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PROGRESS REPORT

Contract No. At(11-1)-2059-1

Numerical and Analytical Investigation of Nonlinear Properties of the Vlasov Equation and of Plasmas

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DISCLAIMER

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On the following pages is a summary of the research which has been carried on under contract No. At(11-1)-2059 at the Department of Physics and Astronomy of the University of Iowa. The requirements of the contract have been complied with.

Since the beginning of the current term of agreement, the principal investigator has devoted 50% of his time to the project; he expects to devote the same amount of time during the remainder of the term.

Professor Glenn Joyce has participated in the project, devoting 50% of his time.

Dr. J. Nuehrenberg has been working on the project since September 1969 and is devoting 100% of his time.

Thomas Burns, a graduate student, has been working for the project full time.

This summer another graduate student, Mark Emery, will begin work on the project.

I. STATEMENT OF THE PROBLEM

The equation which governs the motion of a tenuous collisionless plasma, the Vlasov equation, is given by

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \underline{F} \cdot \frac{\partial f}{\partial v} = 0 \tag{1}$$

for ions and electrons. The force $\underline{F}$ per unit mass is given by $\underline{F} = \frac{q}{m}(\underline{E} + v \times \underline{B})$. $\underline{E}$, the electric field, and $\underline{B}$, the magnetic field, are determined by Maxwell equations. For the electrostatic case, $\underline{B}$ = constant or zero and

$$\underline{E} = -\nabla\varphi \, . \tag{2}$$

The potential φ is then determined self-consistently by

$$+\nabla^2 \varphi = -4\pi\,(n_i - n_e) \tag{3}$$

where $n_{i/e} = \int f_{i/e}\, d^3v$ is the density of the ions or electrons, respectively. The system (1), (2), and (3) is self-consistent if appropriate initial values are given and boundary values are prescribed.

The state of the art of transform methods as compared to the by-now quite advanced and sophisticated particle pushing methods, can still be considered to be in its infancy. In view of this, we felt we should concentrate on developing basic methods which are economical and fast, to solve the Vlasov system, rather than trying to solve a variety of physical conditions with methods which still leave much to be desired.

When reviewing the two existing transform methods, the method of characteristics and the Hermite expansion, it has been been found that actually the two methods are in a way closely related. This is described in Chapter II. A new transform called the "power transform" has been developed.

It was found that the number of terms taken into account can be greatly reduced by introducing a new kind of cut-off procedure in the transformed velocity space. Whereas the number of coefficients necessary in the older work of Armstrong was around 1000, we are now working with a number of coefficients ranging between 40 and 60 without applying any artificial damping to the coefficients. This is described in Chapter III.

When the same methods are applied to the linear theory, it was found that Landau damping can still be represented to better than 1% accuracy by choosing not more than 10 coefficients. Computations are done for times $t = 100\ \omega_{pe}^{-1}$ in a matter of seconds. This work has been reported during the Third Annual Numerical Plasma Simulation Conference at Stanford in 1969.

In Chapter IV a general program, which has been developed by G. Joyce, using the power transform method, is described.

Simultaneously with this full nonlinear program T. Burns has also developed a program using the power transform, which program simulates the quasilinear theory. This quasilinear theory is obtained from the full nonlinear equations by omitting the mode-mode coupling terms. By comparing both, it will be possible to obtain more information about the deficiencies and merits of the quasilinear theory. This is described in Chapter V.

In Chapter VI some theoretical considerations are reported which show that non-resonant particles which are neglected in the ordinary formulation of the quasilinear theory (e.g., Bernstein and Engelmann) do play an important part. It is shown that the quasilinear theory conserves momentum and energy if they are included but not if they are neglected.

In Chapter VII a program is described, which program uses a Fourier transform in velocity space but leaves configuration space unchanged. It is being developed by J. Nuehrenberg. The advantage of such an approach is twofold. All mode coupling terms are combined in one term y E (x, t) F (x, y, t) which allows a faster computation than the corresponding convolution sum

$\sum_{q=-\infty}^{+\infty} y\, E_k\, F_{k-q}$ which appears when the individual modes are computed directly. Second, with this method one should be able to compute shock-like solutions of the Vlasov equation.

