DECEMBER 1975

MATT-1181

TRANSPORT EQUATIONS IN
AXISYMMETRIC TOROIDAL
COORDINATES

BY

SAMUEL L. GRALNICK

PLASMA PHYSICS
LABORATORY

MASTER

PRINCETON UNIVERSITY
PRINCETON, NEW JERSEY

This work was supported by U. S. Energy Research and Development Administration Contract E(11-1)-3073. Reproduction, translation, publication, use and disposal, in whole or in part, by or for the United States Government is permitted.

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

NOTICE

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

Printed in the United States of America.

Available from
National Technical Information Service
U. S. Department of Commerce
5285 Port Royal Road
Springfield, Virginia 22151

Price: Printed Copy \$ * ; Microfiche \$1.45

<u>*Pages</u>	<u>NTIS Selling Price</u>
1-50	\$ 4.00
51-150	5.45
151-325	7.60
326-500	10.60
501-1000	13.60

Transport Equations In Axisymmetric Toroidal Coordinates

Samuel L. Gralnick

Princeton Plasma Physics Laboratory

Princeton, New Jersey 08540

NOTICE
This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

ABSTRACT

This article presents a derivation of the conservation law form of the single energy group transport equation in an axisymmetric toroidal coordinate system formed by rotating a nest of smooth, simply closed, plane curves of arbitrary parametric description about an axis which does not intersect the nest. This general equation may be used for generating equations specific to particular cross section geometries, or as the basis of a finite difference equation for the general case.

The effect of both the toroidal and poloidal curvatures of the system are investigated, and criteria for the validity of cylindrical and planar approximations are established.

The diffusion equation for this geometry is derived, and it is shown to be formally homologous to the "r- θ " cylindrical diffusion equation if the coordinate system is orthogonal and if the azimuthal coordinate, ϕ , is ignorable.

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

I. INTRODUCTION

The solution of neutron and photon transport problems in the analysis and design of proposed fusion reactors¹⁻⁴, and the application of the techniques of transport theory to the problems associated with the transport of neutral particles in fusion reactor plasmas⁵⁻⁷ require that we concern ourselves with axisymmetric toroidal geometries. Existing, discrete ordinate type, transport codes that have been tailored to planar, cylindrical or spherical geometries can, at best, be used to give approximate answers which are sometimes valid but frequently are extended beyond their range of applicability. In fact, a reasonable knowledge of the error introduced by using these codes can only be guessed at in the absence of toroidal analyses. The alternative available is to use Monte Carlo simulation techniques which are inherently suitable to complicated geometries. This approach, however, has not proven to be universally acceptable and the need for a toroidal discrete ordinate transport code still exists.

In this article we will use a tensor transformation formalism developed previously⁸ to transform the streaming term of a single energy group transport equation to a system of axisymmetric, non-orthogonal toroidal coordinates. These coordinates are generated by rotating a nest of simply

closed smooth curves about an axis which does not intersect the nest. Several particular cases have been treated previously by other authors^{9,10,11}. The divergence form of the transport equation for one of these will be derived as an illustrative example; it is readily obtained from the general equation which we will derive.

By casting the general equation in a slightly altered cylindrical form, it is possible to illuminate the roles of both the toroidal and poloidal curvatures of the system and to demonstrate the condition of applicability of the cylindrical approximation in which the toroidal curvature is ignored. Similarly if both the toroidal and poloidal curvatures are ignored a planar approximation will result which will be adequate for some purposes.

Finally, it has been noted that two of the previously treated coordinate systems^{9,10} enjoy the property of yielding diffusion equations which are formal homologs of the cylindrical, "r- θ ", diffusion equation when the toroidal coordinate is ignorable. By using the same tensor formalism that we have applied to the transport equation we have derived the general axisymmetric toroidal diffusion equation. We find that the property of being homologous to the cylindrical "r- θ " diffusion equation is shared by all diffusion equations derived for orthogonal, axisymmetric toroidal coordinate systems in which the toroidal coordinate is ignorable.

II. SINGLE ENERGY GROUP TRANSPORT EQUATION IN AXISYMMETRIC TOROIDAL COORDINATES

Consider a coordinate system in which the coordinate surfaces are formed by rotating a nest of smooth, simply closed, plane curves about an axis which does not intersect the nest. (see Fig. 1.) The equations of transformation between the (x, y, z) Cartesian coordinate system and the (ρ, θ, ϕ) toroidal coordinates are

$$\begin{aligned} T_1: \quad x^1 &= X^1(\rho, \theta) \cos \phi \\ x^2 &= X^2(\rho, \theta) \\ x^3 &= X^1(\rho, \theta) \sin \phi \end{aligned} \quad (1)$$

(Indicial notation will be used throughout with repeated indicies summed.) In equations (1) $x^1=x$, $x^2=y$ and $x^3=z$. ϕ is the toroidal or azimuthal angle. We may chose ρ to be the square root of the area contained within a member of the nest of curves and θ to be the length of arc along the curve normalized to the perimeter of the curve although this is not essential, other selections of ρ and θ being acceptable. The functions X^1 and X^2 give a parametric description of the elements of the nest - they may be arbitrarily specified, and in fact, need not be given by analytical formulas, numerical functions sufficing. We will restrict the choice of X^1 and X^2 such that the nest so described may have at most one singular point at which the Jacobian of T_1 vanishes.

