NOV 19 1963

BNL 7390

This paper was submitted for publication in the open literature at least 6 months prior to the issuance date of this Microcard. Since the U.S.A.E.C. has no evidence that it has been published, the paper is being distributed in Microcard form as a preprint.

MASTER

Analytical Study of Heat Transfer to Liquid Metals
in Cross Flow Through Rod Bundles - II*

by

Chia-Jung Hsu

Brookhaven National Laboratory
Upton, New York

May 22, 1963

(Submitted to International Journal of Heat
and Mass Transfer for publication.)

LEGAL NOTICE

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

*This work was performed under the auspices of the U. S. Atomic Energy Commission.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

ABSTRACT

Analytical expressions for the Nusselt number have been obtained for heat transfer to liquid metals flowing past a single elliptical rod or through elliptical-rod bundles, by assuming inviscid flow. These results can also be used for predicting Nusselt numbers in oblique cross-flow of liquid metals through bundles of circular rods.

Expressions for Nusselt numbers for the following thermal conditions on the surface of a rod have been derived: (a) constant tube-wall temperature, (b) constant surface heat flux, and (c) cosine tube-wall temperature distribution. It was found that Nusselt numbers for normal cross-flow of liquid metals past an elliptic rod (single, or one inside a bundle) bear a certain relationship to those for the case in which the rods are circular. The relationships can be expressed by the following forms:

For a single rod:

$$Nu_{ell}(Pe) = Nu_{cyl}(Pe) \sqrt{\frac{\sqrt{1-e^2} + (1-e^2)}{2-e^2}}$$

For a rod inside a bundle:

$$Nu_{ell}[Pe, \phi_1/(a+b)] = Nu_{cyl}(Pe, \phi_1/D) \sqrt{\frac{\sqrt{1-e^2} + (1-e^2)}{2-e^2}}$$

The above equations show that the average Nusselt number increases as the eccentricity of the ellipse decreases; and, at the limit where the

eccentricity becomes zero, the Nusselt numbers reduce to those for the case of normal flow past a single circular rod or through a bundle of circular rods.

The normalized hydrodynamic potential drop, $\phi_1/(a+b)$, which appears in the Nusselt numbers for elliptical-rod bundles was analytically evaluated by use of conjugate functions. It was found that the numerical values of the corresponding parameter, ϕ_1/D , for flow through circular-rod bundles were applicable here, provided that certain changes in notation were made.

For the case where the liquid metal flows obliquely through the bundle, making a 45 degree angle with circular rods, the values of the parameter, $\phi_1/(a+b)$ are calculated with the aid of a high-speed digital computer.

INTRODUCTION

This paper is a continuation of a previous paper (5) on the subject of heat transfer to liquid metals flowing crosswise through bundles of circular rods. In this paper, heat transfer to liquid metals flowing normally through elliptic-rod bundles, or obliquely through circular-rod bundles, is considered.

In shell-and-tube liquid-metal heat exchangers, oblique cross-flow is always present at certain locations on the shell side. The heat transfer in oblique cross-flow can be considered to be the same as that in flow through elliptic-rod bundles, if the hydrodynamic patterns for the two cases can be assumed to be the same. The present analysis deals mainly with the cross-flow of liquid metals through elliptic-rod bundles. By ignoring the difference in flow patterns, these results are also applicable to oblique cross-flow of liquid metals through circular-rod bundles.

For cross-flow past a single rod, Nusselt numbers expressed as a function of the eccentricity of the ellipse were obtained for the following three different conditions: (a) constant tube-wall temperature, (b) constant surface heat flux, and (c) cosine tube-wall temperature. These results

were extended to the case of flow through elliptic-rod bundles by assuming that the variation of hydrodynamic potential around a rod located in the interior of a bundle was of the cosine type in terms of elliptical coordinates. From a theoretical point of view, this assumption is reasonable.

The theoretical expression for the parameter, $\phi_1/(a+b)$, which appears in the Nusselt numbers for rod bundles was obtained from the results for circular rods by use of conjugate functions. In a previous paper (5), the corresponding parameter, ϕ_1/D , for circular rods was evaluated analytically, using the mathematical functions of Howland and McMullen (4). It will be shown that the computational results for ϕ_1/D which were presented in the previous paper are applicable to the evaluation of $\phi_1/(a+b)$ for normal cross-flow through elliptic-rod bundles. It is only necessary to change ϕ_1/D to $\phi_1/(a+b)$, and replace $D/2P$ by $(a+b)/2P$.

For the case of oblique flow through circular rod bundles, the distance between the centers of the ellipses will be stretched in the direction of flow. Accordingly, the parameter must be modified. For the case of oblique flow making a 45 degree angle with circular rod bundles, the parameter was calculated with the aid of an IBM 7094 computer, and plotted.

THEORETICAL CONSIDERATIONS

The flow field or temperature field around an elliptical rod (alone or located inside a bundle) is most conveniently described in terms of elliptic cylindrical coordinates. The following assumptions are made in this analysis.

(a) For flow around a single elliptical rod.

- i. Fluid properties are constant and there is no energy dissipation.
- ii. Both temperature and velocity fields are two-dimensional.
- iii. The flow is incompressible, non-viscous and irrotational.
- iv. Eddy transport of heat is negligible compared to molecular conduction.
- v. At the solid-liquid interface, there is no contact resistance.

