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AUGUST, 1970

Contract No. AT (11-1)-264

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ABSTRACT

It is proved that conformal invariance of dual dynamics unambiguously fixes the possible quark configurations of all (exotic and nonexotic) hadrons. All these configurations follow a simple star of David rule (rule E of section 1). Rules for computing amplitudes for arbitrary hadronic processes involving external and/or intermediate exotic hadrons are given. The existence of quark diagrams such that one diagram has poles in all three channels of a four point function is pointed out. The total quark number of a hadron is identified with a Casimir operator of a new "D-spin" group.

*Supported in part by the U. S. Atomic Energy Commission.

⁺National Science Foundation Predoctoral Fellow.

1. Introduction

A complete quark model of hadrons must, sooner or later, concern itself with the way in which the apparently complex dynamics of hadrons derives from a simple scheme of quark dynamics. The first steps towards a derivation of hadrodynamics from a quark model have been taken in the framework of the factorized Veneziano model.¹⁾ The crucial feature of duality emerges thereby as a consequence of the conformal invariance²⁾ of the Veneziano amplitudes. In a quark model it is only natural to ask whether in addition to $q\bar{q}$ - and qqq -like hadrons (i.e. of nonexotic mesons and baryons) there could exist also "exotic" hadrons (still with zero triality) with quark structures such as $q^2\bar{q}^2$, $q^4\bar{q}$, $q^3\bar{q}^3$, q^6 , $\bar{q}^6$, In a dual theory the answer to this question is unambiguous. Hadrons of zero triality with arbitrarily high total quark number (i.e. number of quarks + number of antiquarks) must exist.³⁾ Yet in the usual Veneziano model these exotic hadrons have not heretofore been included. The clue towards their inclusion comes from the conformal invariance of the theory. As was demonstrated in the preceeding paper⁴⁾ conformal invariance fixes the way in which arbitrarily many quarks and/or antiquarks in a hadron get dynamically activated in the presence of an arbitrary number of "spectator" quarks and/or antiquarks. It turns out that all active quarks must focus to a point on their world sheet at the moment of their common dynamical activation.

We shall use here this quark focusing principle to derive the "geometry" of hadronic quark structure. By this we mean that we shall find all possible configurations of quarks and antiquarks in free (i.e. asymptotic states of) exotic hadrons. Our result can be stated in the form of a very simple rule.

E. Any possible configuration of an exotic or non-exotic hadron can be visualized as a circular rubber band with all quarks placed at the vertices of one equilateral triangle (T_1) of the regular starred hexagon S_6 ---i.e. "star of David"---inscribed into the circle, and with all antiquarks placed on the other equilateral triangle (T_2) of S_6 , subject only to the restriction that for each q ($\bar{q}$) at a vertex of T_1 (T_2), there must appear either one q ($\bar{q}$) each, at the other two vertices of T_1 (T_2), or one $\bar{q}$ (q) at the diametrically opposed vertex of T_2 (T_1). As an illustration of this principle we show in fig. 1 all possible configurations of hadrons with total quark number $N \leq 9$.

Using the so obtained hadronic quark configurations and the quark focusing and universality principles derived in I we set up a method for calculating the amplitude for any hadronic (exotic or nonexotic) process. In the process of evolving this method a remarkable new phenomenon occurs. We discover the existence of quark diagrams for four point functions that must contain poles in all three channels (s , t and u). These are the quark diagrams exhibiting "crosses" that cannot be eliminated by a change of the cyclic order of the external

hadrons. Therefore they exhibit a duality more closely following the lines suggested by Virasoro⁵⁾ than the 2-channel duality of Veneziano¹⁾ which is typical of quark diagrams⁶⁾ without crosses. As an example the diagram of fig. 2a is redrawn in figures 2b and 2c in all possible cyclic orders. It corresponds to the process $B_3 M_4 \rightarrow B_3 M_4$ ($B_3 = qqq$, $M_4 = qq\bar{q}\bar{q}$) in the s and u channels and to the process $B_3 \bar{B}_3 \rightarrow M_4 M_4$ in the t-channel. It has B_5 ($= qq\bar{q}\bar{q}$) poles each in the s-and u-channels and M_4 poles in the t-channel. For contrast we present in figures 2d-2f the quark diagrams for the process which is $\pi^+\pi^+ \rightarrow \pi^+\pi^+$ in the s-channel, and $\pi^+\pi^- \rightarrow \pi^+\pi^-$ in the t-and u-channels⁶⁾. This process has a cross but it can be eliminated by changing the cyclic ordering (fig. 2f). The corresponding amplitude, as is well known, has poles only in the t-and u-channels and is of the usual Veneziano-type. To fully understand the difference between the two processes in fig. 2 one has to invoke the selection rules of Freund, Waltz and Rosner⁷⁾ which we shall reformulate in this paper in terms of the new concept of "D-spin."

2. The Quark Structure of Exotic Hadrons

From the standard Veneziano model we have the picture of a meson as a "rubber band." The position operator of a point labeled ζ on this rubber band at proper time $\tau = i\lambda$ is given by

$$\varphi_{\mu}(\zeta, \lambda) = \sum_{r=1}^{\infty} \sqrt{\frac{2}{r}} \left[(a_{\mu r} e^{-r\lambda} + a_{\mu r}^+ e^{r\lambda}) \cos r\zeta + (b_{\mu r} e^{-r\lambda} + b_{\mu r}^+ e^{r\lambda}) \sin r\zeta \right] \quad (2.1)$$

so that

$$\varphi_{\mu}(\zeta + 2\pi, \lambda) = \varphi_{\mu}(\zeta, \lambda) \quad (2.2)$$

