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REQUIRED BY THE UNITARITY CONDITION

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THE RESTRICTIONS ON THE ASYMPTOTIC PARAMETERS
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SUMMARY

Making the analytic continuation of the unitarity condition with respect to $\cos\theta$, the relations are derived which connect the asymptotic parameters of large $|\cos\theta|$ and the spectral function of the crossed channel. As the result of these relations, we obtain the threshold behavior of these parameters. Assuming the asymptotic power behavior, we find that there are only a few possibilities for the combination of the asymptotic power series that are allowed by the unitarity condition in the vicinity of the threshold. One of these is the same as the asymptotic form of the Legendre function. Sum rules that relate the total cross section and the asymptotic parameters are derived.

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I. Introduction

In the potential scattering case, Regge¹ showed the meromorphic property of the function $\phi(s, \ell)$ on $\text{Re } \ell > -\frac{1}{2}$ and the exponential damping for large $|\ell|$, to the special class of potentials including the superposition of the Yukawa potential. Using the Sommerfeld-Watson integral, we can separate the scattering amplitude $\phi(s, \cos\theta)$ into the Regge pole terms and the background integral and the former dominate the asymptotic behavior of the large value of $\cos\theta$. But in the more general case of scattering, unfortunately there is no equation that corresponds to the Schrödinger equation, so it is not clear whether we can prove the meromorphic property and the damping for the large value of $|\ell|$ of the function $\phi(s, \ell)$, at least for the small value of $|\text{Re } \ell|$. And so the asymptotic behavior for the large $|\cos\theta|$ is not clear.

On the other hand, the conjecture,^{2,3,4} that the Regge's mechanism works in the general scattering case and also for $s < 0$, succeeded in analyzing⁵ the high energy experiments of the crossed channel. So it is interesting to examine up to what point we can reproduce the Regge's asymptotic behavior for large $|\cos\theta|$ without using the ℓ -plane.

In this note, we study what kind of restriction on the asymptotic behavior is required by the unitarity condition. For this purpose, in section II, we make the analytic continuation of the unitarity condition with respect to $\cos\theta$, fixing s on the real value. In this paper we assume the asymptotic power behavior $\sum_i \beta_i(s) (\cos\theta)^{\alpha_i(s)}$, and we will see that this power behavior is consistent with the unitarity condition. Comparing the coefficient

of $(\cos\theta)^{\alpha_1(s)}$ we obtain sum rules relating the asymptotic parameters and the spectral function of the crossed channel. In section III, using the equation that is derived in the previous section, the threshold behavior of the asymptotic parameters $\alpha_1(s)$ are examined, and we get the same result derived by Barut and Zwanziger⁶ for the case that $(\alpha(4m^2) + \frac{1}{2})$ is not integer, for the integer case, it is the logarithmic type. In section IV, we study what kind of 'grouping' is necessary for the asymptotic power series in order to be consistent with the unitarity in the vicinity of the threshold.

If we write $\phi(s, \cos\theta) \sim \sum_{\beta} \{ (\cos\theta)^{\alpha} + C_1(\cos\theta)^{\alpha-2} + C_2(\cos\theta)^{\alpha-4} + \dots \}$ for large $|\cos\theta|$, then we will find that there is only 2^j possibility of choice for C_j allowed by the unitarity condition. One of the choices in this possibility is the same as the asymptotic expansion of the Legendre function $P_{\alpha}(\cos\theta)$. In section V, sum rules are derived that relate the total cross section with the asymptotic parameter.

II. Analytic Continuation of the Unitarity Condition with respect to $\cos\theta$.

In this section, we shall derive a sum rule which relates the parameters of large momentum transfer variables and the spectral function of the crossed channel, making the analytic continuation of the unitarity condition. For simplicity, we treat the equal mass boson-boson scattering without isospin. But the generalization to the more interesting case will be obvious. Let us start from the usual unitarity condition,⁷ which is true for $-1 \leq x \leq 1$:

$$\Phi_{(s,x)}^{(1)} - \Phi_{(s,x)}^{(2)} = \frac{i}{4} \sqrt{\frac{s-4m^2}{s}} \cdot \int_{-1}^1 dx' \int_{-1}^1 dx'' \Phi_{(s,x')}^{(1)} \Phi_{(s,x'')}^{(2)} \frac{\theta(-k(x; x', x''))}{[-k(x; x', x'')]^{\frac{1}{2}}} \quad (1)$$

where

$$k(x; x', x'') = -1 + x^2 + x'^2 + x''^2 - 2xx'x'' ,$$

$x = \cos\theta$ etc.

s : total energy squared in c.m. system.

$\phi^{(1)}(s,x)$ is the scattering amplitude and $\phi^{(2)}(s,x)$ is the analytically continued function on the s -plane. In particular, for the elastic value of s : $s_0 \leq s < s_1$,

$$\Phi_{(s,x)}^{(2)} = \left(\Phi_{(s,x^*)}^{(1)} \right)^* \quad (2)$$

In this section we restrict ourselves to the elastic region of s , but in section V we shall use the eq.(1) for $0 < s \leq s_0$.

The integration domain of the eq.(1) is inside of the ellipse of the $x'-x''$ plane:

$$-k(x; x', x'') \geq 0 \quad (3)$$

Changing the integration variables such that the domain of the integration is inside of a unit circle, we get

$$\begin{aligned} \Phi_{(s, x)}^{(1)} - \Phi_{(s, x)}^{(2)} &= \frac{i}{4} \sqrt{\frac{s-4m^2}{s}} \\ &\cdot \left(-\frac{1}{2}\right) \int_0^1 dr \int_0^{2\pi} d\varphi \frac{r}{\sqrt{1-r^2}} \Phi_{(s, x')}^{(1)} \Phi_{(s, x'')}^{(2)} \end{aligned} \quad (4)$$

and

$$\begin{aligned} x' &= \frac{r}{\sqrt{2}} \left\{ (1+x)^{\frac{1}{2}} \cos \varphi + (1-x)^{\frac{1}{2}} \sin \varphi \right\} \\ x'' &= \frac{r}{\sqrt{2}} \left\{ (1+x)^{\frac{1}{2}} \cos \varphi - (1-x)^{\frac{1}{2}} \sin \varphi \right\} \end{aligned} \quad (5)$$

For the fixed value of s in the elastic region, eq.(1) is first written down only for the physical value of x , i.e. $-1 \leq x \leq 1$, but since it contains infinitely many points with at least one accumulation point, we can make the analytic continuation to the complex x plane. When we make the value of $|x|$ large;

$$x = A e^{i\phi}, \quad (\phi \neq 0, \pi), \quad A \rightarrow \infty \quad (6)$$

the resulting equation does not depend on the direction of the analytic continuation, i.e. $\emptyset$; therefore we choose $\emptyset = \frac{\pi}{2}$ for simplicity of calculation. As $\emptyset^{(1)}(s, x')$ and $\emptyset^{(2)}(s, x'')$ have cuts on the x' and x'' complex plane, when we make A large, we have to bend the contour of the integration of eq.(4). For fixed value of r in $0 \leq r \leq 1$, the bent contour C of φ is shown in fig. 1. If we assume the power behavior of $\emptyset^{(1)}(s, x)$ and $\emptyset^{(2)}(s, x)$ for large $|x|$:

$$\begin{aligned} \Phi^{(1)}(s, x) &\sim \sum_{\text{Re } \alpha_1(s) > -1} \beta_1(s) \{ (-x)^{\alpha_1(s)} + (\pm 1)_1 x^{\alpha_1(s)} \} + O\left(\frac{1}{|x|}\right) \\ \Phi^{(2)}(s, x) &\sim \sum_{\text{Re } \alpha_2(s) > -1} \beta_2(s) \{ (-x)^{\alpha_2(s)} + (\pm 1)_2 x^{\alpha_2(s)} \} + O\left(\frac{1}{|x|}\right) \end{aligned} \quad (7)$$