(b) For an elliptical rod located in the interior of elliptical rod bundles.

Two additional assumptions are made:

- vi. For an elliptical rod located inside the bundle, the distribution of hydrodynamic potential around the surface of the rod is of the cosine type, in terms of elliptic cylindrical coordinates.

vii. Interaction of the thermal boundary layers of the rods
in a bundle is negligible.

Under these assumptions the energy equation in terms of elliptic cylindrical coordinates can be written as

$$v_{\xi} \frac{\partial T}{\partial \xi} + v_{\eta} \frac{\partial T}{\partial \eta} = \frac{\kappa}{a \sqrt{\sinh^2 \xi + \sin^2 \eta}} \left[\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} \right]. \quad (1)$$

The equation of continuity and the momentum equation can be combined and replaced by the Laplace equation. If the potential and stream functions in elliptical coordinates are defined by:

$$v_{\eta} = \frac{1}{a \sqrt{\sinh^2 \xi + \sin^2 \eta}} \frac{\partial \Psi}{\partial \xi} = \frac{-1}{a \sqrt{\sinh^2 \xi + \sin^2 \eta}} \frac{\partial \Phi}{\partial \eta} \quad (2)$$

and

$$v_{\xi} = \frac{-1}{a \sqrt{\sinh^2 \xi + \sin^2 \eta}} \frac{\partial \Psi}{\partial \eta} = \frac{-1}{a \sqrt{\sinh^2 \xi + \sin^2 \eta}} \frac{\partial \Phi}{\partial \xi}; \quad (3)$$

then,

$$\frac{\partial \Psi}{\partial \xi} = - \frac{\partial \Phi}{\partial \eta}, \quad (4)$$

$$\frac{\partial \Psi}{\partial \eta} = \frac{\partial \Phi}{\partial \xi} \quad (5)$$

and therefore the two-dimensional Laplace's equation retains the form:

$$\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} = 0 \quad (6)$$

or

$$\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial^2 \Phi}{\partial \eta^2} = 0 \quad (7)$$

It is not difficult to show that Boussinesq's transformation (1) is also valid for elliptical coordinates. Thus, if the independent variables, ξ and η , are transformed to ψ and ϕ , Equation (1) can be simplified to:

$$\frac{\partial T}{\partial \phi} = \kappa \left[\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial \psi^2} \right] \quad (8)$$

If the conductive term, $\partial^2 T / \partial \phi^2$, is neglected, Equation (8) further simplifies to

$$\frac{\partial T}{\partial \phi} = \kappa \frac{\partial^2 T}{\partial \psi^2} \quad (9)$$

As was the case for a circular cylinder, this transformation of independent variables causes the ellipse to be mapped into a flat plate, and gives rise to a flow field with uniform velocity, V . To obtain the expressions for Nusselt numbers corresponding to different thermal conditions on the surface of elliptical rods, the dependent variable, T , is changed to T' by letting $T - T_i = T'$. Equation (9) then becomes

$$\frac{\partial T'}{\partial \phi} = \kappa \frac{\partial^2 T'}{\partial \psi^2} \quad (10)$$

A. Constant Tube-Wall Temperature

a. Single elliptical rod.

The solution to Equation (10) which corresponds to a constant temperature, T'_0 , on the surface of the elliptical rod is given (2) as,

$$q''(\phi) = (\rho C_v V k / \pi \phi)^{1/2} T'_0 \quad (11)$$

hence,

$$h''(\phi) = (\rho C_v V k / \pi \phi)^{1/2} \quad (12)$$

Since the above heat transfer coefficient, $h''(\phi)$, is based upon a unit increment ϕ on the surface of a flat plate, it is necessary to convert it to one based upon a unit increment on the surface of the ellipse. Let $h'(\eta)$ be the heat transfer coefficient based upon a unit area on the surface of ellipse in elliptical coordinates, then

$$h'(\eta) = h''(\phi) \frac{d\phi}{ds'} \quad (13)$$

The complex potential for flow around a single elliptical cylinder is given in terms of elliptical coordinates as

$$w = \Phi + i\Psi = V(a+b) \cosh (\zeta - \xi_0) \quad (14)$$

where

$$\zeta = \xi + i\eta .$$

From Equation (14), the potential function can be readily found as

$$\Phi = V(a+b) \cosh (\xi - \xi_0) \cos \eta$$

On the surface of an ellipse, $\xi = \xi_0$, therefore, the distribution of hydrodynamic potential is given by

$$\Phi = V(a+b) \cos \eta \quad (15)$$

For the present purpose, the potential function around a single elliptical cylinder is written as

$$\phi = (a+b)(1 - \cos \eta) \quad (16)$$

The arc length, s' , along the ellipse, $\xi = \xi_0$, can be expressed in elliptical coordinates as $s' = \xi_0 \eta$. Therefore, $d\phi/ds' = (a+b) \sin \eta / \xi_0$, and from Equation (13),

$$h'(\eta) = \frac{\sqrt{a+b}}{\xi_0} \sqrt{\frac{\rho C_v V k}{\pi}} (1 + \cos \eta)^{1/2} \quad (17)$$