The meson's quark and antiquark are located respectively at $\zeta = 0$ and $\zeta = \pi$. Similarly a baryon⁸⁾ has one quark each at $\zeta = 0$, $\zeta = \frac{2\pi}{3}$ and $\zeta = \frac{4\pi}{3}$. Where are the quarks and antiquarks of an exotic meson M_4 made of two quarks and two antiquarks? Naively one would "guess" that they should be located at $\zeta = 0$, $\zeta = \frac{\pi}{2}$, $\zeta = \pi$ and $\zeta = \frac{3\pi}{2}$. We shall now show that the actual locations of the quarks and antiquarks in M_4 are not free for us to fix but follow from duality and in particular the naive guess ventured above is ruled out. M_4 -mesons make their first appearance as intermediate states in $B_3 \bar{B}_3 \rightarrow B_3 \bar{B}_3$. This process has been calculated in I, using the quark focusing principle. One starts from the diagram 3a where the two meson-baryon vertices are known from $M_2 B_3 \rightarrow M_2 B_3$ scattering ($M_2 = q\bar{q}$). This diagram has an M_2 -pole when

$z_2 \rightarrow z_1 = -\infty$ and an M_4 -pole when $z_2 \rightarrow z_3 = i\pi$. One then performs the conformal mapping

$$w = \ln(e^z + 1) \quad (2.3)$$

which leads to diagram 3b. Now observe that the loop L_2 in diagram 3b has the same shape as the loop L_2 in the diagram 3c of $M_2 B_3$ -scattering which is obtained by the same conformal mapping (2.3) from the diagram 3d (where the curves $D_{\pm}$ are the same as in 3a). In diagram 3c the limit $w_2 \rightarrow w_3 = -\infty$ corresponds to the antibaryon pole. Therefore as w_2 approaches $w_3 = -\infty$ and the loop gets stretched, the two curves $D_{\pm}$ (as is shown in detail in Appendix C) will tend to $-\infty \pm \frac{i\pi}{3}$ so that the corresponding two antiquarks be spaced at $\frac{2\pi}{3}$ from each other and from the third antiquark at $-\infty + i\pi$ as befits an antibaryon. The loop in fig. 3b, as was observed above, has the same equation as that in fig. 3c so that as $w_2 \rightarrow w_3$ the two antiquarks of the loop here too go to $-\infty \pm \frac{i\pi}{3}$. The arrangement of quarks and antiquarks in an M_4 is thus a quark each at $\zeta = 2\pi/3$ and $\zeta = 4\pi/3$ and an antiquark each at $\zeta = +\pi/3$ and $\zeta = -\pi/3$. This is, up to an irrelevant overall rotation, precisely the arrangement $N = 4(u)$ of fig. 1. With this arrangement we now consider $B_3 M_4 \rightarrow B_3 M_4$ (fig. 4). It has a B_5 -pole when $w_2 \rightarrow w_3$ in fig. 4a. The loop being again L_2 we find the arrangement of quarks and antiquarks in a B_5 to be two quarks at $\zeta = 0$, one quark each at $\zeta = 2\pi/3$ and $\zeta = -2\pi/3$ and one antiquark at $\zeta = -\pi$. This is the arrangement $N=5$ of fig. 1.

With this arrangement we can now proceed to $B_3 \bar{B}_5 \rightarrow B_3 \bar{B}_5$ whence we get the configurations of M_6 and D_6 and so on.

Thus the recursion procedure is:

R1. Given the configuration of H_N , of total quark number $N \geq 3$, scatter B 's in all possible ways off H_N using the loop L_2 to find configurations of H_{N+1} . Then repeat with $H_{N+1}, \dots$

Observe, though, that at the M_4 level we only got so far the configuration $1.4u$ but not $1.4p$. The latter first appears in $B_5 \bar{B}_5 \rightarrow B_5 \bar{B}_5$ as seen in fig. 5. The two configurations $1.4u$ and $1.4p$ are to be viewed as two varieties of M_4 mesons only one of which (the "unpinched" variety $1.4u$) is produced in $B_3 \bar{B}_3 \rightarrow B_3 \bar{B}_3$. This confirms the statement in I that not all varieties of M_4 can be produced in $B_3 \bar{B}_3 \rightarrow B_3 \bar{B}_3$. More generally, the various configurations corresponding to a given total quark number N and baryon number B correspond to different particles. It is readily checked that the various configurations cannot mix in diagrams without self energy loops.

Just as the recursion procedure R1 skipped the pinched M_4 configuration it will skip the maximally pinched configurations for every exotic meson M_N (i.e. the configuration in which $N/2$ quarks are at $\zeta = 0$ and $N/2$ antiquarks at $\zeta = \pi$). Just as $B_5 \bar{B}_5$ scattering brought into play the M_4 so in general $B_{N+1} \bar{B}_{N+1}$ scattering will have as intermediate state

the maximally pinched M_N meson. So to complete the recursion procedure we have to

R2. Consider all processes $\bar{B}_{N+1}^m B_{N+1}^m \rightarrow \bar{B}_{N+1}^m B_{N+1}^m$, $N = \text{even}$, B_{N+1}^m being the configuration of the baryon with $\frac{N}{2}$ quarks at $\zeta = 0$ one quark each at $\zeta = \pm \frac{2\pi}{3}$ and $\frac{N}{2} - 1$ antiquarks at $\zeta = \pi$. These processes will yield the maximally pinched M'_N s as intermediate states.

