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ENERGY-DEPENDENT PION-NUCLEON

PHASE SHIFT ANALYSIS

0-700 MeV, $\ell_m = 4$ SOLUTION

Livermore, California

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L. David Roper
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ABSTRACT

The equations used in an energy-dependent pion-nucleon phase shift analysis are listed. A list of pion-nucleon scattering data references is included. The best solution obtained using $\ell_m = 4$ for a selected set of 0-700-MeV data is given and graphical comparisons between the solution and the data are given. This solution's predictions for observables that have not yet been measured are given. There are three resonances in this solution: P_{33} at 193 MeV laboratory kinetic energy (mass = 1234 MeV); P_{11} at 575 MeV (mass = 1497 MeV); and D_{13} at 638 MeV (mass = 1534 MeV).

NOTATION

$\mu = \pi^{\pm}$ mass = 139.59 MeV (set equal to one; all energies, momenta, and masses are measured in units of μ except E_{π}).

M = proton mass = 6.7212.

$\mu_p - 1$ = proton anomalous magnetic moment = 1.79275 nuclear magnetons.

$\lambda = \frac{\hbar}{\mu c} = \pi^{\pm}$ Compton wave length = 1.4135×10^{-13} cm.

$\alpha = \frac{e^2}{4\pi\hbar c}$ = fine structure constant = 7.29729×10^{-3} .

$\hbar = c = 1$.

E_{π} = incident pion kinetic energy in the laboratory system (measured in MeV; all other energies are measured in units of μ .)

$\vec{k} = \vec{q}$ = incident pion momentum in center-of-mass system (c. m. s.).

$\vec{p} = -\vec{k}$ = target proton momentum in c. m. s.

$\vec{k}'$ = outgoing c. m. s. pion momentum in an elastic process ($k = k'$).

$\hat{n} = \frac{\vec{k} \times \vec{k}'}{|\vec{k} \times \vec{k}'|}$ = unit vector perpendicular to scattering plane.

θ = c. m. s. angle between incident pion and outgoing pion momenta
($\vec{k} \cdot \vec{k}' = k^2 \cos \theta$).

q_0 = incident pion total energy in c. m. s. ($q_0^2 = k^2 + \mu^2$).

p_0 = target proton total energy in c. m. s. ($p_0^2 = k^2 + M^2$).

$W = p_0 + q_0$ = total reaction energy in c. m. s.

ℓ = relative orbital angular momentum quantum number of π -N system.

j = total angular momentum quantum number of π -N system.

$\ell \pm$ subscript indicates a quantity associated with the state of orbital angular momentum ℓ and total angular momentum $j = \ell \pm 1/2$.

T = total isotopic spin of π -N system ($T = 3/2$ or $1/2$).

$2T$ superscript indicates a quantity associated with the state of total isospin T .

$\pm$ superscript indicates a quantity associated with the $\pi^{\pm} + p \rightarrow \pi^{\pm} + p$ reaction.

cx superscript indicates a quantity associated with the charge-exchange reaction.

$\sigma(\theta)$ = differential cross section (some authors denote this by $d\sigma/d\Omega$).

σ_T = total cross section ($\sigma_T = \sigma_{el} + \sigma_{in}$).

$\mathcal{P}(\theta)$ = recoil-nucleon polarization.

$A(\theta)$ and $R(\theta)$ are the spin-rotation parameters.

$f(\theta)$ = non-spin-flip π -N amplitude.

$g(\theta)$ = spin-flip π -N amplitude.

$\delta_{\ell\pm}^{(2T)}$ = phase shift.

$\eta_{\ell\pm}^{(2T)}$ = absorption parameter or inelastic parameter.

$A_{\ell\pm}^{(2T)}$ = partial-wave amplitude.

Designation of scattering states is identical to Ref. 7, e.g., S_{31} designates the $\ell = 0$, $T = 3/2$, $J = 1/2$ state.

I. INTRODUCTION

In this brief report are listed the various equations used in the PiP (pions on protons) energy dependent phase shift analysis.^{1,2,3} A list of pion-nucleon scattering data references is included. The best solution obtained using $\ell_m = 4$ for a selected set of 0-700-MeV data is given and graphical comparisons between the solution and the data are given. This solution's predictions for observables that have not yet been measured are given.

II. EQUATIONS FOR PION-NUCLEON SCATTERING

A. Basic Equations

We list the basic equations for pion-nucleon scattering here so that our notation will be clear.¹

Total Cross Section

$$\sigma_T = \frac{4\pi\hbar}{k} f(0),$$

where $\hbar = \frac{h}{\mu c} = 1.4135 \times 10^{-13}$ cm and k is the incident pion c.m. momentum.

Differential Cross Section for Unpolarized Target

$$\sigma(\theta) = |f(\theta)|^2 + |g(\theta)|^2,$$

where θ is the c.m. pion scattering angle.

Recoil-Nucleon Polarization for Unpolarized Target

$$\mathcal{P}(\theta) = -2 \operatorname{Im} [f^*(\theta) g(\theta)] \hat{n} / \sigma(\theta)$$

where $\hat{n} = (\vec{k} \times \vec{k}') / |\vec{k} \times \vec{k}'|$ and k' = final pion c.m. momentum.

Differential Cross Section for Polarized Target

$$\sigma(\theta) = |f(\theta)|^2 + |g(\theta)|^2 - 2 \operatorname{Im} [f^*(\theta) g(\theta)] (\hat{n} \cdot \vec{\mathcal{P}}_i)$$

Recoil-Nucleon Polarization for Polarized Target

$$\begin{aligned} \vec{\mathcal{P}}_f(\theta) = & \left\{ -2 \operatorname{Im} [f^*(\theta) g(\theta)] \hat{n} - 2 \operatorname{Re} [f^*(\theta) g(\theta)] (\hat{n} \times \vec{\mathcal{P}}_i) \right. \\ & \left. + (|f(\theta)|^2 + |g(\theta)|^2) (\hat{n} \cdot \vec{\mathcal{P}}_i) \hat{n} - (|f(\theta)|^2 - |g(\theta)|^2) [\hat{n} \times (\hat{n} \times \vec{\mathcal{P}}_i)] \right\} / \sigma(\theta), \end{aligned}$$

where $\vec{\mathcal{P}}_i$ is the initial polarization.

Non-Spin-Flip Amplitude

$$f(\theta) = \frac{\pi}{k} \sum_{\ell=0}^{\ell_m} [(\ell+1)A_{\ell+} + \ell A_{\ell-}] P_{\ell}(\cos \theta),$$

where ℓ_m is the maximum value of the angular momentum ℓ used in the analysis.

Spin-Flip Amplitude

$$g(\theta) = \frac{\pi}{k} \sum_{\ell=0}^{\ell_m} [A_{\ell+} - A_{\ell-}] P_{\ell}^1(\cos \theta)$$

Partial-Wave Amplitude

$$A_{\ell\pm} = \frac{1}{2i} \left[\eta_{\ell\pm} e^{2i\delta_{\ell\pm}} - 1 \right]$$

where $\delta_{\ell\pm}$ is the phase shift and $\eta_{\ell\pm}$ is the absorption parameter for scattering in the states with $J = \ell \pm 1/2$.

Observable Amplitudes in Terms of Isotopic Spin Amplitudes

$$A_{\ell\pm}^+ = A_{\ell\pm}^{(3)}, \quad A_{\ell\pm}^- = \frac{1}{3} \left(A_{\ell\pm}^{(3)} + 2A_{\ell\pm}^{(1)} \right), \quad A_{\ell\pm}^{\text{cx}} = \frac{\sqrt{2}}{3} \left(A_{\ell\pm}^{(3)} - A_{\ell\pm}^{(1)} \right).$$

B. Electromagnetic Effects

To account for electromagnetic effects in $\pi^{\pm} + p \rightarrow \pi^{\pm} + p$ scattering we use the method of Foote et al.⁴ with the following changes: (1) The fully relativistic amplitudes to first order in α are used rather than amplitudes correct only to second order in $\beta_p = k/p_0$. (2) No attempt is made to preserve unitarity. (The addition of the electromagnetic amplitude to the nuclear amplitude causes unitarity to be violated, but only slightly so except at very low energies.)

