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ORTHOGONALITY RELATIONS FOR CHEBYSHEV POLYNOMIALS

by

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January 1970

DEPARTMENT OF COMPUTER SCIENCE
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN · URBANA, ILLINOIS

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Report No. 373

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January 1970

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* This report supported in part by contract U. S. AEC AT(11-1)1469.

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ABSTRACT

Two new discrete point orthogonality relations for the Chebyshev polynomials of the first kind have been obtained. Semi-closed Gauss-Chebyshev quadrature formulas, required for the purpose of obtaining these relations, have also been developed. These orthogonality relations lead to better approximations to the coefficients in the Chebyshev-Fourier expansion of a function and consequently to better values for an integral when the integrand is approximated by a truncated Chebyshev-Fourier series.

1. INTRODUCTION

Let $T_n(x) = \cos(n \arccos x)$, $n = 0, 1, 2, \dots$, be the Chebyshev polynomials of the first kind defined over $[-1, 1]$. These polynomials are known to be orthogonal in two ways (see Lanczos [1], pp. vii, xvii):

$$(1) \quad \int_{-1}^1 \frac{T_r(x) T_s(x)}{(1-x^2)^{1/2}} dx = \begin{cases} \pi & \text{if } r = s = 0 \\ \frac{\pi}{2} & \text{if } r = s \neq 0 \\ 0 & \text{if } r \neq s \end{cases}$$

$$(2) \quad \sum_{j=0}^n {}'' T_r(x_j) T_s(x_j) = \begin{cases} n & \text{if } r = s = 0 \text{ or } n \\ \frac{n}{2} & \text{if } r = s \neq 0 \text{ or } n \\ 0 & \text{if } r \neq s \end{cases}$$

In (2), $x_j = \cos(j\pi/n)$; $n > 0$; $r, s \leq n$. The double prime on sigma indicates that the first and the last terms are to be halved.

The discrete point orthogonality relations (2) have been used by Clenshaw [2] for the approximate computation of the coefficients in the Chebyshev-Fourier expansion of a function and for the approximate integration of a function by means of truncated Chebyshev series.

Our first result establishes that the discrete point orthogonality relations (2) can be obtained from the orthogonality relations (1) by replacing the integral by an appropriate Gauss-Chebyshev quadrature formula.

Theorem 1. For $r, s \leq n$, $(1) \Rightarrow (2)$

Proof. The $(n+1)$ - point Gauss-Chebyshev quadrature formula of the closed type over $[-1, 1]$ with fixed abscissas $x = \pm 1$ (see Chawla [3]) is

$$(3) \quad \int_{-1}^1 \frac{f(x) dx}{(1-x^2)^{1/2}} = \frac{\pi}{n} \sum_{k=0}^n f\left(\cos \frac{k\pi}{n}\right) + E_n(f)$$

where $E_n(f)$ denotes the error of the quadrature formula and is given by

$$(4) \quad E_n(f) = -\frac{\pi}{2^{2n-1}} \frac{f^{(2n)}(\eta)}{(2n)!}$$

for some $\eta \in [-1, 1]$. The quadrature formula (3) is therefore exact for all polynomials of degree $\leq 2n-1$. Replacing the integral in (1) by the quadrature formula (3), together with the error, we obtain

$$(5) \quad \frac{\pi}{n} \sum_{j=0}^n T_r(x_j) T_s(x_j) + E_n(T_r T_s) = \begin{cases} \pi & \text{if } r = s = 0 \\ \frac{\pi}{2} & \text{if } r = s \neq 0 \\ 0 & \text{if } r \neq s \end{cases}$$

If $r + s \leq 2n-1$, then $E_n(T_r T_s) = 0$ and (5) gives

$$(6) \quad \sum_{j=0}^n T_r(x_j) T_s(x_j) = \begin{cases} n & \text{if } r = s = 0 \\ \frac{n}{2} & \text{if } r = s \neq 0 \\ 0 & \text{if } r \neq s \end{cases}$$

Next, let $r = s = n$. Since $T_n^2(x) = \frac{1}{2}(T_{2n}(x) + 1)$ and $T_n(x) = 2^{n-1}x^n$ plus lower degree terms, therefore, $T_n^2(x) = 2^{2n-2}x^{2n}$ plus lower degree terms, and from (4) we obtain $E_n(T_n T_n) = -\frac{\pi}{2}$. Thus, for $r = s = n$ we have from (5)

$$(7) \quad \sum_{j=0}^n T_r(x_j) T_s(x_j) = \frac{n}{\pi} \left(\frac{\pi}{2} - E_n(T_n^2) \right) \\ = n$$

Combining (6) and (7), the orthogonality relation (2) follows. Clearly the argument holds for the converse also.

