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Beta-Function Distortions Due to Linear Coupling

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1. Introduction

Since a coupling between transverse \( x \) and \( y \) degrees of freedom is expected in RHIC it is important to examine its influence on beta-functions.\(^1\) We shall calculate the shifts of the beta-functions produced by point skew-quadrupoles distributed around the ring. The \( X-Y \) coupling is linear in this case\(^2\) and its effect can be calculated exactly assuming that \( k \)-th skew-quadrupole of length \( \ell_k \) is located at \( s_k \) in the ring and has strength \( q_k \).

\[
q_k = (\beta_x \beta_y)^{1/2} \frac{\ell_k}{\rho} a_{1_{x=s_k}}, \quad k = 1, \ldots, N
\]  

where \( \beta_x, \beta_y \) are beta-functions of a perfect machine.\(^3\)

It is known that in the presence of linear coupling there exists a matrix \( R \) such that in passing to new variables \( u, u', v, v' \) the transverse motions are decoupled i.e.,

\[
\begin{bmatrix}
  x \\
  x' \\
  y \\
  y'
\end{bmatrix} = R
\begin{bmatrix}
  u \\
  u' \\
  v \\
  v'
\end{bmatrix},
\]

\[
T = \begin{bmatrix}
  M & m \\
  m & N
\end{bmatrix} = R \begin{bmatrix}
  A & 0 \\
  0 & B
\end{bmatrix} R^{-1}.
\]

Here \( T \) is a single turn \( 4 \times 4 \) symplectic transfer matrix for a coupled machine and \( A, B \) are symplectic \( 2 \times 2 \) submatrices describing uncoupled transverse motions

\[
A = \begin{bmatrix}
  \cos \mu_1 + \alpha_1 \sin \mu_1 & \beta_1 \sin \mu_1 \\
  -\gamma_1 \sin \mu_1 & \cos \mu_1 - \alpha_1 \sin \mu_1
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
  \cos \mu_2 + \alpha_2 \sin \mu_2 & \beta_2 \sin \mu_2 \\
  -\gamma_2 \sin \mu_2 & \cos \mu_2 - \alpha_2 \sin \mu_2
\end{bmatrix},
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]

\[
\]
\[ \alpha_k, \beta_k, \gamma_k, \mu_k, k = 1, 2 \] are the usual Courant-Snyder parameters of decoupled motions.

We would like to calculate the beta-function distortions

\[ \Delta \beta_x = \beta_1 - \beta_x, \] \hspace{1cm} (1.6)
\[ \Delta B_y = \beta_2 - \beta_y, \] \hspace{1cm} (1.7)

assuming that the linear coupling is small. This can be done using general formulae that express the \( A, B \) submatrices in terms of the \( T \) matrix

\[ A = M + (t + \delta)^{-1} (\overline{m} + m) m, \] \hspace{1cm} (1.8)
\[ B = N - (t + \delta)^{-1} (m + \overline{n}) n, \] \hspace{1cm} (1.9)

where

\[ t = \frac{1}{2} \text{Tr} (M - N), \] \hspace{1cm} (1.10)

and

\[ \delta = (t^2 + |\overline{m} + n|)^{1/2}. \] \hspace{1cm} (1.11)

Here \( \overline{m} \) stands for a symplectic conjugate of \( m \)

\[ m = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \rightarrow \overline{m} = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix}. \] \hspace{1cm} (1.12)

The single turn transfer matrix \( T \) can be written as a polynomial

\[ T = \sum_{k=0}^{N} T^{(k)}, \] \hspace{1cm} (1.13)

where \( T^{(k)} \) are given via, \( k \)-th order in the \( q \)'s, driving terms. In this note we shall calculate the beta-function distortions up to the second order in the \( q \)'s.
2. Calculations of the Beta-Function Distortions

According to the formula (1.8), (1.9) one has the relations

\[
A_{12} = \beta_1 \sin \mu_1 = M_{12} + (t + \delta)^{-1} [(m + n) m]_{12},
\]

\[
B_{12} = \beta_2 \sin \mu_2 = N_{12} - (t + \delta)^{-1} [(m + n) m]_{12}.
\]

(2.1)

(2.2)

Taking into account the tune splitting

\[
\mu_1 = \mu_x + \Delta \mu_1,
\]

\[
\mu_2 = \mu_y + \Delta \mu_2,
\]

(2.3)

(2.4)

where \( \Delta \mu_1, \Delta \mu_2 \) are expressed through second-order driving terms, one gets from the expansion (1.13), assuming \( \alpha_x(0) = \alpha_y(0) = 0 \), the results

\[
\frac{\Delta \beta_x}{\beta_x} = \frac{1}{2} \left( d_{cc}^{(2)} - d_{sc}^{(2)} \right) \cot^2 \mu_x + \frac{1}{2} \left( d_{cc}^{(2)} + d_{ss}^{(2)} \right) \cot \mu_x +
\]

\[
+ d_{cs}^{(2)} (t + \delta)^{-1} \left\{ - \left( \left( d_{cc}^{(1)} \right)^2 + \left( d_{ss}^{(1)} \right)^2 \right) \sin \mu_y \cot \mu_x - \left( d_{sc}^{(2)} - d_{cs}^{(2)} \right) \cos \mu_y +
\]

\[
+ \left( d_{cc}^{(1)} d_{sc}^{(1)} + d_{ss}^{(1)} d_{cs}^{(1)} \right) \sin \mu_y \right\} + O(q^4),
\]

