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HIGH ENERGY NEUTRINO AND MUON REACTIONS

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ABSTRACT

The detailed numerical analysis of the weak leptonic processes $\nu + (Z) \rightarrow \nu + \mu + \bar{\mu} + (Z)$ and $\bar{\mu} + (Z) \rightarrow \bar{\mu} + \nu + \bar{\nu} + (Z)$ has been performed based on the conventional V-A Fermi theory.

We found marked asymmetry between μ and $\bar{\mu}$ in the first reaction. The muon is more energetic and appears more in the forward direction compared with the antimuon. The invariant mass of the μ $\bar{\mu}$ pair is found to be small. The antimuon in the second process loses almost all its energy during the reaction. The asymptotic formula at extremely high energy is also derived. We discuss the ratio of the total cross sections for these two processes at high energy. The effects of the deeply inelastic process on these reactions are also discussed. We present a rather detailed phase space reduction.

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I. INTRODUCTION

The self-coupling of weak currents is interesting as a test of the current-current Fermi coupling scheme. Gell-Mann et al.¹⁾ have made an interesting conjecture that the coupling constant of the self-interaction of weak currents (so called diagonal process) could be different from the ordinary Fermi coupling constant $G=10^{-5}/M_p^2$. Also if the commonly assumed weak bosons²⁾ existed with a reasonable mass the point coupling nature of the Fermi theory is modified.

Purely leptonic weak processes may provide a decisive test of these conjectures. We study the consequences of the self-coupling of weak lepton currents

$$\mathcal{L} = \frac{G}{2\sqrt{2}} \left[\bar{\psi}_l \gamma^\mu (1-\gamma_5) \psi_{\nu_l} \bar{\psi}_{\nu_l} \gamma_\mu (1-\gamma_5) \psi_l + h.c. \right]$$

or after the Fierz transformation

$$\mathcal{L} = \frac{G}{\sqrt{2}} \bar{\psi}_l \gamma^\mu (1-\gamma_5) \psi_l \bar{\psi}_{\nu_l} \gamma_\mu (1-\gamma_5) \psi_{\nu_l} \quad (I-1)$$

where ψ_l and ψ_{ν_l} denote the lepton and the corresponding neutrino field. We assume two component neutrinos throughout this study. The most general nonderivative contact four-Fermion coupling which is compatible with the two component neutrino hypothesis is written as

$$\mathcal{L} = \frac{G}{\sqrt{2}} \bar{\psi}_l \gamma^\mu (g_V - g_A \gamma_5) \psi_l \bar{\psi}_{\nu_l} \gamma_\mu (1-\gamma_5) \psi_{\nu_l}$$

The effective Lagrangian (I-1) corresponds to the case $g_V = g_A$. If $g_V \neq g_A$, the original Lagrangian, i.e., the Lagrangian before the Fierz transformation may contain S and P "currents" in addition to the ordinary V and A currents. In particular if $g_V = -g_A$ the original Lagrangian contains only S and P components. Some of the results in the

following depend on the relative magnitude of g_V and g_A . The total cross section usually depends only on the combination $g_V^2 + g_A^2$; we always assume T-invariance, i.e., g_V and g_A are real. However the asymmetry between the lepton and antilepton which will appear in the following depends on the "interference term" $g_V \cdot g_A$; the change of the sign of $g_V \cdot g_A$ corresponds to the replacement of the lepton and the antilepton. Thus we can use the following results to test the equality $g_V = g_A$ as well as the absolute magnitude of the coupling constant.

We largely omitted the study of the "weak bosons". In this respect the purpose of the present study is rather conservative. It does not tell what will be expected if these bosons existed but rather it tells what kind of observation can be regarded as a deviation from the conventional theory.

The content of this paper is as follows:

Section II is devoted to the study of one of the self-couplings of the weak currents, namely the muon pair production by energetic mu-neutrinos in a nuclear Coulomb field, i.e., $\nu_\mu + (Z) \rightarrow \nu_\mu + \mu^+ + \mu^- + (Z)$. Detailed numerical calculation by computer has been employed to study the angular, energy and invariant mass distributions of the muon pairs.

In Section III, we work on the neutrino pair radiation from extremely energetic muons in a nuclear Coulomb field; $\bar{\mu} + (Z) \rightarrow \bar{\mu} + \nu + \bar{\nu} + (Z)$. The energy and angular distributions of the scattered muons are analysed. The close resemblance of these processes to "real neutral weak boson" production processes and bremsstrahlung processes is emphasized.

In Section IV we discuss the effects of the deeply inelastic process on the above reactions. We also derive asymptotic formulas for neutrino radiation from the muon. We discuss the ratio of the two cross sections $\sigma(\nu+Z\rightarrow\nu+\mu+\bar{\mu}+Z)$ and $\sigma(\bar{\mu}+Z\rightarrow\bar{\mu}+\nu+\bar{\nu}+Z)$ at high energy within the frame work of the contact four-Fermion coupling.

In Appendices we discuss the trace calculation and the details of the analytical and numerical integration which have been skipped in the preceding discussions.

II. MUON PAIR RADIATION FROM ENERGETIC NEUTRINO IN A COULOMB FIELD.

1. Introductory Remarks

The total cross section of the muon pair creation by energetic neutrinos in a Coulomb field

$$\nu + (Z) \longrightarrow \nu + \mu + \bar{\mu} + (Z) \quad (\text{II},1)$$

has been calculated by several people³⁾⁻⁵⁾.

The following facts have been established:

(i) the total cross section is very small, namely $10^{-43} \sim 10^{-40} \text{ cm}^2$ depending on the energy ($1 \sim 50 \text{ BeV}$) and target nucleus (or nucleon).

(ii) the asymptotic expression

$$\sigma \sim Z^2 \alpha^2 G_F^2 S \cdot [\log S + \text{const.}] \quad (\text{II},2)$$

becomes valid for incident energies above $50 \sim 100 \text{ BeV}$ depending on the target. G is the usual Fermi coupling constant ($=1.02 \times 10^{-5} / m_p^2$), Z the nuclear charge number, α the fine structure constant ($=1/137$), and s the incident laboratory energy of the neutrino defined by $s = E_\nu \cdot M^*$ = energy of the neutrino times the "effective mass" of the target which is determined by (but not necessarily the same as) the cut-off mass of the nuclear (or nucleon) form factors.

(iii) for light nuclear targets, the incoherent contribution (i.e., Z times cross section on the proton plus N times cross

section on the neutron) amounts to 20~30 percent of the coherent process (i.e., nucleus acts as a whole). Especially at low energies, the incoherent process contributes almost same as the coherent process. The neutron contribution is small, as expected, compared with the proton contribution. For heavy nuclei and at high neutrino energies the coherent process becomes dominant.

(iv) small momentum transfer to the nucleus (or nucleon) is important but the dominant contribution to the cross section does not come from the smallest values of the momentum transfer.

The property (i) is due to the extra factor of the fine structure constant as well as to the weakness of the weak interaction.

It is interesting that the high energy behaviour of the total cross section is what we expect from the naive dimensional analysis, namely, $\sigma \sim \alpha^2 \cdot G^2 \cdot s$, s being the Mandelstam invariant.

The property (iv) comes from the phase space of the multi-particle final state; the cross section is thus free from infrared divergence even when we neglect the muon mass.

The smallness of the cross section has so far prohibited the performance of this interesting experiment. However the development of experimental techniques and the prospect of a high energy accelerator at NAL may make it possible to do this experiment in several years⁶⁾.