Let $f(x)$ be continuous and of bounded variation on $[-1,1]$, then $f(x)$ has a uniformly convergent Chebyshev-Fourier expansion:

$$(8) \quad f(x) = \sum'_{j=0}^{\infty} a_j T_j(x)$$

the prime on the sigma means that the first term is to be halved. The Chebyshev coefficients are given by

$$(9) \quad a_j = \frac{2}{\pi} \int_{-1}^1 \frac{f(x) T_j(x) dx}{(1-x^2)^{1/2}}, \quad j = 0, 1, 2, \dots$$

Replacing the integral in (9) by the Gauss-Chebyshev quadrature formula of the closed type (3) and neglecting the error we obtain

$$(10) \quad a_r \approx \frac{2}{n} \sum_{j=0}^n f(x_j) T_r(x_j)$$

where $x_j = \cos(j\pi/n)$. This is the scheme for the approximate computation of the Chebyshev coefficients described by Clenshaw [2].

If we define the numbers $\alpha_0, \alpha_1, \dots, \alpha_n$ as

$$(11) \quad \alpha_r = \frac{2}{n} \sum_{j=0}^n f(x_j) T_r(x_j)$$

then it is easy to show [2] that

$$(12) \quad \alpha_r = a_r + \sum_{p=1}^{\infty} (a_{2np-r} + a_{2np+r})$$

This shows that α_0 can be taken as an approximation for a_0 with an error of $2(a_{2n} + \dots)$; α_1 can be taken as an approximation for a_1 with an error of $a_{2n-1} + \dots$. However,

$$\alpha_{n-1} = a_{n-1} + a_{n+1} + a_{3n-1} + \dots$$

so that unless the coefficients a_n converge very rapidly, α_{n-1} will not be a good approximation to a_{n-1} . But

$$\alpha_n = 2(a_n + a_{3n} + \dots)$$

so that $\frac{1}{2} \alpha_n$ gives an approximation for a_n with an error of $a_{3n} + \dots$.

2. TWO NEW ORTHOGONALITY RELATIONS FOR THE CHEBYSHEV POLYNOMIALS

We have seen that the approximation (10) for the calculation of the Chebyshev coefficients results from replacing the integral in (9) by the two point closed Gauss-Chebyshev quadrature formula. This raises another possibility of using semi-closed Gauss-Chebyshev quadrature formulas with a fixed abscissas either at $x = +1$ or at $x = -1$ for replacing the integral in (9). Since the semi-closed formulas are more accurate (one degree of precision more) than the fully closed formula, it is expected that the resulting approximation will be better than that given by (10).

For this purpose, in the following, we develop the two semi-closed Gauss-Chebyshev quadrature formulas and use them to obtain two new discrete point orthogonal relations for the Chebyshev polynomials similar to (2).

2.1 Gauss-Chebyshev Quadrature with Fixed Abscissa $x = +1$

Let $w(x) = (x-1) P_n^{(\frac{1}{2}, -\frac{1}{2})}(x)$ where $P_n^{(\frac{1}{2}, -\frac{1}{2})}$ is the Jacobi polynomial on $[-1, 1]$ corresponding to the weight $(\frac{1-x}{1+x})^{1/2}$. Let $x_1, x_2, \dots, x_n$ be the n zeros of $P_n^{(\frac{1}{2}, -\frac{1}{2})}(x)$, and let $L_n(x)$ be the Lagrange interpolation polynomial of degree n for f corresponding to the abscissas $x = 1, x_1, \dots, x_n$,

$$L_n(x) = \frac{w(x)}{w'(1)(x-1)} f(1) + \sum_{k=1}^n \frac{w(x)}{w'(x_k)(x-x_k)} f(x_k)$$

Multiplying both sides by the weight $(1-x^2)^{-1/2}$ and integrating over $[-1,1]$:

$$(13) \quad \int_{-1}^1 \frac{f(x)dx}{(1-x^2)^{1/2}} \approx \int_{-1}^1 \frac{L_n(x)dx}{(1-x^2)^{1/2}} = Af(1) + \sum_{k=1}^n A_k f(x_k)$$

where the weights are given by

$$A = \frac{1}{w'(1)} \int_{-1}^1 \frac{w(x)dx}{(x-1)(1-x^2)^{1/2}}, \quad A_k = \frac{1}{w'(x_k)} \int_{-1}^1 \frac{w(x)dx}{(x-x_k)(1-x^2)^{1/2}},$$