In order to obtain the expression for the Nusselt number, it is necessary to convert h' to rectangular coordinates. The two types of coordinate systems are related by

$$x = c \cosh \xi \cos \eta \quad (18)$$

$$\text{and} \quad y = c \sinh \xi \sin \eta \quad (19)$$

On the surface of an ellipse, $\xi = \xi_0$, therefore,

$$x = c \cosh \xi_0 \cos \eta = a \cos \eta$$

$$y = c \sinh \xi_0 \sin \eta = b \sin \eta$$

The incremental distance, ds , on the surface of an ellipse in rectangular coordinates is

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{a^2 \sin^2 \eta + b^2 \cos^2 \eta} d\eta \quad (20)$$

Let ds' be the incremental distance on the surface of the ellipse in elliptical coordinates, then

$$h(\eta) = h'(\eta) \frac{ds'}{ds} = h'(\eta) \xi_0 \frac{d\eta}{ds} \quad (21)$$

Hence, combining Equations (17), (20), and (21), gives

$$h(\eta) = \sqrt{a+b} \sqrt{\frac{\rho C_v k}{\pi}} \left(\frac{1 + \cos \eta}{a^2 \sin^2 \eta + b^2 \cos^2 \eta} \right)^{1/2} \quad (22)$$

Accordingly, the expression for the local Nusselt number, Nu_L , can be written as

$$Nu_L(\eta) = \frac{2bh}{k} = \sqrt{\frac{2b(a+b)}{\pi}} (Pe)^{1/2} \left(\frac{1 + \cos \eta}{a^2 \sin^2 \eta + b^2 \cos^2 \eta} \right)^{1/2} \quad (23)$$

If "a" and "b," the major and minor axes of an ellipse, are written in terms of the eccentricity, "e," then, since $e = c/a = \sqrt{a^2 - b^2}/a$, Equation (23) can

be written in equivalent form:

$$\text{Nu}_L(\eta) = \sqrt{2/\pi} \sqrt{\frac{(1-e^2) + \sqrt{1-e^2}}{1-e^2 \cos^2 \eta}} (\text{Pe})^{1/2} (1+\cos \eta)^{1/2} \quad (23)'$$

In Figure 1 the local Nusselt number as calculated from Equation (23)' is plotted against the angle measured from the forward stagnation point for a Peclet number of 500. Similar plots are shown in Figure 2 for the case of $e = 0.866$ ($a:b = 2:1$).

To obtain the average Nusselt number, Nu_D , the average heat transfer coefficient in elliptic cylindrical coordinates is first obtained. Thus,

$$\begin{aligned} \bar{h}' &= \frac{\sqrt{a+b}}{\pi \xi_0} \sqrt{\frac{C_v V k \rho}{\pi}} \int_0^\pi (1+\cos \eta)^{1/2} d\eta \\ &= \frac{2 \sqrt{2(a+b)}}{\pi \xi_0} \sqrt{\frac{C_v V k \rho}{\pi}} \end{aligned} \quad (24)$$

To convert the above expression to one in rectangular coordinates, it is noted that

$$\pi \xi_0 \bar{h}' = \pi \bar{h} \sqrt{\frac{a^2 + b^2}{2}}$$

Accordingly,

$$\bar{h} = \frac{4}{\pi} \sqrt{\frac{a+b}{a^2+b^2}} (\rho C_v V k / \pi)^{1/2} \quad (25)$$

and the average Nusselt number, Nu_D , becomes

$$Nu_D = \frac{2b\bar{h}}{k} = 0.718 \sqrt{\frac{2b(a+b)}{a^2+b^2}} (Pe)^{1/2} \quad (26)$$

If $a = b$, the above equation reduces to

$$Nu_D = 1.015 (Pe)^{1/2} \quad (27)$$

which is the Nusselt number for the flow around a circular cylinder (3). If eccentricity, "e," is used, Equation (26) can be written in an alternative form

$$Nu_D = 1.015 \sqrt{\frac{\sqrt{1-e^2} + (1-e^2)}{2-e^2}} (Pe)^{1/2} \quad (28)$$

A circle is an ellipse of zero eccentricity. It is apparent that Equation (28) reduces to Equation (26) if "e" is zero.

b. An elliptical rod located in the interior of an elliptical-rod bundle.

From Equation (11), the rate of heat flow from the entire surface of the elliptical rod can be obtained as:

$$q' = 2 \int_0^{\phi_1} q''(\phi) d\phi = 4T'_o \left(\frac{C_v V \rho k}{\pi} \right)^{1/2} \phi_1^{1/2}$$

The heat transfer coefficient is then given by

$$h_D = \frac{q'}{2\pi T_o \sqrt{\frac{a^2+b^2}{2}}} = \frac{2\sqrt{2}}{\pi^{3/2}} \sqrt{\frac{\phi_1}{a^2+b^2}} \sqrt{kC_v V \rho} \quad (29)$$

and consequently, the Nusselt number, Nu_D , becomes:

$$Nu_D = 0.508 \left(\frac{\phi_1}{a+b}\right)^{1/2} \sqrt{\frac{2b(a+b)}{a^2+b^2}} (Pe)^{1/2} \quad (30)$$

If the Peclet number is based upon the velocity of fluid flowing through the minimum flowing area, Equation (30) can also be written as:

$$Nu_D = 0.718 \left(\frac{\phi_1}{a+b}\right)^{1/2} F(e) (Pe)_{V_{\max}}^{1/2} (V/V_{\max})^{1/2} \quad (31)$$

where

$$F(e) = \sqrt{\frac{\sqrt{1-e^2} + (1-e^2)}{2-e^2}} \quad (32)$$

Throughout this paper, $F(e)$ will be used as an abbreviation to designate the above term containing "e." The plot of $F(e)$ against "e" is shown in Figure 3.