2. General Structure of the Cross Section

We start with the weak purely leptonic "Lagrangian density"
(I-1),

$$\mathcal{L} = \frac{G}{\sqrt{2}} \bar{\psi}_\nu \gamma^\alpha (1 - \gamma_5) \psi_\nu \cdot \bar{\psi}_\mu \gamma_\alpha (1 - \gamma_5) \psi_\mu \quad (\text{II}, 3)$$

With this Lagrangian combined with the one photon exchange approximation we get the invariant matrix element. The corresponding Feynman diagrams are shown in figures 1(a) and 1(b). The corresponding matrix element is

$$\begin{aligned} \mathcal{M} = & \frac{G e^2}{\sqrt{2}} \bar{u}_\nu(k') \gamma^\alpha (1 - \gamma_5) u_\nu(k) \\ & \cdot \bar{u}_\mu(p) \left[\gamma^\mu \frac{\not{p} + \not{q} + \not{\mu}}{(p+q)^2 - \mu^2} \gamma_\alpha (1 - \gamma_5) + \gamma_\alpha (1 - \gamma_5) \frac{-\not{p} - \not{q} + \not{\mu}}{(\bar{p}+q)^2 - \mu^2} \gamma^\mu \right] v_{\bar{\mu}}(\bar{p}) \\ & \cdot \frac{1}{q^2} \cdot \langle Q' | J_\mu(0) | Q \rangle \end{aligned} \quad (\text{II}, 4)$$

Our general notation is

$$\nu(k) + Z(Q) \rightarrow \nu(k') + \mu(p) + \bar{\mu}(\bar{p}) + Z(Q')$$

where k: 4-momentum of the incident neutrino

k': 4-momentum of the outgoing neutrino

p: 4-momentum of the muon

$\bar{p}$: 4-momentum of the antimuon

Q and Q': initial and final momenta of the target nucleus
(or nucleon).

q: 4-momentum transfer to the nucleus (or nucleon)

$$= Q' - Q$$

M: mass of the target nucleus (or nucleon)

μ : Mass of the muon

$u(p)$ and $v(p)$ are the positive and negative energy plane wave solutions of the Dirac equation, respectively.

Our metric is the same as that of Bjorken and Drell⁷⁾

$$k \cdot P = k^0 \cdot P^0 - \vec{k} \cdot \vec{P}.$$

The first factor in (II,4) is the neutrino current. The second factor is the muon current and it contains two terms corresponding to the diagrams 1(a) and 1(b). The third factor comes from the photon propagator and the nuclear (or nucleon) electromagnetic current. The explicit form of the nuclear current depends on the spin of the nucleus⁸⁾.

Spin-0 particle with charge Ze :

$$\langle Q' | J_\mu | Q \rangle = Z (Q + Q')_\mu \cdot F(q^2) \quad (\text{II},5)$$

where $F(q^2)$, $q=Q'-Q$, is a form factor normalized to 1 at $q^2=0$: $F(0)=1$.

Spin-1/2 particle (nucleon):

$$\langle Q' | J_\mu | Q \rangle = \bar{u}(Q') \left[\gamma_\mu F_1(q^2) + i \frac{\kappa \sigma_{\mu\nu} q^\nu}{2M} F_2(q^2) \right] u(Q) \quad (\text{II},6)$$

where $F_1(0)=1$, $F_2(0)=1$, $\kappa=1.79$ for the proton

$F_1(0)=0$, $F_2(0)=1$, $\kappa=-1.91$ for the neutron.

First we rewrite the muon current part slightly. Using the Dirac equations

$$\begin{aligned} (\not{p} - \mu) u(p) &= 0 \\ (\not{p} + \mu) v(\bar{p}) &= 0 \end{aligned} \quad (\text{II},7)$$

the muon current can be written as

$$\begin{aligned} \bar{U}_\mu(p) \left[\frac{2p^\mu + \not{p}}{q^2 + 2q \cdot p} \gamma_5 (1 - \gamma_5) - \gamma_5 (1 - \gamma_5) \frac{2\bar{p}^\mu + \not{\bar{p}}}{q^2 + 2q \cdot \bar{p}} \right] V_\mu(\bar{p}) \\ = \bar{U}_\mu(p) \cdot W_\alpha^\mu \cdot (1 - \gamma_5) \cdot V_\mu(\bar{p}) \end{aligned} \quad (II,8)$$

where we defined

$$W_\alpha^\mu = \frac{2p^\mu + \not{p}}{q^2 + 2q \cdot p} \gamma_\alpha - \gamma_\alpha \frac{2\bar{p}^\mu + \not{\bar{p}}}{q^2 + 2q \cdot \bar{p}} \quad (II,9)$$

Note that this W_α^μ satisfies the "current conservation" condition by itself independently of the spinors,

$$q_\mu W_\alpha^\mu = \frac{2q \cdot p + q^2}{q^2 + 2q \cdot p} \gamma_\alpha - \gamma_\alpha \frac{2q \cdot \bar{p} + q^2}{q^2 + 2q \cdot \bar{p}} = 0 \quad (II,10)$$

Thus we can guarantee the current conservation by inserting this W_α^μ ⁹⁾.

Now we calculate the absolute square of the matrix element $\mathcal{M}$.

We take a summation over all spins involved. The spin averaging will be taken care of later. We calculate only the case of the nucleon target; the result for the spin-0 nucleus target is readily obtained from this result as we will see later.

$$\begin{aligned} \sum_{\text{spins}} |\mathcal{M}|^2 &= \left(\frac{e^4 \cdot G^2}{2 \cdot q^4} \right) \cdot j^{\alpha\beta} \\ &\quad \cdot \text{Tr} \left[(\not{p} + \not{m}) W_\alpha^\mu (1 - \gamma_5) (\not{\bar{p}} - \not{m}) V_\beta^\nu (1 - \gamma_5) \right] \\ &\quad \cdot 2 \cdot J_{\mu\nu} \end{aligned} \quad (II,11)$$

where

$$\begin{aligned} V_\alpha^\mu &\equiv \gamma_0 W_\alpha^{\mu\dagger} \gamma_0 \\ &= \gamma_0 \frac{\not{p} \gamma^\mu + 2p^\mu}{q^2 + 2q \cdot p} - \frac{\gamma^\mu \not{\bar{p}} + 2\bar{p}^\mu}{q^2 + 2q \cdot \bar{p}} \gamma_0 \end{aligned}$$

$j^{\alpha\beta}$ = neutrino current contribution

$$= \text{Tr}[\not{K} \gamma^0 (1 - \gamma_5) \not{K} \gamma^0 (1 - \gamma_5)]$$

$$= 2 \cdot \text{Tr}[\not{K} \gamma^0 \not{K} \gamma^0 (1 - \gamma_5)]$$

$J_{\mu\nu}$ = spin averaged nucleon electromagnetic current contribution

$$= G_1(q^2) \left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right] + G_2(q^2) \left[q_\mu - \frac{1}{2} q_\mu \right] \left[q_\nu - \frac{1}{2} q_\nu \right]$$

with

$$G_1(q^2) = q^2 \cdot [F_1(q^2) + \kappa F_2(q^2)]^2$$

$$G_2(q^2) = 4 F_1(q^2)^2 - q^2 \cdot \frac{\kappa^2}{M^2} \cdot F_2(q^2)^2$$

Note that $J_{\mu\nu} = J_{\nu\mu}$, i.e., $J_{\mu\nu}$ is a symmetric tensor. This property of $J_{\mu\nu}$ is the manifestation of the "parity conservation" of the electromagnetic current. On the other hand, $j^{\alpha\beta} \neq j^{\beta\alpha}$, parity is violated by the weak current. However under the simultaneous exchange

$$\alpha \leftrightarrow \beta \quad \text{and} \quad k \leftrightarrow k', \quad j^{\alpha\beta} \text{ remains unchanged; this}$$

property is used below.

Next we examine the muon current contribution

$$\text{Tr}[(\not{p} + \not{p}') W_\alpha^\mu (1 - \gamma_5) (\not{p} - \not{p}') V_\beta^\nu (1 - \gamma_5)]$$

$$= 2 \cdot \text{Tr}[\not{p} W_\alpha^\mu \not{p}' V_\beta^\nu (1 - \gamma_5)]$$

thus the explicit dependence on the muon mass disappears. First note that under the exchange, $p \leftrightarrow \bar{p}$,

$$W_\alpha^\mu \leftrightarrow -V_\alpha^\mu$$

Thus the muon current part is invariant under the simultaneous exchange,

$\alpha \leftrightarrow \beta$ and $p \leftrightarrow \bar{p}$, if we remember that only the symmetric part under the exchange $\mu \leftrightarrow \nu$ contributes to our matrix element.