Finally, J. Nuehrenberg and the principal investigator considered analytically what simplifications can be introduced in describing the electron component of an ion electron plasma. When the process considered is slow, e.g., ion waves, a natural assumption is to use the constancy of the adiabatic invariant of the electrons. This is an additional constraint which makes the equation of motion of the electrons simpler (in principle). It is shown that the equation of state $n_e(x, t) = n_e(\varphi(x, t))$, i.e., electron density depends on space and time only via the potential $\varphi(x, t)$, which is usually applied for analytical and numerical investigations, does not conserve energy. On the other hand, the actual equation of motion remains so complicated that numerical integration of the electron distribution will still require an appreciable effort.

II. RELATION BETWEEN CHARACTERISTIC TRANSFORM AND HERMITE EXPANSION

We want to show the close relation between the characteristic function and the Hermite transform. We consider the system, consisting of the Vlasov equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial v} = 0 \tag{4}$$

and the Poisson equation

$$\frac{\partial E}{\partial x} = 1 - \int_{-\infty}^{+\infty} f\,(x, v, t)\, dv \,. \tag{5}$$

We apply a Fourier transform in configuration space

$$\begin{aligned} f\,(x, v, t) &= \sum_{-\infty}^{+\infty} f_n\,(v, t)\, e^{ikx} \\ E\,(x, t) &= \sum_{-\infty}^{+\infty} E_n\,(t)\, e^{ikx} \end{aligned} \tag{6}$$

and obtain the system

$$\begin{aligned} \frac{\partial f_n}{\partial t} + ink_o\, v\, f_n - \sum_{q=-\infty}^{+\infty} E_{n-q} \frac{\partial f_q}{\partial v} &= 0 \\ ink_o\, E_n &= - \int_{-\infty}^{+\infty} f_n\,(v, t)\, dv \,. \end{aligned} \tag{7}$$

We now apply another Fourier transform in velocity space

$$F_n (y, t) = \int_{-\infty}^{+\infty} f_n (v, t) \, e^{iyv} \, dv \tag{8}$$

to obtain

$$\frac{\partial F_n}{\partial t} + k_o n \frac{\partial F_n (y, t)}{\partial y} - y \sum_{q=-\infty}^{+\infty} \frac{F_q (0, t)}{q} F_{n-q} (y, t) = 0 \tag{9}$$

and

$$ik_o n E_n (t) = F_n (0, t) . \tag{10}$$

$F_n (y, t)$ is the characteristic function. When we expand $F_n (y, t)$ around $y = 0$ in a power series, we obtain

$$F_n (y, t) = \sum_{v=0}^{\infty} a_{n,\nu} (t) \, g_\nu \, y^\nu \, e^{-1/2 \, y^2} . \tag{11}$$

The factor $g_\nu = 2^{\nu/2} \frac{\Gamma (\frac{v+2}{2})}{\Gamma (v+1)}$ has been added for convenience. It has the effect that in a numerical computation $a_{n,\nu}$, $\nu = 0, 1 \ldots$ are all essentially of the same order of magnitude. The factor $\exp (- \frac{1}{2} y^2)$ has to be added in order to assure convergence of the expression for $y \to \infty$.

Inserting Eq. (8) into Eq. (6) and comparing equal powers of y leads to the following set of equations:

$$\dot{a}_{n,v} + nk_o \left[a_{n,v+1} \frac{g_{v+1}}{g_v} (v+1) - a_{n,v-1} \frac{g_{v-1}}{g_v}\right]$$

$$- 1/k_o \frac{g_{v-1}}{g_v} \sum_{m=-\infty}^{+\infty} 1/m \, a_{m,o} \, a_{n-m,v-1} = 0 . \qquad (12)$$

This is the power transformation. From its derivation it is clear that it is closely related to the characteristic function method. It can also be shown, using the theorem that the Fourier transformation of a cylinder function is again, apart from a factor, the same cylinder function, that the $a_{n,\nu}$ are proportional to the coefficients $Z_{n,\nu}$, when we write:

$$f_n (v, t) = \sum_{\nu=0}^{\infty} Z_{n,\nu} \, He_\nu (v) \, e^{-1/2 \, v^2} .$$

This is clearly the Hermite expansion.