$$k = 1, \dots, n.$$

The precision of the quadrature formula (13) is clearly $2n$. Since

$$P_n\left(\frac{1}{2}, -\frac{1}{2}\right)(x) = c_n \frac{T_n(x) - T_{n+1}(x)}{1-x}$$

where c_n is a positive constant, the n free abscissas of the formula (13)

are the zeros of $P_n\left(\frac{1}{2}, -\frac{1}{2}\right)(x)$,

$$x_k = \cos \frac{2k\pi}{2n+1}, \quad k = 1, \dots, n$$

Therefore,

$$A = \frac{1}{2n+1} \int_{-1}^1 \frac{T_n(x) - T_{n+1}(x)}{(1-x)(1-x^2)^{1/2}} dx = \frac{\pi}{2n+1}$$

and $A_k = \frac{\mu_k}{(1-x_k)}$ where we have put

$$\mu_k = \frac{1}{P_n \left(\frac{1}{2}, -\frac{1}{2} \right) (x_k)} \int_{-1}^1 \left(\frac{1-x}{1+x} \right)^{1/2} \frac{P_n \left(\frac{1}{2}, -\frac{1}{2} \right) (x) dx}{x - x_k}, \quad k = 1, \dots, n$$

But μ_k , $k = 1, \dots, n$ are the weights in the Gauss-Jacobi quadrature formula

$$(14) \quad \int_{-1}^1 \left(\frac{1-x}{1+x} \right)^{1/2} f(x) dx \approx \sum_{k=1}^n \mu_k f(x_k)$$

and are given (see [4] by

$$\mu_k = \frac{4\pi}{2n+1} \sin^2 \left(\frac{k\pi}{2n+1} \right)$$

and thus

$$A_k = \frac{2\pi}{2n+1}, \quad k = 1, \dots, n$$

The Gauss-Chebyshev quadrature formula with fixed abscissa at $x = +1$ is, therefore,

$$(15) \quad \int_{-1}^1 \frac{f(x) dx}{(1-x^2)^{1/2}} = \frac{2\pi}{2n+1} \sum_{k=0}^n f\left(\cos \frac{2k\pi}{2n+1}\right) + E_n^+(f)$$

where $E_n^+(f)$ denotes the error of the formula.

2.2 Gauss-Chebyshev Quadrature Formula with Fixed Abscissa $x = -1$

Let $v(x) = (x+1) P_n \left(-\frac{1}{2}, \frac{1}{2} \right) (x)$ where $P_n \left(-\frac{1}{2}, \frac{1}{2} \right) (x)$ is the Jacobi polynomial of degree n over $[-1, 1]$ corresponding to the weight $\left(\frac{1+x}{1-x} \right)^{1/2}$.

Let the zeros of $P_n^{(-\frac{1}{2}, \frac{1}{2})}(x)$ be denoted by $x_1, \dots, x_n$, and let $M_n(x)$ be the Lagrange interpolation polynomial of degree n for f corresponding to the abscissas $x = -1, x_1, \dots, x_n$:

$$M_n(x) = \frac{v(x)}{v'(-1)(x+1)} f(-1) + \sum_{k=1}^n \frac{v(x)}{v'(x_k)(x-x_k)} f(x_k)$$

Integrating over $[-1, 1]$ with weight $(1-x^2)^{-1/2}$,

$$(16) \quad \int_{-1}^1 \frac{f(x) dx}{(1-x^2)^{1/2}} \approx \int_{-1}^1 \frac{M_n(x) dx}{(1-x^2)^{1/2}} = B f(-1) + \sum_{k=1}^n B_k f(x_k)$$

where

$$B = \frac{1}{v'(-1)} \int_{-1}^1 \frac{v(x) dx}{(x+1)(1-x^2)^{1/2}}, \quad B_k = \frac{1}{v'(x_k)} \int_{-1}^1 \frac{v(x) dx}{(x-x_k)(1-x^2)^{1/2}},$$

$$k = 1, \dots, n.$$

Since

$$P_n^{(-\frac{1}{2}, \frac{1}{2})}(x) = c_n \frac{T_n(x) + T_{n+1}(x)}{1+x}$$

the abscissas $x_k = \cos((2k-1)\pi/(2n+1))$, $k = 1, \dots, n$. Now,

$$B = \frac{(-1)^n}{2^{n-1}} \int_{-1}^1 \frac{T_n(x) + T_{n+1}(x)}{(1+x)(1-x^2)^{1/2}} dx = \frac{\pi}{2^{n+1}}$$

and

$$B_k = \frac{1}{1+x_k} \lambda_k$$

where we have put

$$\lambda_k = \frac{1}{P_n \left(-\frac{1}{2}, \frac{1}{2} \right) (x_k)} \int_{-1}^1 \left(\frac{1+x}{1-x} \right)^{1/2} \frac{P_n \left(-\frac{1}{2}, \frac{1}{2} \right) (x) dx}{x - x_k}$$