From these considerations we get the following result:

The matrix squared is, after spin summation, invariant under the simultaneous exchange $k \leftrightarrow k'$ and $p \leftrightarrow \bar{p}$, but not invariant under the single exchange $p \leftrightarrow \bar{p}$.¹⁰⁾

This property is responsible for the asymmetry of the energy and angular distributions between the muon and the antimuon as we will see in the following sections. We write the matrix element slightly more explicitly as,

$$\sum_{\text{spino}} |M|^2 = \left(\frac{e^4 G^2}{2 q^4} \right) \cdot \text{Tr} [K \gamma^\mu K \gamma^\nu (1 - \gamma_5)] \cdot \left\{ \frac{1}{(q^2 + 2 q \cdot p)^2} \text{Tr} [\gamma (2 p^\mu + \gamma^\mu \not{p}) \gamma_\alpha \not{p} \gamma_\beta (\gamma^\nu + 2 p^\nu) (1 - \gamma_5)] \right. \\ + \frac{1}{(q^2 + 2 q \cdot \bar{p})^2} \text{Tr} [\gamma \gamma_\alpha (2 \bar{p}^\mu + \gamma^\mu \not{\bar{p}}) \not{p} (\gamma^\nu + 2 \bar{p}^\nu) \gamma_\beta (1 - \gamma_5)] \\ \left. - \frac{2}{(q^2 + 2 q \cdot p)(q^2 + 2 q \cdot \bar{p})} \text{Tr} [\gamma \gamma_\alpha (2 \bar{p}^\mu + \gamma^\mu \not{\bar{p}}) \not{p} \gamma_\beta (\gamma^\nu + 2 p^\nu) (1 - \gamma_5)] \right\} \cdot J_{\mu\nu} \quad (\text{II}, 12)$$

This form of the matrix element is enough for our argument in the following. The explicit calculation of the trace is not difficult and we will give the result in Appendix A. From this expression we see that complex mixing of the nucleon and lepton variables takes place in the denominators which are written explicitly. The mixing of the variables in the numerator does not offer any difficulty in the phase space integration.

The differential cross section is given by

$$d\sigma = \frac{1}{|\vec{v}_k - \vec{v}_q| \cdot 4 E_k E_q} \cdot \sum_{\text{spino}} |M|^2 \cdot (2\pi)^4 \cdot \delta(k - k' - p - \bar{p} - q) \cdot \frac{d^3 k'}{(2\pi)^3 \cdot 2 E_{k'}} \cdot \frac{d^3 p}{(2\pi)^3 \cdot 2 E_p} \cdot \frac{d^3 \bar{p}}{(2\pi)^3 \cdot 2 E_{\bar{p}}} \cdot \frac{d^3 q'}{(2\pi)^3 \cdot 2 E_{q'}} \cdot \int$$

The last factor S is the spin average factor. In the present reaction $S=1/2$ corresponding to nucleon spin $1/2^{11}$; we do not take an average over the incident neutrino spin.

Hence

$$d\sigma = \frac{1}{(2\pi)^8} \cdot \frac{1}{P(k,Q)} \sum_{\text{spins}} |M|^2 \cdot \delta(k-k'-p-\bar{p}-q) \cdot \frac{d^3k'}{2E_{k'}} \cdot \frac{d^3p}{2E_p} \cdot \frac{d^3\bar{p}}{2E_{\bar{p}}} \cdot \frac{d^3q'}{2E_{q'}} \quad (\text{II},13)$$

Next we try to simplify the phase space integration. We start with 7-dimensional phase space; 4-dimensional delta-function and the symmetry around the incident neutrino momentum reduce the original 12-dimensional phase space to 7-dimensional one. We treat the lepton part of the phase space separately from the nucleon part.

First observe that any two-particle phase space can be transformed into "one particle phase space" plus "internal degrees of freedom of the combined particle." The internal degrees of freedom will be best defined in the center of momentum of the two-particle system. We can show that the integration over the "internal variables" in our problem can be done analytically for any combination of two of the three leptons which appear in the final state; this is due to the locality of the lepton couplings, i.e., no form factors at all. Before we enter the detailed calculation we express the above mentioned phase space "reduction" explicitly.

First we consider the case where the two muon variables are combined. The following relation can be readily established

$$\frac{d^3p}{2E_p} \cdot \frac{d^3\bar{p}}{2E_{\bar{p}}} = \frac{\lambda(m^2, \mu^2, \mu^2)}{8\pi^2} \cdot d^4q_1 \cdot \delta(m^2 - q_1^2) dm^2 d\Omega_{q_2} \quad (\text{II}, 14)$$

where

$$q_1 = p + \bar{p}$$

$$q_2 = p - \bar{p}$$

$d\Omega_{q_2}$ = solid angle of q_2 in the C. M. of q_1 , i.e.,

where $q_1 = 0$.

$$\lambda(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2xy - 2yz - 2zx}$$

Similarly if we combine the antimuon and the neutrino, we get

$$\frac{d^3\bar{p}}{2E_{\bar{p}}} \cdot \frac{d^3k'}{2E_{k'}} = \frac{\lambda(m^2, \mu^2, 0)}{8\pi^2} \cdot d^4q_1 \cdot \delta(m^2 - q_1^2) dm^2 d\Omega_{q_2} \quad (\text{II}, 15)$$

where

$$q_1 = \bar{p} + k'$$

$$q_2 = \bar{p} - k'$$

In the following we discuss the integration over "these internal angular variables $d\Omega$ " plus another phase space variable. Thus we will be finally left with 4-dimensional phase space. We perform the remaining integration by the computer.

We first consider the energy and angular distribution of the muon.

Our starting expression is

$$d\sigma = \frac{1}{(2\pi)^8} \cdot \frac{1}{f(k, Q)} \sum_{\text{spins}} |M|^2 \cdot \delta(k - p - q_1 - q_2) \cdot \frac{d^3p}{2E_p} \cdot \frac{d^3Q'}{2E_{Q'}} \cdot \frac{\lambda(m^2, \mu^2, 0)}{8\pi^2} \cdot d^4q_1 \cdot \delta(m^2 - q_1^2) dm^2 \cdot d\Omega_{q_2} \quad (\text{II}, 16)$$

where we used Eq. (II,15). We perform the integration over $d\Omega_2$.

After the integration we get

$$d\sigma = f(k, p, Q, q) \frac{d^3 p}{2E_p} \cdot dm^2 \cdot \delta(m^2 - (k-p-q)^2) \cdot \frac{d^3 Q'}{2E_{Q'}}$$

where f is an invariant function of the arguments shown. Next we go

to the lab. frame (note that the cross section is Lorentz invariant).

In this frame $\vec{Q}' = \vec{q}$ due to the definition of q , $q = Q' - Q$, and $t = -q^2 = -(Q' - Q)^2 = 2M(E_{Q'} - M) \geq 0$

Thus

$$\frac{d^3 Q'}{2E_{Q'}} = \frac{|\vec{q}|}{4M} dt d\Omega_q$$

and

$$d\sigma = f(k, p, Q, q) \frac{d^3 p}{2E_p} \cdot dm^2 \cdot \delta(m^2 - (k-p-q)^2) \frac{|\vec{q}|}{4M} dt d\Omega_q \quad (\text{II,17})$$

Finally we want to perform the integration $\int d\Omega_q$. The kinematical constraint, which is solely expressed by the remaining delta-function $\delta[m^2 - (k-p-q)^2]$, contains q only as a combination $q \cdot (k-p)$. This suggests that we should take the direction $k-p$ as the z -direction when we integrate over $d\Omega_q$. By this choice, the variable φ_q has no constraint; it can vary from $-\pi$ to $+\pi$.

Next we examine the q -dependence of the function $f(k, p, Q, q)$. The invariant $q \cdot Q$ is related to $t = -q^2$ by $q \cdot Q = t/2$. Thus we have only two nontrivial inner products which contain q , i.e., (qk) and (qp) , or equivalently, $(k-p) \cdot q$ and (qp) . This means $f(k, p, Q, q)$ depends on φ_q only through the combination (qp) .