The right hand side of the unitarity condition (4) becomes

$$\begin{aligned} & -\frac{1}{8} \sqrt{\frac{s-4m^2}{s}} \left(\beta_2 (e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2}) \cdot A^{\alpha_2} \left[I_2 + \frac{\sqrt{\pi}}{2} \beta_1 (e^{-i\pi\alpha_1} - e^{+i\pi\alpha_1}) \frac{1}{\alpha_2 - \alpha_1} \left(\frac{1}{2}\right)^{\alpha_2} \right. \right. \\ & \cdot \frac{\Gamma(\alpha_2 + 1)}{\Gamma(\alpha_2 + \frac{3}{2})} \cdot \left\{ (\chi'_{L1})^{\alpha_1 - \alpha_2} + (\pm 1)_1 (\pm 1)_2 (-\chi'_{L5})^{\alpha_1 - \alpha_2} \right\} \\ & + \beta_1 (e^{-i\frac{\pi}{2}\alpha_1} + (\pm 1)_1 e^{+i\frac{\pi}{2}\alpha_1}) \cdot A^{\alpha_1} \left[I_1 + \frac{\sqrt{\pi}}{2} \beta_2 (e^{-i\pi\alpha_2} - e^{+i\pi\alpha_2}) \frac{1}{\alpha_1 - \alpha_2} \left(\frac{1}{2}\right)^{\alpha_1} \right. \\ & \cdot \left. \left. \frac{\Gamma(\alpha_1 + 1)}{\Gamma(\alpha_1 + \frac{3}{2})} \cdot \left\{ (\chi''_{L7})^{\alpha_2 - \alpha_1} + (\pm 1)_1 (\pm 1)_2 (-\chi''_{L3})^{\alpha_2 - \alpha_1} \right\} \right] \right) \end{aligned}$$

where

(8)

The detailed derivation is given in the Appendix.

$$I_1 = \int_0^1 dr \frac{r}{\sqrt{1-r^2}} \left[\int_{x''_{L7}}^{\infty} dx'' \frac{[x'' - \sqrt{x''^2 - r^2}]^{\alpha_1}}{\sqrt{x''^2 - r^2}} (\hat{\Phi}_{(s, x''+i\eta)}^{(2)} - \hat{\Phi}_{(s, x''-i\eta)}^{(2)}) \right. \\ \left. - (\pm 1)_1 \int_{-\infty}^{x''_{L3}} dx'' \frac{[-x'' - \sqrt{x''^2 - r^2}]^{\alpha_1}}{\sqrt{x''^2 - r^2}} (\hat{\Phi}_{(s, x''+i\eta)}^{(2)} - \hat{\Phi}_{(s, x''-i\eta)}^{(2)}) \right] \quad (9)$$

$$I_2 = \int_0^1 dr \frac{r}{\sqrt{1-r^2}} \left[\int_{x'_{L1}}^{\infty} dx' \frac{[x' - \sqrt{x'^2 - r^2}]^{\alpha_2}}{\sqrt{x'^2 - r^2}} (\hat{\Phi}_{(s, x'+i\eta)}^{(1)} - \hat{\Phi}_{(s, x'-i\eta)}^{(1)}) \right. \\ \left. - (\pm 1)_2 \int_{-\infty}^{x'_{L5}} dx' \frac{[-x' - \sqrt{x'^2 - r^2}]^{\alpha_2}}{\sqrt{x'^2 - r^2}} (\hat{\Phi}_{(s, x'+i\eta)}^{(1)} - \hat{\Phi}_{(s, x'-i\eta)}^{(1)}) \right] \quad (10)$$

and ϕ 's are defined in (A5), (A5') and (A5''), (A5'''). For simplicity, we wrote down only the highest terms of A. The leading terms of the left-hand side of the unitarity condition (4) are:

$$-\beta_2 (e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2}) A^{\alpha_2} + \beta_1 (e^{-i\frac{\pi}{2}\alpha_1} + (\pm 1)_1 e^{+i\frac{\pi}{2}\alpha_1}) A^{\alpha_1} \quad (11)$$

In comparing (8) and (11), we can see the A-dependency of the analytically continued unitarity condition is quite consistent. By equating the coefficients, final results are obtained.

$$1 = \frac{1}{8} \sqrt{\frac{s-4m^2}{s}} \left[I_2 + \sqrt{\pi} \beta_1 \sin \pi \alpha_1 \cdot \frac{1}{i(\alpha_2 - \alpha_1)} \right. \\ \left. \cdot \left(\frac{1}{2}\right)^{\alpha_2} \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2+\frac{3}{2})} \cdot \left\{ (x'_{L1})^{\alpha_1-\alpha_2} + (\pm 1)_1 (\pm 1)_2 (-x'_{L5})^{\alpha_1-\alpha_2} \right\} \right] \quad (12)$$

and

$$1 = \frac{1}{8} \sqrt{\frac{s-4m^2}{s}} \left[-I_1 + \sqrt{\pi} \beta_2 \sin \pi \alpha_2 \cdot \frac{1}{i(\alpha_2 - \alpha_1)} \cdot \left(\frac{1}{2}\right)^{\alpha_1} \cdot \frac{\Gamma(\alpha_1 + 1)}{\Gamma(\alpha_2 + \frac{3}{2})} \left\{ (\chi_{L\eta}'')^{\alpha_2 - \alpha_1} + (\pm 1)_1 (\pm 1)_2 (-\chi_{L3}'')^{\alpha_2 - \alpha_1} \right\} \right]_{(13)}$$

where $\Gamma(z)$ is the usual gamma function. In our case where $s_0 \leq s < s_1$, eq. (12) and eq. (13) are complex conjugates of each other, but for $s < s_0$ these two equations are independent. We only treated the leading power of A . However, A^α dependency of the lower powers of the unitarity condition is consistent, and we can easily obtain similar but independent equations as eq.(12) and eq.(13) by comparing the coefficients of A^α for $\alpha > -1$. In the following section we shall examine the implication of these equations.