The recursion procedure $R = R_1 + R_2$ will yield all possible configurations of all possible hadrons. Now we are going to prove the rule E stated in the introduction. We proceed by complete induction. We know that the three quarks of a baryon are one each at the three vertices of one triangle of the star of David S_6 inscribed in the circle so that the rule E holds. Now assume that the rule E holds for all configurations of H_N . The recursion R requires that we scatter B_N^i s off H_N . This means that one of the quarks*) say a of H_N will undergo the process of fig. 6. Thereby, in the asymptotic configuration ($w_2 \rightarrow -\infty$) of the H_{N+1} intermediate state the quark a will be replaced by the anti-quarks $\bar{b}_+$ and $\bar{b}_-$. If a had the ordinate ζ_A , $\bar{b}_\pm$ will have the asymptotic ordinate $\zeta_A \pm \pi/3$ as we have seen above. Now if in H_N the quark a was obeying the rule E by being matched with two quarks a_+ and a_- on one of the triangles (T_1) of S_6

*) We can without any restriction of generality assume that it is a quark. If it were a $\bar{q}$ just consider $\bar{H}_N$.

then in H_{N+1} the quarks a_+ and a_- will obey the rule E as they will be diametrically opposed to the antiquarks $\bar{b}_+$ and $\bar{b}_-$. If on the other hand a was matched in H_N with a diametrically opposed antiquark $\bar{b}$, then in H_{N+1} the antiquarks $\bar{b}$, $\bar{b}_+$ and $\bar{b}_-$ will form a triangular configuration (one at each vertex of T_2) just as required by E. So if the rule E holds for H_N it must hold for H_{N+1} . As the rule E obviously holds for all maximally pinched mesons M_N , we have just proved that all hadrons obtained through the recursion R^{*)} obey the rule E. Now we still have to make sure that processes other than those included in the recursion R do not upset the star of David configurations required by rule E. The most general process involves finite and infinite focusing curves (see appendices A and B). The only 4-body processes that admit a configuration that does not involve any focusing are $M_2 H_N \rightarrow M_2 H_N$. These processes are therefore unambiguous. Making conformal mappings on the unfocused configuration of $M_2 H_N \rightarrow M_2 H_N$ we find specific combinations of finite or infinite focusing curves for any combination (i.e. any number and any focusing angles) of quark lines. Whenever any combination of quark lines has to be focused in a process for which there exists no totally unfocused configuration (like $B_3 \bar{B}_3 \rightarrow B_3 \bar{B}_3$, etc...) we have to use that combination of focusing lines that is determined by $M_2 H_N \rightarrow M_2 H_N$ scattering. With this observation it is then readily checked that no process can possibly lead to deviations from the star of David rule E.

*) It is readily seen that all configurations compatible with rule E are obtained by the recursion procedure R.

3. D-spin. 4-point Quark Diagrams with Poles in all Three Channels

Certain two hadron systems such as pp or $\pi^+\pi^+$, etc... are totally void of resonances. The Veneziano model for $\pi^+\pi^+$ -scattering has no poles in the s-channel (it is an entire function of s). While no $q\bar{q}$ -state could ever lead to a $\pi^+\pi^+$ resonance a $q^2\bar{q}^2$ state could have the right quantum numbers to decay into $\pi^+\pi^+$. Nevertheless, neither are such states experimentally observed, nor does the Veneziano model allow them. A selection rule seems to be operative. This selection rule has been formulated in full generality by Freund, Waltz and Rosner.⁷ They require that A) at any 3-hadron vertex each hadron must exchange at least one quark line with each one of the other two hadrons and that B) no $q\bar{q}$ pair of any one hadron should annihilate at the vertex. For example the M_4 intermediate state in $\pi^+\pi^+$ scattering is forbidden by rule A for (as can be seen in fig. 2d) the two π^+ 's do not exchange even a single quark line. In ref. 7 (but are not implied by) it was shown that requirements A) and B) imply the selection rule S) given three hadrons A, B and C of total quark numbers N_A , N_B and N_C , the vertex $A \rightarrow B+C$ allowed by the space-time (angular momentum, parity, etc...) and internal (isospin, hypercharge, baryon number) symmetries of strong interactions is forbidden unless

$$\left| \frac{N_A}{2} - 1 - \left(\frac{N_C}{2} - 1 \right) \right| \leq \frac{N_B}{2} - 1 \leq \frac{N_A}{2} - 1 + \frac{N_C}{2} - 1 \quad (3.1)$$

Bayon number conservation further requires that

$$\frac{N_B}{2} - 1 - \left| \frac{N_A}{2} - 1 \pm \left(\frac{N_C}{2} - 1 \right) \right| = \text{integer} \quad (3.2)$$

The relations (3.1) and (3.2) obviously suggest that the quantity

$$D = \frac{N}{2} - 1 \quad (3.3)$$

be interpreted as a spin.^{*)} We shall call D the dual spin or for short, D -spin of the hadron H with total quark number N . The selection rule S expresses the conservation of D -spin. As an immediate example $D_{M_2} = 0, D_{M_4} = 1$ whence $M_4 \rightarrow M_2 + M_2$ is forbidden by D -spin conservation which explains why $\pi^+\pi^+$ is free of resonances. The fact that the usual $q\bar{q}$ -mesons M_2 (like π, ρ , etc....) have D -spin zero has further consequences. Thus, e.g., scattering these mesons off an arbitrary target can not lead to the formation of a resonance of different total quark number than the target. Exotic hadrons, though, can be associatedly produced in such a process. As was explained in ref. 7 the selection rule S holds only for tree diagrams. Final state interactions violate it. Diagrams with one or more loops violate D -spin conservation. In terms of the Veneziano model this means that vertices and propagators are D -spin invariant but the operation of taking the trace is not.

^{*)} The possibility of interpreting the selection rule S in terms of a spin-like quantity was first realized in 1969 by P. G. O. Freund, H. J. Lipkin and J. L. Rosner (unpublished).

Through final state interactions, for instance, dibaryons could (weakly) couple to pp so that there may be small resonance effects in say $pp \rightarrow \pi d$.

At this point we ought to ask what is the symmetry of the dual theory that leads to the conservation of D-spin? Moreover what are the various D_3 states of a D-spin multiplet? For the time being, all we can say about these questions is that as N increases, so does the number of quark arrangements in the hadron. D-spin is very likely to be related to some transformation shuffling the oscillator modes that go with these various configurations. The facts observed above that different configurations mix only in diagrams with closed loops, and that closed loops violate D-spin conservation lend support to the idea that the various configurations corresponding to a given N are like the various D_3 components of a D-multiplet.