III. CUT-OFF PROCEDURE IN TRANSFORMED VELOCITY SPACE. APPLICATIONS TO LINEAR AND NONLINEAR THEORY.

When system (12) is integrated, it has to be cut off at some ν_{max}. However, in order to compute $a_{n,\nu}$ for the next time step, the knowledge of $a_{n,\nu+1}$ is required. When doing an actual calculation for the linearized Vlasov equation, it was found that the $a_{n,\nu}$'s for $\nu \geq 5$ appear to be arranged in a very regular form. It was, therefore, natural to guess $a_{n,\nu_{max}+1}$ by putting an interpolation polynomial through $a_{n,\nu_{max}-\mu}$, $\mu = 1,2,3,4$. The result was that Landau damping could be recovered from numerical calculations using as few as 10 coefficients $a_{n,\nu}$. The deviation of the damping decrement turned out to be only $\delta\gamma/\gamma = 0.8\%$. The mathematical reason for this phenomenon, which saves a fantastic amount of computer time, has been clarified by H. Meier, ORNL. The linear part of system (12) without the nonlinear last term on the right can be solved as an eigenvalue problem. It turns out that

$$a_{n,\nu} \sim He_{\nu}\left(\frac{\omega}{n\,k_o}\right)$$

where ω is the continuous, real eigenvalue. The general solution can be written as

$$a_{n,\nu} = \int_{-\infty}^{+\infty} d\omega\, h(\omega)\, He_\nu \left(\frac{\omega}{n k_o}\right) e^{-1/2\, \omega^2 + i\omega t}$$

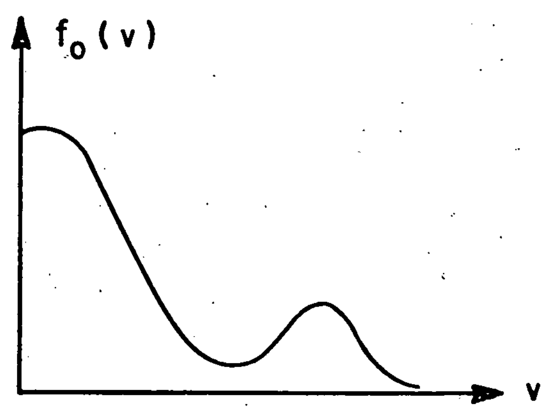
where h (ω) is an arbitrary function. $a_{n,v}$ can thus be represented as an integral over a continuous eigenvalue spectrum. It can be shown that $a_{n,\nu}$ tends to zero like $t^\alpha e^{-1/2\, t^2}$ when $t \rightarrow \infty$. When system (9) is cut off at some ν_{max}, the resulting solution is no longer an integral but a sum over a finite number of eigenvalues:

$$a_{n,v} = \sum_{\mu=1}^{N} h\,(\omega_\mu)\, He_v \left(\frac{\omega_\mu}{n\, k_o}\right) e^{-1/2\, \omega_\mu^2 + i\omega t} \quad .$$

This is an almost periodic function in time and $a_{n,v}$ does no longer tend to zero as $t \rightarrow \infty$. The recurrence is interpreted as "instability". The remedy is to replace ω by $\omega + i\,\lambda$. This can most easily be done by an appropriately chosen extrapolation formula.

IV. NONLINEAR PROGRAM USING POWER TRANSFORM. COMPARISON BETWEEN NONLINEAR AND QUASILINEAR THEORY

The truncation methods discussed in Chapter II have been implemented in nonlinear programs which are being used to test the validity of the quasilinear theory of Drummond and Pines,[1] and to see the effect of including mode coupling on the "bump-on-the-tail" distribution. The distribution of particles has the form



where the bump occurs at 4 times the thermal velocity. The results of these calculation are still in a preliminary stage, but certain results are beginning to emerge. For example, some modes which are linearly stable begin to grow after the electric field becomes appreciable. This may be due to instabilities induced by the trapped particles.[2] This implies that the region (in k space) of large amplitude waves may be much larger than is predicted by the quasilinear theory.

The advantage of the truncation method can be seen in this calculation. Previously a similar problem has been investigated by Armstrong and Montgomery.[3] Their calculation took 8 hours on the University of Iowa IBM 360. A typical computer run using the truncation method takes about four minutes.