III. The threshold behavior of the parameter α .

In order to study the threshold behavior of the asymptotic parameters $\alpha(s)$ and $\beta(s)$, let us change the variable x' , x'' to the standard variable t , u of the Mandelstam representation, in eq.(12) and (10).

$$\text{As } t = -2q^2 (1 - x)$$

$$u = -2q^2 (1 + x)$$

(14)

and

$$s = 4(q^2 + m^2) ,$$

for small q ,

$$\begin{aligned}
 I_2 &= \int_0^1 dr \frac{r^{2\alpha_2+1}}{\sqrt{1-r^2}} \left(\frac{1}{2}\right)^{\alpha_2} \cdot \left\{ \int_{t'_{L1}}^{+\infty} dt' (2q^2)^{\alpha_2} \cdot t'^{-\alpha_2-1} (\hat{\Phi}_{(s,t'+i\eta)}^{(1)} - \hat{\Phi}_{(s,t'-i\eta)}^{(1)}) \right. \\
 &\quad \left. + (\pm 1)_2 \cdot \int_{u'_{L5}}^{+\infty} du' (2q^2)^{\alpha_2} u'^{-\alpha_2-1} (\hat{\Phi}_{(s,u'+i\eta)}^{(1)} - \hat{\Phi}_{(s,u'-i\eta)}^{(1)}) \right\} \\
 &= \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2+\frac{3}{2})} \cdot q^{2\alpha_2} \cdot \left\{ \int_{t'_{L1}}^{+\infty} dt' \frac{(\hat{\Phi}_{(s,t'+i\eta)}^{(1)} - \hat{\Phi}_{(s,t'-i\eta)}^{(1)})}{t'^{\alpha_2+1}} \right. \\
 &\quad \left. + (\pm 1)_2 \cdot \int_{u'_{L5}}^{+\infty} du' \frac{(\hat{\Phi}_{(s,u'+i\eta)}^{(1)} - \hat{\Phi}_{(s,u'-i\eta)}^{(1)})}{u'^{\alpha_2+1}} \right\}
 \end{aligned} \tag{15}$$

where t'_{L1} and u'_{L5} are the lower limit of the spectral function of $\phi^{(1)}$. As the integral converges because of (A11), the threshold behavior of I_2 is (const.) $\cdot q^{2\alpha_2}$. Equation (12) becomes:

$$1 = O(q^{2\alpha_2+1}) + \frac{1}{s} \frac{q}{m} \sqrt{\pi} \beta_1 \sin \pi \alpha_1 \cdot \left(\frac{1}{2}\right)^{\alpha_2-1} \cdot \frac{\Gamma(\alpha_2+1)}{\Gamma(\alpha_2+\frac{3}{2})} \cdot \frac{1}{i(\alpha_2-\alpha_1)} \tag{16}$$

If we assume from the Regge theory that the asymptotic form of $\phi^{(1)}(s, x')$ is determined by the poles of the $\phi^{(1)}(s, \ell')$ -- the analytically continued partial wave amplitude -- and $q^{-2\ell'} \cdot \phi^{(1)}(s, \ell')$ gives constant threshold behavior, we can easily obtain the threshold behavior of β_1 ,

$$\beta_1(s) = q^{2\alpha_1(s)} \cdot b_1(s) \tag{17}$$

where $b(s)$ is constant around the threshold. Therefore eq.(16) becomes, for $\alpha_2(s) > -\frac{1}{2}$,

$$i(\alpha_2 - \alpha_1) = C \cdot (s - 4m^2)^{\alpha_1(4m^2) + \frac{1}{2}} \quad (18)$$

where

$$C = \frac{1}{8m} \sqrt{\pi} b_1(4m^2) \cdot \sin \pi \alpha_1(4m^2) \cdot \left(\frac{1}{2}\right)^{3\alpha_2(4m^2)} \cdot \frac{\Gamma(\alpha_2(4m^2) + 1)}{\Gamma(\alpha_2(4m^2) + \frac{3}{2})} \quad (19)$$

If we define an analytic function $\alpha(s)$ on the s -plane with cut from $s = 4m^2$ to $s = +\infty$, and $\alpha(s + i\eta) = \alpha_1(s)$, $\alpha(s - i\eta) = \alpha_2(s)$ for real s , then for $\alpha_1(4m^2) + \frac{1}{2}$ is not integer

$$\alpha(s) = C' (4m^2 - s)^{\alpha_1(4m^2) + \frac{1}{2}} \quad (20)$$

and C' is determined by

$$\begin{aligned} & \alpha(s+i\eta) - \alpha(s-i\eta) \\ &= C' \left\{ e^{-i\pi \left\{ \alpha_1(4m^2) + \frac{1}{2} \right\}} - e^{+i\pi \left\{ \alpha_1(4m^2) + \frac{1}{2} \right\}} \right\} \cdot (s - 4m^2)^{\alpha_1(4m^2) + \frac{1}{2}} \\ &= -2i C' \cdot \cos \pi \alpha_1(4m^2) \cdot (s - 4m^2)^{\alpha_1(4m^2) + \frac{1}{2}} \end{aligned}$$

so,

$$C' = \frac{1}{2 \cos \pi \alpha_1(4m^2)} \cdot C \quad (21)$$

The threshold behavior of the $\text{Re } \alpha(s)$ is

$$\text{Re } \alpha(s) = - \frac{\sin \pi \alpha_1(4m^2)}{2 \cos \pi \alpha_1(4m^2)} \cdot C \cdot (s - 4m^2)^{\alpha_1(4m^2) + \frac{1}{2}} \quad (22)$$

This behavior is exactly the same as that which was derived by Barut and Zwanziger.⁶ For $\alpha_1(4m^2) + \frac{1}{2}$ is integer, from eq. (18)

$$\alpha(s) = -\frac{C}{2\pi} (s-4m^2)^{\alpha_1(4m^2)+\frac{1}{2}} \cdot \log(4m^2-s) \quad (23)$$

and so

$$\text{Re } \alpha(s) = -\frac{C}{2\pi} (s-4m^2)^{\alpha_1(4m^2)+\frac{1}{2}} \cdot \log(s-4m^2) \quad (24)$$

These results are true not only for the leading $\alpha(s)$, but also for the parameter $\alpha'(s)$ of the second term except when $\alpha'(s)$ is equal to $\alpha(s)$, $\alpha(s) - 2$ or $\alpha(s) - 4$, for all s , and so on.

IV. The 'Grouping' of the Asymptotic Power Series Required by the Unitarity Condition.

In this section we shall examine what kind of restriction from the unitarity condition we can obtain for $\beta(s)$'s and $\alpha(s)$'s which differ from each other by even integers in the vicinity of the threshold. First, we do not assume the Regge theory -- the Legendre grouping -- but only assume the power asymptotic behavior of eq. (7). Since the existence of the other terms which are different from χ^α , $\chi^{\alpha-2}$, $\chi^{\alpha-4}$..., in the $\phi(s, \chi)$ does not interfere with the following arguments, we put

$$\begin{aligned} \Phi_{(s,x)}^{(1)} = \beta_1(s) & \left[\left\{ (-x)^{\alpha_1(s)} + (\pm 1)_1 (x)^{\alpha_1(s)} \right\} + C_1(s) \left\{ (-x)^{\alpha_1(s)-2} + (\pm 1)_1 (x)^{\alpha_1(s)-2} \right\} \right. \\ & \left. + C_2(s) \left\{ (-x)^{\alpha_1(s)-4} + (\pm 1)_1 (x)^{\alpha_1(s)-4} \right\} + \dots + O(x^{-\frac{1}{2}}) \right] \end{aligned} \quad (25)$$

and

$$\begin{aligned} \Phi_{(s,x)}^{(2)} = \beta_2(s) & \left[\left\{ (-x)^{\alpha_2(s)} + (\pm 1)_2 (x)^{\alpha_2(s)} \right\} + C'_1(s) \left\{ (-x)^{\alpha_2(s)-2} + (\pm 1)_2 (x)^{\alpha_2(s)-2} \right\} \right. \\ & \left. + C'_2(s) \left\{ (-x)^{\alpha_2(s)-4} + (\pm 1)_2 (x)^{\alpha_2(s)-4} \right\} + \dots + O(x^{-\frac{1}{2}}) \right] \end{aligned} \quad (26)$$