B. Constant Surface Heat Flux

a. Single elliptical rod.

(i) Nusselt number, Nu_D . For a constant surface heat flux, q'' , from the surface of an ellipse, the expression for the heat flux as a function of ϕ in (ϕ, ψ) coordinates can be written as:

$$q''(\phi) = q'' \frac{ds}{d\phi} = \frac{\xi_0 q''}{\sqrt{\phi \{2(a+b) - \phi\}}}$$

The solution of Equation (10), corresponding to the case where the surface heat flux is $q''(\phi)$, is given as (2)

$$\begin{aligned} T'(\eta) &= \frac{\xi_0 q''}{k} \sqrt{\frac{\kappa}{V\pi}} \int_0^\phi \frac{d\delta}{\sqrt{\delta \{2(a+b) - \phi + \delta\} (\phi - \delta)}} \\ &= \frac{q'' \xi_0}{k} \sqrt{\frac{\kappa}{\pi V}} \sqrt{\frac{2}{a+b}} K \left(\sqrt{\frac{1 - \cos \eta}{2}} \right) \end{aligned} \quad (33)$$

where K denotes an elliptic integral of the first kind. The local heat transfer coefficient, $h'(\eta)$, in elliptical coordinates is therefore:

$$h'(\eta) = \frac{q''}{T'} = \frac{k}{\xi_0} \sqrt{\frac{\pi V}{\kappa}} \sqrt{\frac{a+b}{2}} \frac{1}{K \left(\sqrt{\frac{1 - \cos \eta}{2}} \right)}$$

Transforming to rectangular coordinates:

$$h(\eta) = h'(\eta) \frac{ds'}{ds} = \frac{k}{\sqrt{a^2 \sin^2 \eta + b^2 \cos^2 \eta}} \sqrt{\frac{\pi V}{\kappa}} \sqrt{\frac{a+b}{2}} \frac{1}{K\left(\sqrt{\frac{1-\cos \eta}{2}}\right)} \quad (34)$$

The local Nusselt number then becomes:

$$Nu_D = \frac{2bh}{k} = \sqrt{\frac{\pi}{2}} \left(\frac{2b(a+b)}{a^2 \sin^2 \eta + b^2 \cos^2 \eta} \right)^{1/2} (Pe)^{1/2} \frac{1}{K\left(\sqrt{\frac{1-\cos \eta}{2}}\right)} \quad (35)$$

To obtain the average Nusselt number, the average heat transfer coefficient in elliptical coordinates is first obtained. Thus,

$$\bar{h}' = \frac{k}{\pi \xi_0} \sqrt{\frac{a+b}{2}} \sqrt{\frac{\pi C_v \rho V}{k}} \int_0^\pi \frac{d\eta}{K\left(\sqrt{\frac{1-\cos \eta}{2}}\right)}$$

The average heat transfer coefficient in rectangular coordinates then becomes:

$$\bar{h} = \frac{k}{\pi} \sqrt{\frac{\pi C_v \rho V}{k}} \sqrt{\frac{a+b}{2}} \int_0^\pi \frac{d\eta}{K\left(\sqrt{\frac{1-\cos \eta}{2}}\right)}$$

Hence:

$$\text{Nu}_D = \frac{2b\bar{h}}{k} = \frac{1}{\sqrt{\pi}} (\text{Pe})^{1/2} \sqrt{\frac{2b(a+b)}{a^2+b^2}} \int_0^\pi \frac{d\eta}{K\left(\sqrt{\frac{1-\cos \eta}{2}}\right)} \quad (36)$$

After the integral in Equation (36) is graphically evaluated, the Nusselt number can be finally written as:

$$\text{Nu}_D = 0.9474 \sqrt{\frac{2b(a+b)}{a^2+b^2}} (\text{Pe})^{1/2} \quad (37)$$

or

$$\text{Nu}_D = 1.34 F(e) (\text{Pe})^{1/2} \quad (38)$$

(ii) Nusselt number, Nu_t . The average temperature excess is first obtained as:

$$\bar{T}' = \frac{q'' \xi_0}{\pi k} \sqrt{\frac{\kappa}{\pi V}} \sqrt{\frac{2}{a+b}} \int_0^\pi K\left(\sqrt{\frac{1-\cos \eta}{2}}\right) d\eta \quad (39)$$

from which the heat transfer coefficient, h'_t , becomes:

$$h'_t = \frac{\pi k}{\xi_0} \sqrt{\frac{\pi V}{\kappa}} \sqrt{\frac{a+b}{2}} \int_0^\pi K\left(\sqrt{\frac{1-\cos \eta}{2}}\right) d\eta$$

Transforming to rectangular coordinates and then combining with the expression for Nusselt number, yields:

$$\text{Nu}_t = \frac{2bh_t}{k} = \pi^{3/2} \sqrt{\frac{2b(a+b)}{a^2 + b^2}} (\text{Pe})^{1/2} \left/ \int_0^\pi K\left(\sqrt{\frac{1-\cos \eta}{2}}\right) d\eta \right. \quad (40)$$

After graphically integrating the integral in the denominator, Eq. (40) can finally be written as:

$$\text{Nu}_t = 0.81(\text{Pe})^{1/2} \sqrt{\frac{2b(a+b)}{a^2 + b^2}} \quad (41)$$

or

$$\text{Nu}_t = 1.145 F(e)(\text{Pe})^{1/2} \quad (42)$$

In Figure 4 the plots of Nu_t against Peclet numbers are shown for different eccentricities.

b. A rod in a bundle.