The relativistic electromagnetic amplitudes to first order in α are¹

$$f_{\text{rem}}^{\pm}(\theta) = \frac{\mp a \lambda}{2W(1 - \cos \theta)} \left[\frac{W - M}{p_0 - M} + \frac{W + M}{p_0 + M} \cos \theta - (\mu_p - 1) \frac{q_0}{M} (1 - \cos \theta) - (\mu_p - 1) \frac{p_0 - M}{2M} \sin^2 \theta \right] \quad (1)$$

and

$$g_{\text{rem}}^{\pm}(\theta) = \frac{\pm a \lambda \sin \theta}{2W(1 - \cos \theta)} \left[\frac{W + M}{p_0 + M} + (\mu_p - 1) \frac{W + q_0 + M}{2M} + (\mu_p - 1) \frac{p_0 - M}{2M} \cos \theta \right],$$

where the proton total c.m. energy is $p_0 = (k^2 + M^2)^{1/2}$, the pion total c.m. energy is $q_0 = (k^2 + 1)^{1/2}$, and the total energy is $W = p_0 + q_0$. These equations agree with Solmitz' equations⁵ in the approximation that Solmitz uses.

The nonrelativistic electromagnetic (Coulomb) amplitudes correct to all orders in a are

$$f_{\text{Coul}}^{\pm}(\theta) = \frac{\mp a \lambda (q_0 p_0 + k^2)}{k^2 W (1 - \cos \theta)} \exp \left[\mp i \frac{a (q_0 p_0 + k^2)}{kW} \ln \left(\frac{1 - \cos \theta}{2} \right) \right] \quad \text{and } g_{\text{Coul}}^{\pm}(\theta) = 0. \quad (2)$$

(The kinematic factors in $f_{\text{Coul}}^{\pm}(\theta)$ reduce to the usual ones in the non-relativistic limit.) These amplitudes are obtained by solving Schroedinger's equation with a Coulomb potential. To first order in a the Coulomb amplitudes are

$$f_{\text{Coul } a}^{\pm}(\theta) = \frac{\mp a \lambda (q_0 p_0 + k^2)}{k^2 W (1 - \cos \theta)} \quad \text{and } g_{\text{Coul } a}^{\pm}(\theta) = 0. \quad (3)$$

Taking the nonrelativistic limit of the relativistic amplitudes (Eqs.(1)) yields the same amplitudes given in Eq. (3), as it should. Thus, electromagnetic amplitudes correct nonrelativistically to all orders in a and relativistically to first order in a can be obtained by subtracting Eqs. (3) from the sum of Eqs. (1) and (2):

$$f_{em}^{\pm}(\theta) = f_{rem}^{\pm}(\theta) + f_{Coul}^{\pm}(\theta) - f_{Coul a}^{\pm}(\theta)$$

and

$$g_{em}^{\pm}(\theta) = g_{rem}^{\pm}(\theta) + g_{Coul}^{\pm}(\theta) - g_{Coul a}^{\pm}(\theta) = g_{rem}^{\pm}(\theta).$$

These are the electromagnetic amplitudes used in this work.

The total amplitudes for $\pi^{\pm} + p \rightarrow \pi^{\pm} + p$ scattering are now

$$f^{\pm}(\theta) = f_N^{\pm}(\theta) + f_{em}^{\pm}(\theta) \quad \text{and} \quad g^{\pm}(\theta) = g_N^{\pm}(\theta) + g_{em}^{\pm}(\theta).$$

The nuclear amplitudes (subscript N) modified to include the Coulomb phase shifts Φ_{ℓ} are⁶:

$$f_N^{\pm}(\theta) = \frac{\pi}{k} \sum_{\ell=0}^{\ell_m} e^{\pm 2i\Phi_{\ell}} \left[(\ell+1) A_{\ell+}^{\pm} + \ell A_{\ell-}^{\pm} \right] P_{\ell}(\cos \theta)$$

and

$$g_N^{\pm}(\theta) = \frac{\pi}{k} \sum_{\ell=0}^{\ell_m} e^{\pm 2i\Phi_{\ell}} \left[A_{\ell+}^{\pm} - A_{\ell-}^{\pm} \right] P_{\ell}^1(\cos \theta),$$

where

$$\Phi_{\ell} = \sum_{n=1}^{\ell} \tan^{-1} \left[\frac{a(q_0 p_0 + k^2)}{k W n} \right] \quad \text{for } \ell > 0 \quad (\Phi_0 = 0).$$

For charge-exchange scattering we follow the method used by Rugge and Vik² (cf. Sorenson's work⁸) in which a Coulomb phase shift of one-half of the $\pi^{-} + p \rightarrow \pi^{-} + p$ phase shift is used. There is no electromagnetic amplitude. Thus, for the charge-exchange amplitudes we use

$$f^{cx}(\theta) = f_N^{cx}(\theta) = \frac{\pi}{k} \sum_{\ell=0}^{\ell_m} e^{-i\Phi_{\ell}} \left[(\ell+1) A_{\ell+}^{cx} + \ell A_{\ell-}^{cx} \right] P_{\ell}(\cos \theta)$$

and

$$g^{cx}(\theta) = g_N^{cx}(\theta) = \frac{\pi}{k} \sum_{\ell=0}^{\ell_m} e^{-i\Phi_{\ell}} \left[A_{\ell+}^{cx} - A_{\ell-}^{cx} \right] P_{\ell}^1(\cos \theta). \quad (4)$$

C. Parametrization of Partial-Wave Amplitudes

Rather than doing single energy phase shift analyses and then trying to continuously connect the single energy phases together as functions of energy, we chose to parametrize the phase shifts and absorption parameters as functions of energy with variable parameters. The success achieved by a similar scheme for proton-proton scattering gave us confidence that such a scheme would work in this case. The computer code is set up such that many types of energy dependent forms can be easily inserted. As a first try we chose the following parametrization. We parametrized the partial-wave amplitudes as follows:

$$A_{\ell\pm}^{(2T)} = \epsilon A_{\ell\pm}^{(2T)}(\text{res.}) + A_{\ell\pm}^{(2T)}(\text{nonres.});$$

where

$$A_{\ell\pm}^{(2T)}(\text{res.}) = \frac{-(1/2) \Gamma_{el\ell\pm}^{(2T)}}{(q_0 - q_{0\ell\pm}^{(2T)}) + i(1/2) \Gamma_{t\ell\pm}^{(2T)}} \quad (\text{see Layson's work}^9),$$

$$A_{\ell\pm}^{(2T)}(\text{nonres.}) = \frac{1}{2i} \left(\frac{\bar{\eta}_{\ell\pm}^{(2T)}}{\eta_{\ell\pm}^{(2T)}} e^{2i\delta_{\ell\pm}^{(2T)}} - 1 \right),$$

$$\Gamma_{el\ell\pm}^{(2T)} = \frac{4M}{q_0 + q_{0\ell\pm}^{(2T)}} k r_{0\ell\pm}^{(2T)} y_{\ell\pm}^{(2T)^2} \mathcal{V}_{\ell} \left(k r_{0\ell\pm}^{(2T)} \right),$$

$$\mathcal{V}_{\ell} \left(k r_{0\ell\pm}^{(2T)} \right) = 1/k^2 r_{0\ell\pm}^{(2T)^2} \left[j_{\ell}^2 \left(k r_{0\ell\pm}^{(2T)} \right) + n_{\ell}^2 \left(k r_{0\ell\pm}^{(2T)} \right) \right],$$

$$\Gamma_{t\ell\pm}^{(2T)} = \Gamma_{el\ell\pm}^{(2T)} + \Gamma_{in\ell\pm}^{(2T)}, \quad \Gamma_{in\ell\pm}^{(2T)} = \bar{\Gamma}_{in\ell\pm}^{(2T)} (k - k_0)^{2\ell+1},$$

$$k_0 = \text{threshold pion c. m. momentum} = 1.479,$$

$$q_0 = \sqrt{1 + k^2} = \text{total pion c. m. energy},$$

$$\epsilon = \begin{cases} 1 & \text{for states in which we chose to put a resonance,} \\ 0 & \text{for states in which we do not put a resonance,} \end{cases}$$

$$\bar{\delta}_{\ell\pm}^{(2T)} = k^{2\ell+1} \sum_{n=0}^{\ell-m-\ell} \left[a_{\ell\pm}^{(2T)} \right]_n k^n,$$

$$\eta_{\ell\pm}^{(2T)} = e^{-2\gamma},$$

and

$$\gamma = \begin{cases} 0 & \text{for } k < k_0 \\ (k - k_0)^{2\ell+1} \sum_{n=0}^{\ell-m-\ell} \left[b_{\ell\pm}^{(2T)} \right]_n (k - k_0)^n & \text{for } k \geq k_0. \end{cases}$$

Thus the correct threshold behavior is guaranteed for the phase shifts and approximated for the absorption parameters.