In our case,

$$\beta_2(s) = \beta_1^*(s) \quad , \quad \alpha_2(s) = \alpha_1^*(s) \quad , \quad C'_j(s) = C_j^*(s). \quad (27)$$

We must now calculate in the same manner as shown in section II by retaining the lower power terms of A. But if we are only interested in the vicinity of the threshold, the terms that correspond to I_1 and I_2 in (8) become arbitrarily small. For example, in the integration on C_1 -- eq. (A1) is exact in our case -- by expanding the integrand of eq. (A1) in A^{-2} , we get

$$\begin{aligned} & \int_{C_1} d\varphi \Phi_{(s,x'')}^{(2)} \Phi_{(s,x')}^{(1)} \\ & = -i \beta_2 (e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2}) A^{\alpha_2} \cdot \int_{x'_{L1}}^{x'_{U1}} dx' \left\{ (x' - \sqrt{x'^2 - r^2})^{\alpha_2} + \right. \\ & \quad \left. + (-\frac{1}{A^2}) \left[\frac{\alpha_2}{2} (x' - \sqrt{x'^2 - r^2})^{\alpha_2-1} \sqrt{x'^2 - r^2} + C'_1 (x' - \sqrt{x'^2 - r^2})^{\alpha_2-2} \right] \right. \\ & \quad \left. + O(\frac{1}{A^4}) \right\} \cdot (\Phi_{(s,x'+i\eta)}^{(1)} - \Phi_{(s,x'-i\eta)}^{(1)}) \cdot \frac{1}{\sqrt{x'^2 - r^2}}. \end{aligned} \quad (28)$$

The coefficient of the term $-i\beta_2(e^{-i\frac{\pi}{2}(\alpha_2-2)} + (+1)_2 e^{+i\frac{\pi}{2}(\alpha_2-2)}) A^{\alpha_2-2}$ in the r.h.s. of eq.(28) is

$$\begin{aligned} & \int_{x'_{L1}}^{x'_{u1}} dx' \left\{ \frac{\alpha_2}{2} (x' - \sqrt{x'^2 - r^2})^{\alpha_2-1} + c'_1 (x' - \sqrt{x'^2 - r^2})^{\alpha_2-2} \cdot \frac{1}{\sqrt{x'^2 - r^2}} (\phi_{(s, x'+i\eta)}^{(1)} - \phi_{(s, x'-i\eta)}^{(1)}) \right\} \\ &= \int_{x'_{L1}}^{x'_{u1}} dx' F_c(x') (\phi_{(s, x'+i\eta)}^{(1)} - \phi_{(s, x'-i\eta)}^{(1)}) \\ &+ \int_{x'_{L1}}^{x'_{u1}} dx' F_d(x') \left[(\hat{\phi}_{(s, x'+i\eta)}^{(1)} - \hat{\phi}_{(s, x'-i\eta)}^{(1)}) + \beta_1 (e^{-i\pi\alpha_1} - e^{+i\pi\alpha_1}) x'^{\alpha_1} \right. \\ &\quad \left. + \beta_1 \cdot c'_1 (e^{-i\pi(\alpha_1-2)} - e^{+i\pi(\alpha_1-2)}) x'^{\alpha_1-2} \right] \quad (29) \end{aligned}$$

where we separate the first factor of the integrand of the l.h.s. into two terms

$$\left\{ \frac{\alpha_2}{2} (x' - \sqrt{x'^2 - r^2})^{\alpha_2-1} + c'_1 (x' - \sqrt{x'^2 - r^2})^{\alpha_2-2} \cdot \frac{1}{\sqrt{x'^2 - r^2}} \right\} = F_c(x') + F_d(x') \quad (30)$$

and

$$\begin{aligned} F_d(x') &= \frac{\alpha_2}{2} \left(\frac{r^2}{2} \right)^{\alpha_2-1} \left(\frac{1}{x'} \right)^{\alpha_2-1} \left\{ 1 + \frac{1}{2} (\alpha_2-1) \left(\frac{r^2}{2} \right) \left(\frac{1}{x'^2} \right) \right\} \\ &+ c'_1 \left(\frac{r^2}{2} \right)^{\alpha_2-2} \left(\frac{1}{x'} \right)^{\alpha_2-1} \left\{ 1 + \frac{\alpha_2}{2} \left(\frac{r^2}{2} \right) \left(\frac{1}{x'^2} \right) \right\} \end{aligned} \quad (31)$$

By definition the leading term of $F_c(x')$ is order of $(\frac{1}{x'})^{\alpha_2+3}$, so the first integral of eq.(29) converges when we put $x'_{u1} \rightarrow +\infty$, and we obtain the factor $q^{2\alpha_2+4}$ if the integration variable x' is changed to the normal variable t' . The terms in the second integral which contain $\phi^{(1)}$ also cannot contribute to the unitarity

at threshold, if $(\alpha_2 - 2) > -\frac{1}{2}$. So the remaining contribution is

$$\begin{aligned}
 & \int_0^1 dr \frac{r}{\sqrt{1-r^2}} \int_{C_1} d\varphi \Phi_{(s, x'')}^{(2)} \Phi_{(s, x')}^{(1)} \\
 &= -i\beta_2 (e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2}) A^{\alpha_2} \frac{\sqrt{\pi}}{2} \beta_1 (e^{-i\pi\alpha_1} - e^{+i\pi\alpha_1}) \frac{1}{\alpha_2 - \alpha_1} \\
 & \cdot \left\{ (\chi'_{L1})^{\alpha_1 - \alpha_2} - (\chi'_{u1})^{\alpha_1 - \alpha_2} \right\} \cdot \left(\left(\frac{1}{2} \right)^{\alpha_2} \frac{\Gamma(\alpha_2 + 1)}{\Gamma(\alpha_2 + \frac{3}{2})} + \left(-\frac{1}{A^2} \right) \left\{ \frac{\alpha_2}{2} \left(\frac{1}{2} \right)^{\alpha_2 - 1} \left[(\alpha_2 - 1) \frac{\Gamma(\alpha_2 - 1)}{\Gamma(\alpha_2 + \frac{1}{2})} \cdot C_1 \right. \right. \right. \\
 & \left. \left. + (\alpha_2 - 1) \frac{1}{2^2} \cdot \frac{\Gamma(\alpha_2 + 1)}{\Gamma(\alpha_2 + \frac{3}{2})} \right] + C'_1 \left(\frac{1}{2} \right)^{\alpha_2 - 2} \left[\frac{\Gamma(\alpha_2 - 1)}{\Gamma(\alpha_2 - \frac{1}{2})} \cdot C_1 + \frac{1}{2^2} \frac{\Gamma(\alpha_2 + 1)}{\Gamma(\alpha_2 + \frac{1}{2})} \right] \right\} \Bigg)
 \end{aligned} \tag{32}$$

But the terms with χ'_{u1} are cancelled by the terms from C_2 and C_8 . And the contribution from C_5 is the same except for the factor $(\pm 1)_1 (\pm 1)_2$.