Hence if $f(k, p, Q, q)$ can be expressed as a rational function of (qp) , the integration over $\mathcal{Q}_2$ is doable, however complicated it may be. This is actually our case.¹²⁾ When we integrate over $d\Omega_q$, we use the delta-function

$$\int \delta[m^2 - (k-p-q)^2] d\cos\theta_q = \frac{1}{2|\vec{k}-\vec{p}| \cdot |\vec{q}|}$$

This result is valid only on a special condition; $\cos(\theta_q)$ must lie between -1 and +1. This condition introduces the following constraint among phase space variables.

$$-2|\vec{q}| \cdot |\vec{k}-\vec{p}| \leq m^2 - q^2 + 2q^0(k^0 - p^0) - (k-p)^2 \leq 2|\vec{q}| \cdot |\vec{k}-\vec{p}|$$

or

$$m^2 + \lambda + \frac{\lambda}{M^2} [(k \cdot Q) - (p \cdot Q)] - (k-p)^2 \leq \frac{p}{M^2} \cdot \sqrt{4M^2\lambda + \lambda^2} \quad (\text{II}, 18)$$

where

$$p = M |\vec{k}-\vec{p}| = \sqrt{[(k \cdot Q) - (p \cdot Q)]^2 + 2(k \cdot p) \cdot M^2 - \mu^2 M^2}$$

Thus finally,

$$d\omega = \frac{1}{\pi} g(k \cdot Q, (p \cdot Q), (k \cdot p), m^2, \lambda) \frac{d^3p}{\partial E_p} \cdot dm^2 \cdot d\lambda \quad (\text{II}, 19)$$

where g is a function of the arguments shown. Due to the rotational symmetry around the momentum of the incident neutrino, the phase space has only four nontrivial variables.

$$d\omega = g(k \cdot Q, (p \cdot Q), (k \cdot p), m^2, \lambda) |\vec{p}| dE_p d\cos\theta_p dm^2 d\lambda \quad (\text{II}, 20)$$

We have to do two dimensional numerical integration over $dm^2 dt$ to get the energy and angular distribution. Note that we did not assume any special form for the nuclear form factors; the integration so far can be done independently of the form factors. The integration over the momentum transfer t depends on the form factors explicitly. We assume the same form factors as in reference¹³⁾(4), namely "dipole" form factors

$$F(t) = [1 + t/M_\rho^2]^{-2}$$

where M_ρ^2 is the square of the rho-meson mass for nucleons, and is equal to $0.2 \times (1.64/\sqrt[3]{A})^2$ for a nucleus with a mass number A . This choice has been taken rather arbitrarily, but it makes it easier to compare our total cross section with the previously known values.

We get a similar result for the energy and angular distribution of the antimuon. The only difference is that the invariant mass m now corresponds to the mass of the muon-neutrino system instead of the anti-muon-neutrino system. The dependence of the matrix element on these phase space variables is, of course, different.

Finally, if we combine the muon pair first, the result becomes something like,

$$d\sigma = h(k, Q, (k', Q), (kk'), m^2, t) \frac{d^3k'}{2k_0} dm^2 dt \quad (\text{II}, 21)$$

where h is an invariant function and

k' : 4-momentum of the outgoing neutrino

m : invariant mass of the muon pair.

Furthermore, as we will see in the next section, the energy loss of the neutrino (i.e., $k_0 - k'_0$) can be identified with the total energy of the muon pair; this comes from the fact that the energy transfer to the nucleon (or nucleus) is extremely small. Thus we can get a correlation

between the invariant mass and the total energy of the di-muon system.

We will discuss the results of the numerical calculation in the following sections.

3. Energy and Angular Distribution of the Muon and the Antimuon; Numerical Calculation.

In the previous section we found that the energy and angular distribution of the muon can be written as¹⁴⁾

$$\frac{d^2\sigma}{dE d\cos\theta} = \int f(kQ, pQ, kp, t, m^2) dt dm^2$$

with a kinematical constraint,

$$m^2 + t + \frac{t}{M^2} [(kQ) - (pQ)] - (k-p)^2 \leq \frac{p}{M^2} \sqrt{4M^2 t + t^2}$$

where $p = M \cdot |\vec{k} - \vec{p}| = \sqrt{[(kQ) - (pQ)]^2 + 2(kp) \cdot M^2 - M^2 \mu^2}$ and

k: 4-momentum of the incident neutrino = $(k_0, 0, 0, k_0)$

p: 4-momentum of the outgoing muon = $(E, |\vec{p}| \cdot \sin\theta, 0, |\vec{p}| \cdot \cos\theta)$

Q: 4-momentum of the target nucleon (or nucleus)

m: invariant mass of the antimuon-neutrino system

t: square of the momentum transfer to the nucleon system

We first find the shape of the phase space of these four remaining variables.

For this purpose we introduce several new notations;

$$s = (kQ) = k^0 \cdot M$$

$$x = (pQ) = p^0 \cdot M$$

$$y = (kp) = k^0 \cdot p^0 - |\vec{k}| |\vec{p}| \cos\theta$$

in the laboratory frame.

We consider the combination $x+y$

$$x+y = (kp) + (pQ)$$

$$= [p \cdot (k+Q)].$$

This means $x+y$ reduces to

$$p \cdot (k+Q) = p_0 \cdot \sqrt{s_c} \quad (15)$$

in the C. M. Now p_0 is the energy of the muon in the C. M., thus satisfies a relation $p_0 \geq$ muon mass. This gives a condition

$$x+y \geq \mu \sqrt{s_c} = \mu \cdot \sqrt{2s+M^2} \quad (II,22)$$

The maximum value of p_0 is obtained from the co-linear configuration in the C. M. We get an equation

$$\begin{aligned} \sqrt{s_c} \cdot p_0 &= [s_c + \mu^2 - (M+m)^2]/2 \\ &\leq [s_c + \mu^2 - (M+\mu)^2]/2 \quad \text{for } m \geq \mu. \end{aligned}$$

Thus we get another constraint

$$x+y \leq [2s-2M\mu]/2 = s-M\mu$$

Combining with (II,22) we finally get the phase space for (x, y) ,

$$\mu \cdot \sqrt{2s+M^2} \leq x+y \leq s-M\mu \quad (II,23)$$

Remembering the definition of x and y , we can convert this condition into that on E and $\cos \theta$ in the laboratory frame. The result is shown in figure (2)¹⁶⁾.

The kinematical constraint on (t, m^2) is given by Eq. (II,19) and the condition $m \geq \mu$ for the fixed point (x, y) .

$$0 \leq m^2 - \mu^2 \leq \frac{p}{M^2} \cdot \sqrt{4M^2 x + x^2} - x - \frac{x}{M^2} (s-x) - 2y$$

This is shown in figure (3).

We first integrate over (t, m^2) ¹⁷⁾ and get the distribution,

$$\frac{d^2 \sigma}{d\cos \theta dE} = \int f(s, x, y, t, m^2) dt dm^2 \equiv G(s, E, \cos \theta) \quad (II,24)$$

The computer printed out $G(s, E, \cos \theta)$ for about 300 points in the two dimensional phase space $(E, \cos \theta)$. We first fixed $\cos \theta$ and calculated the energy spectrum of the muon at that fixed angle. We calculated these distributions for the proton, neutron, Fe^{56} and Pb^{208} targets with incident neutrino energy at 10 BeV and 40 BeV.

We repeated the exactly same procedure for the antimuon.

We found the following interesting features of these distributions:

(i) The muon and the antimuon have quite different energy and angular distributions. The muon is more energetic and appears more in the forward direction¹⁸⁾. This is best seen from the energy spectrum with fixed angles. See figures (5), (6), (7), (8).

(ii) The angular distribution, i.e. $d\sigma/d\cos\theta$, decreases rapidly when we go off-forward direction. However, the long tail of this distribution gives an important contribution to the total cross section. The angular distribution at a fixed energy, i.e. $d\sigma/(d\cos\theta dE)$ with fixed E , decreases faster for larger muon energy E . This is due to the combined effects of the narrow phase space and small momentum transfer¹⁹⁾. See figures (9), (10), (11), (12), (13), (14).

(iii) The energy spectrum after the integration over the angular variables, i.e. $d\sigma/dE$, has a peak at low energies²⁰⁾.

This, combined with (ii), means that the peak of the forward energy spectrum at high energy side dies off quite rapidly with the deviation of $\cos \theta$ from 1. The energy of the muon (not antimuon) is rather uniformly distributed over the phase space except a peak at very low energies. See figures (15), (16), (17).

(iv) Momentum transfer dependence:

We also checked the momentum transfer dependence of the distribution at fixed $\cos \theta$ and E . Figures (18) and (19) show this dependence for the proton target at 10 BeV and the Fe^{56} target at 40 BeV. The points $(\cos \theta, E)$ were chosen at the region of the phase space which gives an important contribution to the total cross section.

From these figures one may guess that the major contribution to the distribution comes from the region where $t \leq 0.1 (\text{BeV})^2$. This is actually correct; we printed out every momentum transfer dependence during the numerical integration procedure. We can also estimate the energy transfer to the nucleon or to the nucleus.