We now write the single energy group transport equation as a conservation law in a 5-dimensional Riemannian space, R_5 , having coordinates $\chi^1=x$, $\chi^2=y$, $\chi^3=z$, $\chi^4=\tan^{-1}(\lambda_z/\lambda_x)$, $\chi^5=\lambda_y$, (where λ_x , λ_y and λ_z are the x, y, and z Cartesian components of $\underline{\Omega}$, the propagation vector), and having the metric tensor,

$$g_{ij} = \begin{Bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & (1 - (\chi^5)^2) & \\ & & & & (1 - (\chi^5)^2)^{-1} \end{Bmatrix}, \quad (2)$$

introduced by Drawbaugh¹²;

$$(|g|)^{-1/2} \frac{\partial}{\partial \chi^i} \left[(|g|)^{1/2} \Omega^i \Psi \right] + \sigma \Psi = S. \quad (3)$$

$|g|$ denotes the determinant of the metric tensor, Ψ is the angular flux, σ is the total cross section and S is the source function.

Following the prescription given in reference (8) we apply the transformation T_1 to equation (3). Under this transformation,

$$\bar{g}_{ij} = g_{mn} \frac{\partial \chi^m}{\partial \bar{\chi}^i} \frac{\partial \chi^n}{\partial \bar{\chi}^j}, \quad (4)$$

and

$$\bar{\Omega}^i = \bar{g}^{ij} \bar{\Omega}_j = \bar{g}^{ij} \Omega_k \frac{\partial \chi^k}{\partial \bar{\chi}^j} \quad (5)$$

In equations (4) and (5) the unbarred χ 's refer to the Cartesian coordinates and the barred χ 's refer to the toroidal coordinates. The contravariant tensor $\bar{g}^{ij}$ is found by solving

$$\bar{g}^{ij} \bar{g}_{jk} = \delta_k^i, \quad (6)$$

where δ_k^i is a Kronecker delta. The solution of equations (6) and the evaluation of equations (4) and (5) yield the following results:

$$g_{ij} = \left\{ \begin{array}{ll} \left(\frac{\partial \chi^1}{\partial \rho} \right)^2 + \left(\frac{\partial \chi^2}{\partial \rho} \right)^2 & \frac{\partial \chi^1}{\partial \rho} \frac{\partial \chi^1}{\partial \theta} + \frac{\partial \chi^2}{\partial \rho} \frac{\partial \chi^2}{\partial \theta} \\ \frac{\partial \chi^1}{\partial \rho} \frac{\partial \chi^1}{\partial \theta} + \frac{\partial \chi^2}{\partial \rho} \frac{\partial \chi^2}{\partial \theta} & \left(\frac{\partial \chi^1}{\partial \theta} \right)^2 + \left(\frac{\partial \chi^2}{\partial \theta} \right)^2 \\ & (\chi^1)^2 \\ & (1 - (\chi^5)^2) \\ & (1 - (\chi^5)^2)^{-1} \end{array} \right\} \quad (7)$$

$$\begin{aligned}
 \bar{\Omega}^1 &= \left[(1-(\gamma)^2)^{1/2} \sin(\omega+\phi) \frac{\partial x^2}{\partial \theta} - \gamma \frac{\partial x^1}{\partial \theta} \right] (J)^{-1} \\
 \bar{\Omega}^2 &= - \left[(1-(\gamma)^2)^{1/2} \sin(\omega+\phi) \frac{\partial x^2}{\partial \rho} - \gamma \frac{\partial x^1}{\partial \rho} \right] (J)^{-1} \\
 \bar{\Omega}^3 &= (1-(\gamma)^2)^{1/2} \cos(\omega+\phi) / x^1 \\
 \bar{\Omega}^4 &= \bar{\Omega}^5 = 0 \quad ,
 \end{aligned} \tag{8}$$

where $\omega \equiv x^4$, $\gamma \equiv x^5$ and

$$J = \frac{\partial x^1}{\partial \rho} \frac{\partial x^2}{\partial \theta} - \frac{\partial x^1}{\partial \theta} \frac{\partial x^2}{\partial \rho} \quad . \tag{9}$$

$$\bar{g}^{ij} = \left\{ \begin{array}{cc} \frac{\left(\frac{\partial x^1}{\partial \theta}\right)^2 + \left(\frac{\partial x^2}{\partial \theta}\right)^2}{(J)^2} & - \frac{\frac{\partial x^1}{\partial \rho} \frac{\partial x^1}{\partial \theta} + \frac{\partial x^2}{\partial \rho} \frac{\partial x^2}{\partial \theta}}{(J)^2} \\ - \frac{\frac{\partial x^1}{\partial \rho} \frac{\partial x^1}{\partial \theta} + \frac{\partial x^2}{\partial \rho} \frac{\partial x^2}{\partial \theta}}{(J)^2} & \frac{\left(\frac{\partial x^1}{\partial \rho}\right)^2 + \left(\frac{\partial x^2}{\partial \rho}\right)^2}{(J)^2} \\ & \frac{1}{(x^1)^2} \\ & (1-(x^5)^2)^{-1} \\ & (1-(x^5)^2) \end{array} \right\} . \tag{10}$$