The results of keeping the full nonlinear set of Eqs. (12) have been compared with the equation without the mode coupling terms. This simplification is represented by

$$\dot{a}_{n,v} + nk_o \left[a_{n,v+1} \frac{g_{v+1}}{g_v} (v+1) - a_{n,v-1} \frac{g_{v-1}}{g_v}\right] - 1/k_o \frac{g_{v-1}}{g_v} 1/n \, a_{n,o} \, a_{o,v-1} = 0 \,. \tag{13}$$

$$\dot{a}_{o,v} - 1/k_o \frac{g_{v-1}}{g_v} \sum_{m=-\infty}^{+\infty} 1/m \, a_{m,o} \, a_{-m,v-1} = 0 \,. \tag{14}$$

While the qualitative features of the two sets of equations are similar, the second set does not represent the interaction of particles with the wave correctly, and the appearance of the trapped particle instability is modified considerably.

Considerable analysis of these results remains before concrete statements can be made regarding the effect of omitting the mode coupling terms.

V. NON-RESONANT PARTICLE CONTRIBUTIONS IN QUASILINEAR THEORY

We want to show that non-resonant particles may contribute significantly in the quasilinear theory. We first derive the equations of the quasilinear theory. Omitting the mode coupling terms for $n \neq 0$ in Eq. (7) results in

$$\frac{\partial fu}{\partial t} + ink_o\, vf_n - E_n \frac{\partial f_o}{\partial v} = 0$$

$$ink_o\, E_n = - \int_{-\infty}^{+\infty} f_n(v_1 t)\, dv \tag{15}$$

and

$$\frac{\partial f_o}{\partial t} = \sum_{n=-\infty}^{+\infty} E_{-n}\, f_n \quad . \tag{16}$$

From Eq. (15) we derive with the WKB Ansatz $f_n \sim f_n \exp(-i\int\omega(t')dt')$

$$f_n = E_n^o \frac{\partial f_o/\partial v}{i(k_o nv-\omega_n)} \quad . \tag{17}$$

With the help of Eq. (17) we can rewrite Eq. (16) as

$$\frac{\partial f_o}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f_o}{\partial v} \tag{18}$$

where $D = \sum_{m=-\infty}^{+\infty} \frac{|E_m|^2}{ik(v-\omega/k)} = \sum_{m=o}^{\infty} \frac{2\gamma_m |E_m|^2}{(k_m\ v-\mathrm{Re}\omega_m)^2 + \gamma_m{}^2}$.

These are the familiar equations of the quasilinear theory. Usually D is replaced by a delta function, which takes into account the resonant particles only. A more careful expansion of $g(v)/(v-\omega/k)$ leads to the following expansion with $u = v-i\gamma/k$, where u is real.

$$\frac{g(v)}{v-\omega/k} = \frac{g(u+i\gamma/k)}{u-\mathrm{Re}\omega/k} .$$

This expression can be rewritten as

$$\{ P \frac{1}{u-\mathrm{Re}\omega/k} [1 + i\gamma/k \frac{d}{du} +\dots] g(u) + i\pi[\delta(u-\mathrm{Re}\omega/k)$$

$$- i\gamma/k\ \delta'(u-\mathrm{Re}\omega/k)]\} g(u)$$

where P refers to the principle value when integrated over v. This result, applied to the dispersion equation leads to the well known formulae:

$$k^2 = P\int \frac{\partial f_o/\partial u}{u-\mathrm{Re}\,\omega/k}\, du$$

$$\gamma/k\ P\int du\, \frac{\partial^2 f_o/\partial u^2}{u-\mathrm{Re}\,\omega/k} = -\pi\, \partial f_o/\partial u\ (\mathrm{Re}\,\omega/k) \quad .$$

When applied to Eq. (4) we obtain

$$D = \sum_{m=-\infty}^{+\infty} \frac{2}{k_o m} |E_m|^2 \{P\, \frac{\gamma m/k_o m}{u-\mathrm{Re}\,\omega_m/k_o m}\, \frac{d}{du} + \pi\delta(u-\mathrm{Re}\,\omega/k)\} \quad .$$

It can be shown that momentum and energy are conserved with this form of the diffusion coefficient. The appearance of a principal value integral indicates that contributions of particles which are not resonant are important.

