEN, J. D., Proceedings of Annual Meeting on Theoretical Aspects of Controlled Thermonuclear Research, Arlington, VA., U.S.A., April, 1975.
- [11] The justification for treating this effect perturbatively is given in ROSENBLUTH, M. N. and CATTO, P. J., to be published in Nuclear Fusion.
- [12] BHATNAGAR, P. L., GROSS, E. P., KROOK, M., Phys. Rev. 94, 511 (1954); and GROSS, E. P., KROOK, M., Phys. Rev. 102, 593 (1956).