Also,

$$|\bar{g}| = (x^1)^2 (J)^2 \quad . \quad (11)$$

The appropriate orthogonal representation of $\underline{\Omega}$ for this coordinate system will be in terms of its components in the ϕ direction, in the direction normal to the toroidal surface and in the direction orthogonal to these two which is the direction of increasing θ . These components are formed by taking the inner product of $\underline{\Omega}$ and the unit vector in the particular direction. We denote the unit vectors (see Fig. 1) by $\hat{v}$ for the unit vector normal to the toroidal surface, and by $\hat{\theta}$ and $\hat{\phi}$ for the unit vectors in the θ and ϕ directions. The components of $\underline{\Omega}$ we will call $\bar{\lambda}_v$, $\bar{\lambda}_\theta$ and $\bar{\lambda}_\phi$. In the $\bar{x}^1$ coordinate system,

$$\hat{v}: \quad \bar{v}_1 = J \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{-1/2} \quad (12)$$

$$\bar{v}_2 = \bar{v}_3 = 0 \quad ,$$

$$\begin{aligned} \bar{\theta}_1 &= \left(\frac{\partial x^1}{\partial \theta} \frac{\partial x^1}{\partial \rho} + \frac{\partial x^2}{\partial \theta} \frac{\partial x^2}{\partial \rho} \right) \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{-1/2} \\ \hat{\theta}: \quad \bar{\theta}_2 &= \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2} \\ \bar{\theta}_3 &= 0 \quad , \end{aligned} \quad (13)$$

$$\begin{aligned} \bar{\phi}_1 &= \bar{\phi}_2 = 0 \\ \hat{\phi}: \quad \bar{\phi}_3 &= x^1 \end{aligned} \quad (14)$$

and consequently,

$$\begin{aligned} \bar{\lambda}_v &= \left[(1-(\gamma)^2)^{1/2} \sin(\omega+\phi) \frac{\partial x^2}{\partial \theta} - \gamma \frac{\partial x^1}{\partial \theta} \right] \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{-1/2} \\ \bar{\lambda}_\theta &= \left[(1-(\gamma)^2)^{1/2} \sin(\omega+\phi) \frac{\partial x^1}{\partial \theta} + \gamma \frac{\partial x^2}{\partial \theta} \right] \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{-1/2} \\ \bar{\lambda}_\phi &= (1-\gamma^2)^{1/2} \cos(\omega+\phi) \end{aligned} \quad (15)$$

Using this orthogonal representation of $\underline{\Omega}$ we introduce angular coordinates $\bar{x}^4$ and $\bar{x}^5$ by the transformation

$$\begin{aligned} T_2: \quad \bar{x}^4 &= \tan^{-1} \left(\frac{\bar{\lambda}_v}{\bar{\lambda}_\theta} \right) \\ \bar{x}^5 &= \bar{\lambda}_\phi \end{aligned} \quad (16)$$

Three systems of coordinates in R_5 have been used. The original coordinate system denoted by χ^i , (without overbars) consists of the x , y and z Cartesian spatial coordinates and ω and γ as the angular variables. The coordinate system $\bar{\chi}^i$ (single overbar) results when transformation T_1 is applied to the spatial variables. This coordinate system has ρ , θ and ϕ for spatial variables but uses the ω and γ angular variables. Finally when T_2 is applied the $\bar{\bar{\chi}}^i$ coordinate system (double overbars) results. In this coordinate system, the spatial variables are ρ , θ and ϕ and the angular variables are

$$\chi = \bar{\bar{\chi}}^4, \quad (17)$$

and

$$\zeta = \bar{\bar{\chi}}^5.$$

The ν , θ and ϕ components of $\underline{\Omega}$ in the $\bar{\bar{\chi}}^i$ coordinate system are

$$\begin{aligned} \bar{\bar{\lambda}}_\nu &= (1-(\zeta)^2) \sin \chi \\ \bar{\bar{\lambda}}_\theta &= (1-(\zeta)^2) \cos \chi \\ \bar{\bar{\lambda}}_\phi &= \zeta \end{aligned} \quad (18)$$

The 5 components of $\bar{\Omega}^i$ (i.e. written in $\bar{\chi}^i$ coordinates) are then,

$$\begin{aligned}
 \bar{\Omega}^1 &= \bar{\lambda}_\nu \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2} / J \\
 \bar{\Omega}^2 &= \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{-1/2} \left[\bar{\lambda}_\theta - \bar{\lambda}_\nu \left(\frac{\partial x^1}{\partial \theta} \frac{\partial x^1}{\partial \rho} + \frac{\partial x^2}{\partial \theta} \frac{\partial x^2}{\partial \rho} \right) / J \right] \\
 \bar{\Omega}^3 &= \bar{\lambda}_\phi / x^1 \\
 \bar{\Omega}^4 &= \left\{ \left(\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \theta^2} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \theta^2} \right) / \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right] \right\} \times \\
 &\quad \left\{ \bar{\lambda}_\theta - \frac{\bar{\lambda}_\nu}{J} \left[\left(\frac{\partial x^1}{\partial \theta} \frac{\partial x^1}{\partial \rho} + \frac{\partial x^2}{\partial \theta} \frac{\partial x^2}{\partial \rho} \right) \right. \right. \\
 &\quad \left. \left. - \frac{\left(\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \rho \partial \theta} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \rho \partial \theta} \right) \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]}{\left(\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \theta^2} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \theta^2} \right)} \right] \right\} \\
 &\quad + \left\{ \frac{(\bar{\lambda}_\phi)^2}{(\bar{\lambda}_\nu)^2 + (\bar{\lambda}_\theta)^2} \frac{\bar{\lambda}_\theta \frac{\partial x^2}{\partial \theta} - \bar{\lambda}_\nu \frac{\partial x^1}{\partial \theta}}{x^1 \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2}} \right\} \\
 \bar{\Omega}^5 &= - \frac{\bar{\lambda}_\phi}{x^1} \frac{\bar{\lambda}_\nu \frac{\partial x^2}{\partial \theta} + \bar{\lambda}_\theta \frac{\partial x^1}{\partial \theta}}{\left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2}} \quad (19)
 \end{aligned}$$