The left-hand side of the unitarity eq.(4) is

$$-\beta_2(s) (e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2}) A^{\alpha_2} \cdot \left\{ 1 + C'_1 \left(-\frac{1}{A^2} \right) + \dots \right\} \tag{33}$$

Using the result of eq.(16) and Eq.(4), we obtain the quadratic equation for C_1

$$C'_1 = (\alpha_2 + \frac{1}{2}) C_1 + \frac{\alpha_2(\alpha_2 - 1)}{4} + 4 C'_1 \left\{ \frac{(\alpha_2 + \frac{1}{2})(\alpha_2 - \frac{1}{2})}{\alpha_2(\alpha_2 - 1)} C_1 + \frac{1}{4} (\alpha_2 + \frac{1}{2}) \right\} \tag{34}$$

since at the threshold

$$C'_j(4m^2) = C_j(4m^2) \tag{35}$$

we get two solutions for $C_1(4m^2)$

$$C_1(4m^2) = -\frac{(\alpha_2 - 1)\alpha_2}{2^2(\alpha_2 + \frac{1}{2})} \quad \text{or,} \quad C_1(4m^2) = -\frac{(\alpha_2 - 1)\alpha_2}{2^2(\alpha_2 - \frac{1}{2})} \quad (36)$$

where $\alpha_2 = \alpha_2(4m^2)$.

In the same way, we can calculate $C_2, C_3 \dots C_j$, if $(\alpha - 2j) > -\frac{1}{2}$. We can calculate them in a more systematic way, and the result is:

$$\begin{array}{ccccccc}
 C_0 & & C_1 & & C_2 & & \dots \\
 & & & & & & \\
 1 & \begin{array}{l} \nearrow \\ \searrow \end{array} & \begin{array}{l} - \frac{\alpha_2(\alpha_2-1)}{2^2 \cdot (\alpha_2 - \frac{1}{2})} \\ - \frac{\alpha_2(\alpha_2-1)}{2^2 (\alpha_2 + \frac{1}{2})} \end{array} & \begin{array}{l} \nearrow \\ \searrow \end{array} & \begin{array}{l} + \frac{\alpha_2(\alpha_2-1)(\alpha_2-2)(\alpha_2-3)}{2^4 \cdot 2! (\alpha_2 - \frac{1}{2})(\alpha_2 - \frac{3}{2})} \\ + \frac{\alpha_2(\alpha_2-1)(\alpha_2-2)(\alpha_2-3)}{2^4 \cdot 2! (\alpha_2 + \frac{1}{2})(\alpha_2 - \frac{5}{2})} \\ + \frac{\alpha_2(\alpha_2-1)(\alpha_2-2)(\alpha_2-3)}{2^4 \cdot 2! (\alpha_2 + \frac{1}{2})(\alpha_2 - \frac{1}{2})} \\ + \frac{\alpha_2(\alpha_2-1)(\alpha_2-2)(\alpha_2-3)(\alpha_2 - \frac{7}{2})}{2^4 \cdot 2! (\alpha_2 + \frac{1}{2})(\alpha_2 - \frac{3}{2})(\alpha_2 - \frac{5}{2})} \end{array} & \begin{array}{l} \nearrow \\ \searrow \end{array} & \begin{array}{l} \dots \\ \dots \\ \dots \\ \dots \end{array} \\
 & & & & & & (37)
 \end{array}$$

where C_j was defined by eq.(25).

It is interesting to compare these results with the asymptotic expansion of the Legendre function.

$$P_{\alpha}(z) = \frac{1}{\cos \pi \alpha} \cdot \frac{1}{\Gamma(\frac{1}{2})} \left\{ \frac{\sin \pi \alpha \cdot \Gamma(d+1)}{2^{\alpha+1} \Gamma(d+\frac{3}{2}) \cdot z^{\alpha+1}} \cdot {}_2F_1\left(\frac{\alpha+1}{2}, 1+\frac{\alpha}{2}; \alpha+\frac{3}{2}; \frac{1}{z^2}\right) \right. \\ \left. - \frac{\sin \pi \alpha \cdot 2^{\alpha} \cdot \Gamma(-\alpha)}{\Gamma(\frac{1}{2}-\alpha) z^{-\alpha}} \cdot {}_2F_1\left(-\frac{\alpha}{2}, \frac{1-\alpha}{2}; \frac{1}{2}-\alpha; \frac{1}{z^2}\right) \right\} \quad (38)$$

where

$${}_2F_1(a, b; c; \xi) = 1 + \sum_{\kappa=1}^{\infty} \frac{a(a+1) \cdots (a+\kappa-1) \cdot b(b+1) \cdots (b+\kappa-1)}{1 \cdot 2 \cdots (\kappa-1) \kappa \cdot c(c+1) \cdots (c+\kappa-1)}$$

Therefore

$$(\text{const.}) \cdot P_{\alpha}(z) = z^{\alpha} - \frac{\alpha(\alpha-1)}{2^2(\alpha-\frac{1}{2})} z^{\alpha-2} + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{2^4 \cdot 2! (\alpha-\frac{1}{2})(\alpha-\frac{3}{2})} z^{\alpha-4} - \dots \quad (39)$$

This is the same as the first line of the coefficients of (37). Here we notice that if the power asymptotic behavior is assumed, then only a few alternative asymptotic forms are allowed by the unitarity condition (at least in the vicinity of the threshold). In these allowed asymptotic forms, there is the Legendre 'grouping' (39).

All through this paper, we assumed the power behavior βx^{α} , but it is interesting to assume other types of asymptotic behavior -- for example $x^{\alpha}(\log x)^{\beta}$ -- and to study the consistency of the unitarity condition and what kind of 'grouping' is required. This problem will be treated in the separate note.

V. The Sum Rules for the Total Cross Section

In this section we use the unitarity condition eq.(2) for $0 < s < s_0$, where the amplitudes $\phi^{(1)}(s, x')$ and $\phi^{(2)}(s, x'')$ are defined by the analytic continuation on the s -plane. We also assume the power behavior for large x' and x'' ; eq.(7). It is not known if there is some relation between the parameter $\alpha_2(s)$ for $0 < s < s_0$ and the analytically continued function $\alpha(s)$ from $s_0 \leq s < s_1$ because of the difficulty to define the asymptotic behavior of $\phi^{(2)}(s, x'')$ for complex s , as in this case there appear branch points on the x'' -plane on the spiral.⁸

We can obtain the same equation as eq.(12) and eq.(13) with the definition of I_1 and I_2 in eq. (9) and eq.(10). If we take the limit $s \rightarrow +0$, and change the integration variable from x' to t' and u' , eq. (12) becomes