Energy transfer $\leq t/2M \leq 50\text{MeV}$ for nucleon

$\leq 1\text{MeV}$ for Fe^{56} or Pb^{208}

This suggests that it is hard to detect the recoiled nucleon or nucleus, although one may observe some of the recoiled nucleons under very favourable conditions.

(v) The dependence on the invariant mass of the neutrino-antimuon system (The invariant mass of the neutrino-muon system has no connection with the mass of the commonly assumed weak boson):

We checked this invariant mass dependence of the distributions at fixed $\cos \theta$, E , and momentum transfer t . These special points in the phase space are the same as in (iv). We consider only the case of Fe^{56} target and the energy of the incident neutrino at 40 BeV. We fixed the energy of the outgoing muon at 8 BeV. The invariant mass distribution remains almost unchanged when this energy is lowered to 4 BeV. The invariant mass is generally small and is limited by the kinematical upper bound set by the small momentum transfer. See figure (20)²¹⁾.

(vi) The total cross section integrated over these variables agrees well with previously known results for the incident energy of the neutrino at 10 BeV.

	PRESENT RESULT		RESULT BY OTHER AUTHORS	
Energy of Neutrino = 10 BeV				
Target	$\sigma'(-)$	$\sigma'(+)$	$\sigma'(a)$	$\sigma'(b)$
Proton	7.63	7.59	$\sim 7(*)$	8.9(**)
Neutron	1.0	0.98	$\sim 1(*)$	
Fe ⁵⁶	1.5Z ²	1.5Z ²	1.4Z ²	
Pb ²⁰⁸	0.70Z ²	0.69Z ²	0.68Z ²	

Energy of Neutrino = 40 BeV

	$\sigma'(-)$	$\sigma'(+))$	$\sigma'(a)$
Proton	71	67	$\sim 70(*)$
Neutron	8.8	8.6	$\sim 10(*)$
Fe ⁵⁶	$13.8Z^2$	$13.4Z^2$	$18Z^2(***)$
Pb ²⁰⁸	$72.7Z^2$	$71.6Z^2$	$96Z^2(***)$

Note 1 : The unit of the cross section is $3.03 \times 10^{-44} \text{ cm}^2$.

Note 2 : $\sigma'(-)$ and $\sigma'(+))$ denote the total cross sections obtained from the muon and antimuon distributions, respectively. $\sigma'(a)$ is from Reference (4). $\sigma'(b)$ is from Reference (5). For the iron and lead targets, we show only the coherent contribution. The total cross section is obtained when one adds incoherent contributions from individual protons and neutrons to the coherent cross section.

Note 3 : We used a "dipole" type form factor.

$$F(t) = (1 + t/M_\rho^2)^{-2}$$

where M_ρ^2 is the square of the rho-meson mass for nucleons, and is equal to $0.2 \times (1.64/\sqrt[3]{A})^2$ for a nucleus with a mass number A.

(*) Reference (4) gives no numerical values for nucleons. We read these numbers from their graph.

(**) Reference (5) uses form factors given by L. N. Hand, D. G. Miller, and R. Wilson, Rev. Mod. Phys., 35, 335 (1963).

(***) Results for incident neutrinos of 50 BeV.

This agreement of our result with the previously known values and the agreement among $\sigma(+)$ and $\sigma(-)$ suggest the correctness of our calculation. Especially the latter agreement is rather impressive when one recalls the quite different phase space distribution of these two cross sections. From the discrepancy among $\sigma(+)$ and $\sigma(-)$, and also from the dependence of the distribution of the change of the integration mesh, we estimate the error in our calculation to be less than 5 percent²²⁾.

The energy dependence of the total cross section shows that the neutrino energy of $10 \sim 40$ BeV is not yet in the asymptotic region.

In figure 21, we give a "picture" which shows the general tendency of the phase space distribution we discussed above. Note that the scales in this picture are quite arbitrary.

4. Invariant Mass Distribution of the Muon Pair.

We also performed a calculation of the invariant mass distribution of the muon pair produced by the neutrino. We recall the expression in Section II - 2

$$d\sigma = h(k, Q, (k', Q), (kk'), m^2, t) \frac{d^3k'}{2E_{k'}} dm^2 dt \quad (\text{II}, 25)$$

where

k: 4-momentum of the incident neutrino

k': 4-momentum of the outgoing neutrino

Q: 4-momentum of the target nucleus (or nucleon)

m: invariant mass of the muon pair

t: momentum transfer to the nucleon system.

We first rewrite (II,25) slightly to

$$\frac{d\sigma}{dm d\Delta k} = \int h(k, Q, (k', Q), (kk'), m^2, t) \pi \cdot k'_0 d\cos\theta_{k'} dt \quad (\text{II}, 26)$$

where $\Delta k = k_0 - k'_0$ = energy loss of the neutrino. We do this because Δk is "measureable" even when we do not measure the recoil nucleon nor the energies of the neutrinos. Δk can be identified with the total energy of the muon pair (i.e., just the sum of two muon energies) due to the smallness of the energy transfer to the nucleon system. Thus we can get an interesting two dimensional correlation.

We first fix the invariant mass of the muon pair and then calculate the variation of the remaining phase space variables. This can be compared with the procedure in the previous section where we first fixed the energy and angle of the muon, which corresponds to the neutrino in the present section, and then looked for the conditions on the remaining variables. Besides the difference of the masses of the muon and the neutrino, we follow a different procedure in this section than in the previous section.

Now we define

$$s = (kQ)$$

$$x = (k' \cdot Q) = k'_0 \cdot M$$

$$y = (k \cdot k') = k_0 \cdot k'_0 (1 - \cos \theta_k)$$

where k , k' and Q are the 4-momenta of the incident neutrino, the outgoing neutrino, and the target nucleus (or nucleon), respectively.

The condition $x+y$ now becomes

$$0 \leq x + y \leq \frac{1}{2} [M^2 + 2s - (M+m)^2] \quad (\text{II},27)^{23})$$

where M is the mass of the target nucleus (or nucleon). This immediately gives a condition on the invariant mass m .

$$0 \leq M^2 + 2s - (M+m)^2$$

Thus m must lie in the region

$$2\mu \leq m \leq \sqrt{M^2 + 2s} - M \quad (\text{II},28)$$

The condition (II,27) can be converted into that on E_k , and $\cos \Theta_{k_1}$.

We show the phase space $(\cos \Theta_{k_1}, E_{k_1})$ in figure (4). The upper and lower limits of the momentum transfer t are now given by two roots of the quadratic equation,

$$m^2 + 2y + [(s-x)/M^2 + 1] \cdot x = \frac{p}{M^2} \cdot \sqrt{4M^2x + x^2}$$

where $p = M \cdot |\vec{k} - \vec{k}'| = \sqrt{(s-x)^2 + 2yM^2}$.

The integration procedure is as follows:

We first integrate over the momentum transfer t . Then we integrate over the angular distribution of the outgoing neutrino. Thus we get the distribution,

$$\frac{d^2\sigma}{dm d\Delta k} = H(s, \Delta k, m)$$

with H a certain invariant function. Next we calculate $H(s, \Delta k, m)$ for fixed m and varying Δk , i.e., the "energy loss spectrum." We integrate the energy loss spectrum over Δk to get the invariant mass spectrum $d\sigma/dm$. Finally we integrate over the invariant mass to get the total cross section. We did this calculation only for the proton and Fe^{56} targets and incident energy of the neutrino at 10 BeV²⁴⁾.

The results show several interesting features:

- (i) The invariant mass is very small. The invariant mass of the muon pair produced coherently from the Fe^{56} target is about half of that produced incoherently from the individual nucleons²⁵. See figures (22) (23).

In both cases the major part of the distribution of the invariant mass is less than 1 BeV.

(ii) The total energy of the dimuon system (=energy loss of the incident neutrino) is evenly distributed over phase space for small invariant mass. See figure (24). However the distribution has a peak on the high energy side for big invariant mass. See figure (25).

(iii) The larger the invariant mass becomes, the larger momentum transfer to the nucleon system is required.

The total cross sections after the integration over the invariant mass agree fairly well with those obtained in the previous section. The incoherent cross section on the proton target is in excellent agreement with the previous value, however the coherent cross section on the Fe^{56} target is about 10 percent smaller than that in the previous section.