Since the choice of angular coordinates used gives an orthogonal representation of $\underline{\Omega}$, the Jacobian of transformation T_2 is unity⁸. Consequently,

$$|\bar{g}| = |\bar{g}| \quad (20)$$

We now substitute in the divergence expression of equation (3) written in the $\bar{x}^i$ coordinates to give the single energy group transport equation in axisymmetric toroidal coordinates. (Note: This equation and the remaining content of this paper will be with reference to the $\bar{x}^i$ system. The overbars will now be dropped.)

$$\begin{aligned}
 & \frac{1}{(x^1_J)} \left\{ \frac{\partial}{\partial \rho} \left\{ \lambda_v \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2} x^1 \Psi \right\} + \right. \\
 & \quad \frac{\partial}{\partial \theta} \left\{ x^1 \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{-1/2} \left[\lambda_\theta J - \lambda_v \left(\frac{\partial x^1}{\partial \theta} \frac{\partial x^1}{\partial \rho} + \frac{\partial x^2}{\partial \theta} \frac{\partial x^2}{\partial \rho} \right) \right] \Psi \right\} + \\
 & \quad \frac{\partial}{\partial \phi} \left\{ \lambda_\phi J \Psi \right\} + \\
 & \quad \frac{\partial}{\partial x} \left\{ x^1_J \left\{ \frac{\left(\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \theta^2} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \theta^2} \right)}{\left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{3/2}} \times \right. \right. \\
 & \quad \left. \left[\lambda_\theta - \frac{\lambda_v}{J} \left(\frac{\partial x^1}{\partial \theta} \frac{\partial x^1}{\partial \rho} + \frac{\partial x^2}{\partial \theta} \frac{\partial x^2}{\partial \rho} \right) \right. \right. \\
 & \quad \left. \left. - \frac{\left(\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \rho \partial \theta} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \rho \partial \theta} \right) \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]}{\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \theta^2} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \theta^2}} \right] \right\} \\
 & \quad + \left(\frac{(\lambda_\phi)^2}{(\lambda_v)^2 + (\lambda_\theta)^2} \frac{\lambda_\theta \frac{\partial x^2}{\partial \theta} - \lambda_v \frac{\partial x^1}{\partial \theta}}{x^1 \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2}} \right) \Psi \Bigg\} + \\
 & \quad \frac{\partial}{\partial \zeta} \left\{ - \lambda_\phi J \frac{\lambda_v \frac{\partial x^2}{\partial \theta} + \lambda_\theta \frac{\partial x^1}{\partial \theta}}{\left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2}} \Psi \right\} + \sigma \Psi = S
 \end{aligned}
 \tag{21}$$

III. A PARTICULAR COORDINATE SYSTEM OF INTEREST

Here we will display the form of the transport equation for a particular coordinate system of interest. Such a coordinate system is specified when $X^1(\rho, \theta)$ and $X^2(\rho, \theta)$ are chosen. The geometric factors in the terms of the transport equation (21) involve X^1 , X^2 and derivatives of these functions up to and including order two. For analytically described X^1 and X^2 , these factors are evaluated and substituted into the equation. For purely numerical applications it is possible to specify X^1 and X^2 as tabular functions and to generate derivative tables that may then be used in the solution of the equation. The geometric factors that must be evaluated are:

$$\begin{aligned}
 G_i(\rho, \theta) &\equiv \left(\frac{\partial X^1}{\partial \theta} \right)^2 + \left(\frac{\partial X^2}{\partial \theta} \right)^2 \\
 G_{ii}(\rho, \theta) &\equiv \frac{\partial X^1}{\partial \rho} \frac{\partial X^2}{\partial \theta} - \frac{\partial X^2}{\partial \rho} \frac{\partial X^1}{\partial \theta} = J \\
 G_{iii}(\rho, \theta) &\equiv \frac{\partial X^1}{\partial \rho} \frac{\partial^2 X^1}{\partial \theta^2} + \frac{\partial X^2}{\partial \rho} \frac{\partial^2 X^2}{\partial \theta^2} \\
 G_{iv}(\rho, \theta) &\equiv \frac{\partial X^1}{\partial \theta} \frac{\partial^2 X^2}{\partial \theta^2} - \frac{\partial X^2}{\partial \theta} \frac{\partial^2 X^1}{\partial \theta^2} \\
 G_v(\rho, \theta) &\equiv \frac{\partial X^1}{\partial \theta} \frac{\partial^2 X^2}{\partial \rho \partial \theta} - \frac{\partial X^2}{\partial \theta} \frac{\partial^2 X^1}{\partial \rho \partial \theta} \\
 G_{vi}(\rho, \theta) &\equiv \frac{\frac{\partial X^1}{\partial \theta}}{X^1 \left[\left(\frac{\partial X^1}{\partial \theta} \right)^2 + \left(\frac{\partial X^2}{\partial \theta} \right)^2 \right]^{1/2}} \\
 G_{vii}(\rho, \theta) &\equiv \frac{\frac{\partial X^2}{\partial \theta}}{X^1 \left[\left(\frac{\partial X^1}{\partial \theta} \right)^2 + \left(\frac{\partial X^2}{\partial \theta} \right)^2 \right]^{1/2}}
 \end{aligned} \tag{22}$$