The actual phase shifts and absorption parameters are

$$\delta_{\ell\pm}^{(2T)} = 1/2 \tan^{-1} \left[\frac{2 \operatorname{Re} A_{\ell\pm}^{(2T)}}{1 - 2 \operatorname{Im} A_{\ell\pm}^{(2T)}} \right]$$

and

$$\eta_{\ell\pm}^{(2T)} = 2 \sqrt{\left(\operatorname{Re} A_{\ell\pm}^{(2T)} \right)^2 + \left(1/2 - \operatorname{Im} A_{\ell\pm}^{(2T)} \right)^2}.$$

In the case of $\epsilon = 0$ these expressions are redundant. In order that unitarity be preserved we make the restrictions:

$$-1/2 \leq \operatorname{Re} A_{\ell\pm}^{(2T)} \leq 1/2, \quad 0 \leq \operatorname{Im} A_{\ell\pm}^{(2T)} \leq 1, \quad \text{and } 0 \leq \eta_{\ell\pm}^{(2T)} \leq 1.$$

The reduced elastic width $\gamma_{\ell\pm}^{(2T)^2}$ and the interaction range $r_{0\ell\pm}^{(2T)}$ could possibly be different for different states that resonate.

However, for simplicity, and because Layson found it to be a good approximation, we initially fixed them at the values given by Layson,⁹ $\gamma_{\ell\pm}^{(2T)^2} = 0.133$ and $r_{0\ell\pm}^{(2T)} = 0.71$. Later we allowed $\gamma_{\ell\pm}^{(2T)^2}$ to vary for each resonant state. It may be that the way we put in the background to a resonance is not a good way. Other ways will be tried in the future.

We input the coefficients of the lowest powers of k (scattering lengths) in the energy dependent form for $\bar{\delta}$ at the values given by Hamilton and Woolcock,¹⁰ except for the $T = 3/2$, $P_{3/2}$ phase shift, which was made resonant. They are

$$\begin{aligned} \left[a_0^{(1)} \right]_0 &= 0.171, \quad \left[a_0^{(3)} \right]_0 = -0.088, \\ \left[a_{1-}^{(1)} \right]_0 &= -0.101, \quad \left[a_{1-}^{(3)} \right]_0 = -0.038, \text{ and } \left[a_{1+}^{(1)} \right]_0 = -0.029. \end{aligned}$$

The 0-350-MeV analysis has been done with these scattering lengths fixed at the Hamilton-Woolcock values and, also, with their being allowed to vary. All other a and b coefficients were input as zero. A 0-350-MeV analysis was also done in which all a and b coefficients were input as zero. The former procedure yielded the best solution. A report describing the 0-350-MeV analysis is in preparation. The Hamilton-Woolcock 0-350-MeV solution was used as input in the 0-700-MeV analysis.

III. DATA

Figure 1 shows the energies at which data of the various kinds were available as of January, 1964. The data that are available are the total cross sections σ_T^+ and σ_T^- ; the elastic and charge exchange differential cross sections $\sigma^+(\theta)$, $\sigma^-(\theta)$, and $\sigma^{cx}(\theta)$; and the recoil-nucleon polarizations $P^+(\theta)$, $P^-(\theta)$, and $P^{cx}(\theta)$. (Only one value of $P^{cx}(\theta)$ is presently available.¹¹) No measurements have yet been made of the spin rotation parameters.¹²

One can treat the real part of the forward scattering amplitude at a given energy as datum when it is calculated from the dispersion relation integral over total cross sections at all energies. Cronin¹³ and, more recently, Niederer¹⁴ have done the necessary calculations. We use Niederer's preliminary values. Actually, we use the forward scattering differential cross section rather than the real part of the amplitude. The equations relating them are

$$\sigma^\pm(0) = \left[\text{Re } f^\pm(0) \right]^2 + \left[\frac{k}{4\pi\lambda} \sigma_T^\pm \right]^2$$

and

$$\sigma^{cx}(0) = \frac{1}{2} \left[\text{Re } f^+(0) - \text{Re } f^-(0) \right]^2 + \frac{1}{2} \left[\frac{k}{4\pi\lambda} (\sigma_T^+ - \sigma_T^-) \right]^2.$$

Values of these "data" were included at approximate 30-MeV intervals. A large error of 3 mb/sr was used in the least-squares fitting at all energies, but the final solution falls well inside this error. Later an error of 1 mb/sr was used at all energies.

A. Total Cross Sections

The authors have leaned heavily upon the work of Klepikov et al.¹⁵ at Dubna in obtaining the total cross section values. This Dubna work contains almost all of the total cross sections available in March, 1960. Several corrections to the original data were made by Klepikov, including "Coulomb corrections."

The total cross section references are listed in approximate chronological order in Appendix I. The numbers 1 and 2 after a reference indicate $\pi^+ - p$ and $\pi^- - p$ respectively.

B. $\pi^{\pm} + p$ Differential Cross Sections

The differential cross section references for $\pi^{\pm} + p \rightarrow \pi^{\pm} + p$ are given in approximate chronological order in Appendix II. The numbers 3, 4, and 5 after a reference indicate $\pi^{+} - p$ elastic, $\pi^{-} - p$ elastic, and $\pi^{-} - p$ charge-exchange, respectively.

It was desired to obtain all of the data in terms of values at each angle, rather than in terms of coefficients of $\cos \theta$ or Legendre polynomial series. It is not possible to do this in every case. Data obtainable only in terms of the coefficients were not used. It was necessary to extract some of the data from graphs, as several references contain the data at each angle in graph form only. For graphed data that do not contain error brackets, the errors were determined by consideration of the number of events observed at each angle. For angular intervals at which no events were observed in bubble chamber experiments, the error was taken as the error when one event is observed.

A few data were renormalized such that their elastic cross sections are more in conformity with the large body of data.

Originally only those data have been rejected whose original authors later stated that they were incorrect or later gave final versions. Later we found it necessary to discard more data because of computer restrictions. In this case new data were given preference to old.

C. Charge-Exchange Differential Cross Sections

The charge-exchange differential cross section references are given along with the $\pi^{\pm} + p \rightarrow \pi^{\pm} + p$ references in Appendix II. The number 3 after a reference indicates charge exchange. Only the recent data are obtainable in terms of values at each angle. The rest are given as coefficients of power series in $\cos \theta$, coefficients of Legendre polynomial series, values of the gamma-ray production differential cross section at each angle, or combinations of these three ways.

The relevant equations are

$$\sigma^{\text{cx}}(\theta) = \sum_{n=0}^{2\ell_m} a_n \cos^n \theta = \sum_{n=0}^{2\ell_m} b_n P_n(\cos \theta)$$

and

$$\sigma^Y(\theta) = \sum_{n=0}^{2\ell_m} k_n b_n P_n(\cos \theta),$$

where

$$k_n = \frac{\mu'^2}{k'^2} \int_{-1}^1 \frac{P_n(Z) dZ}{\left(\frac{q'_0}{k'} - Z\right)^2} = 2(n+1) \left[\frac{q'_0}{k'} Q_n\left(\frac{q'_0}{k'}\right) - Q_{n+1}\left(\frac{q'_0}{k'}\right) \right],$$

where q'_0 and k' are the total energy and momentum of π^0 in the c.m.