Another very important application of D-spin is the identification of quark diagrams with poles in all three channels in 4-point amplitudes involving exotic (external or intermediate) particles. Consider the diagrams of fig. 2. For the diagrams $2a \equiv 2b \equiv 2c$ the s and u channels are $B_3 + M_4 \rightarrow B_3 + M_4$. In the s - and u -channels one has D-spin $1/2 + 1 \rightarrow 3/2 \rightarrow 1/2 + 1$ and in the t -channel $1/2 + 1/2 \rightarrow 1 \rightarrow 1 + 1$ so all these processes are allowed by D-spin conservation and one is led to the conclusion that the diagram 2a has poles in all three channels. We shall call it an "s-t-u quark diagram." On the other hand the diagrams $2d \equiv 2e \equiv 2f$ while obviously having M_2 poles in

the t-and u-channels cannot have an M_4 pole in the s-channel and conserve D-spin. Indeed a diagram in which all external particles are M_2 and therefore have D-spin zero cannot have an intermediate state of nonvanishing D-spin like M_4 . We shall see in the next section how to calculate the amplitude of the "s-t-u" diagram 2a.

It is worth remarking that should one not distinguish between quarks and antiquarks (i.e. not orient the quark diagrams), then s-t-u quark diagrams appear already at the level of $B_3 B_3 \rightarrow B_3 B_3$, fig. 7 but this diagram is nonorientable and therefore does not appear in the real world.

4. Calculation of Amplitudes for Processes Involving Exotic Hadrons

We now proceed to use our knowledge about the quark arrangement in exotic hadrons to get simple operatorial expressions for amplitudes involving exotic hadrons. The procedure to be presented can be recursively pushed to arbitrarily high total quark numbers, though as we shall see, in its present form the combinatorics involved is non-trivial. We shall consider a few typical cases involving hadrons with low D-spin and a few processes involving hadrons of arbitrarily high D-spin to illustrate the details and power of our technique.

In all examples of this section we shall only consider exotic mesons and baryons so that the total quark number unambiguously characterizes a hadron (if it is even $B = 0$; if it is odd $B = 1$). We shall label hadrons simply by their total quark number. Thus M_2 will be called 2, $B_3 \rightarrow 3$, $M_4 \rightarrow 4$, $B_5 \rightarrow 5$, etc.... By $23 \rightarrow 23$ we mean nonexotic meson-baryon scattering, etc...

We start with the process $22 \rightarrow 22$. The vertex is ^{*)}

$$\Gamma_{22}^2(k_i, \xi_i, \xi_{i_2}, \lambda_i) = G \sum_{\gamma=1}^2 : e^{ik_i \phi(\xi_{i\gamma}, \lambda_i)} : \quad (4.1)$$

^{*)} Here and below we use the notation $\Gamma_{AB}^C(k_i, \xi_i, \dots, \xi_{i_N}, \lambda_i)$ for the vertex by which the hadron A emits the hadron C of momentum k_i at any of the points $\lambda_1 + i \xi_{i\gamma}$ of the world sheet while getting transformed into the final hadron B.

where $z_{i\gamma} = \lambda_i + i z_{i\gamma}$ ($\gamma=1,2$) are the two points on the world sheet at which the meson of momentum k_i can be absorbed at proper time $\tau_i = i\lambda_i$ and G is the universal (see I) normalization of the hadronic vertex. The $22 \rightarrow 22$ amplitude is then

$$\begin{aligned} T(22 \rightarrow 22) &= \int d\mathbf{v} \langle 0 | T \Gamma_{22}^2(k_2, 0, \pi, \lambda_2) \Gamma_{22}^2(k_3, 0, \pi, \lambda_3) | 0 \rangle \\ &= 2G^2 [B_{st}^0 + B_{tu}^0 + B_{us}^0] \end{aligned} \quad (4.2)$$

where

$$B_{st}^\theta = \int_0^1 du \int_0^1 dv \ u^{-\alpha(s)-1} v^{-\alpha(t)-1} \delta(u^2 + v^2 - 2uv \cos \theta - 1) \quad (4.3)$$

and similar definitions for B_{tu}^θ and B_{us}^θ . Of course for $\theta = 0$, B_{st}^θ becomes the usual Beta-function. Finally $d\mathbf{v}$ in eq. (4.2) is the well known volume element. The factor 2 comes from the two possible quark diagrams for each cyclic ordering of the external lines, or formally from the summation over γ in the vertex (4.1). For non-exotic meson-baryon scattering ($23 \rightarrow 23$) the relevant vertex is

$$\Gamma_{33}^2(k_i, z_{i1}, z_{i2}, z_{i3}, \lambda_i) = G \sum_{\gamma=1}^3 : e^{ik_i \varphi(z_{i\gamma}, \lambda_i)} : \quad (4.4)$$

in an obvious notation, and we find

$$T(23 \rightarrow 23) = \int dV \langle 0 | T \Gamma_{33}^2(k_2, 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \lambda_2) \Gamma_{33}^2(k_3, 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \lambda_3) | 0 \rangle$$

$$= 3G^2 [B_{st}^0 + B_{tu}^0 + 2 B_{us}^{2\pi/3}] \quad (4.5)$$

there $B_{us}^{2\pi/3}$ is the dual function first considered by Mandelstam.⁸⁾

We can now compute the general process of the type $N + 2 \rightarrow N+2+2+\dots+2$. With the vertex (here we temporarily suppress the variables $\{i\gamma$ and λ_i in Γ_{NN}^2)

$$\Gamma_{NN}^2(i) = G \sum_{\gamma=1}^N : e^{ik_i \varphi(\{i\gamma, \lambda_i\})} : \quad (4.6)$$

the amplitude becomes

$$T(2N \rightarrow \underbrace{N+2+2+\dots+2}_{n-2}) =$$

$$= \int dV \langle 0 | T \Gamma_{NN}^2(2) \Gamma_{NN}^2(3) \dots \Gamma_{NN}^2(n-1) | 0 \rangle \quad (4.7)$$

In particular, for the four-point function

$$T(24 \rightarrow 24) = 4G^2 [B_{st}^0 + B_{tu}^0 + B_{us}^0 + B_{us}^{\pi/3} + B_{us}^{2\pi/3}] \quad (4.8)$$

Observe that a Mandelstam term $B_{us}^{2\pi/3}$ again appears, but also a new term $B_{us}^{\pi/3}$ appears. As N increases no new terms appear.