Several geometries have been considered previously by other authors and the present writer. Pomraning and Stevens⁹ derived the gradient form of the transport equation and an appropriate diffusion equation for nested circular toroids. The divergence form for this system of coordinates has been derived by the present writer⁸ and independently by Jung¹¹ who has also derived a set of appropriate finite difference equations. Pomraning¹⁰ has considered two systems of elliptical-toroidal coordinates. We will now use the general equation (21) to derive the divergence form for the latter of the two illustrating its utility for this task.

$$\begin{aligned}x^1(\rho, \theta) &= [\sinh \rho \sin \theta + R] \\x^2(\rho, \theta) &= [\cosh \rho \cos \theta]\end{aligned}\tag{23}$$

The scale factor, s , used in reference (10) is set to 1 here without loss of generality, and we note that in this case ρ and θ are not to be interpreted as the square root of the area and fractional arc length. We find that for this geometry,

$$\begin{aligned}
 G_i &= [\sinh^2 \rho \cos^2 \theta + \cosh^2 \rho \sin^2 \theta] \\
 G_{ii} &= - [\sinh^2 \rho \cos^2 \theta + \cosh^2 \rho \sin^2 \theta] \\
 G_{iii} &= 0 \\
 G_{iv} &= - \sinh \rho \cosh \rho \\
 G_v &= - \sin \theta \cos \theta \\
 G_{vi} &= \frac{\tanh \rho}{[\sinh \rho \sin \theta + R]} \frac{1}{\sqrt{\tanh^2 \rho + \tan^2 \theta}} \\
 G_{vii} &= \frac{\tan \theta}{[\sinh \rho \sin \theta + R]} \frac{1}{\sqrt{\tanh^2 \rho + \tan^2 \theta}}
 \end{aligned} \tag{24}$$

and substituting into the transport equation,

$$\begin{aligned}
 & \frac{-1}{(\sinh \rho \sin \theta + R) (\sinh^2 \rho \cos^2 \theta + \cosh^2 \rho \sin^2 \theta)} \times \\
 & \left\{ \frac{\partial}{\partial \rho} \left\{ [\lambda_v (\sinh^2 \rho \cos^2 \theta + \cosh^2 \rho \sin^2 \theta)^{1/2} (\sinh \rho \sin \theta + R) \Psi] \right\} + \right. \\
 & \frac{\partial}{\partial \theta} \left\{ \frac{(\sinh \rho \sin \theta + R)}{(\sinh^2 \rho \cos^2 \theta + \cosh^2 \rho \sin^2 \theta)^{1/2}} [\lambda_\theta (\sinh^2 \rho \cos^2 \theta + \cosh^2 \rho \sin^2 \theta) \Psi] \right\} + \\
 & \frac{\partial}{\partial \phi} \left\{ \lambda_\phi (\sinh^2 \rho \cos^2 \theta + \cosh^2 \rho \sin^2 \theta) \Psi \right\} + \\
 & \frac{\partial}{\partial \chi} \left\{ - (\sinh \rho \sin \theta + R) (\sinh^2 \rho \cos^2 \theta + \cosh^2 \rho \sin^2 \theta) \times \right. \\
 & \left(\frac{1}{(\sinh^2 \rho \cos^2 \theta + \cosh^2 \rho \sin^2 \theta)^{3/2}} [\lambda_v (\sin \theta \cos \theta) - \lambda_\theta (\sinh \rho \cosh \rho)] \right. \\
 & \left. \left. + \frac{(\lambda_\phi)^2}{(\lambda_v)^2 + (\lambda_\theta)^2} \frac{(\lambda_\theta \tan \theta - \lambda_v \tanh \rho)}{(\sinh \rho \sin \theta + R) (\tanh^2 \rho + \tan^2 \theta)^{1/2}} \right) \Psi \right\} \\
 & \left. \frac{\partial}{\partial \zeta} \left\{ \lambda_\phi (\sinh^2 \rho \cos^2 \theta + \cosh^2 \rho \sin^2 \theta) \frac{(\lambda_v \tan \theta + \lambda_\theta \tanh \rho)}{(\tanh^2 \rho + \tan^2 \theta)^{1/2}} \Psi \right\} + \right\}
 \end{aligned}$$

$$\sigma \Psi = S$$

(25)

This result may be compared to that of Pomraning by noting that,

$$\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial \chi^i} \sqrt{|g|} \Omega^i = 0, \quad (26)$$

and forming the gradient streaming operator,

$$\Omega^i \frac{\partial}{\partial \chi^i}.$$

IV. LIMITING CASES - THE EFFECT OF CURVATURE

To date, approximate analyses using "r- θ " cylindrical transport codes have been employed in the solution of toroidal fusion reactor problems. These solutions ignore effects introduced by the non-circular characteristic of the minor cross section and are only valid when the physical scale length of the problem, λ , is small compared to the inverse of the toroidal curvature, κ_T , at the point in question. (We define the toroidal curvature at a point, $P(\rho, \theta, \phi)$, as the curvature of the right circular cylinder passing through that point and having the principal axis of the torus as its axis. Referring to Fig. 1, the toroidal curvature is equal to the curvature of the circle labelled curve of constant θ). The statement regarding the ratio of

λ and $1/\kappa_T$, to our knowledge, has not been rigorously demonstrated. We offer a rigorous proof of its validity here. We also show that, when in addition, λ is small compared to the inverse of κ_p , the poloidal curvature at P, a planar approximation is valid. (κ_p is the curvature at a point of the curve labelled, "curve of constant ϕ " in Fig. 1). Finally, we note, that as before we consider arbitrarily chosen cross-section shapes imposing only those limitations discussed previously.