Target	Present Result	Result in Section 3
Proton	7.4	7.5
Fe^{56}	$1.3Z^2$	$1.5Z^2$

where Z = charge number of $\text{Fe}^{56} = 26$, and the unit of the cross section is $\sigma_c = 3.03 \times 10^{-44} \text{ cm}^2$. We ascribe this large discrepancy in the case of Fe^{56} to the singular behaviour of the integrand and rather rough mesh of the numerical integration. Actually the integrand for the proton target is better behaved (especially angular dependence) and the agreement is reasonable. The discrepancy between two different methods still lies within about 10 percent error; this is consistent, although not satisfactory, with our error estimate in the numerical calculation, i.e., about 5 percent error.

Thus we conclude that our program is working well.

III. NEUTRINO PAIR RADIATION FROM ENERGETIC LEPTONS IN A COULOMB FIELD.

1. Introductory Remarks

The universal Fermi interaction allows charged leptons to radiate neutrino pairs in a Coulomb field just as leptons can radiate photons in the bremsstrahlung process. This problem has been studied by several authors in connection with its astrophysical implications²⁶⁾⁻²⁸⁾.

Walker and Ebel²⁹⁾ already discussed the possibility of an experiment on the neutrino pair radiation from energetic muons, about 10 years ago. They discussed the problem based on the charged vector boson hypothesis. The prospect of energetic muon beams at NAL also revived interest in the experiment.³⁰⁾

If one assumes the existence of the weak boson with a reasonable mass, say, 10 BeV, the rate of the neutrino radiation is enormously increased (roughly by a factor of $10^3 \sim 10^4$). In the following, however, we study this problem based on the usual contact four-Fermion interaction.

The following facts about the process are already known:

(i) The energy loss of the charged lepton during the neutrino radiation process is substantial. In the nonrelativistic region, the lepton loses about 70 percent of its energy on the average.

(ii) At high energies the asymptotic behaviour

$$\sigma \sim Z^2 \cdot \alpha^2 \cdot G^2 \cdot s \cdot [\log s + \text{const.}]$$

is expected, where s =lepton energy times the "effective mass" of the target which is determined by the nuclear (or nucleon) form factors.

If one assumes the existence of charged bosons,

the asymptotic form becomes instead

$$\sigma \sim Z^2 \cdot \alpha^2 \cdot g^2 / M_B^2$$

where M_B is the mass of the boson, and g is the dimensionless weak coupling constant defined by $g^2/M_B^2 = G/\sqrt{2}$, the Fermi constant.

(iii) The total cross section is therefore of the order of 10^{-40} cm^2 for the conventional Fermi theory, and 10^{-35} cm^2 for the weak boson hypothesis.

The property (i) is rather encouraging when one remembers the formidable background of the usual bremsstrahlung process. The bremsstrahlung radiates mainly soft photons. If the property (i) is preserved at extremely relativistic energies, we may have some hope to detect the small neutrino radiation process among the dominant backgrounds. The angular distribution of the charged leptons after the neutrino radiation process is expected to be distinct from those leptons in the bremsstrahlung process under these circumstances.

The asymptotic behaviour of the total cross section (ii) is expected on the basis of a dimensional analysis. The smallness of the cross section makes this experiment in the laboratory very difficult. However, if one uses 150 BeV muons the test of the weak vertex up to "effective energy" 20 BeV or so is possible.

2. General Structure of the Cross Section

We start with the, by now familiar, effective Lagrangian

$$\mathcal{L} = \frac{G}{\sqrt{2}} \cdot \bar{\psi}_\nu \gamma_\alpha (1 - \gamma_5) \psi_\nu \cdot \bar{\psi}_\ell \gamma^\alpha (1 - \gamma_5) \psi_\ell \quad (\text{III}, 1)$$

The Feynman diagrams we have in mind are shown in figures 26(a) and 26(b).

Our notation is

$$\bar{\mu}(p) + Z(Q) \rightarrow \bar{\mu}(\bar{p}) + \nu(k) + \bar{\nu}(k') + Z(Q')$$

where k : 4-momentum of the neutrino

k' : 4-momentum of the antineutrino

Q : 4-momentum of the nucleon (or nucleus)

Q' : 4-momentum of the scattered nucleon (or nucleus)

p : 4-momentum of the incident antimuon

$\bar{p}$: 4-momentum of the outgoing antimuon

q : momentum transfer to the nucleon system = $Q' - Q$

The corresponding invariant amplitude is

$$\begin{aligned} \mathcal{M} = & \frac{G e^2}{\sqrt{2}} \cdot \bar{u}(k) \gamma^\alpha (1 - \gamma_5) v(k') \\ & \cdot \bar{v}(p) \left\{ \frac{-2 p^\mu + \gamma^\mu \not{q}}{(p - q)^2 - \mu^2} \gamma_\alpha (1 - \gamma_5) - \gamma_\alpha (1 - \gamma_5) \frac{\not{q} \gamma^\mu + 2 \bar{p}^\mu}{(\bar{p} + q)^2 - \mu^2} \right\} \cdot v(\bar{p}) \\ & \cdot \frac{1}{q^2} \cdot \langle Q' | J_\mu | Q \rangle \end{aligned} \quad (\text{III}, 2)$$

where we have already slightly modified the lepton weak current using the Dirac equation. See Eq. (II,8). The lepton weak current can be further modified as

$$\bar{U}(p) W'_\alpha (1 - \gamma_5) U(\bar{p})$$

where

$$W'_\alpha = \frac{-2p^\mu + \gamma^\mu \not{p}}{q^2 - 2q \cdot p} \gamma_\alpha - \gamma_\alpha \frac{2\bar{p}^\mu + \gamma^\mu \not{\bar{p}}}{q^2 + 2q \cdot \bar{p}} \quad (\text{III},3)$$

The only difference of (II,9) and (III,3) is the sign of the momentum p .

Note also that W'_α satisfies the gauge condition,

$$q_\mu W'_\alpha = 0$$

independent of the spinors. If we remember that the explicit dependence on the lepton mass disappears after the spin average, we can obtain the matrix element for the present process from the expression in the preceding section. However we are also interested in the polarization dependence of the cross section in the present calculation. The incident muon in the laboratory is usually strongly polarized through the meson decay process. Thus we need extra trace calculation.

We first notice that the neutrino contribution to the matrix element is already separated from other contributions. This is the special property of the point coupling scheme. One can therefore expect that the cross section after integration over the neutrino variables will have a structure quite similar to that of the real neutral vector meson production. We show this in the following.

First we calculate the square of the matrix element, and take a summation over the spins of the final state particles and the target nucleon³¹⁾. We introduce the polarization vector of the incident lepton explicitly. The result can be written as

$$\begin{aligned} \sum_{\text{spins}} |M|^2 &= \frac{G^2 e^4}{2 q^4} \cdot \text{Tr}[\not{k} \gamma^\alpha (1 - \gamma_5) \not{k}' \gamma^\beta (1 - \gamma_5)] \\ &\quad \cdot \text{Tr}[(\not{p} - \not{\mu}) \left(\frac{1 + \gamma_5 \not{x}}{2} \right) W_\alpha^\mu (1 - \gamma_5) (\not{p}' - \not{\mu}) V_\beta^\nu (1 - \gamma_5)] \\ &\quad \cdot 2 \cdot J_{\mu\nu} \end{aligned} \quad (\text{III},4)$$

where

$$\begin{aligned} V_\beta^\nu &= \gamma_0 W_\beta^{\nu\dagger} \gamma_0 \\ &= \gamma_\beta \frac{-2p^\nu + \not{x} \gamma^\nu}{q^2 - 2(\not{x} p)} - \frac{2\bar{p}^\nu + \gamma^\nu \not{x}}{q^2 + 2(\not{x} \bar{p})} \gamma_\beta \end{aligned} \quad (\text{III},5)$$

and $J_{\mu\nu}$ is the nucleon electromagnetic current identical to (II,11).