(2.5)

\[
\frac{\Delta \beta_y}{\beta_y} = \Delta \beta_x \bigg|_{x \rightarrow y}
\]

(2.6)

Here the driving terms of the first order \( d^{(1)} \) and of the second order \( d^{(2)} \) are defined as follows:

\[
\begin{bmatrix}
    d_{ss}^{(1)} \\
    d_{sc}^{(1)} \\
    d_{cs}^{(1)} \\
    d_{cc}^{(1)}
\end{bmatrix}
= \sum_{r=1}^{N} q_r
\begin{bmatrix}
    \sin \mu_x^r \sin \mu_y^r \\
    \sin \mu_x^r \cos \mu_y^r \\
    \cos \mu_x^r \sin \mu_y^r \\
    \cos \mu_x^r \cos \mu_y^r
\end{bmatrix},
\]

(2.7)

and

\[
\begin{bmatrix}
    d_{ss}^{(2)} \\
    d_{sc}^{(2)} \\
    d_{cs}^{(2)} \\
    d_{cc}^{(2)}
\end{bmatrix}
= \sum_{1 \leq r < s \leq N} q_r q_s \sin (\mu_x^s - \mu_y^s)
\begin{bmatrix}
    \sin \mu_x^r \sin \mu_y^r \\
    \sin \mu_x^r \cos \mu_y^r \\
    \cos \mu_x^r \sin \mu_y^r \\
    \cos \mu_x^r \cos \mu_y^r
\end{bmatrix},
\]

(2.8)

and \( \mu_x^r, \mu_y^r \) are phase advances

\[
\mu_x^r = \int_0^{S_r} \frac{ds}{\beta_x}, \quad \mu_y^r \int_0^{S_r} \frac{ds}{\beta_y}.
\]

(2.9)
Additional sets of driving terms, denoted as \( \tilde{d}^{(1)}_{ss}, \tilde{d}^{(2)}_{ss} \), etc, are obtained from the above equations by simply exchanging \( x \) and \( y \),

\[
\tilde{d}^{(k)}(x, y) = d^{(k)}(y, x), \quad k = 1, 2
\]  

(2.10)

It is easy to notice the relations

\[
\tilde{d}^{(1)}_{ss} = d^{(1)}_{ss},
\]

\[
\tilde{d}^{(1)}_{sc} = d^{(1)}_{cs},
\]

\[
\tilde{d}^{(1)}_{cc} = d^{(1)}_{cc},
\]  

(2.11)

It is interesting to check if the beta-function distortions disappear after correction of the tune-splitting which requires, among others, that the following equalities hold

\[
d^{(2)}_{cc} - d^{(2)}_{ss} = 0,
\]

\[
\tilde{d}^{(2)}_{cc} - \tilde{d}^{(2)}_{ss} = 0,
\]

\[
d^{(2)}_{cs} - d^{(2)}_{gc} = 0,
\]

\[
\tilde{d}^{(2)}_{cs} - \tilde{d}^{(2)}_{gc} = 0.
\]  

(2.12)

Applying them to the formula (2.5), (2.6) one finds that residual beta-function distortions are present

\[
\frac{\Delta \beta_x}{\beta_x} = -d^{(2)}_{ss} \cot \mu_x + \tilde{d}^{(2)}_{cs} + (t + \delta)^{-1} \left\{ - \left[ \left( d^{(1)}_{sc} \right)^2 + \left( d^{(1)}_{ss} \right)^2 \right] \sin \mu_y \cot \mu_x + \right.
\]

\[
+ \left( e^{(1)}_{cc} e^{(1)}_{sc} + e^{(1)}_{ss} e^{(1)}_{cs} \right) \sin \mu_y \} + O \left( q^4 \right),
\]  

(2.13)

\[
\frac{\Delta \beta_y}{\beta_y} \big|_{\Delta \nu = 0} = \frac{\Delta \beta_x}{\beta_x} \big|_{\Delta \nu = 0, x \rightarrow y},
\]  

(2.14)

and, according to the formula (1.10), (1.11)

\[
t + \delta \big|_{\Delta \nu = 0} = 2 \left( \cos \mu_x - \cos \mu_y \right) + O \left( q^4 \right).
\]  

(2.15)

One sees that passing to the limit \( \nu_x - \nu_y \rightarrow 0 \) is delicate here, and higher order terms in the last expansion should be included. This is essential since RHIC is designed to operate at almost equal tunes: \( \nu_x = 28.826, \nu_y = 28.821 \).
Our results should be compared with perturbative calculations of the beta-function distortions.\textsuperscript{4,5} Clearly, the residual beta-function distortions can be removed if we correct driving terms which appear in the formula (2.13), (2.14).

Assuming that skew-quadrupole errors are randomly distributed around the ring and taking into account that

\[ N < q^2 >= G_0^2 \]  

(2.16)

where, for RHIC we take

\[ G_0 \approx 0.25, \]  

(2.17)

we get the estimate for the average distortion

\[ < \frac{\Delta \beta_x}{\beta_x} > \approx -0.25, \]  

(2.18)

and similar for \( < \frac{\Delta \beta_y}{\beta_y} > \).

Even larger beta-function distortions are expected in SSC in which case \( G_0 \approx 0.5 \) which yields for the average beta-function distortion \( \approx -1.2 \).

3. References

5. G. Parzen, BNL Report, AD/RHIC-102, July 1991, and references to earlier work on the subject given there.