To begin with, for comparison, we will need the divergence form of the transport equation for a cylindrical coordinate system of arbitrary cross sectional shape. This equation is readily derived by introducing the transformation,

$$\begin{aligned}x^1 &= x^1(\rho, \theta) \\x^2 &= x^2(\rho, \theta) \\x^3 &= \bar{x}^3 \equiv z\end{aligned}\tag{27}$$

and employing the same technique as used previously. We find,

$$\begin{aligned}
 & \frac{1}{J} \left\{ \frac{\partial}{\partial \rho} \left\{ \lambda_v \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2} \Psi \right\} + \right. \\
 & \left. \frac{\partial}{\partial \theta} \left\{ \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{-1/2} \left[\lambda_\theta J - \lambda_v \left(\frac{\partial x^1}{\partial \rho} \frac{\partial x^1}{\partial \theta} + \frac{\partial x^2}{\partial \rho} \frac{\partial x^2}{\partial \theta} \right) \right] \Psi \right\} + \right. \\
 & \left. \frac{\partial}{\partial z} \left\{ \lambda_z J \Psi \right\} + \right. \\
 & \left. \frac{\partial}{\partial x} \left\{ \frac{\left(\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \theta^2} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \theta^2} \right)}{\left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{3/2}} \left[\lambda_\theta J - \lambda_v \left(\frac{\partial x^1}{\partial \theta} \frac{\partial x^1}{\partial \rho} + \frac{\partial x^2}{\partial \theta} \frac{\partial x^2}{\partial \rho} \right) - \right. \right. \right. \\
 & \left. \left. \left. \frac{\left(\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \rho \partial \theta} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \rho \partial \theta} \right) \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right] \right]}{\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \theta^2} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \theta^2}} \right] \right\} \Psi \right\} +
 \end{aligned}$$

$$\sigma \Psi = S \quad (28)$$

Although certain similarities between this equation and the toroidal equation (21) are apparent, a more striking similarity can be achieved by putting the toroidal equation into what we shall call cylindrical form. We accomplish this by introducing the transformation,

$$z = x^1(\rho, \theta) \phi \quad (29)$$

Under this transformation the toroidal equation becomes,

$$\begin{aligned}
 & \frac{1}{J} \left\{ \frac{\partial}{\partial \rho} \left\{ \lambda_v \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2} \Psi \right\} + \right. \\
 & \frac{\partial}{\partial \theta} \left\{ \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{-1/2} \left[\lambda_\theta J - \lambda_v \left(\frac{\partial x^1}{\partial \theta} \frac{\partial x^1}{\partial \rho} + \frac{\partial x^2}{\partial \theta} \frac{\partial x^2}{\partial \rho} \right) \right] \Psi \right\} + \\
 & \frac{\partial}{\partial z} \left\{ \left[\lambda_z + z \left\{ \frac{\lambda_\theta \frac{\partial x^1}{\partial \theta} + \lambda_v \frac{\partial x^2}{\partial \theta}}{x^1 \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2}} \right\} \right] J \Psi \right\} + \\
 & \frac{\partial}{\partial x} \left\{ J \left[\frac{\left(\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \theta^2} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \theta^2} \right)}{\left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{3/2}} \left[\lambda_\theta - \frac{\lambda_v}{J} \left(\frac{\partial x^1}{\partial \theta} \frac{\partial x^1}{\partial \rho} + \frac{\partial x^2}{\partial \theta} \frac{\partial x^2}{\partial \rho} \right) \right. \right. \right. \\
 & \left. \left. \left. - \frac{\left(\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \rho \partial \theta} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \rho \partial \theta} \right) \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]}{\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \theta^2} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \theta^2}} \right] \right\} + \\
 & \left. \frac{(\lambda_z)^2}{(\lambda_v^2) + (\lambda_\theta)^2} \frac{\lambda_\theta \frac{\partial x^2}{\partial \theta} - \lambda_v \frac{\partial x^1}{\partial \theta}}{x^1 \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2}} \right\} \Psi \Bigg\} + \\
 & \frac{\partial}{\partial \xi} \left\{ \lambda_z J \frac{\lambda_v \frac{\partial x^2}{\partial \theta} + \lambda_\theta \frac{\partial x^1}{\partial \theta}}{x^1 \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{1/2}} \Psi \right\} +
 \end{aligned}$$

$$\sigma \Psi = S$$

(30)

Equation (30) is the same as equation (28) save for the appearance of additional terms due to the effect of toroidal curvature. To facilitate our discussion of these toroidal effects we state the following information about the surface $\rho=\text{const.}$ (see Fig. 1): The curves $\theta=\text{const.}$ and $\phi=\text{const.}$ are lines of curvature of this surface and the directions defined by the unit tangent vectors to these curves are the principal directions of normal curvature of the surface. The principal curvatures are,

$$\kappa_1 = \frac{\left[\frac{\partial x^1}{\partial \theta} \frac{\partial^2 x^2}{\partial \theta^2} - \frac{\partial x^2}{\partial \theta} \frac{\partial^2 x^1}{\partial \theta^2} \right]}{\left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{3/2}} \quad (31)$$

and

$$\kappa_2 = - \frac{\partial x^2 / \partial \theta}{x^1} \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{-1/2} \quad (32)$$