The lepton (muon) current part can be written as

$$\text{Tr}[(\not{p} + \not{\mu} \not{x}) W_\alpha^\mu \not{x} V_\beta^\nu (1 - \gamma_5)]$$

Thus finally we get

$$\begin{aligned} \sum_{\text{spins}} |M|^2 &= \frac{2G^2 e^4}{q^4} \text{Tr}[\not{k} \gamma^\alpha \not{k}' \gamma^\beta (1 - \gamma_5)] \\ &\quad \cdot \text{Tr}[(\not{p} + \not{\mu} \not{x}) W_\alpha^\mu \not{x} V_\beta^\nu (1 - \gamma_5)] \\ &\quad \cdot J_{\mu\nu} \end{aligned} \quad (\text{III},6)$$

The differential cross section is expressed as

$$d\sigma = \frac{2}{(2\pi)^8 \cdot \int \sqrt{(PQ)^2 - M^2 \mu^2}} \cdot \sum_{\text{spins}} |M|^2 \cdot \delta(p - \bar{p} - k - k' - q) \cdot \frac{d^3 k'}{2E_{k'}} \cdot \frac{d^3 k}{2E_k} \cdot \frac{d^3 \bar{p}}{2E_{\bar{p}}} \cdot \frac{d^3 Q'}{2E_{Q'}} \quad (\text{III},7)$$

The derivation of this expression is identical to that of Eq. (II,13).

Next we integrate over the neutrino phase space variables because we do not measure them. The integration over the neutrino variables becomes from Eq. (III,6) and Eq. (III,7),

$$\int \text{Tr}[\not{K} \gamma^\alpha \not{K}' \gamma^\beta (1 - \gamma_5)] \frac{d^3 k}{2E_k} \cdot \frac{d^3 k'}{2E_{k'}} \quad (\text{III},8)$$

Now we use Eq. (II,14) in Section II,

$$\frac{d^3 k}{2E_k} \cdot \frac{d^3 k'}{2E_{k'}} = \frac{1}{8} \cdot d^4 q_1 \cdot \delta(m^2 - q_1^2) dm^2 d\Omega_{q_2} \quad (\text{III},9)$$

where

$$q_1 = k + k'$$

$$q_2 = k - k'$$

$$d\Omega_{q_2} = \text{solid angle of } q_2 \text{ in the frame where } \vec{q}_1 = 0.$$

Note that the "Jacobian" $\lambda(m^2, 0, 0)/m^2$ becomes the identity in the present problem.

Thus Eq. (III,8) becomes

$$\frac{1}{8} \int \text{Tr} [K \gamma^\alpha K' \gamma^\beta (1 - \gamma_5)] d\Omega_{q_2} d^4 q_1 \delta(q_1^2 - m^2) dm^2 \quad (\text{III}, 10)$$

We first perform the integration over $d\Omega_{q_2}$. After the integration we have only one vector, namely, q_1 . Thus we have only two second rank tensors available, i.e., $g^{\alpha\beta}$ and $q_1^\alpha q_1^\beta$.

Eq. (III,10) now becomes

$$\frac{2\pi}{3} \left[\frac{q_1^\alpha q_1^\beta}{m^2} - g^{\alpha\beta} \right] d^4 q_1 \delta(m^2 - q_1^2) m^2 dm^2 \quad (\text{III}, 11)$$

Eq. (III,11) shows that the neutrino pair acts as a "vector meson" with a variable mass m . The extra factor $m^2 dm^2$ means that the "vector meson" has the coupling constant $G \sqrt{\langle m^2 \rangle \Delta m^2}$. $\langle m^2 \rangle$ is the average value of the square of the invariant mass and Δm^2 the "width" of the mass spectrum.

The remaining phase space integration is done identically as in Section II. We first rewrite the nucleon phase space in the lab. frame as (see Eq. (II,17)),

$$\frac{d^3 q'}{2E_{q'}} = \frac{|\vec{q}|}{4M} dt d\Omega_q \quad (\text{III}, 12)$$

where t = square of the momentum transfer to the nucleon system, and $\vec{q}$ the spatial part of the momentum transfer. The cross section (III,7) now becomes

$$\begin{aligned} d\sigma = & \frac{2 \cdot G^2 \cdot e^4}{(2\pi)^7 \cdot 24 \sqrt{(pq)^2 - M^2 \mu^2}} \cdot \left[\frac{q_1^\alpha q_1^\beta}{m^2} - g^{\alpha\beta} \right] \\ & \cdot \text{Tr} [(\not{p} + \not{M}) W^\mu \not{q} V_\mu] \cdot \frac{1}{8^4} \cdot \frac{|\vec{q}|}{4M} dt d\Omega_q \\ & \cdot \frac{d^3 \bar{p}}{2E_{\bar{p}}} \cdot \delta(p - \bar{p} - q_1 - q) d^4 q_1 \delta(q_1^2 - m^2) m^2 dm^2 \end{aligned} \quad (\text{III}, 13)$$

We need the explicit form for the polarization vector n .

$$\mu n = \frac{\phi}{\sqrt{(p \cdot Q)^2 - M^2 \mu^2}} \cdot [(p \cdot Q) p - \mu^2 Q] \quad (\text{III}, 14)$$

where $\phi = \pm 1$ corresponds to positive and negative helicity state of the lepton in the C.M., respectively. Inserting this expression, the trace for the lepton current becomes

$$\begin{aligned} & \left[1 + \frac{(p \cdot Q)}{\sqrt{(p \cdot Q)^2 - M^2 \mu^2}} \phi \right] \text{Tr}[\not{p} W^\mu_\alpha \not{p} V^\nu_\beta] \\ & - \frac{\mu^2}{\sqrt{(p \cdot Q)^2 - M^2 \mu^2}} \phi \text{Tr}[\not{Q} W^\mu_\alpha \not{p} V^\nu_\beta] \end{aligned} \quad (\text{III}, 15)$$

or for energetic incident leptons,

$$[1 + \phi] \text{Tr}[\not{p} W^\mu_\alpha \not{p} V^\nu_\beta] - \frac{\mu^2}{M \cdot E_p} \phi \cdot \text{Tr}[\not{Q} W^\mu_\alpha \not{p} V^\nu_\beta] \quad (\text{III}, 16)$$

where E_p is the incident energy of the lepton³³⁾. We notice that the second term is multiplied by a small factor $\mu^2/M \cdot E_p$. For the "good polarization", i.e., $P=+1$, the second term will not be important, but for the "wrong polarization", i.e., $P=-1$, the second term cannot be neglected. We can also interpret this polarization P as the average polarization of the incident lepton, e.g., $P=0$ corresponds to an unpolarized beam³⁴⁾. In any case we have now two terms which we call the "unpolarized term" (the first term) and the "polarization term" (the second term). The trace calculation of the polarization term is slightly complicated. The explicit result of the trace calculation is given in Appendix A. When we perform the integration over $d^4 q_1 d\Omega q$ in (III,13) we finally get the result

$$\frac{d\sigma(S, E, \cos\theta, m, t)}{dE d\cos\theta dm dt} = (1 + f) \frac{d\sigma_{unpol.}}{dE d\cos\theta dm dt} - f \frac{d\sigma_{pol.}}{dE d\cos\theta dm dt} \quad (III, 17)$$

with a kinematical constraint

$$m^2 + 2y - 2\mu^2 + t + \frac{t}{M^2}(s-x) \leq \frac{\rho}{M^2} \cdot \sqrt{4M^2 t + t^2}$$

where

$$s = (p \cdot Q)$$

$$x = (\bar{p} \cdot Q) = M \cdot E$$

$$y = (p \cdot \bar{p})$$

E = energy of the outgoing lepton

$\cos\theta$ = scattering angle of the lepton in the lab. frame

m = invariant mass of the radiated neutrino pair

t = square of the momentum transfer to the nucleon

$$\rho = \sqrt{(s-x)^2 + 2M^2(y-\mu^2)}$$

We use the computer to do the integration over $dm dt$ to get the energy and angular distribution of the scattered lepton³⁵⁾.

Before we go to the numerical integration, we determine the phase space for $(E, \cos\theta)$. The constraint on these two variables is obtained considering the combination $x+y$ as in Eq. (II,24).

$$\mu \sqrt{M^2 + \mu^2 + 2s} \leq x + y \leq s + \mu^2 \quad (III, 18)$$

This is shown in figure (27).

The available phase space for (m^2, t) is shown in figure (28).