Assumption vi leads to the expression for the hydrodynamic potential distribution around a single rod located inside a bundle,

$$\phi = \frac{\phi_1}{2(a+b)} (a+b)(1-\cos \eta) = \frac{\phi_1}{2} (1-\cos \eta) \quad (43)$$

If this expression is used, the local surface temperature excess in elliptical coordinates can be obtained as:

$$T'(\eta) = \frac{2\xi_o q''}{k\sqrt{\phi_1}} (\kappa/\pi V)^{1/2} K\left(\sqrt{\frac{1-\cos \eta}{2}}\right) \quad (44)$$

From Equation (44), the two types of Nusselt number can be derived in exactly the same manner as before. The final results are:

$$\begin{aligned} \text{Nu}_D &= 0.669 \left(\frac{\phi_1}{a+b} \right)^{1/2} \sqrt{\frac{2b(a+b)}{a^2+b^2}} (\text{Pe})^{1/2} \\ &= 0.946 \left(\frac{\phi_1}{a+b} \right)^{1/2} F(e) (\text{Pe})_{V_{\max}}^{1/2} (V/V_{\max})^{1/2} \end{aligned} \quad (45)$$

$$\begin{aligned} \text{Nu}_t &= 0.573 \left(\frac{\phi_1}{a+b} \right)^{1/2} \sqrt{\frac{2b(a+b)}{a^2+b^2}} (\text{Pe})^{1/2} \\ &= 0.81 \left(\frac{\phi_1}{a+b} \right)^{1/2} F(e) (\text{Pe})_{V_{\max}}^{1/2} (V/V_{\max})^{1/2} \end{aligned} \quad (46)$$

C. Cosine Tube-Wall Temperature Distribution

In the previous analysis for cross flow of liquid metal through circular rod bundles (5), it was pointed out that the assumption of cosine tube-wall temperature distribution led to Nusselt numbers which agreed well with experimental results. To the author's knowledge, there have been no experimental observations reported concerning the circumferential distribution of tube-wall temperature around an elliptical rod located in a bundle, in liquid metal heat transfer. The following derivations are based

upon the assumption that the circumferential distribution of both the tube-wall temperature and the hydrodynamic potential on the surface of an elliptical rod located in the interior of a bundle are of the cosine type in terms of elliptical coordinates.

Under these assumptions, the distribution of hydrodynamic potential can be expressed by Equation (43), and the excess tube-wall temperature distribution can be written as:

$$T' = \theta_1 (1 - \cos \eta) \quad (47)$$

Combining Equations (43) and (47), gives:

$$T' = (2\theta_1 / \phi_1) \phi \quad (48)$$

It was shown in a previous study (5) that, for the above type of temperature distribution, the total rate of heat transfer and the average tube-wall temperature can be written, respectively, as:

$$q' = \frac{16\theta_1}{3} \sqrt{\frac{kC_v \rho V}{\pi}} \phi_1^{1/2} \quad (49)$$

and

$$T'_m = \theta_1 \quad (50)$$

The heat transfer coefficient for the ellipse can, therefore, be obtained as:

$$h_t = \frac{q'}{2\pi\theta_1 \sqrt{\frac{a^2+b^2}{2}}} = \frac{16\sqrt{2}}{6\pi \sqrt{a^2+b^2}} \sqrt{\frac{kC_v V \rho \phi_1}{\pi}} \quad (51)$$

This leads to the following expressions for Nusselt numbers:

$$Nu_t = \frac{2bh_t}{k} = 0.677(\phi_1/a+b)^{1/2} \sqrt{\frac{2b(a+b)}{a^2+b^2}} (Pe)^{1/2} \quad (52)$$

or

$$Nu_t = 0.958(\phi_1/a+b)^{1/2} F(e)(Pe)_{V_{\max}}^{1/2} (V/V_{\max})^{1/2} \quad (53)$$

To obtain the expression for the Nusselt number, Nu_D , based upon the average heat transfer coefficient, the average heat transfer coefficient in rectangular coordinates is obtained as:

$$\bar{h} = \frac{4\sqrt{2}}{\pi \sqrt{a^2+b^2}} \sqrt{\frac{kC_v V \rho \phi_1}{\pi}} \quad (54)$$

from which

$$Nu_D = 1.016(\phi_1/a+b)^{1/2} \sqrt{\frac{2b(a+b)}{a^2+b^2}} (Pe)^{1/2} \quad (55)$$

or

$$Nu_D = 1.437(\phi_1/a+b)^{1/2} F(e)(Pe)_{V_{\max}}^{1/2} (V/V_{\max})^{1/2} \quad (56)$$