We note that,

$$\kappa_1 = \kappa_p \quad (33)$$

The geodesic curvature of the $\theta=\text{const.}$ curve is,

$$\kappa_g = - \frac{\partial x^1 / \partial \theta}{x^1} \left[\left(\frac{\partial x^1}{\partial \theta} \right)^2 + \left(\frac{\partial x^2}{\partial \theta} \right)^2 \right]^{-1/2} \quad (34)$$

Finally and importantly we point out that,

$$\left[\kappa_2 \right]^2 + \left[\kappa_g \right]^2 = \frac{1}{\left[x^1 \right]^2} \equiv \left[\kappa_T \right]^2 \quad (35)$$

After scaling all lengths to λ , the physical scale length of the problem, we see from equation (35) which contains a sum of square terms on the left hand side that the necessary and sufficient condition for the neglect of the toroidal effects is that,

$$\frac{\lambda}{x^1} \ll 1 \quad (36)$$

If in addition,

$$\lambda \kappa_p \ll 1, \quad (37)$$

all of the effects of curvature vanish and a planar problem results.

V. DIFFUSION EQUATIONS

The general form of the diffusion equation for an axisymmetric toroidal geometry may be found by substituting the appropriate quantities into,

$$\left(|d| \right)^{-1/2} \frac{\partial}{\partial x^1} \left\{ \left(|d| \right)^{1/2} d^{ij} J_j \right\} + \sigma_a \cdot T = S, \quad (38)$$

where J_j , are the covariant components of the current, related to the scalar flux, T , by,

$$J_j = - \frac{1}{3\sigma} \frac{\partial T}{\partial x^j} \quad (39)$$

σ_a is the absorption cross section and the metric tensor d_{ij} is formed by applying T_1 to the metric for Cartesian three space. In terms of the (ρ, θ, ϕ) coordinates the equation is

$$\begin{aligned} & \frac{1}{x^1 J} \left\{ \frac{\partial}{\partial \rho} \left\{ x^1 \left(\frac{d_{22}}{J} J_\rho + \frac{d_{12}}{J} J_\theta \right) \right\} + \right. \\ & \quad \left. \frac{\partial}{\partial \theta} \left\{ x^1 \left(\frac{d_{11}}{J} J_\theta + \frac{d_{12}}{J} J_\rho \right) \right\} + \right. \\ & \quad \left. \frac{\partial}{\partial \phi} \left\{ \frac{J}{x^1} J_\phi \right\} \right\} + \\ & \sigma_a T = S . \end{aligned} \tag{40}$$

J defined as before in equation (9). We now turn our attention to the (2×2) submatrix,

$$\hat{d}_{ij} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} , \tag{41}$$

of the metric d_{ij} . First we note that,

$$|\hat{d}_{ij}| = J^2 , \tag{42}$$

and now assume that it is possible, by a suitable change of variables, to write

$$d_{ij} = \mu^2 \delta_{ij} , \quad (43)$$

where δ_{ij} is a Kronecker delta. This can always be accomplished for an orthogonal coordinate system, (i.e. one for which $\hat{d}_{12} = \hat{d}_{21} = 0$), by introducing variables ρ' and θ' defined by

$$d\rho = \sqrt{d_{22}} d\rho' , \quad (44)$$

and

$$d\theta = \sqrt{d_{11}} d\theta' . \quad (45)$$

The element of arc for these coordinates is

$$ds^2 = d_{11}d_{22} (d\rho')^2 + d_{11}d_{22} (d\theta')^2 , \quad (46)$$

from which we see that

$$\mu^2 = d_{11}d_{22} . \quad (47)$$

For such a system of orthogonal coordinates the diffusion equation is

$$\frac{1}{x^1 J} \left\{ \frac{\partial}{\partial \rho'} \left\{ x^1 J_{\rho'} \right\} + \frac{\partial}{\partial \theta'} \left\{ x^1 J_{\theta'} \right\} + \frac{\partial}{\partial \phi} \left\{ \frac{J}{x^1} J_{\phi} \right\} \right\} + \sigma_{\alpha} T = S \quad (48)$$

Noting that $J = \mu^2$ and eliminating the independent variable ρ' in favor of μ by a logarithmic transformation,

$$\rho' = \ln \mu \quad , \quad (49)$$

yields

$$\frac{-1}{x^1 \mu^2} \left\{ \mu \frac{\partial}{\partial \mu} \left\{ x^1 D \mu \frac{\partial T}{\partial \mu} \right\} + \frac{\partial}{\partial \theta'} \left\{ x^1 D \frac{\partial T}{\partial \theta'} \right\} + \frac{\partial}{\partial \phi} \left\{ \frac{\mu^2}{x^1} D \frac{\partial T}{\partial \phi} \right\} \right\} + \sigma_{\alpha} T = S \quad , \quad (50)$$

where we have used $D \equiv (1/3\sigma)$. Pomraning^{9,10} has noted that if the ϕ derivative term is dropped and if the diffusion coefficient absorption coefficient, and source function are redefined as

$$D' = x^1 D \quad , \quad (51)$$

$$s' = x^1 s \quad (52)$$

and

$$\sigma_{a'} = x^1 \sigma_a, \quad (53)$$

that a formally identical structure to the "r- θ " cylindrical diffusion equation is obtained for the two cases of concentric circular and orthogonal-elliptical toroidal coordinates. We have proven that this homology is obtained for all orthogonal, axisymmetric toroidal coordinate systems providing the toroidal coordinate is ignorable in the problem of interest.