The amplitudes for all 4-point functions no matter what the four external particles are can be expressed as sums of B^0 , $B^{\pi/3}$ and $B^{2\pi/3}$ functions with appropriate coefficients.

The common feature of all processes discussed so far in this section is that they were described in terms of quark diagrams without any focused lines. We now start considering graphs that have focused lines. As a first case consider again the process $23 \rightarrow 23$ but now in the description corresponding to fig. 8a. As opposed to the processes without focusing the number of quark lines along the world sheet is not a constant in proper time. For instance at $\lambda < 0$ we have one quark and one antiquark whereas for $\lambda > 0$ there are three quarks. It is therefore useful to introduce an indicator function

$$\mathcal{V}(\lambda) = \text{number of quark and antiquark lines intersected by the line } \text{Re } z = \lambda \quad (4.9)$$

We now write down the $23 \rightarrow 23$ amplitude

$$T(23 \rightarrow 23) = \int d\mathcal{V} \langle 0 | \Gamma_{\nu u_3, \nu u_3}^2(k_3, \mathcal{Z}_1, \dots, \mathcal{Z}_\nu, \lambda_3) \Gamma_{23}^3(k_1, \pi, 0) | 0 \rangle \quad (4.10)$$

and find^{*} (using the results of I) again the expression (4.5) provided

$$\Gamma_{23}^3(k_i, \mathcal{Z}_i, \lambda_i) = 3G : e^{ik_i \varphi(\mathcal{Z}_i, \lambda_i)} : \quad (4.11)$$

In (4.10) $\mathcal{V}(\lambda_3) = 2\theta(\lambda_1 - \lambda_3) + 3\theta(\lambda_3 - \lambda_1)$ and $\mathcal{Z}_\nu$ takes the values 0, π for $\lambda_3 < \lambda_1$ and 0, or that of one of the two focusing curves for $\lambda_3 > \lambda_1$. In much the same way we can deduce

$$\Gamma_{24}^4(k_i, \mathcal{Z}_i, \lambda_i) = 4G : e^{ik_i \varphi(\mathcal{Z}_i, \lambda_i)} : \quad (4.12)$$

^{*}) Note that $d\mathcal{V}$ should reflect the character of the line element which in some time orderings is curved and others is straight.

for an "unpinched" M_4 (fig. 1, 4u).

It is instructive to calculate $T(23 \rightarrow 23)$ in yet another way, namely with the two mesons at $z_2 = -\infty$ and $z_3 = +\infty$ (figs. 8b, c). At this point finite focusing curves enter our picture. Only the Γ_{23}^3 and its hermitian conjugate appear in the diagrams of fig. 8b, c. So we can calculate

$$T(23 \rightarrow 23) = w_b \int dV \langle 0 | \Gamma_{23}^3(k_1, \pi, \lambda_1) \Gamma_{23}^{3+}(k_4, \pi, \lambda_4) | 0 \rangle + \\ + w_c \int dV \langle 0 | T \Gamma_{23}^3(k, \frac{2\pi}{3}, \lambda_1) \Gamma_{23}^{3+}(k_4, 0, \lambda_4) | 0 \rangle \quad (4.13)$$

In order that this agree with (4.5) the weight factors w_b and w_c have to take the values

$$w_b = 1/3 \quad w_c = 2/3 \quad (4.14)$$

These factors can be easily understood. For instance in fig. 8b there is because of eq. (4.11) a factor 3 for each vertex, while once the vertex at λ_1 is fixed, the fate of all quarks at the vertex at λ_4 is determined so that there should be no additional factor of 3. One therefore has to isolate the structures of fig. 8b', c' and attach to them the weights (4.14) whenever they appear in a more complicated graph. One can proceed in this way to $3\bar{3} \rightarrow 3\bar{3}$ (fig. 8d).

We have the vertex

$$\Gamma_{34}^3(k, \frac{2\pi}{3}, \lambda) = 3G : e^{ik\varphi(\frac{2\pi}{3}, \lambda)} : \quad (4.15)$$

which is obtained by noting that there are six possible pairs of spectator quarks at the focused vertex. We then find a weight $w_d = 1/3$ associated with the structure fig. 8d'. One can proceed in this way to ever more exotic processes, determining the vertices step by step as well as the weights for structures containing finite focusing curves. In particular these weights and vertices as determined from different processes have to be compatible. We have successfully performed a number of such compatibility checks, but we do not find it worthwhile to go into the details, because it is quite obvious that a more elegant operatorial formalism (that puts the focusing into the propagators rather than the vertices, as we have done) ought to exist and these cumbersome combinatorial details ought to emerge automatically from it.

We still treat here the problem of s-t-u diagrams mentioned in the previous section. Take the quark diagram 2a. We can draw corresponding to it the three evolutions of the world sheets, one is an st, one an su and one a tu term (see figs. 2a,2b,2c). The sum of these diagrams will have poles in all three channels and the corresponding amplitudes will have a weighting-factor that can be determined by quark counting. Indeed all quark diagrams for $34 \rightarrow 34$ are presented in fig. 9 and it is easy (by just counting quarks) to see that the complete $34 \rightarrow 34$ amplitude will be

$$\begin{aligned}
 T(34 \rightarrow 34) = & 3G^2 \left[B_{st}^{\circ}(3,4) + B_{ut}^{\circ}(3,4) + 2B_{st}^{\circ}(5,2) + \right. \\
 & 2B_{ut}^{\circ}(5,2) + 2B_{su}^{\frac{2\pi}{3}}(3,5) + 2B_{su}^{\frac{2\pi}{3}}(5,3) + \\
 & \left. + 8C \{ B_{st}^{\circ}(5,4) + B_{ut}^{\circ}(5,4) + B_{su}^{\circ}(5,5) \}^2 \right]_{(4.16)}
 \end{aligned}$$

where $B_{st}^{\circ}(i,j)$ is given by the expression (4.3) and the s-channel trajectory is that of total quark number i and the t-channel trajectory that of total quark number j. The last three terms here, come from the s-t-u-diagram. Observe that we have assumed that three world-sheet evolutions for this diagram are possible and that we have chosen for it the B° configuration (fig. 10). It is not obvious whether this is a unique prescription. It is also not completely clear how to normalize this term whence the constant C in eq. (4.16). We are presently studying this problem. The symmetric quark model may be of help in uniquely nailing down this term.