The relationships between the a_n and the b_n are

$$b_{2r} = (4r+1) \sum_{n=r}^{\ell_m} a_{2n} \frac{2n(2n-2) \dots (2n-2r+2)}{(2n+2r+1)(2n+2r-1) \dots (2n+1)}$$

and

$$b_{2r+1} = (4r+3) \sum_{n=r}^{\ell_m-1} a_{2n+1} \frac{2n(2n-2) \dots (2n-2r+2)}{(2n+2r+3)(2n+2r+1) \dots (2n+3)}$$

The relationship between an a_n and the partial-wave amplitudes $A_{\ell\pm}$ is

$$a_n = \frac{\pi^2}{k^2} \sum_{i \geq j} a_{ijn} \text{Re} \left[A_i A_j^* \right]$$

where the a_{ijn} are given in Table 1 for values of n from 0 to 8. (This corresponds to ℓ_m ranging from 0 to 4.) A few data were renormalized to conform with recent charge-exchange cross section measurements.

D. Recoil-Nucleon Polarization

The recoil-nucleon polarization references are listed in chronological order in Appendix III. The numbers 6, 7, and 8 after a reference indicate π^+ - p elastic, π^- - p elastic, and π^- - p charge-exchange, respectively.

The main consideration here is to maintain a fixed reference direction for the polarization. We shall use the usual direction, namely $\hat{n} = \vec{k} \times \vec{k}' / |\vec{k} \times \vec{k}'|$. All of the data are given in terms of this direction except that of Ref. 1 of Appendix III, which uses $\hat{n} = \vec{k}' \times \vec{k} / |\vec{k}' \times \vec{k}|$.

It was necessary to convert the angles from the laboratory system to the c. m. s. for the data of Refs. 1 and 3 of Appendix III.

Table 1. The numbers that relate the coefficients of $\cos \theta$ power series for the differential cross sections to the partial-wave amplitudes.

i	j	0	1	2	3	4	5	6	7	8
0	0	1								
0	1-		2							
0	1+		4							
0	2-	-2		6						
0	2+	-3		9						
0	3-		-9		15					
0	3+		-12		20					
0	4-	3		-30		35				
0	4+	15/4		-75/2		175/4				
1-	1-	1								
1-	1+	-2		6						
1-	2-		4							
1-	2+		-9		15					
1-	3-	-3		9						
1-	3+	3		-30		35				
1-	4-		-12		20					
1-	4+		75/4		-175/2		315/4			
1+	1+	1		3						
1+	2-		-10		18					
1+	2+				12					
1+	3-	3		-36		45				
1+	3+	-3		-6		25				
1+	4-		21		-110		105			
1+	4+		-15/2		-25		105/2			
2-	2-	1		3						
2-	2+	3		-36		45				
2-	3-				12					
2-	3+		21		-110		105			
2-	4-	-3		-6		25				
2-	4+	-15/4		375/4		-1225/4		945/4		
2+	2+	9/4		-9/2		45/4				
2+	3-		45/2		-117		225/2			
2+	3+		9		-30		45			
2+	4-	-9/2		207/2		-675/2		525/2		
2+	4+	-45/8		225/8		-675/8		735/8		
3-	3-	9/4		-9/2		45/4				
3-	3+	-9/2		207/2		-675/2		525/2		
3-	4-		9		-30		45			
3-	4+		-315/8		3075/8		-7245/8		4725/8	
3+	3+	9/4		45/4		-165/4		175/4		
3+	4-		-81/2		795/2		-1875/2		1225/2	
3+	4+				150/2		-210		175	
4-	4-	9/4		45/4		-165/4		175/4		
4-	4+	45/8		-225		5325/4		2450		11025/8
4+	4+	225/64		-225/16		3675/32		-4025/16		11025/64

IV. 0-700-MeV, $\ell_m = 4$ SOLUTION

A. The $T = 1/2$ and $T = 3/2$ phase shifts and absorption parameters are given in Figs. 2 and 3. The $T = 1/2$ and $T = 3/2$ real and imaginary parts of the partial wave amplitudes are given in Figs. 4 and 5. The values of all of these at various energies are listed in Tables 2 through 5.

B. The values of the coefficients in the parametrization are given in Tables 6 and 7. Note that the resonance reduced widths are allowed to vary.³

C. Comparison of the observables as calculated from the solution with some of the experimental data are given in Figs. 6 through 32. The values (N) by which the computer chose to divide the differential cross section data (renormalization) are placed on each graph. The data as shown have not been divided by N. The worse agreement with data occurs near the end of the energy range, a common feature of energy dependent analyses. It is worse for π^+ and charge-exchange, forward scattering, and differential cross sections. The fact that π^- forward scattering agrees with the dispersion relation calculation and that the fit to most of the π^- data is good indicates that the $T = 1/2$ part of the interaction is probably well determined. Further refinements in the analysis should improve the $T = 3/2$ situation.

D. Prediction of the solution for the partial cross sections σ_{el}^+ , σ_{el}^- , and σ^{cx} are shown in Fig. 33. Predictions of the inelastic cross sections σ_{in}^+ and σ_{in}^- are shown in Fig. 34. Agreement with experimental values is good.¹⁶ We plan to include these values as data in future analyses. Graphs of the $T = 1/2$ total cross section $\sigma_T^{(1/2)}$, elastic cross section $\sigma_{el}^{(1/2)}$, and inelastic cross section $\sigma_{in}^{(1/2)}$ are given in Fig. 35.

The relevant equations are:

$$\sigma_T^{(1/2)} = -\frac{1}{2} \sigma_T^+ + \frac{3}{2} \sigma_T^-$$

$$\sigma_{el}^{(1/2)} = -\frac{1}{2} \sigma_{el}^+ + \frac{3}{2} \sigma_{el}^- + \frac{3}{2} \sigma^{cx}$$

and

$$\sigma_{in}^{(1/2)} = -\frac{1}{2} \sigma_{in}^+ + \frac{3}{2} \sigma_{in}^-$$

Table 2a. Phase shifts at several energies (degrees) $T = 1/2$.

E_{π} (MeV)	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}
6	2.743	-0.082	-0.015	0	0	0	0	-0	0
25	5.001	-0.522	-0.123	0.003	0.002	0	0	-0	0
53	6.646	-1.132	-0.364	0.020	0.012	0	0	-0	0
78	7.621	-1.400	-0.632	0.057	0.032	0.002	0.001	-0	0
98	8.248	-1.335	-0.871	0.109	0.059	0.004	0.001	-0	0
128	9.058	-0.673	-1.258	0.230	0.121	0.011	0.003	-0	0
150	9.603	0.294	-1.557	0.361	0.186	0.021	0.006	-0	0.001
170	10.087	1.554	-1.836	0.516	0.261	0.034	0.010	-0.001	0.001
200	10.827	4.146	-2.262	0.824	0.405	0.066	0.018	-0.002	0.003
224	11.453	6.834	-2.604	1.146	0.548	0.104	0.028	-0.003	0.005
251	12.216	10.505	-2.988	1.607	0.742	0.166	0.043	-0.005	0.009
277	13.027	14.667	-3.354	2.169	0.961	0.250	0.061	-0.009	0.015
300	13.821	18.838	-3.672	2.784	1.183	0.347	0.082	-0.014	0.023
324	14.736	23.647	-4.000	3.569	1.443	0.476	0.108	-0.021	0.035
345	15.619	28.211	-4.282	4.404	1.692	0.618	0.136	-0.030	0.049
376	17.076	35.489	-4.873	5.955	2.098	0.882	0.184	-0.047	0.079
410	18.909	44.104	-5.132	8.241	2.591	1.261	0.248	-0.076	0.125
427	19.926	48.614	-5.350	9.691	2.855	1.492	0.284	-0.094	0.156
450	21.415	54.884	-5.643	12.080	3.229	1.853	0.338	-0.125	0.207
469	22.748	60.176	-5.886	14.525	3.551	2.197	0.387	-0.156	0.258
500	25.135	68.945	-6.285	19.757	4.099	2.861	0.474	-0.220	0.365
523	27.087	75.487	-6.586	24.989	4.518	3.444	0.545	-0.280	0.464
550	29.583	83.119	-6.949	33.132	5.022	4.236	0.633	-0.367	0.608
572	31.789	89.234	-7.253	41.735	5.437	4.976	0.709	-0.453	0.751
600	34.830	96.793	-7.658	55.682	5.965	6.054	0.809	-0.585	0.970
626	37.898	103.494	-8.053	73.137	6.451	7.202	0.904	-0.734	1.218
650	40.947	109.326	-8.440	98.336	6.889	8.366	0.992	-0.898	1.489
670	43.653	113.872	-8.780	124.224	7.243	9.505	1.065	-1.055	1.750
700	+48.003	120.057	-9.327	148.310	7.748	11.362	1.170	-1.333	2.211

Table 2b. Phase shifts at several energies (degrees) $T=3/2$.