But λ_k are the weights in the Gauss-Jacobi quadrature formula

$$(17) \quad \int_{-1}^1 \left(\frac{1+x}{1-x} \right)^{1/2} f(x) dx \approx \sum_{k=1}^n \lambda_k f(x_k)$$

and are given by

$$\lambda_k = \frac{4}{2n+1} \cos^2 \left(\left(k - \frac{1}{2} \right) \pi / (2n+1) \right), \quad k = 1, \dots, n.$$

Therefore,

$$B_k = \frac{2\pi}{2n+1}, \quad k = 1, \dots, n.$$

The Gauss-Chebyshev quadrature formula with fixed abscissa $x = -1$ is,

therefore,

$$(18) \quad \int_{-1}^1 \frac{f(x) dx}{(1-x)^{1/2}} = \frac{2\pi}{2n+1} \sum_{k=1}^{n+1} f \left(\cos \frac{(2k-1)\pi}{2n+1} \right) + E_n^-(f)$$

where $E_n^-(f)$ denotes the error of the formula. The lower prime on the sigma indicates that the last term is to be halved. The formula (18) is of precision $2n$.

An error analysis of the semi-closed Gauss-Chebyshev quadrature formulas (15) and (18) with fixed abscissa at $x = 1$ and -1 , respectively, will be given in Section 3.

2.3 New Discrete Point Orthogonality Relations

From the orthogonality relations (1) for the Chebyshev polynomials we deduce the following two discrete point orthogonality relations similar to (2).

Replacing the integral in (1) by the quadrature formula (15) of precision $2n$, we obtain:

Theorem 2. If $x_k = \cos \left(\frac{2k\pi}{2n+1} \right)$, $k = 0, 1, \dots, n$, then for $r, s \leq n$,

$$(19) \quad \sum_{k=0}^n T_r(x_k) T_s(x_k) = \begin{cases} (2n+1)/2 & \text{if } r = s = 0 \\ (2n+1)/4 & \text{if } r = s \neq 0 \\ 0 & \text{if } r \neq s \end{cases}$$

Again, replacing the integral in (1) by the quadrature formula (18) of precision $2n$, we obtain

Theorem 3. If $x_k = \cos ((2k-1)\pi/(2n+1))$, $k = 1, \dots, n+1$, then for $r, s \leq n$,

$$(20) \quad \sum_{k=1}^{n+1} T_r(x_k) T_s(x_k) = \begin{cases} (2n+1)/2 & \text{if } r = s = 0 \\ (2n+1)/4 & \text{if } r = s \neq 0 \\ 0 & \text{if } r \neq s \end{cases}$$

3. NEW SCHEMES FOR THE APPROXIMATE COMPUTATION OF CHEBYSHEV-FOURIER COEFFICIENTS

We have seen that the Clenshaw scheme for calculating approximately the Chebyshev-Fourier coefficients a_j consists of replacing the integral in (9) by the fully closed Gauss-Chebyshev quadrature formula (3). Alternative better schemes can be obtained by replacing the integral by either of the semi-closed formulas; however, these formulas lack in symmetry.

Replacing the integral in (9) by the semi-closed Gauss-Chebyshev quadrature formula (15) and neglecting the error, we obtain the approximation

$$(21) \quad a_r \approx \frac{4}{2n+1} \sum_{j=0}^n f(x_j) T_r(x_j)$$

where $x_j = \cos(2j\pi/(2n+1))$, $j = 0, 1, \dots, n$. If we define for $r = 0, 1, \dots, n$, the numbers

$$(22) \quad \beta_r = \frac{4}{2n+1} \sum_{j=0}^n f(x_j) T_r(x_j)$$

then substituting (8) in (22) and using the orthogonality relations (19) we obtain

$$(23) \quad \beta_r = a_r + \sum_{p=1}^{\infty} (a_{(2n+1)p-r} + a_{(2n+1)p+r}), \quad r = 0, 1, \dots, n$$

Note that for $r = 0$, (23) reduces to the quadrature formula (15) so that the error is given by

$$(24) \quad E_n^+(f) = -\pi \sum_{p=1}^{\infty} a_{(2n+1)p}$$

and for $r = 0, 1, \dots, n$, $\beta_r \approx a_r$ with an error of $\sum_{p=1}^{\infty} (a_{(2n+1)p-r} + a_{(2n+1)p+r})$.