5. Degeneracy of Exotic Hadron Spectrum

Each quark arrangement for a given hadron is displayed on one rubber band and therefore has the same spectral degeneracy as the baryons.*) Since as we have seen the various configurations do not mix, the spectra for fixed total quark number N , has a degeneracy for very large mass of the Veneziano form. The detailed rules of section 4 correspond to this degeneracy.

There is another model that one can envisage in which one attaches to each quark arrangement as many world sheets as there are distinct combinations of the individual quarks and antiquarks. For instance for the unpinched M_4 one considers all the arrangements of fig. 11 calling $q_1 q_2$ and $\bar{q}_1 \bar{q}_2$ the quarks and antiquarks of M_4 . One then attaches one field to each configuration $\phi_{\mu 1}$ and $\phi_{\mu 2}$ and the vertex for "kicking" the quark q_i is postulated to be

$$\Gamma_{44}^2(q_i) = G: e^{i \frac{k^r}{\sqrt{2}} \sum_{j=1}^2 \phi_{rj}(\xi_{q_i}, \lambda)} : \quad (5.1)$$

Similarly the vertex for kicking the antiquark $\bar{q}_i$ is

$$\Gamma_{44}^2(\bar{q}_i) = G: e^{i \frac{k^r}{\sqrt{2}} \sum_{j=1}^2 \phi_{rj}(\xi_{\bar{q}_i}, \lambda)} : \quad (5.2)$$

So e.g. (see fig. 11)

*) With the exception of maximally pinched mesons which have the same spectrum as M_2 's.

$$\Gamma_{44}^2(\bar{q}_2) = G : e^{i \frac{k}{r_2} [\varphi_{\mu_1}(-\frac{\pi}{3}, \lambda) + \varphi_{\mu_2}(\pi, \lambda)]} : \quad (5.3)$$

The vertices (5.1) (5.2) are automatically symmetric in all quarks and in all antiquarks. This picture has

$$H = \sum_{\sigma \text{ all sheets}} \sum_r r (a_{\mu\sigma r}^\dagger a_{\mu\sigma r} + b_{\mu\sigma r}^\dagger b_{\mu\sigma r}) \quad (5.4)$$

and the degeneracy increases exponentially with N since the number of sheets does so. This spectrum, we believe, is much too rich and we prefer the spectrum that increases linearly with N and that corresponds to the detailed rules discussed in the previous sections.

6. Conclusions

We have shown in this paper that the conformal invariance of dual dynamics, as expressed in the quark focusing principle⁴⁾ unambiguously fixes the possible quark configurations of all hadrons. The simple star of David rule (rule E of section 1) for the construction of hadrons from quarks was obtained. Rules to calculate arbitrary hadron processes were found and the problem of "stu quark diagrams" investigated. Here we would like to add one observation of a more philosophical nature. While our star of David arguments were given in a quark language they have a simple meaning in the orthodox bootstrap⁹⁾ language as well. Essentially the star of David rule says that all configurations of a hadron can be obtained by constructing in all possible ways from nonexotic baryons (three quark-triangles), nonexotic antibaryons (three antiquark-triangles), and nonexotic mesons ($q\bar{q}$ -strings). The role of baryons and antibaryons is to provide their half unit of D-spin towards an explosion of exoticity. While if quarks existed, baryons and exotic hadrons would have a common origin as multiquark systems, in the bootstrapper's world it is the baryons that provide the possibility of nonvanishing D-spin as well as that of half odd integral ordinary spin. The bootstrap counterpart of quarks, in a certain sense, are the nonexotic baryons and nonexotic mesons.

Acknowledgements

We would like to thank Dr. L. N. Chang and Dr. R. J. Rivers for valuable discussions.

Appendix A Infinite Focusing Curves (IFC)

All focusing curves can be divided into two classes: those of infinite and those of finite extent. The latter will be discussed in Appendix B.

In $23 \rightarrow 23$ an infinite focusing curve which changes ordinate by $\pi/3$ (IFC ($\pi/3$)) occurs and its detailed equation was given in Paper I. There it was shown that the quark trajectories $\text{Im} z = \pi \pm \pi/3$ are taken by the conformal mapping

$$w = \ln(e^z - 1) \quad (\text{A.1})$$

into the curves (with $\text{Re } w = \mu$)

$$w_{\pm} = \mu + i \arccos\left(-\frac{3 + (4e^{2\mu} - 3)^{1/2}}{4e^{\mu}}\right) \quad (\text{A.2})$$

selected by the asymptotic criteria

$$\text{Im}(w_{\pm})(\mu \rightarrow +\infty) = \pi \pm \pi/3 \quad (\text{A.3})$$

respectively.