$$\begin{aligned}
 1 = & \frac{i2m}{8} \cdot \frac{1}{\sqrt{s}} \left[\int_0^1 dr \frac{r}{\sqrt{1-r^2}} \left\{ \int_{u_L'}^\infty du' \cdot \frac{1}{2m^2} \cdot \frac{\left[\left(\frac{u'}{2m^2} - 1 \right) - \sqrt{\left(\frac{u'}{2m^2} - 1 \right)^2 - r^2} \right]^{\alpha_2}}{\sqrt{\left(\frac{u'}{2m^2} - 1 \right)^2 - r^2}} \cdot \left(\hat{\phi}_{(0, u'+i\eta)}^{(1)} - \hat{\phi}_{(0, u'-i\eta)}^{(1)} \right) \right. \right. \\
 & + (\pm 1)_2 \int_{t_L'}^\infty dt' \cdot \frac{1}{2m^2} \cdot \frac{\left[\left(\frac{t'}{2m^2} - 1 \right) - \sqrt{\left(\frac{t'}{2m^2} - 1 \right)^2 - r^2} \right]^{\alpha_2}}{\sqrt{\left(\frac{t'}{2m^2} - 1 \right)^2 - r^2}} \cdot \left. \left(\hat{\phi}_{(0, t'+i\eta)}^{(1)} - \hat{\phi}_{(0, t'-i\eta)}^{(1)} \right) \right\} \\
 & + \sum_{\alpha_1 \geq \text{Re} \alpha_2} \sqrt{\pi} \cdot \beta_i \cdot \sin \pi \alpha_1 \cdot \frac{1}{i(\alpha_2 - \alpha_1)} \cdot \left(\frac{1}{2} \right)^{\alpha_2} \cdot \frac{\Gamma(\alpha_2 + 1)}{\Gamma(\alpha_2 + \frac{3}{2})} \cdot \left\{ \left(\frac{u_L'}{2m^2} - 1 \right)^{\alpha_1 - \alpha_2} + (\pm 1)_1 (\pm 1)_2 \left(\frac{t_L'}{2m^2} - 1 \right)^{\alpha_1 - \alpha_2} \right\} \Bigg] \quad (40)
 \end{aligned}$$

where u_L' and t_L' are the lower limit of the spectral function of the u and t channels respectively. If $\text{Re } \alpha_2 > \alpha_1$, the separation of $\phi^{(1)}(0, t')$ and $\phi^{(1)}(0, u')$ is not necessary, so $\phi^{(1)} = \hat{\phi}^{(1)}$, and the last terms of eq.(40) do not appear. By the optical the-

orem, $\phi^{(1)}(0, u' + i\eta) - \phi^{(1)}(0, u' - i\eta)$ can be written by using the total cross section, so eq.(40) is a sum rule with which we can determine $\alpha_2(0)$. If $\alpha_2(0)$ is large enough, (this can occur because the Froissart condition⁹ at $s = 0$ does not put the upper limit on the asymptotic behavior of $\phi^{(2)}(0, x'')$), we obtain other sum rules like eq.(40) comparing the coefficient of A^{α_2-2} , A^{α_2-4} ... A^{α_2-2K} ... and A^{α_2-2j} of the unitarity condition eq.(1), where $(\alpha_2 - 2j) > -1$. To obtain these sum rules, it is necessary to assume some 'grouping' of the asymptotic form of $\phi^{(1)}(s, x')$ and $\phi^{(2)}(s, x'')$; for example, the Legendre grouping. Separation of $\phi^{(1)}(0, u')$ must be done such that $\int_{-\infty}^{\infty} du' u'^{-(\alpha_2-2K)-1} \phi^{(1)}(0, u')$ converges.

Using the $\alpha_2(0)$ determined by eq.(40), the other sum rules may serve to check the consistency of the total cross section, or to determine some parameters of $(\phi^{(1)}(0, u' + i\eta) - \phi^{(1)}(0, u' - i\eta))$; for example, the parameters of the second vacuum trajectory at $s = 0$.

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Appendix

We divide the φ integration contour into eight parts as shown in fig. 1, where $|x'_u|$ and $|x''_u|$ are chosen large enough so that $\phi^{(1)}(s, x'_u)$ and $\phi^{(2)}(s, x''_u)$ can be expressed in the form of eq.(7). This choice is possible if $(\frac{s-4m^2}{4m^2}) \cdot r \cdot A \gg 1$; therefore, the threshold is excluded. We can, however, fix s as close to the threshold as we would like because A can be made arbitrarily large. Also the integration range of r must be $\varepsilon \leq r \leq 1$. The condition of $\varepsilon \rightarrow 0$ will be discussed later.

$$\begin{aligned} & \int_{C_1} d\varphi \Phi^{(2)}(s, x'') \Phi^{(1)}(s, x') \\ &= -i \sum_{\text{Re } \alpha_2 > -1} \beta_2 \cdot (e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2}) \cdot A^{\alpha_2} \int_{x'_{L1}}^{x'_{u1}} dx' [x' - \sqrt{1 + \frac{1}{A^2}} \cdot \sqrt{x'^2 - r^2}]^{\alpha_2} \\ & \quad \cdot (\Phi^{(1)}(s, x' + i\eta) - \Phi^{(1)}(s, x' - i\eta)) \cdot \frac{1}{\sqrt{x'^2 - r^2}} \end{aligned} \quad (A1)$$

where x'_{L1} is the lower limit of the right-hand cut of the $\phi^{(1)}(s, x')$, and η is positive small number. Here, we used the fact that on C_1

$$x'' = iA [x' - \sqrt{1 + \frac{1}{A^2}} \cdot \sqrt{x'^2 - r^2}] \quad (A2)$$

On C_2 , both $\phi^{(1)}(s, x')$ and $\phi^{(2)}(s, x'')$ have the asymptotic form of eq. (7)

$$\int_{C_2} d\varphi \Phi_{(s,x'')}^{(2)} \Phi_{(s,x')}^{(1)} = \sum_{\substack{\alpha_2 \\ \text{Re } \alpha_2 > -1}} \sum_{\substack{\alpha_1 \\ \text{Re } \alpha_1 > -1}} \int_{C_2} d\varphi \beta_1 \beta_2 \cdot \left[\left(-\frac{r}{\sqrt{2}} \right)^{\alpha_2} + (\pm 1)_2 \left(\frac{r}{\sqrt{2}} \right)^{\alpha_2} \right] \cdot \left[\left(-\frac{r}{\sqrt{2}} \right)^{\alpha_1} + (\pm 1)_1 \left(\frac{r}{\sqrt{2}} \right)^{\alpha_1} \right] \cdot \left\{ (1+iA)^{\frac{1}{2}} \cos \varphi + (1-iA)^{\frac{1}{2}} \sin \varphi \right\}^{\alpha_1} \cdot \left\{ (1+iA)^{\frac{1}{2}} \cos \varphi - (1-iA)^{\frac{1}{2}} \sin \varphi \right\}^{\alpha_2} \quad (A3)$$

The contribution from other segments of contour is the same as eq.(A1) and eq.(A3), except for the trivial change of the sign. When we sum up all the contributions, χ'_{u1} must disappear. This really occurs; for example, the terms with χ'_{u1} of C_1 cancel with the terms from C_2 and C_8 . In order to illustrate this cancellation and how to separate the functions $\phi^{(1)}(s,x')$ and $\phi^{(2)}(s,x'')$, we treat the highest order term of A , to avoid the algebraic complexity. The highest term of eq.(A1) is

$$\int_{C_1} d\varphi \Phi_{(s,x'')}^{(2)} \Phi_{(s,x')}^{(1)} = -i \beta_2 \left(e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2} \right) \cdot A^{\alpha_2} \cdot \int_{\chi'_{L1}}^{\chi'_{u1}} dx' \frac{[x' - \sqrt{x'^2 - r^2}]^{\alpha_2}}{\sqrt{x'^2 - r^2}} \left(\Phi_{(s,x'+i\eta)}^{(1)} - \Phi_{(s,x'-i\eta)}^{(1)} \right) \quad (A4)$$