It was pointed out earlier that the present analysis deals with the cross-flow of liquid metals normally through elliptical-rod bundles. As illustrated in Figure 5, oblique cross-flow through circular-tube bundles takes place in baffled heat exchangers. For this type of flow, the cross-sectional areas formed by the flowing fluid and the circular rods are ellipses. If the hydrodynamic patterns for the two types of flow, i.e., normal cross-flow through elliptic rods and oblique cross-flow through circular rods, can be assumed to be identical, the above analyses are valid for both types of flow. Under the assumption of identical flow patterns, the Nusselt numbers for liquid metals flowing at a 45-degree angle with circular rods can be expressed by the following equations. The cross-sectional area of flow, in this case, corresponds to an ellipse of eccentricity 0.707; thus, from Equations (31), (46), and (53):

(i) Constant tube-wall temperature:

$$Nu_D = 0.644(\phi_1/a+b)^{1/2}(\overline{Pe})_{V_{max}}^{1/2} (V/V_{max})^{1/2} \quad (57)$$

(ii) Constant surface heat flux:

$$Nu_t = 0.727(\phi_1/a+b)^{1/2}(\overline{Pe})_{V_{max}}^{1/2} (V/V_{max})^{1/2} \quad (58)$$

(iii) Cosine surface temperature:

$$Nu_t = 0.859(\phi_1/a+b)^{1/2}(\overline{Pe})_{V_{max}}^{1/2} (V/V_{max})^{1/2} \quad (59)$$

D. Theoretical Evaluation of the Parameter, $\phi_1/(a+b)$

In the expressions for the Nusselt number presented in the previous sections, a term, $\phi_1/(a+b)$, appears in each of the equations for rod bundles. This term represents the difference in hydrodynamic potential between the front and rear stagnation points of an elliptical rod located inside a bundle, in terms of elliptical coordinates. The analytical evaluation of this parameter will now be given by applying the principles of conformal transformation and by using conjugate functions.

In a previous paper (5), it was shown that the potential field around a circular cylinder located in the interior of a bundle could be calculated analytically, using the mathematical functions of Howland and McMullen (4). The potential field around the circular rod was found to be given by the expression:

$$\Phi = VR_o \left\{ \sum_{n=0}^{\infty} [A_{2n+1} \lambda^{-2n} \{ (R_o/r)^{2n+1} + (r/R_o)^{2n+1} \} \sin(2n+1)\theta] \right. \\ \left. - \sum_{n=1}^{\infty} [B_{2n} \lambda^{-2n+1} \{ (R_o/r)^{2n} + (r/R_o)^{2n} \} \cos 2n\theta] \right\} \quad (60)$$

where λ is $D/2P$, and the constants, A_{2n+1} and B_{2n} , are as given in the previous paper.

To calculate the potential field in terms of elliptical coordinates around an elliptical rod located inside a bundle, the following consecutive conformal transformations are made:

$$Z = \frac{1}{2} (z + \sqrt{z^2 - c^2}) \quad (61)$$

and

$$z = c \cosh \zeta = c \cosh(\xi + i\eta) \quad (62)$$

The first transformation maps the group of circular cylinders defined by Howland and McMullen (4) into a group of elliptical cylinders. The second transformation changes the variables from rectangular coordinates to elliptical cylindrical coordinates (6). From Equation (62) it can be readily seen that:

$$\sqrt{z^2 - c^2} = c \sinh \zeta \quad (63)$$

and accordingly:

$$z + \sqrt{z^2 - c^2} = c(\sinh \zeta + \cosh \zeta) = c e^{\zeta} \quad (64)$$

$$z - \sqrt{z^2 - c^2} = c(\cosh \zeta - \sinh \zeta) = c e^{-\zeta} \quad (65)$$

On an ellipse, $\xi = \xi_0$, the major and minor axes of the ellipse, "a" and "b," respectively, can be expressed by

$$a = c \cosh \xi_0, \quad b = c \sinh \xi_0 \quad (66)$$

Therefore:

$$a + b = c(\cosh \xi_0 + \sinh \xi_0) = c e^{\xi_0} \quad (67)$$

$$a - b = c(\cosh \xi_0 - \sinh \xi_0) = c e^{-\xi_0} \quad (68)$$

The relationship between the original variables and the final transformed variables can be found by noting that $Z = r e^{i\theta}$, and then combining this expression with Equations (61) and (64). Thus:

$$Z = r e^{i\theta} = \frac{1}{2} (z + \sqrt{z^2 - c^2}) = \frac{1}{2} c e^{\xi} = (c e^{\xi}/2) e^{i\zeta} \quad (69)$$

It is seen, therefore, that the two consecutive transformations cause the following transformations of variables:

$$r \rightarrow c e^{\xi}/2 \quad \theta \rightarrow \eta \quad (70)$$

It is also noted that the circles, $r = R_0$, in the original rectangular coordinates are mapped into ellipses, $\xi = \xi_0$. Therefore, from Equations (70) and (67), one can write:

$$R_0 \rightarrow c e^{\xi_0}/2 = (a+b)/2 \quad (71)$$

Since the solution to the Laplace equation,

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0, \quad (72)$$

with the appropriate boundary conditions, is known to be (4,5):

$$\begin{aligned}\Psi = VR_0 \left\{ \sum_{n=0}^{\infty} [A_{2n+1} \lambda^{-2n} \{ (R_0/r)^{2n+1} - (r/R_0)^{2n+1} \} \cos(2n+1)\theta] \right. \\ \left. + \sum_{n=1}^{\infty} [B_{2n} \lambda^{-2n+1} \{ (R_0/r)^{2n} - (r/R_0)^{2n} \} \sin 2n\theta] \right\} \quad (73)\end{aligned}$$

the stream function for the flow around the elliptical rods in elliptical coordinates can be written as:

$$\begin{aligned}\Psi = V(a+b) \left\{ \sum_{n=0}^{\infty} [A_{2n+1} \lambda^{-2n} \sinh(2n+1)(\xi_0 - \xi) \cos(2n+1)\eta] \right. \\ \left. + \sum_{n=1}^{\infty} [B_{2n} \lambda^{-2n+1} \sinh 2n(\xi_0 - \xi) \sin 2n\eta] \right\} \quad (74)\end{aligned}$$

where λ is now $(a+b)/2P$. From Equation (74), it can be readily seen that on the surfaces of the ellipses, $\xi = \xi_0$, $\Psi = 0$.