To derive the equation for IFC ($2\pi/3$) we consider $24 \rightarrow 24$ scattering in the configuration of Figures 12a, 12b. Here the appropriate conformal mapping is again Eq. (A.1) and the image of the line $\text{Im} z = 5\pi/3$ gives the equation for IFC ($2\pi/3$). From this discussion the method for finding IFC ($p\pi/3$) for all integer p is already clear, so we will present only the general formula, for the curve starting at $w = 0$ and with asymptote

$$\operatorname{Im} W_{\pm} (\mu \rightarrow +\infty) = \pm p\pi/3 \quad (\text{A.4})$$

Putting $w = \mu + i\eta$ we find therefore

$$W = \ln (e^{\lambda + i\pi p/3} - 1) \quad (\text{A.5})$$

and hence in turn

$$\eta = \tan^{-1} \frac{(e^{\lambda} \sin \pi p/3)}{(-1 + e^{\lambda} \cos \pi p/3)} \quad (\text{A.6})$$

$$e^{2\mu} = e^{2\lambda} - 2 e^{\lambda} \cos \pi p/3 + 1 \quad (\text{A.7})$$

$$e^{\lambda} = \cos \pi p/3 + \sqrt{\cos^2 \pi p/3 - 1 + e^{2\mu}} \quad (\text{A.8})$$

and then

$$\cos \eta_{\pm} = \pm 1 / \sqrt{1 + \tan^2 \eta} \quad (\text{A.9})$$

leads to the final form, to wit

$$W_{\pm} = \mu + i \arccos \left(\mp e^{-\mu} \left[1 - \cos^2 \frac{2\pi}{3} p - \cos \pi p/3 \cdot \sqrt{\cos^2 \frac{\pi p}{3} - 1 + e^{2\mu}} \right] \right) \quad (\text{A.10})$$

It is amusing to note that when p is a multiple of 3 the curve IFC ($p\pi/3$) adopts a "hairpin" structure. An example of this occurs in $2^4_p \rightarrow 2^4_p$ scattering, see Figure 13. When we transform from the configuration 13a to the configuration 13b the quark line from particle 1 to particle 4 executes a hairpin curve IFC (π); the quark proceeds from particle 1 at $z = 0$, along the negative real axis to $z = -\infty$ where it changes its course to return via the line $\text{Im } z = \pi$ to the particle 4 at $+\infty$. The dashed portion of the curve in Figure 13b should not of course be regarded as a constituent of particle 2; on the other hand between $z = i\pi$ and $z = z_3$ this focused quark is a constituent of the 4_p intermediate state. The reason for this is that along the full portion the quark only appears once at a given proper time, while on the dotted portion it appears twice and the same quark cannot appear twice in the same hadron at a given proper time. It is important to remark that upon a conformal mapping a dashed portion of a quark line can become a full portion and vice versa.

Appendix B. Finite Focusing Curves (FFC)

Focused curves of finite length require a separate analysis. We first derive a general equation for FFC which do not change the ordinate such as the FFC appearing in Figs. 3b, 3c. We find it here useful to employ the notation FFC $(\Lambda, 0, p\frac{\pi}{3})$ in which the first argument Λ represents the difference in abscissa between the ends, the second argument gives the difference in ordinate, and the third gives the asymptotic position (as $\Lambda \rightarrow \infty$) of the mid-point's ordinate.

To illustrate the procedure we derive the equation of FFC $(\Lambda, 0, \frac{\pi}{3})$ for the $23 \rightarrow 23$ diagram, Figure 3c. We perform the mapping

$$w = -\ln(e^{-z} + 1) \quad (B.1)$$

to go from 14a to 14b. The curve marked X which is of course IFC $(\frac{\pi}{3})$ is mapped into $\tilde{X}$ which is FFC $(\mu_2, 0, \frac{\pi}{3})$. The work of Appendix A gives for X the form, Equation (A.2), (with $\lambda' = \lambda - \lambda_2$)

$$z = \lambda + i \arccos \left(e^{-\lambda'} \left[\frac{3}{4} + \frac{1}{2} \sqrt{e^{2\lambda'} - \frac{3}{4}} \right] \right) \quad (B.2)$$

which transforms into $\tilde{X}$ given by the parametric form

$$\mu(\lambda) = \operatorname{Re} w(\lambda) \quad (B.3a)$$

$$\eta(\lambda) = \operatorname{Im} w(\lambda) \quad (B.3b)$$

with

$$W(\lambda) = -\ln \left[1 + \exp \left\{ -\lambda - i \arccos \left(e^{-\lambda'} \left[\frac{3}{4} + \frac{1}{2} (e^{2\lambda'} - \frac{3}{4})^{1/2} \right] \right) \right\} \right]. \quad (\text{B.4})$$

Clearly we may use IFC ($p\pi/3$) to find similarly a parametric form of FFC ($\mu_2, 0, p\pi/3$) with (B.4) replaced by its immediate generalization, to wit

$$W(\lambda) = -\ln \left[1 + \exp \left\{ -\lambda - i \arccos \left(e^{-\lambda'} \left[1 - \cos^2 \frac{\pi}{3} p - \cos \frac{\pi}{3} (\cos^2 \frac{\pi}{3} p - 1 + e^{2\lambda'})^{1/2} \right] \right) \right\} \right]. \quad (\text{B.5})$$

In some cases two or more FFC's can have both end points in common, as in the Figures 3b, 3c.

Next we turn to those FFC where the ordinate difference between the ends is non-zero. Here the analysis becomes more involved. We shall be content to give a simple illustrative example. A similar approach can be applied almost line-by-line for other ordinate-changing FFC's.

The su quark diagram for $23 \rightarrow 23$ scattering can be represented by the graph of Figure 8c in which an ordinate-changing FFC appears. To find the equation of this curve we need to make two successive mappings on Figure 15. Since we ultimately wish to map the point 3 to $w_3 = +\infty$, it is

useful to map first to a v-plane in which 3 is at the origin by

$$v = z - z_3 \quad (\text{B.6})$$

and then finally from v to w by the map

$$w = -\ln(e^{-v} - 1) \quad (B.7)$$

Thus the curve y of Figure 15, given by $z_y(\lambda)$, is mapped into

$w_y(\lambda)$ satisfying

$$\begin{aligned} e^{-w_y(\lambda)} + 1 &= \exp[-z_y(\lambda) + z_3] = \\ &= \exp\left[-\lambda - i\pi + i \arccos\left(e^{-\lambda}\left[\frac{3}{4} + \frac{1}{2}(e^{2\lambda} - \frac{3}{4})^{\frac{1}{2}}\right] + z_3\right)\right] \end{aligned} \quad (B.8)$$

where in turn

$$\begin{aligned} z_3 &= \lambda_3 + i\pi + i \arccos\left(e^{-\lambda_3}\left[\frac{3}{4} + \right. \right. \\ &\quad \left. \left. + \frac{1}{2}(e^{2\lambda_3} - \frac{3}{4})^{\frac{1}{2}}\right]\right) \end{aligned} \quad (B.9)$$

Thus Eq. (B.8) gives parametrically (in λ) the equation of

$w_y(\lambda)$ for any pre-assigned λ_3 . We note that for $z_3 = \frac{1}{2} \ln 3 +$

$+ i \frac{7\pi}{6}$ the FFC in the w -plane becomes a vertical

line from $w_4 = -i\pi$ to $w_1 = -i\pi/3$.