Here we separate the asymptotic terms from $\phi^{(1)}(s,x')$ which increase equally or more rapidly than x'^{α_2} . For $x' > 0$

$$\Phi_{(s,x'\pm i\eta)}^{(1)} = \hat{\Phi}_{(s,x'\pm i\eta)}^{(1)} + \sum_{\substack{\alpha_1 \\ \text{Re } \alpha_1 \geq \text{Re } \alpha_2}} \beta_1 \cdot (x' e^{\mp i\pi\alpha_1}) \cdot \frac{\sqrt{x'^2 - r^2}}{[x' - \sqrt{x'^2 - r^2}]^{\alpha_2}} \cdot \left(\frac{r^2}{2} \right)^{\alpha_2} \cdot \left(\frac{1}{x'} \right)^{\alpha_2+1} \quad (A5)$$

Since we are limiting ourselves to the elastic scattering, and so $\alpha_1^* = \alpha_2$, there is only one term to be separated

$$\begin{aligned}
 \int_{C_1} d\varphi \Phi_{(s,x'')}^{(2)} \Phi_{(s,x')}^{(1)} &= -i\beta_2 (e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2}) \cdot A^{\alpha_2} \\
 &\cdot \int_{x'_{L1}}^{x'_{u1}} dx' \frac{[x' - \sqrt{x'^2 - r^2}]^{\alpha_2}}{\sqrt{x'^2 - r^2}} \cdot (\hat{\Phi}_{(s,x'+i\eta)}^{(1)} - \hat{\Phi}_{(s,x'-i\eta)}^{(1)}) \\
 &- i\beta_2\beta_1 (e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2}) (e^{-i\pi\alpha_1} - e^{+i\pi\alpha_1}) A^{\alpha_2} \cdot \left(\frac{r^2}{2}\right)^{\alpha_2} \cdot \int_{x'_{L1}}^{x'_{u1}} dx' x'^{\alpha_1 - \alpha_2 - 1}
 \end{aligned} \tag{A6}$$

On the other hand

$$\begin{aligned}
 &\int_{C_2} d\varphi \Phi_{(s,x'')}^{(2)} \Phi_{(s,x')}^{(1)} \\
 &= \int_{C_2} d\varphi \beta_2 \left(\frac{r}{\sqrt{2}} \sqrt{A}\right)^{\alpha_2} \cdot \left\{ e^{i(\varphi - \frac{3\pi}{4})\alpha_2} + (\pm 1)_2 e^{i(\varphi + \frac{\pi}{4})\alpha_2} \right\} \\
 &\quad \cdot \beta_1 \left(\frac{r}{\sqrt{2}} \sqrt{A}\right)^{\alpha_1} \cdot \left\{ e^{i(-\varphi + \frac{5\pi}{4})\alpha_1} + (\pm 1)_1 e^{i(-\varphi + \frac{\pi}{4})\alpha_1} \right\} \\
 &= \beta_1\beta_2 \cdot \frac{1}{i(\alpha_2 - \alpha_1)} \cdot (e^{i\frac{5\pi}{4}\alpha_1} + (\pm 1)_1 e^{i\frac{\pi}{4}\alpha_1}) \cdot (e^{-i\frac{3\pi}{4}\alpha_2} + (\pm 1)_2 e^{i\frac{\pi}{4}\alpha_2}) \\
 &\quad \cdot \left\{ (-x''_{u3})^{\alpha_2 - \alpha_1} \cdot \left(\frac{r^2}{2}\right)^{\alpha_1} \cdot A^{\alpha_1} e^{i(\alpha_2 - \alpha_1)\frac{3\pi}{4}} - (x'_{u1})^{-(\alpha_2 - \alpha_1)} \left(\frac{r^2}{2}\right)^{\alpha_2} \cdot A^{\alpha_2} e^{i(\alpha_2 - \alpha_1)\frac{\pi}{4}} \right\}
 \end{aligned} \tag{A7}$$

and similarly we get

$$\begin{aligned}
 & \int_{C_8} d\varphi \Phi_{(s, x'')}^{(2)} \Phi_{(s, x')}^{(1)} \\
 &= \beta_1 \beta_2 \frac{1}{i(d_2 - d_1)} \cdot (e^{i\frac{5}{4}\pi d_1} + (\pm 1)_1 e^{i\frac{9}{4}\pi d_1}) (e^{-i\frac{11}{4}\pi d_2} + (\pm 1)_2 e^{-i\frac{7}{4}\pi d_2}) \\
 & \cdot \left\{ (x'_{u1})^{-(d_2 - d_1)} \left(\frac{r^2}{2}\right)^{d_2} A^{d_2} e^{i(d_2 - d_1)\frac{9\pi}{4}} - (x''_{u1})^{d_2 - d_1} \left(\frac{r^2}{2}\right)^{d_1} A^{d_1} e^{i(d_2 - d_1)\frac{7\pi}{4}} \right\} \quad (A8)
 \end{aligned}$$

Collecting the terms that contain x'_{u1} in $\int_C \Phi^{(1)} \Phi^{(2)} d\varphi$, we obtain

$$\begin{aligned}
 & -i\beta_2 (e^{-i\frac{\pi}{2}d_2} + (\pm 1)_2 e^{i\frac{\pi}{2}d_2}) A^{d_2} \left[\int_{x'_{L1}}^{x'_{u1}} dx' \frac{[x' - \sqrt{x'^2 - r^2}]^{d_2}}{\sqrt{x'^2 - r^2}} (\hat{\Phi}_{(s, x' + i\eta)}^{(1)} - \hat{\Phi}_{(s, x' - i\eta)}^{(1)}) \right. \\
 & \quad \left. + \beta_1 (e^{-i\pi d_1} - e^{i\pi d_1}) \left(\frac{r^2}{2}\right)^{d_2} \int_{x'_{L1}}^{x'_{u1}} dx' x'^{d_1 - d_2 - 1} \right] \\
 & -i\beta_1 \beta_2 \frac{1}{d_2 - d_1} (e^{-i\frac{\pi}{2}d_2} + (\pm 1)_2 e^{i\frac{\pi}{2}d_2}) \cdot (x'_{u1})^{-(d_2 - d_1)} \left(\frac{r^2}{2}\right)^{d_2} A^{d_2} \\
 & \cdot \left\{ (e^{-i\pi d_1} + (\pm 1)_1, 1) - (e^{i\pi d_1} + (\pm 1)_1, 1) \right\} \quad (A9)
 \end{aligned}$$

As for large value of x'

$$\frac{[x' - \sqrt{x'^2 - r^2}]^{d_2}}{\sqrt{x'^2 - r^2}} \sim \left(\frac{r^2}{2}\right)^{d_2} x'^{-d_2 - 1} \quad (A10)$$

and

$$\lim_{x' \rightarrow \infty} \hat{\Phi}_{(s, x')}^{(1)} \cdot x'^{-\alpha_2} = 0 \quad (A11)$$

The upper limit of the first integral of (A9) can be taken to be infinity. So the final form of (A9) is

$$\begin{aligned} & -i\beta_2 (e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2}) A^{\alpha_2} \left[\int_{x'_{L1}}^{\infty} dx' \frac{[x' - \sqrt{x'^2 - r^2}]^{\alpha_2}}{\sqrt{x'^2 - r^2}} (\hat{\Phi}_{(s, x'+i\eta)}^{(1)} - \hat{\Phi}_{(s, x'-i\eta)}^{(1)}) \right. \\ & \left. + \beta_1 (e^{-i\pi\alpha_1} - e^{+i\pi\alpha_1}) \frac{1}{\alpha_2 - \alpha_1} \cdot \left(\frac{r^2}{2}\right)^{\alpha_2} \cdot (x'_{L1})^{\alpha_1 - \alpha_2} \right] \end{aligned} \quad (A12)$$