Similarly, the potential function for the flow around the elliptical rods can be found in terms of elliptical coordinates as:

$$\begin{aligned}\Phi = V(a+b) \left\{ \sum_{n=0}^{\infty} [A_{2n+1} \lambda^{-2n} \cosh(2n+1)(\xi - \xi_0) \sin(2n+1)\eta] \right. \\ \left. - \sum_{n=1}^{\infty} [B_{2n} \lambda^{-2n+1} \cosh 2n(\xi - \xi_0) \cos 2n\eta] \right\} \quad (75)\end{aligned}$$

The distribution of hydrodynamic potential on the surface of the elliptical rods can be obtained by letting $\xi = \xi_0$ in Equation (75). Thus:

$$\Phi_s = V(a+b) \left\{ \sum_{n=0}^{\infty} [A_{2n+1} \lambda^{-2n} \sin(2n+1)\eta] - \sum_{n=1}^{\infty} [B_{2n} \lambda^{-2n+1} \cos 2n\eta] \right\} \quad (76)$$

and accordingly the difference in hydrodynamic potential between the forward and rear stagnation points of the elliptical cylinder can be obtained by forming the difference of Φ_s at $\eta = \pi/2$ and $\eta = 3\pi/2$. Finally, this can be written as:

$$\phi_1/(a+b) = 2 \sum_{n=0}^{\infty} (-1)^n A_{2n+1} \lambda^{-2n} \quad (77)$$

From the above equation, it can be seen that the final mathematical form for the parameter is identical to that for flow around circular cylinders. The only differences are that the term ϕ_1/D is replaced by $\phi_1/(a+b)$, and λ now stands for $(a+b)/2P$.

In a previous paper (5), theoretical values of ϕ_1/D were presented as a function of λ . These values are, therefore, still useful for finding $\phi_1/(a+b)$, provided that the above changes are made. The plots of $\phi_1/(a+b)$ against $(a+b)/2P$ are shown in Figure 6, for two different tube-bank geometries, i.e., the centers of the ellipses are arranged in square and equilateral triangular fashions. For oblique cross-flow through circular-rod bundles, the distance between the centers of the ellipses will be stretched in the

direction of flow, as compared to the cases of square spacing or equilateral triangular spacing. For this reason, certain constants in the potential function must be modified. For a particular case of 45-degree oblique cross-flow through circular-rod bundles, the values of $\phi_1/(a+b)$ are calculated with the aid of an IBM 7094. The results are plotted in Figure 7.

In deriving the Nusselt numbers for rod bundles, it was assumed that the distribution of hydrodynamic potential along the surface of an elliptical rod located in the interior of rod bundles can be expressed by a cosine function, in terms of elliptical coordinates. This assumption can be analytically justified by using Equation (76). In a previous paper (5), similar justification was shown for flow through circular-rod bundles. Since Equation (76) has an identical form to the corresponding equation for flow around circular rods, Figures 8 and 9 in the aforementioned reference (5) can now be interpreted as the plots of $\phi_1/(a+b)$ against the angle measured from the forward stagnation point. It is thus seen that the said assumption is reasonable.

SUMMARY

(1) Analytical expressions for Nusselt numbers for liquid metals flowing past a single elliptical rod or through elliptical-rod bundles were obtained by assuming inviscid flow. The Nusselt numbers were expressed as a function of the eccentricity of the ellipse. It was found that the Nusselt numbers bear

constant relationship to those for cross-flow through circular-rod bundles.

(2) By assuming identical hydrodynamic flow patterns, the above-mentioned Nusselt numbers are also applicable to oblique cross-flow of liquid metals past a single circular rod or through circular-rod bundles. The Nusselt numbers for 45-degree oblique cross-flow through circular-rod bundles are shown.

(3) The parameter, $\phi_1/(a+b)$, which represents the hydrodynamic potential drop in terms of elliptical coordinates is analytically obtained by using conjugate functions. It was found that the numerical results of ϕ_1/D presented in the previous paper are still applicable, provided that ϕ_1/D is changed to $\phi_1/(a+b)$ and $D/2P$ to $(a+b)/2P$. For 45-degree oblique cross-flow through circular-rod bundles, the values of $\phi_1/(a+b)$ are calculated and plotted as a function of $(a+b)/2P$.

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Dr. O. E. Dwyer who suggested the problem and gave kind advice during the course of this study.