E_{π} (MeV)	S_{31}	P_{31}	P_{33}	D_{33}	D_{35}	F_{35}	F_{37}	G_{37}	G_{39}
6	-0.804	-0.028	0.228	0	-0	0	0	-0	-0
25	-2.502	-0.233	2.012	0.002	-0.003	0	0	-0	-0
53	-4.805	-0.704	6.707	0.011	-0.023	0	0	-0	-0
78	-6.801	-1.237	13.183	0.031	-0.062	0	0	-0	-0
98	-8.364	-1.717	20.475	0.055	-0.111	0	0	-0	-0
128	-10.642	-2.497	36.512	0.111	-0.223	0.001	0	-0	-0
150	-12.251	-3.095	53.045	0.167	-0.336	0.002	0	-0.001	-0
170	-13.659	-3.644	70.594	0.230	-0.463	0.004	0	-0.001	-0.001
200	-15.658	-4.459	95.414	0.348	-0.700	0.007	0.001	-0.002	-0.001
224	-17.147	-5.085	110.265	0.461	-0.930	0.011	0.002	-0.004	-0.002
251	-18.689	-5.741	121.859	0.610	-1.232	0.018	0.003	-0.008	-0.004
277	-20.029	-6.309	129.532	0.772	-1.563	0.027	0.005	-0.014	-0.007
300	-21.087	-6.746	134.494	0.929	-1.887	0.038	0.007	-0.021	-0.011
324	-22.056	-7.125	138.470	1.105	-2.253	0.053	0.010	-0.031	-0.017
345	-22.783	-7.384	141.248	1.267	-2.595	0.069	0.013	-0.044	-0.024
376	-23.641	-7.627	144.517	1.516	-3.127	0.100	0.019	-0.070	-0.038
410	-24.273	-7.679	147.312	1.794	-3.737	0.145	0.029	-0.111	-0.060
427	-24.462	-7.614	148.485	1.932	-4.048	0.173	0.036	-0.138	-0.074
450	-24.579	-7.423	149.890	2.114	-4.468	0.216	0.046	-0.183	-0.099
469	-24.553	-7.171	150.920	2.257	-4.811	0.258	0.056	-0.229	-0.123
500	-24.263	-6.566	152.395	2.470	-5.353	0.340	0.076	-0.323	-0.174
523	-23.846	-5.956	153.256	2.606	-5.731	0.412	0.094	-0.412	-0.222
550	-23.131	-5.055	154.366	2.734	-6.136	0.511	0.120	-0.540	-0.291
572	-22.365	-4.168	155.110	2.807	-6.428	0.605	0.144	-0.666	-0.359
600	-21.147	-2.833	155.970	2.852	-6.736	0.743	0.181	-0.860	-0.463
626	-19.768	-1.382	156.694	2.836	-6.945	0.891	0.221	-1.080	-0.582
650	-18.279	0.145	157.308	2.763	-7.061	1.046	0.265	-1.320	-0.711
670	-16.878	1.558	157.785	2.656	-7.092	1.191	0.705	-1.552	-0.836
700	-14.499	3.921	158.448	2.404	-7.013	1.436	0.376	-1.961	-1.056

Table 3a. Absorption parameters at several energies $T = 1/2$.

E_{π} (MeV)	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}
6	1	1	1	1	1	1	1	1	1
25	1	1	1	1	1	1	1	1	1
53	1	1	1	1	1	1	1	1	1
78	1	1	1	1	1	1	1	1	1
98	1	1	1	1	1	1	1	1	1
128	1	1	1	1	1	1	1	1	1
150	1	1	1	1	1	1	1	1	1
170	0.999	1	1	1	1	1	1	1	1
200	0.998	0.997	1	0.999	1	1	1	1	1
224	0.999	0.990	0.999	0.999	1	1	1	1	1
251	1.000	0.971	0.997	0.998	1	1	1	1	1
277	1.000	0.941	0.995	0.997	1	1	1	1	1
300	0.998	0.905	0.991	0.995	1	1	1	1	1
324	0.994	0.856	0.987	0.993	1	1	1	1	1
345	0.988	0.807	0.983	0.991	1	0.999	1	1	1
376	0.975	0.725	0.976	0.987	1	0.998	1	1	1
410	0.953	0.628	0.968	0.978	1	0.996	1	1	1
427	0.938	0.581	0.963	0.973	0.999	0.994	0.999	1	1
450	0.916	0.519	0.958	0.962	0.999	0.991	0.999	1	1
469	0.894	0.471	0.953	0.950	0.999	0.988	0.999	1	1
500	0.852	0.401	0.947	0.918	0.999	0.981	0.997	1	0.999
523	0.816	0.359	0.944	0.875	0.999	0.974	0.996	1	0.999
550	0.770	0.318	0.942	0.789	0.998	0.966	0.993	1	0.997
572	0.729	0.293	0.943	0.678	0.998	0.958	0.990	1	0.996
600	0.673	0.273	0.945	0.484	0.998	0.947	0.985	1	0.994
626	0.619	0.266	0.952	0.294	0.998	0.938	0.979	1	0.991
650	0.567	0.272	0.963	0.178	0.998	0.930	0.971	1	0.987
670	0.524	0.289	0.975	0.168	0.999	0.925	0.963	1	0.983
700	0.460	0.343	1	0.234	1	0.923	0.947	1	0.974

Table 3b. Absorption parameters at several energies $T = 3/2$.

E_{π} (MeV)	S_{31}	P_{31}	P_{33}	D_{33}	D_{35}	F_{35}	F_{37}	G_{37}	G_{39}
6	1	1	1	1	1	1	1	1	1
25	1	1	1	1	1	1	1	1	1
53	1	1	1	1	1	1	1	1	1
78	1	1	1	1	1	1	1	1	1
98	1	1	1	1	1	1	1	1	1
128	1	1	1	1	1	1	1	1	1
150	1	1	1	1	1	1	1	1	1
170	0.998	1	1	1	1	1	1	1	1
200	0.993	1	1	1	1	1	1	1	1
224	0.992	1	1	1	1	1	1	1	1
251	0.993	0.999	1	1	1	1	1	1	1
277	0.995	0.997	1	1	0.999	1	1	1	1
300	0.996	0.993	1	1	0.999	1	1	1	1
324	0.997	0.988	1	1	0.997	1	1	1	1
345	0.998	0.981	1	1	0.996	1	1	1	1
376	0.999	0.968	1	1	0.992	1	1	1	1
410	0.999	0.946	1	1	0.986	1	1	1	1
427	0.999	0.932	1	1	0.983	1	1	1	1
450	0.998	0.910	1	1	0.977	1	1	1	1
469	0.997	0.889	1	1	0.972	1	1	1	1
500	0.995	0.849	1	1	0.963	1	1	1	1
523	0.994	0.815	1	0.999	0.957	1	1	1	1
550	0.993	0.770	1	0.999	0.950	1	1	1	1
572	0.992	0.730	1	0.999	0.946	1	1	1	1
600	0.992	0.675	1	0.998	0.943	1	1	1	1
626	0.992	0.621	1	0.998	0.944	1	1	1	1
650	0.994	0.569	1	0.997	0.950	1	1	1	1
670	0.996	0.525	1	0.995	0.959	1	1	1	1
700	1	0.458	1	0.994	0.983	1	1	1	1

Table 4a. Real parts of partial-wave amplitudes at several energies $T = 1/2$.