VI. A FINITE DIFFERENCE SCHEME FOR VLASOV'S EQUATION

For the numerical solution of the one-dimensional Vlasov equation, a difference scheme approach in x-space seems to be more adequate than a Fourier analysis if the solution is not periodic in x-space or if a periodic solution contains many Fourier modes. Since a difference scheme in v-space would fail because of the so-called free-streaming term (see Ref. 4) a Fourier transformation of velocity space is made, so that Vlasov's equation

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial v} = 0 \tag{19}$$

is transformed into $$\frac{\partial F}{\partial t} - i \frac{\partial^2 F}{\partial x \partial y} + iy\, EF = 0 \tag{20}$$

where $$F = \int_{-\infty}^{\infty} f e^{ivy}\, dv \quad . \tag{21}$$

In the following, a difference scheme in x and y is developed for Eq. (2). Since f is real, the reality condition for F is

$$F(x, -y, t) = F^*(x, y, t) \quad . \tag{22}$$

So that it is sufficient to consider $y > 0$. The difference equations have to be supplemented by a cut-off prescription at the largest value of y, which is retained in the numerical procedure.

A. The Difference Scheme

In order to get a difference approximation of second order accuracy in time an approach is used which is analogous to the Lax-Wendroff technique (see Ref. 5). In the Taylor expansion to second order in Δt of $F(x, y, t + \Delta t)$

$$F(x, y, t + \Delta t) = F(x, y, t) + \Delta t \frac{\partial F}{\partial t}(x, y, t) + \frac{\Delta t^2}{2} \frac{\partial^2 F}{\partial t^2}(x, y, t)$$

$\frac{\partial F}{\partial t}$ is inserted from Eq. (20) and $\frac{\partial^2 F}{\partial t^2}$ from the time derivative of Eq. (20). The result is

$$F(x,u,t + \Delta t) = F(x,y,t) + i\Delta t \left(\frac{\partial^2 F}{\partial x \partial y} - y\, EF \right)$$

$$- \frac{(\Delta t)^2}{2} \left(\frac{\partial^4 F}{\partial x^2 \partial y^2} - \frac{\partial^2}{\partial x \partial y}(Y\, EF) + iy \frac{\partial}{\partial t}(EF) \right). \tag{23}$$

Since a zeroth order approximation of the term $\frac{\partial}{\partial t}(EF)$ is sufficient, this term can be approximated by

$$\frac{\partial}{\partial t}(EF) = \frac{E(t + \Delta t/2)F(t + \Delta t/2) - E(t)F(t)}{\Delta t/2} .$$

Therefore, the two terms ΔtEF and $(\Delta t^2/2)\frac{\partial}{\partial t}(EF)$ can be combined, so that

$$F(x,y,t+\Delta t) = F(x,y,t) + i\Delta t\frac{\partial^2 F}{\partial x\partial y} - \frac{(\Delta t)^2}{2}\left(\frac{\partial^4 F}{\partial x^2 \partial y^2} - \frac{\partial^2}{\partial x\partial y}(y\,Ef)\right)$$

$$- iy\Delta tE(t+\Delta t/2)\cdot F(x,y,t+\Delta t/2)\,. \tag{24}$$

It can be shown that the following two-step procedure

$$F^{n+1/2}_{j,k} = F^n_{j,k} + \frac{i}{8}\frac{\Delta t}{\Delta x\Delta y}\left(::F^n_{j,k} - 4\Delta x\Delta y yE^n_j F^n_{j,k}\right)$$

$$F^{n+1}_{j,k} = F^n_{j,k} + \frac{i}{4}\frac{\Delta t}{\Delta x\Delta y}\left(::F^{n+1/2}_{j,k}\right) - iy\Delta t\,E^{n+1/2}_j F^{n+1/2}_{j,k} \tag{25}$$

where the indices n, j, k correspond to t, x, y, respectively and

$$::H_{j,k} = H_{j+1,k+1} - H_{j+1,k-1} - H_{j-1,k+1} + H_{j-1,k-1}$$

is a difference approximation of Eq. (24). Considering the free-streaming case, i.e., the simplified equation

$$F(x,y,t+\Delta t) = F(x,y,t) + i\Delta t \frac{\partial^2 F}{\partial x \partial y} - \frac{(\Delta t)^2}{2} \frac{\partial^4 F}{\partial x^2 \partial y^2} ,$$

one can show that the correspondingly simplified version of Eq. (25) is stable according to the von Neumann stability condition (see Ref. 5), if Δt, Δx, $\Delta y \to 0$ in such a way that

$$\frac{\Delta t}{(\Delta x \Delta y)^{4/3}} = \text{const} . \tag{26}$$

A numerical program for Eq. (25) can be written so as to contain only four multiplications per mesh point and time step.