Other three contributions are nearly the same. The final result is:

$$\begin{aligned} & \int_C d\varphi \Phi_{(s, x'')}^{(2)} \Phi_{(s, x')}^{(1)} \\ & = -i\beta_2 (e^{-i\frac{\pi}{2}\alpha_2} + (\pm 1)_2 e^{+i\frac{\pi}{2}\alpha_2}) A^{\alpha_2} \cdot \left[\int_{x'_{L1}}^{\infty} dx' \frac{[x' - \sqrt{x'^2 - r^2}]^{\alpha_2}}{\sqrt{x'^2 - r^2}} (\hat{\Phi}_{(s, x'+i\eta)}^{(1)} - \hat{\Phi}_{(s, x'-i\eta)}^{(1)}) \right. \\ & \quad - (\pm 1)_2 \cdot \int_{-\infty}^{x'_{L5}} dx' \frac{[-x' - \sqrt{x'^2 - r^2}]^{\alpha_2}}{\sqrt{x'^2 - r^2}} (\hat{\Phi}_{(s, x'+i\eta)}^{(1)} - \hat{\Phi}_{(s, x'-i\eta)}^{(1)}) \\ & \quad \left. + \beta_1 (e^{-i\pi\alpha_1} - e^{+i\pi\alpha_1}) \frac{1}{\alpha_2 - \alpha_1} \left(\frac{r^2}{2}\right)^{\alpha_2} \left\{ (x'_{L1})^{\alpha_1 - \alpha_2} + (\pm 1)_1 (\pm 1)_2 (-x'_{L5})^{\alpha_1 - \alpha_2} \right\} \right] \end{aligned}$$

$$\begin{aligned}
 & -i\beta_1 (e^{-i\frac{\pi}{2}\alpha_1} + (\pm 1)_1 e^{i\frac{\pi}{2}\alpha_1}) A^{\alpha_1} \left[\int_{x''_{L1}}^{\infty} dx'' \frac{[x'' - \sqrt{x''^2 - r^2}]^{\alpha_1}}{\sqrt{x''^2 - r^2}} (\hat{\phi}_{(s, x''+i\eta)}^{(2)} - \hat{\phi}_{(s, x''-i\eta)}^{(2)}) \right. \\
 & \quad \left. - (\pm 1)_1 \int_{-\infty}^{x''_{L3}} dx'' \frac{[-x'' - \sqrt{x''^2 - r^2}]^{\alpha_1}}{\sqrt{x''^2 - r^2}} (\hat{\phi}_{(s, x''+i\eta)}^{(2)} - \hat{\phi}_{(s, x''-i\eta)}^{(2)}) \right] \\
 & + \beta_2 (e^{-i\pi\alpha_2} - e^{i\pi\alpha_2}) \frac{1}{\alpha_1 - \alpha_2} \left(\frac{r^2}{2} \right)^{\alpha_1} \left\{ (x''_{L1})^{\alpha_2 - \alpha_1} + (\pm 1)_1 (\pm 1)_2 (-x''_{L3})^{\alpha_2 - \alpha_1} \right\} \Bigg]
 \end{aligned}$$

(A13)

where ϕ 's are defined by eq.(A5) and by:

for $x' < 0$

$$\phi_{(s, x' \pm i\eta)}^{(1)} = \hat{\phi}_{(s, x' \pm i\eta)}^{(1)} + (\pm 1)_1 \beta_1 (-x' e^{\pm i\pi})^{\alpha_1} \frac{\sqrt{x'^2 - r^2}}{[-x' - \sqrt{x'^2 - r^2}]^{\alpha_2}} \cdot \left(\frac{r^2}{2} \right)^{\alpha_2} \cdot (-x')^{-\alpha_2 - 1}$$

(A5')

for $x'' > 0$

$$\phi_{(s, x'' \pm i\eta)}^{(2)} = \hat{\phi}_{(s, x'' \pm i\eta)}^{(2)} + \beta_2 (x'' e^{\mp i\pi})^{\alpha_2} \frac{\sqrt{x''^2 - r^2}}{[x'' - \sqrt{x''^2 - r^2}]^{\alpha_1}} \cdot \left(\frac{r^2}{2} \right)^{\alpha_1} \cdot (x'')^{-\alpha_1 - 1}$$

(A5'')

for $x'' < 0$

$$\phi_{(s, x'' \pm i\eta)}^{(2)} = \hat{\phi}_{(s, x'' \pm i\eta)}^{(2)} + (\pm 1)_2 \beta_2 (-x'' e^{\pm i\pi})^{\alpha_2} \frac{\sqrt{x''^2 - r^2}}{[-x'' - \sqrt{x''^2 - r^2}]^{\alpha_1}} \cdot \left(\frac{r^2}{2} \right)^{\alpha_1} \cdot (-x'')^{-\alpha_1 - 1}$$

(A5''')

In the calculation of

$$\int_{\epsilon}^1 \frac{r}{\sqrt{1-r^2}} dr \int_C d\varphi \Phi_{(s, x'')}^{(2)} \Phi_{(s, x')}^{(1)},$$

the condition $\epsilon \rightarrow 0$ is

$$\operatorname{Re} \alpha_1(s) > -1, \quad \text{and} \quad \operatorname{Re} \alpha_2(s) > -1, \quad (\text{A14})$$

because

$$\begin{aligned} \int_{\epsilon}^1 dr \frac{r}{\sqrt{1-r^2}} \frac{[x' - \sqrt{x'^2 - r^2}]^{\alpha_2}}{\sqrt{x'^2 - r^2}} &= \int_{\epsilon}^1 dr \frac{r^{2\alpha_2+1}}{\sqrt{1-r^2}} \cdot \left(\frac{1}{2}\right)^{\alpha_2} \cdot x'^{-\alpha_2-1} \\ &= \int_0^1 dr \frac{r^{2\alpha_2+1}}{\sqrt{1-r^2}} \left(\frac{1}{2}\right)^{\alpha_2} \cdot x'^{-\alpha_2-1}. \end{aligned}$$

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Figure Caption

Fig. 1. As $|x|$ increases, cuts come down to the real axis of the

φ -plane. So we have to bend the integration contour.

This graph shows the bent contour C of eq.(4). C is

divided into $C_1 C_2 \dots C_8$. φ_{Lj} and φ_{uj} are related

to X'_{Lj} , x'_{uj} by eq. (5) and to X''_{Lj} , x''_{uj} by eq.(5).

If we neglect the lower order of $\frac{1}{A^2}$,

$$\varphi_{L1} \text{ and } \varphi_{L7} = \log \frac{X'_{Lj} + \sqrt{X'^2_{Lj} - r^2}}{\sqrt{2A} \cdot r}, \quad \varphi_{L3} \text{ and } \varphi_{L5} = \log \frac{-X'_{Lj} + \sqrt{X'^2_{Lj} - r^2}}{\sqrt{2A} \cdot r}$$

In order to obtain φ_{uj} , X'_{Lj} must be changed to X'_{uj} .

Fig. 1