E_{π} (MeV)	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}
6	0.0478	-0.00143	-0.00027	0	0	0	0	-0	0
25	0.0869	-0.00912	-0.00215	0.00005	0.00003	0	0	-0	0
53	0.115	-0.0198	-0.00636	0.00035	0.00020	0.00001	0	-0	0
78	0.131	-0.0244	-0.01103	0.00100	0.00056	0.00003	0.00001	-0	0
98	0.142	-0.0233	-0.01519	0.00190	0.00103	0.00007	0.00002	-0	0
128	0.155	-0.0118	-0.02194	0.00402	0.00212	0.00019	0.00006	-0	0
150	0.164	0.00514	-0.0272	0.00630	0.00325	0.00036	0.00011	-0.00001	0.00001
170	0.172	0.0271	-0.0320	0.00900	0.00456	0.00059	0.00017	-0.00001	0.00002
200	0.184	0.0736	-0.0394	0.01436	0.00706	0.00115	0.00032	-0.00003	0.00005
224	0.194	0.1169	-0.0453	0.01998	0.00957	0.00182	0.00049	-0.00005	0.00009
251	0.207	0.174	-0.0519	0.0280	0.0129	0.00290	0.00074	-0.00009	0.00016
277	0.220	0.231	-0.0581	0.0377	0.01678	0.00435	0.00107	-0.00016	0.00027
300	0.231	0.276	-0.0634	0.0483	0.0207	0.00605	0.00143	-0.00025	0.00041
324	0.245	0.315	-0.0687	0.0618	0.0252	0.00831	0.00189	-0.00037	0.00062
345	0.256	0.336	-0.0732	0.0759	0.0295	0.01077	0.00237	-0.00052	0.00086
376	0.274	0.342	-0.0796	0.102	0.0366	0.0154	0.00321	-0.00083	0.00137
410	0.292	0.314	-0.0862	0.139	0.0451	0.0219	0.00432	-0.00132	0.00219
427	0.301	0.288	-0.0894	0.161	0.0497	0.0259	0.00496	-0.00164	0.00272
450	0.311	0.244	-0.0937	0.197	0.0562	0.0320	0.00590	-0.00218	0.00361
469	0.319	0.203	-0.0973	0.231	0.0618	0.0378	0.00675	-0.00272	0.00451
500	0.328	0.135	-0.103	0.292	0.0712	0.0489	0.00826	-0.00384	0.00636
523	0.331	0.0870	-0.108	0.335	0.0784	0.0584	0.00947	-0.00489	0.00809
550	0.331	0.0378	-0.113	0.361	0.0871	0.0711	0.0110	-0.00640	0.01059
572	0.326	0.00392	-0.118	0.337	0.0941	0.0828	0.0123	-0.00790	0.01306
600	0.316	-0.0320	-0.125	0.225	0.103	0.0993	0.0139	-0.0102	0.01683
626	0.300	-0.0603	-0.132	0.0816	0.111	0.117	0.0154	-0.0128	0.02105
650	0.281	-0.0849	-0.140	-0.0255	0.119	0.134	0.0168	-0.0157	0.02563
670	0.262	-0.1068	-0.147	-0.0783	0.125	0.151	0.0179	-0.0184	0.03000
700	0.228	-0.149	-0.160	-0.105	0.134	0.178	0.0193	-0.0233	0.03754

Table 4b. Real parts of partial-wave amplitudes at several energies $T = 3/2$.

E_{π} (MeV)	S_{31}	P_{31}	P_{33}	D_{33}	D_{35}	F_{35}	F_{37}	G_{37}	G_{39}
6	-0.0140	-0.00049	0.00398	0	-0	0	0	-0	-0
25	-0.0436	-0.00406	0.0351	0.00003	-0.00006	0	0	-0	-0
53	-0.0835	-0.01229	0.116	0.00020	-0.00040	0	0	-0	-0
78	-0.118	-0.02158	0.222	0.00053	-0.00108	0	0	-0	-0
98	-0.144	-0.02995	0.326	0.00096	-0.00194	0.00001	0	-0	-0
128	-0.182	-0.0435	0.478	0.00193	-0.00389	0.00002	0	-0	-0
150	-0.207	-0.0539	0.480	0.00291	-0.00586	0.00004	0	-0.00001	-0
170	-0.229	-0.0634	0.313	0.00402	-0.00808	0.00006	0.00001	-0.00002	-0.00001
200	-0.258	-0.0775	-0.0939	0.00607	-0.0122	0.00012	0.00002	-0.00004	-0.00002
224	-0.280	-0.0883	-0.325	0.00805	-0.0162	0.00019	0.00003	-0.00008	-0.00004
251	-0.301	-0.0994	-0.448	0.01064	-0.0215	0.00031	0.00005	-0.00014	-0.00008
277	-0.320	-0.109	-0.491	0.01347	-0.0273	0.00047	0.00008	-0.00024	-0.00013
300	-0.334	-0.116	-0.500	0.01621	-0.0329	0.00067	0.00012	-0.00036	-0.00019
324	-0.347	-0.122	-0.496	0.0193	-0.0392	0.00092	0.00017	-0.00055	-0.00029
345	-0.356	-0.125	-0.488	0.0221	-0.0450	0.00121	0.00023	-0.00077	-0.00041
376	-0.367	-0.127	-0.473	0.0265	-0.0540	0.00175	0.00034	-0.00122	-0.00066
410	-0.374	-0.125	-0.455	0.0313	-0.0641	0.00253	0.00051	-0.00194	-0.00104
427	-0.376	-0.122	-0.446	0.0337	-0.0692	0.00301	0.00062	-0.00241	-0.00130
450	-0.377	-0.117	-0.434	0.0369	-0.0759	0.00377	0.00080	-0.00320	-0.00172
469	-0.377	-0.110	-0.425	0.0394	-0.0813	0.00451	0.00098	-0.00400	-0.00215
500	-0.373	-0.096	-0.411	0.0431	-0.0895	0.00593	0.00132	-0.00564	-0.00304
523	-0.368	-0.084	-0.401	0.0454	-0.0951	0.00719	0.00164	-0.00719	-0.00387
550	-0.359	-0.0650	-0.390	0.0476	-0.101	0.00893	0.00209	-0.00942	-0.00507
572	-0.349	-0.0529	-0.382	0.0489	-0.105	0.01056	0.00251	-0.01162	-0.00626
600	-0.334	-0.0333	-0.372	0.0496	-0.110	0.0130	0.00316	-0.01502	-0.00809
626	-0.316	-0.0150	-0.363	0.0493	-0.113	0.0155	0.00386	-0.01884	-0.01015
650	-0.296	0.001	-0.356	0.0480	-0.116	0.0183	0.00462	-0.02304	-0.01241
670	-0.277	0.0143	-0.350	0.0461	-0.117	0.0208	0.00533	-0.02708	-0.01459
700	-0.242	0.0313	-0.342	0.0417	-0.11917	0.0251	0.00656	-0.03420	-0.01843

Table 5a. Imaginary parts of partial-wave amplitudes at several energies $T = 1/2$.