Similarly, replacing the integral in (9) by the semi-closed Gauss-Chebyshev quadrature formula (18) and neglecting the error, we obtain the approximation

$$(25) \quad a_r \approx \frac{4}{2n+1} \sum_{j=0}^n f(x_j) T_r(x_j)$$

where $x_j = \cos \left(\frac{(2k-1)\pi}{2n+1} \right)$ $j = 1, \dots, n+1$. For $r = 0, 1, \dots, n$ define the numbers

$$(26) \quad \gamma_r = \frac{4}{2n+1} \sum_{j=1}^{n+1} f(x_j) T_r(x_j)$$

Substituting (8) in (26) and using the orthogonality relations (20), we obtain

$$(27) \quad \gamma_r = a_r + \sum_{p=1}^{\infty} (-1)^p (a_{(2n+1)p-r} + a_{(2n+1)p+r})$$

For $r = 0$, (27) leads to the quadrature formula (18), and the error is given by

$$(28) \quad E_n^-(f) = \pi \sum_{p=1}^{\infty} (-1)^{p+1} a_{(2n+1)p}$$

Also for $r = 0, 1, \dots, n$, (27) shows that $\gamma_r \approx a_r$ with an error of

$$\sum_{p=1}^{\infty} (-1)^p (a_{(2n+1)p-r} + a_{(2n+1)p+r}).$$

4. NUMERICAL INTEGRATION BY TRUNCATED CHEBYSHEV-FOURIER EXPANSION

Let $f(x)$ have the Chebyshev-Fourier expansion (8). A useful polynomial approximation to $f(x)$ can be found by truncating the expansion at some $j = N$:

$$(29) \quad f(x) \approx \sum_{k=0}^N a_k T_k(x)$$

and therefore,

$$(30) \quad \int_{-1}^1 f(x) dx \approx \sum_{k=0}^N a_k \left(\int_{-1}^1 T_k(x) dx \right)$$

But

$$(31) \quad \int_{-1}^1 T_k(x) dx = \begin{cases} -\frac{2}{k^2-1} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

so that

$$(32) \quad \int_{-1}^1 f(x) dx \approx \sum_{k=0}^{[N/2]} \frac{-2}{4k^2-1} a_{2k}$$

where $[k]$ denotes the largest integer contained in k . The Chebyshev coefficients a_{2k} can be computed through either (21) or (25).

Note that as far as approximate integration over $[-1,1]$ is concerned, the lack of symmetry of the approximations (21) or (25) does not affect as only even coefficients are needed in (32). The improved

accuracy of these algorithms for the approximate computation of the Chebyshev coefficients will therefore produce better approximations to the value of the integral than that resulting from the use of the algorithm (10).

5. EXAMPLES

The Chebyshev coefficients for $(1-x^2)^{1/2}$ and $\tan^{-1}x$ calculated from (22) are compared in Tables 1 and 2 respectively, with those calculated from (11) as also with their exact values. In each case $n = 9$. In Table 3 we compare the approximate values I_{α} and

$$I_{\beta} \text{ for } I(f) = \int_{-1}^1 f(x) dx \text{ for } f(x) = (1-x^2)^{1/2} \text{ and } \log(1.01+x) \text{ as}$$

obtained from (32) when the a_k 's are approximated, respectively, by α_k 's and β_k 's with the exact values for these integrals. It will be observed that the β_k 's produce better approximations.

Table 1
 $f(x) = (1-x^2)^{1/2}$

r	β_r	a_r (exact)	α_r
0	+1.2703378	+1.2732395	+1.2602859
2	-0.4273309	-0.4244132	-0.4376913
4	-0.0878491	-0.0848826	-0.0992158
6	-0.0394289	-0.0363783	-0.0527911
8	-0.0233851	-0.0202102	-0.0404401

Table 2
 $f(x) = \tan^{-1}x$

r	β_r	a_r (exact)	α_r
1	+0.82842712	+0.82842712	+0.82842716
3	-0.04737854	-0.04737854	-0.04737878
5	+0.00487732	+0.00487732	+0.00487895
7	-0.00059773	-0.00059773	-0.00060892

Table 3

f(x)	N	I_β	I	I_α
$(1-x^2)^{1/2}$	9	1.5699337	1.5707963	1.5696093
$\log(1.01+x)$	5	-0.5613405	-0.5506975	-0.5662477

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