VI. CONCLUDING REMARKS

In this paper we have presented several results of importance in the application of discrete ordinate methods to the solution of transport problems in toroidal geometries. Such problems arise in the design and analysis of the blanket structures of proposed fusion reactors, as well as in attempting to apply these techniques to the analysis of the transport of neutral particles in a tokamak plasma.

The axisymmetric toroidal coordinates system in which we have written the single energy group transport equation, allows for the selection of coordinate surfaces coincident with surfaces delimiting regions of constant physical properties in typical tokamak devices. This eliminates the need to model toroidal regions with complicated combinations of non-toroidal geometric elements. Equation (21) has been written in conservation law form appropriate to the derivation of finite difference equations by integration. The general nature of the equation, $X^1(\rho, \theta)$ and $X^2(\rho, \theta)$ are unspecified, allows one to consider the design of a general purpose toroidal transport code which would accept as input data the specification of X^1 and X^2 . Alternatively equation (21) can be used to quickly derive transport equations for particular geometries of interest. This has been illustrated in Section III.

It is clear from the analysis given in Section IV. that many circumstances can be envisioned in which the neglect of the effects of one or both of the curvatures of the system would lead to serious errors. For example, Fig. 2 is a schematic representation of a set of superconducting toroidal field coils for a future tokamak device. Accurate determinations will have to be made of the radiation and neutron fluxes to determine the heat load on the cryostats, especially in the central core region where

the space available for shielding is at a premium. Particularly here, where X^1 is small, (comparable to characteristic mean-free paths). Accurate answers will require that the curvature effects not be neglected. Equation (21) is suited to the solution of this type of problem.

In Section V. an important conclusion about diffusion equations is reached. The diffusion equation for any orthogonal, axisymmetric toroidal coordinate system in which there are no variations in the toroidal direction, can be written in a form which is formally homologous to the "r- θ " cylindrical diffusion equation. This similarity of structure was first pointed out by Pomraning and Stevens⁹ for a toroidal system having circular cross sections and later Pomraning¹⁰ found the same result for an orthogonal elliptical toroidal system. Our analysis extends this result to a general class of toroidal coordinate systems.

REFERENCES

¹R. G. Mills, ed., "A Fusion Power Plant", Princeton Plasma Physics Laboratory Report MATT-1050 (1974).

²B. Badger et. al., "Wisconsin Tokamak Reactor Design", Vol. 1, Report UWFD-68, University of Wisconsin, Madison, Wisconsin (1974).

³W. M. Stacey, Jr. et.al., Tokamak Experimental Power Reactor Studies Argonne National Laboratory CTR Technical Memorandum-75-2 (1975).

⁴General Atomic Engineering Staff, Experimental Power Reactor Conceptual Design Study, (Progress Report for July 1, 1974 - June 30, 1975) GA-A13534 (1975).

⁵E. Greenspan, Calculation of the Transport of Neutral Atoms in Highly Ionized Plasmas Using Neutron Transport Methods, Nuclear Fusion 14, 771-778 (1974).

⁶J. G. Gilligan, S. L. Gralnick, W. G. Price, Jr., Transport of Neutral Impurity Atoms through a Plasma, Bull. Amer. Phys. Soc. II, 19, 853, (1974).

⁷R. L. Miller and G. H. Miley, Monte Carlo Simulation of Neutral Beam Injection Into Fusion Reactors, Second International Conference on Plasma Science (Ann Arbor, Michigan, May 1975), to be published.

⁸S. L. Gralnick, A Tensor Transformation Technique for the Transport Equation, (Submitted for publication in Nuclear Science & Engineering); Princeton Plasma Physics Laboratory Report MATT-1154 (1975).

⁹G. C. Pomraning and C. A. Stevens, Transport and Diffusion Equations in Toroidal Geometry, Nuclear Science and Engineering, 55, 359-367 (1974).

¹⁰G. C. Pomraning, Elliptical Toroidal Geometry - A Geometric Transport Correction, Nuclear Science and Engineering 57, 188-195 (1975).

¹¹J. Jung, Finite Difference Equations For Transport Equation In Toroidal Geometry, Argonne National Laboratory CTR Technical Memorandum No. 52 (1975).

¹²D. W. Drawbaugh, The Tensor Form of the Neutron-Transport Equation with Application to Finite Differencing, Nuclear Science and Engineering 44, 58-65 (1971).

ACKNOWLEDGEMENT

The author wishes to thank his many colleagues at the Princeton Plasma Physics Laboratory for their continuing assistance. In particular, I wish to thank Dr. William G. Price, Jr. for many helpful and illuminating discussions. This research was supported by the Energy Research and Development Administration under contract E(11-1)-3073.

754403

Fig. 1. Geometry of the ρ, θ, ϕ axisymmetric toroidal coordinate system generated by rotating a nest of smooth curves about the Y axis.

744497

Fig. 2. Toroidal Field Coils proposed for a future Tokamak power reactor (see Reference 1).