E_{π} (MeV)	S_{11}	P_{11}	P_{13}	D_{13}	D_{15}	F_{15}	F_{17}	G_{17}	G_{19}
6	0.00229	0	0	0	-0	-0	-0	-0	-0
25	0.00760	0.00008	0	0	-0	-0	-0	-0	-0
53	0.0134	0.00039	0.00004	0	0	-0	-0	-0	-0
78	0.0176	0.00060	0.00012	0	0	-0	-0	-0	-0
98	0.0206	0.00054	0.00023	0.00001	0	0	-0	-0	-0
128	0.0248	0.00014	0.00048	0.00005	0	0	-0	-0	-0
150	0.0278	0.00003	0.00074	0.00011	0.00001	0	-0	-0	-0
170	0.0311	0.00075	0.00103	0.00022	0.00002	0	0	-0	0
200	0.0360	0.00650	0.00170	0.00052	0.00005	0	0	-0	0
224	0.0398	0.01922	0.00259	0.00095	0.00009	0	0	-0	0
251	0.0448	0.0467	0.00411	0.00175	0.00017	0.00001	0	-0	0
277	0.0510	0.0897	0.00613	0.00298	0.00029	0.00005	0	0	0
300	0.0580	0.142	0.00838	0.00463	0.00045	0.00012	0	0	0
324	0.0674	0.210	0.0112	0.00717	0.00068	0.00027	0.00001	0	0
345	0.0776	0.277	0.0139	0.01035	0.00095	0.00052	0.00003	0	0
376	0.0967	0.382	0.0185	0.0175	0.00148	0.00120	0.00007	0	0.00002
410	0.124	0.490	0.0239	0.03095	0.00228	0.00261	0.00019	0	0.00005
427	0.140	0.537	0.0267	0.04119	0.00278	0.00366	0.00029	0	0.00008
450	0.164	0.588	0.0304	0.0609	0.00356	0.00556	0.00051	0	0.00015
469	0.187	0.619	0.0334	0.0846	0.00431	0.00762	0.00077	0.00001	0.00024
500	0.228	0.649	0.0378	0.146	0.00574	0.01205	0.00144	0.00001	0.00050
523	0.261	0.657	0.0405	0.219	0.00694	0.01630	0.00221	0.00002	0.00081
550	0.303	0.654	0.0429	0.341	0.00851	0.0224	0.00351	0.00004	0.00137
572	0.338	0.647	0.0441	0.461	0.00989	0.0283	0.00499	0.00006	0.00206
600	0.383	0.633	0.0442	0.588	0.0117	0.0369	0.00758	0.00010	0.00332
626	0.424	0.618	0.0426	0.622	0.0135	0.0459	0.01088	0.00016	0.00504
650	0.460	0.606	0.0395	0.585	0.0151	0.0548	0.0149	0.00025	0.00724
670	0.488	0.597	0.0354	0.531	0.0164	0.0626	0.0191	0.00034	0.00965
700	0.524	0.586	0.0263	0.448	0.0182	0.0743	0.0270	0.00054	0.01448

Table 5b. Imaginary parts of partial-wave amplitudes at several energies $T = 3/2$.

E_{π} (MeV)	S_{31}	P_{31}	P_{33}	D_{33}	D_{35}	F_{35}	F_{37}	G_{37}	G_{39}
6	0.00020	0	0.00002	-0	-0	-0	-0	-0	-0
25	0.00191	0.00002	0.00123	-0	-0	-0	-0	-0	-0
53	0.00702	0.00015	0.0136	0	0	-0	-0	-0	-0
78	0.01402	0.00047	0.0520	0	0	-0	-0	-0	-0
98	0.02116	0.00090	0.122	0	0	-0	-0	-0	-0
128	0.0341	0.00190	0.354	0	0.00002	-0	-0	-0	-0
150	0.0450	0.00291	0.639	0.00001	0.00003	-0	-0	-0	-0
170	0.0569	0.00404	0.890	0.00002	0.00007	0	-0	-0	0
200	0.0758	0.00606	0.991	0.00004	0.00015	0	-0	-0	0
224	0.0901	0.00799	0.880	0.00006	0.00028	0	-0	-0	0
251	0.1054	0.01059	0.721	0.00011	0.00056	0	-0	0	0
277	0.1194	0.01368	0.595	0.00018	0.00106	0	-0	0	0
300	0.131	0.01699	0.509	0.00026	0.00177	0	-0	0	0
324	0.142	0.02115	0.440	0.00037	0.00289	0	0	0	0
345	0.151	0.02549	0.392	0.00049	0.00424	0	0	0	0
376	0.161	0.0333	0.337	0.00071	0.00698	0	0	0	0
410	0.169	0.0440	0.292	0.00100	0.0111	0.00001	0	0	0
427	0.172	0.0503	0.273	0.00117	0.0136	0.00001	0	0.00001	0
450	0.174	0.0601	0.252	0.00142	0.0174	0.00001	0	0.00001	0
469	0.174	0.0690	0.236	0.00164	0.0208	0.00002	0	0.00002	0
500	0.170	0.0865	0.215	0.00203	0.0267	0.00004	0	0.00003	0.00001
523	0.1654	0.101	0.201	0.00233	0.0311	0.00005	0	0.00005	0.00002
550	0.157	0.121	0.187	0.00268	0.0358	0.00008	0	0.00009	0.00003
572	0.148	0.139	0.177	0.00297	0.0390	0.00011	0.00001	0.00014	0.00004
600	0.133	0.164	0.166	0.00333	0.0417	0.00017	0.00001	0.00023	0.00007
626	0.117	0.190	0.157	0.00365	0.0419	0.00024	0.00001	0.00036	0.00010
650	0.101	0.216	0.149	0.00394	0.0396	0.00033	0.00002	0.00053	0.00016
670	0.086	0.238	0.143	0.00419	0.0351	0.00043	0.00003	0.00073	0.00022
700	0.0627	0.273	0.135	0.00460	0.0230	0.00063	0.00004	0.00117	0.00035

Table 6. Phase shift coefficients.

$$\delta = k^{2\ell+1} \sum_{n=0}^{\ell} a_n k^n$$

Phase	a_0	a_1	a_2	a_3	a_4	q_{0r}	γ^2
S_{11}	+0.211817	-0.106994	+0.034278	-0.006008	+0.001537		
S_{31}	-0.027933	-0.105959	+0.005968	+0.005283	+0.000145		
P_{11}^a	-0.108652	+0.098605	-0.014148	-0.000356			
P_{31}	-0.031195	+0.007950	+0.000605	-0.667×10^{-4}			
P_{13}	-0.017529	+0.006562	-0.000563	-0.523×10^{-4}			
P_{33}^b	-0.897×10^{-10}	$+0.119 \times 10^{-8}$	$+0.644 \times 10^{-9}$	-0.185×10^{-9}		1.923	0.133
D_{13}^b	-0.001147	+0.000359	$+0.460 \times 10^{-6}$			3.493	0.0857
D_{33}	+0.000761	-0.000171	-0.637×10^{-5}				
D_{15}	+0.000723	-0.959×10^{-4}	-0.111×10^{-4}				
D_{35}	-0.001564	+0.000390	-0.293×10^{-5}				
F_{15}	$+0.323 \times 10^{-4}$	-0.103×10^{-5}					
F_{35}	$+0.285 \times 10^{-5}$	$+0.219 \times 10^{-6}$					
F_{17}	$+0.135 \times 10^{-4}$	-0.297×10^{-5}					
F_{37}	$+0.893 \times 10^{-8}$	$+0.265 \times 10^{-6}$					
G_{17}	-0.269×10^{-6}						
G_{37}	-0.395×10^{-6}						
G_{19}	$+0.445 \times 10^{-6}$						
G_{39}	-0.213×10^{-6}						

^aResonates without resonance form.

^bBackground to the resonances.

Table 7. Absorption coefficients.

$$\eta = e^{-2Y}, \quad Y = \theta(k - k_0) (k - k_0)^{2\ell+1} \sum_{n=0}^{\ell} b_n k^n$$

Phase	b_0	b_1	b_2	b_3	b_4	$\bar{\Gamma}_{in}$
S_{11}	+0.01037	-0.040059	+0.032726	+0.011179	+0.001392	
S_{31}	+0.029141	-0.069753	+0.054314	-0.013257	-0.417×10^{-5}	
P_{11}	+0.153236	+0.045552	+0.005506	-0.023979		
P_{31}	-0.002736	+0.022741	-0.001462	+0.000792		
P_{13}	+0.018173	-0.003617	-0.432×10^{-5}	-0.001229		
P_{33}^a	-0.328×10^{-8}	$+0.164 \times 10^{-8}$	-0.106×10^{-8}	-0.145×10^{-8}		0
D_{13}^a	+0.000726	-0.00352	-0.381×10^{-6}			0.01226
D_{33}	-0.148×10^{-4}	$+0.276 \times 10^{-5}$	$+0.204 \times 10^{-4}$			
D_{15}	+0.000211	0	-0.498×10^{-4}			
D_{35}	+0.006901	-0.001048	-0.001067			
F_{15}	+0.002158	-0.000924				
F_{35}	0	0				
F_{17}	$+0.392 \times 10^{-7}$	-0.85×10^{-4}				
F_{37}	0	0				
G_{17}	0					
G_{37}	0					
G_{19}	$+0.199 \times 10^{-4}$					
G_{39}	$+0.813 \times 10^{-8}$					

^aBackground to the resonances.