B. The Conservation Laws

There are several ways of determining the time dependence of the electric field. One way would be to use Poisson's equation; another one is to use Maxwell's equation $j = i \frac{\partial E}{\partial t}$. In terms of F the latter can be written as

$$\frac{\partial F}{\partial y}(x,0,t) = + i \frac{\partial E}{\partial t}$$

so that (using Eq. (22)

$$\frac{F_{j,1}^{n+1/2} - F_{j,1}^{n+1/2*}}{2\Delta y} = i \frac{E_j^{n+1} - E_j^n}{\Delta t} \tag{27}$$

is a possible way of integrating the electric field in time.

The first application of Eq. (24) and (27) will be the strongly nonlinear, spatially periodic case in which many Fourier modes exist. In this case (spatial periodicity), it can be shown that using Eq. (27) to supplement Eq. (24) the following expressions are exactly conserved by the difference scheme

$$\sum_j F^n_{j,0}$$

$$\sum_j F^n_{j,1}$$

$$\frac{1}{(\Delta y)^2} \sum_j (F^n_{j,1} - 2F^n_{j,0} + F^{n*}_{j,1}) + \sum_j {E^n_j}^2$$

$$0 \le j \le j_{max} \qquad F^n_{j,o} + \frac{1}{2\Delta x} (E^n_{j+1} - E^n_{j-1}) \, . \tag{28}$$

Since the expressions (28 a-d) correspond to difference approximations for

$$\int_o^L F(x,0,t)\, dx$$

$$\int_o^L \frac{\partial F}{\partial y} (x,0,t)\, dx \quad \text{(because of Eq. (22))}$$

$$-\int_o^L \frac{\partial^2 F}{\partial y^2} (x,0,t)\, dx + \int_o^L E^2(x)\, dx$$

$$(n + \frac{\partial E}{\partial x})_j = 1 \text{ according to Poisson's equation.}$$

n is the electron density. The conservation of expressions (28 a-c) corresponds to conservation of the total number of electrons, momentum, and energy. The accuracy with which Eq. (28 d) is equal to one for t = 0, indicates the accuracy with which Poisson's equation is solved. Thus, Eq. (24) supplemented by Eq. (27) has exact conservation laws which would not be the case if the time integration of the electric field were performed in a different way.

C. Numerical Calculations

Preliminary numerical calculations show the validity of the stability condition Eq. (26) in the free-streaming case and that the cut-off prescription at $y = y_{max}$ has some influence on the numerical result. So far, extrapolations of zeroth and first order in Δy have been used; the latter seems to be preferable. Further investigations have shown that the difference scheme exhibits correct linear Landau damping (or a linearly instable behavior). In the near future, the scheme will be applied to a variety of strongly nonlinear periodic and spatially nonperiodic cases.

VII. ELECTRON MOTION FOR SLOW PROCESSES IN PLASMAS

When we follow ion and electron trajectories on a computer, the time step must be adjusted in such a way that a plasma oscillation can still be represented. This means, however, that we spend most of our computing effort on the time scale of the electrons. Following the evolution of an ion wave, for example, it becomes either extremely time consuming or practically impossible. When the electron gas is collision free, the adiabatic invariant of the electrons is conserved if the potential change is slow in a certain sense. The adiabatic invariant is given by

$$J = \int v dx = \oint \sqrt{2/m \, (W + e\varphi(x_1 t))} \; dx \,, \qquad (29)$$

W being the total energy of an electron. We can express the constancy of J by

$$\frac{\partial}{\partial t} f_e(J,t) = 0 \quad .$$

If we want to express the distribution function by the total energy W of the electrons, we can write the above equation as

$$\frac{\partial f(W,t)}{\partial t} = - \frac{\Delta W}{\Delta t} \frac{\partial f(W,t)}{\partial W} \tag{30}$$

where ΔW represents the energy a particle has gained in time Δt, Δt comprising many bounce periods. Explicitly, we obtain:

$$f_e = f_e \left[W - \overline{W(t)} - \overline{W(0)}\right] \quad . \tag{31}$$

$\overline{W(t)}$ is given by the relation

$$\oint \sqrt{\overline{W(0)} + e\varphi(x,0)}\; dx = \oint \sqrt{\overline{W(t)} + e\varphi(x,t)}\; d\gamma \quad . \tag{32}$$

It is important to realize that Eq. (31) and Eq. (32) are indeed much simpler than the full Vlasov equation because W(t) changes on the time scale of the slow motion.

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