NOMENCLATURE

A_{2n+1}, B_{2n}	Coefficients in Equation (60)
C_v	Specific heat at constant volume, Btu/lb-°F
D	Diameter of a cylinder, ft
$F(e)$	As defined by Equation (32)
K	Elliptic function of the first kind
Nu_D	Over-all Nusselt number, $2bh_D/k$, dimensionless
Nu_L	Local Nusselt number, $2bh/k$, dimensionless
Nu_t	Over-all Nusselt number, $2bh_t/k$, dimensionless
P	Pitch, ft
Pe	Over-all Peclet number, $2b\rho C_v V/k$, dimensionless
$(Pe)_{V_{max}}$	Over-all Peclet number, $2b\rho C_v V_{max}/k$, dimensionless
R_o	Radius of a circle, ft
T	Temperature, °F
T'	Temperature excess, °F
T'_o	A constant temperature excess, °F
T_i	Uniform upstream temperature, °F
T'_m	Average temperature excess, °F
V	Uniform upstream fluid velocity, ft/sec

$V_{\max}$	Shell-side fluid velocity across tube bank and based on minimum flow area, ft/sec
Z	Complex function as defined by Equation (61)
a, b	Major and minor axis of an ellipse, ft
c	$= \sqrt{a^2 - b^2}$
e	Eccentricity of an ellipse, c/a
h	Local heat transfer coefficient in rectangular coordinates, Btu/ft ² -hr-°F
h'	Local heat transfer coefficient in elliptic coordinates, Btu/ft ² -hr-°F
h''	Local heat transfer coefficient in ϕ, ψ coordinates, Btu/ft ² -hr-°F
$\bar{h}'$	Average heat transfer coefficient in elliptic coordinates, Btu/ft ² -hr-°F
$\bar{h}$	Average heat transfer coefficient in rectangular coordinates, Btu/ft ² -hr-°F
h'_t	Over-all heat transfer coefficient based on a specified surface temperature in elliptic coordinates, Btu/ft ² -hr-°F
h_t	Over-all heat transfer coefficient based on a specified surface temperature in rectangular coordinates, Btu/ft ² -°F
i	$= \sqrt{-1}$
k	Thermal conductivity, Btu/ft ² -hr-°F

n	An integer
q'	Rate of heat flow per unit length of elliptic cylinder perpendicular to the direction of flow, Btu/ft-hr
q''	Surface heat flux, Btu/ft ² -hr
r	Radial distance, ft
s	Arc length of an ellipse in rectangular coordinates, ft
s'	Arc length of an ellipse in elliptic coordinates, ft
v_ξ, v_η	Velocity components in ξ and η directions
w	Complex potential, $\Phi + i\Psi$
x, y, z	Distance coordinates, ft
δ	A parameter
ζ	$\xi + i\eta$, Complex function
η	Imaginary part of the complex function ζ , or angle measured from the front stagnation point, degree
θ	Angle measured from the front stagnation point of a circle, degree
θ_1	Temperature excess at $\eta = \pi/2$
κ	Thermal diffusivity, ft ² /hr
λ	$(a+b)/2P$ or $D/2P$
ξ	Real part of the complex function
π	= 3.1416.....

ρ	Density of fluid, lb_m/ft^3
Φ	Hydrodynamic potential function
Φ_s	Hydrodynamic potential on the surface of an elliptical cylinder
ϕ_1	Normalized hydrodynamic potential at the rear stagnation point of an elliptical or circular cylinder, dimensionless
ϕ	Normalized hydrodynamic potential, $\Phi/-V$
Ψ	Hydrodynamic stream function
ψ	Normalized hydrodynamic stream function, $\Psi/-V$

REFERENCES

1. M. J. Boussinesq, Calcul du pouvoir refroidissant des courants fluids. Journal de Mathematique 1, 285 (1905).
2. H. S. Carslaw and J. C. Jeager, Conduction of heat in solids. Oxford University Press, London (1958).
3. R. J. Grosh and R. D. Cess, Heat transfer to fluids with low Prandtl numbers for flow across plates and cylinders of various cross section. Trans. ASME 80, 667 (1958).
4. R. C. J. Howland and B. W. McMullen, Potential functions related to groups of circular cylinders. Cambridge Phil. Soc. 32, 402 (1936).
5. C. J. Hsu, Analytical study of heat transfer to liquid metals in cross-flow through rod bundles. BNL 7053, Submitted to International Jour. of Heat and Mass Transfer, April 1963.
6. L. M. Milne-Thomson, Theoretical hydrodynamics. The MacMillan Company (1957).

FIGURE CAPTIONS

- Figure 1 Local Nusselt number as a function of η , the angle measured from the forward stagnation point. (Constant surface temperature, $Pe = 500$.)
- Figure 2 Local Nusselt number as a function of η . (Constant surface temperature, $a:b = 2:1$.)
- Figure 3 A plot of $F(e)$ versus eccentricity, "e."
- Figure 4 Plots of average Nusselt number, Nu_t , as a function of Peclet number (constant surface heat flux).
- Figure 5 Schematic diagram showing the direction of fluid flow inside a baffled heat exchanger.
- Figure 6 Plots of normalized hydrodynamic potential drop, $\phi_1/(a+b)$, versus $(a+b)/2P$.
- Figure 7 Plots of $\phi_1/(a+b)$ versus $(a+b)/2P$ for 45-degree oblique cross-flow through circular rod bundles.

33

34

C. J. Hsu International Journal of Heat and Mass Transfer Fig. 3

C. J. Hsu International Journal of Heat and Mass Transfer Fig. 4

C. J. Hsu International Journal of Heat and Mass Transfer Fig. 5