Predictions of $\sigma^{\text{cx}}(\theta)$ are shown in Fig. 36. It appears that measurements at backward angles would yield the most profitable results.

Predictions of $A(\theta)$ and $R(\theta)$ for π^+ and π^- elastic and charge-exchange scattering are shown in Figs. 37 through 42. The equations for $A(\theta)$ and $R(\theta)$ are¹²:

$$A(\theta) = \frac{-(|f(\theta)|^2 - |g(\theta)|^2) \sin \theta - 2 \operatorname{Re}[f^*(\theta) g(\theta)] \cos \theta}{|f(\theta)|^2 + |g(\theta)|^2}$$

and

$$R(\theta) = \frac{(|f(\theta)|^2 - |g(\theta)|^2) \cos \theta - 2 \operatorname{Re}[f^*(\theta) g(\theta)] \sin \theta}{|f(\theta)|^2 + |g(\theta)|^2}.$$

One must have a polarized target or polarized beam in order to measure these quantities.¹² It is seen that $A^+(\theta)$ and $R^+(\theta)$ vary considerably less rapidly with energy than do $A^{\text{cx}}(\theta)$ and $R^{\text{cx}}(\theta)$.

Facility is available for making movies of the angular observables where energy changes with time. One has already been made of $\sigma^{\text{cx}}(\theta)$ and $\sigma^{\text{cx}}(\theta)$ plotted together.

IV. CONCLUSIONS AND ACKNOWLEDGMENTS

A solution for $\ell_m = 4$ has been obtained that fits the data quite well. (There are 1173 data in the data selection, and the ratio of X^2 to the expected X^2 is 2.70.) It fits the high-energy polarization data significantly better than the $\ell_m = 3$ solution.³ Three resonances occur in this solution:

- 1) P_{33} at 193 MeV lab kinetic energy (1234 MeV c.m. total energy);
- 2) P_{11} at 575 MeV lab kinetic energy (1497 MeV c.m. total energy);
- 3) D_{13} at 638 MeV lab kinetic energy (1534 MeV c.m. total energy).

The absorption parameter at the P_{11} resonance position is ~ 0.3 and at the D_{13} resonance position is ~ 0.2 .

In the 0-700-MeV energy range experimentalists should concentrate on measurement of recoil-nucleon polarization and the spin rotation parameters. Very little is to be gained by further measurement of elastic differential cross sections.

Professor B. T. Feld of Massachusetts Institute of Technology first suggested this problem to one of us (L.D.R.) as a thesis dissertation. He will join us in a detailed account of the work to be published soon. Dr. M. J. Moravcsik of the Lawrence Radiation Laboratory, Livermore, offered many profitable suggestions during the course of the analysis. One of us (R. M. W.) wrote the computer code modeled after a code originally written by R. A. Arndt for the Livermore proton-proton phase shift analysis (MIDPOP). Eldon Halda wrote the very useful codes for plotting CRT curves of the results.

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APPENDIX I.

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APPENDIX III.

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AVAILABLE DATA 1/64

Fig. 1. Energies at which pion-nucleon scattering data were available in January, 1964.

Fig. 2a. 0-700 MeV, $\ell_m = 4$ phase shifts.

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Fig. 2b. 0-700 MeV, $l_m = 4$ phase shifts.

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Fig. 3a. 0-700 MeV, $l_m = 4$ absorption parameters.

Fig. 3b. 0-700 MeV, $\ell_m = 4$ absorption parameters.

Fig. 4a. 0-700 MeV, $\ell_m = 4$ real parts of the partial-wave amplitudes.

Fig. 4b. 0-700 MeV, $\ell_m = 4$ real parts of the partial-wave amplitudes.

Fig. 5a. 0-700 MeV, $\ell_m = 4$ imaginary parts of the partial-wave amplitudes.

Fig. 5b. 0-700 MeV, $\ell_m = 4$ imaginary parts of the partial-wave amplitudes.

Fig. 6. The solution versus some of the total cross section data.

Fig. 7. The solution (solid curve) versus Niederer's calculation¹⁴ (dotted curve) of the forward dispersion relations.

Fig. 8. The solution versus charge-exchange differential cross section coefficients (see Data section).

Fig. 9. The solution versus data from Appendix II Ref. 49 (24.8 MeV, π^+) and Appendix II Ref. 82 (31.1 MeV, π^-). Note: In Figs. 9-17, the solid curves include electromagnetic effects and the dashed curves exclude electromagnetic effects. In Figs. 9-32 the dot at zero-degrees is the value given by Niederer's forward dispersion relation calculation¹⁴ and the number N is the value by which the data were divided in the renormalization.

Fig. 10. The solution versus data from Appendix II Refs.80 (97.1 MeV, π^+) and 45 (98 MeV, π^-).

Fig. 11. The solution versus data from Appendix II Refs. 67 (120 MeV, π^+) and 77 (152 MeV, π^-).

Fig. 12. The solution versus data from Appendix II Ref. 36 (170 MeV, π^+ and π^-).

Fig. 13. The solution versus data from Appendix II Refs. 30 (220 MeV, π^+) and 77 (226 MeV, π^-).

Fig. 14. The solution versus data from Appendix III Refs. 9 (246 MeV, π^+) and 1 (224 MeV, π^-).

Fig. 15. The solution versus data from Appendix II Refs. 35 (270 MeV, π^+) and 57 (270 MeV, π^-).

Fig. 16. The solution versus data from Appendix II Refs. 61 (310 MeV, π^+) and 74 (310 MeV, π^-).

Fig. 17. The solution versus data from Appendix III Refs. 3 (300 MeV, π^-); 2 (307 MeV, π^+); 4 (310 MeV, π^+); 6 (310 MeV, π^-); and 7 (310 MeV, cx).

Fig. 18. The solution versus data from Appendix II Ref. 89 (370 MeV, π^+ and π^-). Note: In Figs. 18-32 the curves exclude electromagnetic effects.

Fig. 19. The solution versus data from Appendix II Ref. 89 (410 MeV, π^+ and π^-).

Fig. 20. The solution versus data from Appendix II Ref. 89 (450 MeV, π^+ and π^-).

Fig. 21. The solution versus data from Appendix II Ref. 89 (490 MeV, π^+ and π^-).

Fig. 22. The solution versus data from Appendix III Ref. 5 (500 MeV, π^-). GLL-647-2182

Fig. 23. The solution versus data from Appendix III Ref. 8 (523 MeV, π^+ and π^-).

Fig. 24. The solution versus data from Appendix II Refs. 76 (533 MeV, π^+ and π^-) and 88 (533 MeV, cx).

Fig. 25. The solution versus data from Appendix II Ref. 89 (550 MeV, π^+ and π^-).

Fig. 26. The solution versus data from Appendix III Ref. 8 (572 MeV, π^+ and π^-).

Fig. 27. The solution versus data from Appendix II Refs. 76 (581 MeV, π^+ and π^-) and 88 (581 MeV, cx).

Fig. 28. The solution versus data from Appendix II Ref. 89 (600 MeV, π^+ and π^-).

Fig. 29. The solution versus data from Appendix II Ref. 89 (650 MeV, π^+ and π^-).

Fig. 30. The solution versus data from Appendix III Ref. 5 (616, 667 and 695 MeV; π^-).

Fig. 31. The solution versus data from Appendix III Ref. (689 MeV, π^+ and π^-).

Fig. 32. The solution versus data from Appendix II Refs. 76 (698 MeV, π^+ and π^-) and 88 (698 MeV, cx).

Fig. 33. Prediction of the solution for the partial cross sections.

Fig. 34. Prediction of the solution for inelastic cross sections.

Fig. 35. Graphs of the $T = 1/2$ cross sections for the solution.

Fig. 36. Prediction of the solution for $P^{CX}(\theta)$.

Fig. 37. Prediction of the solution for $A^+(\theta)$.

Fig. 38. Prediction of the solution for $A^{-}(\theta)$.

Fig. 39. Prediction of the solution for $A^{cx}(\theta)$.

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Fig. 40. Prediction of the solution for $R^+(\theta)$.

Fig. 41. Prediction of the solution for $R^-(\theta)$.

Fig. 42. Prediction of the solution for $R^{cx}(\theta)$.

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