Appendix C. Focusing Curves Relevant to Discussion of "Star of David" in Section II

The baryon intermediate state in $2\bar{3} \rightarrow 2\bar{3}$ scattering for the focused configuration in Fig. 5a can become an on-mass-shell s-channel pole when $(\lambda_4 - \lambda_1) \rightarrow \infty$. We must therefore confirm that the quarks are located at the appropriate sites of the star of David.

A simple proof begins with the graph of Fig. 16a in the u-plane. Making the conformal mapping

$$v = \ln(-e^{-u} - 1) \quad (C.1)$$

we observe that the points $u_2 = i\pi$, $u_3 = -\infty$, $u_4 = +\infty$ are mapped into the points $v_2 = -\infty$, $v_3 = +\infty$, $v_4 = i\pi$ (see Figure 16b). The point $u_1 = i\pi + \mathcal{J}_1$ with $\mathcal{J}_1 = \text{Re } \mathcal{J}_1 > 0$ is mapped into $v_1 = i\pi - \mathcal{J}'_1$ with $\mathcal{J}'_1 = \text{Re } \mathcal{J}'_1 > 0$. The s-channel pole in $(k_1 + k_2)^2$ arises from the limit $\mathcal{J}'_1 \rightarrow +\infty$ which corresponds to $\mathcal{J}_1 \rightarrow 0$.

To find the quark positions consider the points $u_{\pm} = i\pi + \mathcal{J}_1 + \delta e^{\pm i\pi/3} + O(\delta^2)$ which for infinitesimal δ are on the upper and lower focusing curves. They map into the points

$$\begin{aligned} v_{\pm} &= \ln(-e^{-i\pi - \mathcal{J}_1 - \delta e^{\pm i\pi/3}} - 1) + O(\delta^2) \\ &\xrightarrow{\mathcal{J}_1 \rightarrow 0} \ln(-\mathcal{J}_1 - \delta e^{\pm i\pi/3}) \\ &\longrightarrow +\ln \delta + i\pi \pm i\pi/3 \end{aligned} \quad (C.2)$$

This demonstrates that for $\gamma'_1 \rightarrow \infty$, the quarks take up positions with ordinate $\text{Im } v = \pi \pm \pi/3$ respectively.

Exactly the same focusing curves occur in Figure 3b and this leads to the star of David structure for the unpinched 4_u as discussed in Section 2.

Next we must treat the intermediate state for graphs with ordinate-changing FFC's such as in Figures 8c. Although the finite curve has a rather cumbersome form the following argument shows directly that at the pole there is consistency with the star of David rule.

We consider Fig. 17a for $23 \rightarrow 23$ scattering and let $u_3 \rightarrow u_2$. The mapping

$$v = -\ln(e^{-(u-u_3)} - 1) \quad (C.3)$$

transforms to the graph of Fig. 17b with $v_3 = +\infty$. The pole in $(k_1 + k_4)^2$ arises from the limit $\mu_2 \rightarrow +\infty$ (fig. 17b), which corresponds to $u_3 \rightarrow i\pi$. In this limit, Eq. (C.3) becomes Eq. (C.1), and the arguments leading to Eq. (C.2) can be applied here to derive the asymptotic quark positions to be $\text{Im } w = \pm \frac{\pi}{3}$ as required. It can be similarly shown that all other ordinate-changing FFC's are consistent with the star of David rule.

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Figure Captions

1. Star of David configurations for $N \leq 9$.
2. Quark diagrams with crosses.
3. Quark loops and the star of David.
4. $B_3 M_4 \rightarrow B_3 M_4$ scattering.
5. $B_5 \bar{B}_5 \rightarrow B_5 B_5$ scattering.
6. $B_3 H_N \rightarrow B_3 H_N$ scattering.
7. Nonorientable s-t-u quark diagram for $B_3 B_3 \rightarrow B_3 B_3$.
8. Graphs with focused lines.
9. Quark diagrams for $34 \rightarrow 34$ scattering.
10. A possible world sheet evolution for s-t-u graph in $34 \rightarrow 34$ scattering. Two more diagrams corresponding to the cyclic ordering 1 3 2 4 and 1 2 4 3 (the latter as usual has $\Theta = \pi$) have to be added.
11. An alternative model.
12. IFC ($2\pi/3$) in $24 \rightarrow 24$ scattering.
13. Hairpin structure of IFC (π).
14. FFC ($\mu_2, 0, \pi/3$) in $23 \rightarrow 23$ scattering.
15. Starting point for calculation of an ordinate-changing FFC.
16. Star of David and non-ordinate-changing FFC.
17. Star of David and ordinate-changing FFC.

$N=2 \quad B=0$

$N=3 \quad B=1$

$N=4 \quad B=0$

$N=5 \quad B=1$

Figure 1 (part one of three)

Figure I (part two of three)

B=2

N=9 B=1

B=3

Figure I (part three of three)

Figure 2

a

b

c

d

Figure 3

Figure 4

Figure 5

Figure 6

Figure 7

Figure 8

a

b

c

d

e

Figure 9

Figure 10

Figure II

Figure 12

Figure 13

$$\underline{Z} = \lambda + i\zeta$$

$$\underline{W} = \mu + i\eta$$

Figure 14

Figure 15

Figure 16

Figure 17