3. Neutrino Pair Radiation from Muons; Numerical Calculation

We calculate the energy and angular distribution of the muon after the neutrino pair radiation in a Coulomb field. We first recall the main result in the previous section.

$$\frac{d\sigma(s, E, \cos\theta)}{dE d\cos\theta} = (1 + f) \frac{d\sigma_{unpol}}{dE d\cos\theta} - f \frac{d\sigma_{pol}}{dE d\cos\theta} \quad (III,19)$$

where

$$s = (p \cdot Q)$$

$E, \cos\theta$: energy and angle of the outgoing muon

P : polarization of the incident muon

We did the two dimensional numerical integration over $dmdt$ to get Eq. (III,19). See Appendix C for the numerical integration.

We are mainly interested in the cross section in the energy region of the accelerator at NAL. We expect the energy of the muon beam at 120~150 BeV. The most energetic muons have the "wrong polarization" due to the pion or kaon weak decay process which produces them. Thus for 150 BeV incident muons we calculate the "polarization term" in Fe^{56} target as well as the "unpolarized term". For 120 BeV incident muons, which are expected to have the "good polarization," we calculate only the unpolarized term in Fe^{56} . The incoherent process at 150 BeV amounts to ~15 percent of the total cross section.

We found the following properties of the cross section:

(i) Total cross section.

The coherent process is dominant over the incoherent process and the "polarization term" gives a small contribution at 150 BeV. Our result³⁶⁾ gives

$$\sigma = (1 + P) \cdot [12 Z^2 + 50 Z] \sigma_0 - P \cdot \frac{Z^2}{70} \sigma_0$$

at 150 BeV. $\sigma_0 = 3.03 \times 10^{-44} \text{ cm}^2$ and P is the polarization of the incident muon. $Z = 26$.

We calculated only the coherent contribution for a 120 BeV unpolarized beam. The result is $\sigma = 9.0 Z^2 \sigma_0$. The ratio of the coherent cross section at 150 BeV to that at 120 BeV is $12 Z^2 \sigma_0 / 9.0 Z^2 \sigma_0 = 4/3 \approx 150/120$. This shows that the coherent cross section approximately follows the asymptotic formula

$$\sigma \sim E \cdot [\log E + \text{constant}]$$

(ii) Energy spectrum.

The forward energy spectrum of the scattered muon, i.e., $d\sigma/dE \cos\theta$ at $\cos\theta = 1$, shows a rather uniform distribution. See figure (29). The energy distribution $d\sigma/dE$ has a sharp peak at small energies. The "average energy" of the outgoing muon $\langle E \rangle = \int E d\sigma(E) / \int d\sigma(E)$ is about 20 BeV. This means that the muon loses energy by more than 130 BeV during the neutrino radiation process.

See figures (30), (31), (32).

(iii) Angular distribution.

The angular distribution at fixed energy, i.e., $d\sigma'/dE d\cos\theta$ at fixed E, shows a predominantly forward distribution of the scattered muon. This is particularly true for the energetic muons. For those muons which lose almost all their energies, the angular distribution is rather uniform. See figures (33), (34).

(iv) Radiated mass and momentum transfer to the target. The invariant mass of the radiated neutrino pair was also checked. The distribution covers larger masses in the incoherent process than in the coherent process. See figures (35), (36). The limit of the invariant mass is estimated by ³⁸⁾

$$m \sim \sqrt{2E \cdot m^*}$$

where m^* is the "effective mass" of the target which is about same as the cut-off mass of the form factors. This estimate gives

$m \sim 15$ Bev for proton target

7 BeV for Fe^{56} target

This estimate and also the actual numerical calculation show that the coherent process is more effective to produce the "massive neutral vector meson" if it existed.

The momentum transfer dependence is more or less the same as in the calculation in Section II. The major contribution

comes from the region where $t \lesssim 0.1 (\text{BeV})^2$. This upper limit is slightly smaller for the coherent process. See figure (37). This is expected from the momentum dependence of the form factors. The form factor dependence of the incoherent process is reasonable.³⁹⁾ The infinite momentum cut-off (i.e., point nucleon), however, increases the incoherent cross section by a factor of 10.

We also checked the energy and angular distribution of the muon in the bremsstrahlung process. The analytic expression for these two processes are almost identical; the major difference is the invariant mass of the radiated "particle". The cross section for the bremsstrahlung is obtained from our cross section by fixing the invariant mass of the radiated neutrino pair at zero if one neglects small terms which are proportional to the muon mass.⁴⁰⁾ The bremsstrahlung mainly produces energetic muons (infrared divergence). On the contrary our process produces "low energy muons" as we discussed in (ii). This difference makes it slightly easier to do the actual experiment. The calculation, however, shows that the bremsstrahlung is still about 10^9 times larger than the neutrino radiation process even when we detect the muons which lost 140 BeV out of their 150 BeV incident energy. The angular distribution is slightly more forward oriented in the bremsstrahlung.

The error due to the numerical calculation in this section is estimated at about 5%. Note that the semi-singular integration over momentum transfer in the present calculation is almost identical to that in Section II.

IV. THE EFFECTS OF THE DEEPLY INELASTIC PROCESS AND THE ASYMPTOTIC FORMULAS AT HIGH ENERGY.

It may be interesting to calculate the effects of the deeply inelastic process⁴¹⁾ on the weak leptonic processes we discussed in the preceding sections.

We only calculate the total cross section of the process $\bar{\mu} + p \rightarrow \bar{\mu} + \nu + \bar{\nu} + \Gamma$ where Γ is some system of hadrons (see figure (38)). The calculation of the process $\nu + p \rightarrow \nu + \mu + \bar{\mu} + \Gamma$ is much more difficult. See, however, the following discussions.

In the deeply inelastic process the momentum transfer to the hadronic system is generally large compared with the muon mass squared. Thus we can neglect the muon mass during the calculation. With this approximation and without any further approximation, we can show that the cross section for the unpolarized incident muon is expressed as (see Eq. (III,13))

$$d\sigma = \frac{G^2 \alpha^2}{96 (2\pi)^3} \cdot \frac{1}{s^2} \cdot \frac{1}{t^2} \cdot L^{\mu\nu}(v, t) H_{\mu\nu}(u, t) dv du dt$$

where

v =invariant mass squared of the final leptonic system

u =invariant mass squared of the final hadronic system

t =momentum transfer squared.

$H^{\mu\nu}(u, t)$ is the hadronic vertex function;

$$H^{\mu\nu}(u, t) = g_1(u, t) \left(g^{\mu\nu} - \frac{\partial^\mu \partial^\nu}{q^2} \right) + g_2(u, t) \left[q^\mu - \frac{(q \cdot q)}{q^2} \partial^\mu \right] \left[q^\nu - \frac{(q \cdot q)}{q^2} \partial^\nu \right]$$

The elastic process is obtained when we put

$$g_1(u, x) = G_1(x) \cdot \delta(u - M^2)$$

$$g_2(u, x) = G_2(x) \cdot \delta(u - M^2)$$

where $G_1(t)$ and $G_2(t)$ are defined in Eq. (II-11). For the inelastic process, our $g_1(u, t)$ and $g_2(u, t)$ are related to the familiar $W_1(\nu, t)$ and $W_2(\nu, t)$ by

$$g_1(u, x) = -2M \cdot W_1(\nu, x)$$

$$g_2(u, x) = \frac{2}{M} \cdot W_2(\nu, x)$$

or in the scaling limit ⁴²⁾

$$g_1(u, x) = -\frac{M^2}{1+R} \left(1 + \frac{\nu^2}{M^2 x}\right) \cdot g_2(u, x)$$

$$g_2(u, x) = \frac{2}{\nu} \cdot F(\omega)$$

$$F(\omega) \sim 0.35 \text{ for } \omega \geq 2$$

$$\nu = (q \cdot Q) = \frac{1}{2}(u + x - M^2)$$

$$\omega = \nu/x \geq \frac{1}{2} \quad (\omega = 1/2 \text{ corresponds to the elastic process})$$

$$R = \frac{\sigma_S}{\sigma_T} \geq 0$$

The leptonic vertex function is given by

$$L^{\mu\nu}(\nu, x) = f_1(\nu, x) \left(g^{\mu\nu} - \frac{g^\mu g^\nu}{g^2} \right) + \frac{4x}{(\nu+x)^2} f_2(\nu, x) \left[p^\mu - \frac{(g \cdot p)}{g^2} g^\mu \right] \left[p^\nu - \frac{(g \cdot p)}{g^2} g^\nu \right]$$

where

(IV,1)

$$f_1(\nu, x) = \frac{(\nu+x)}{6} \left\{ 13\nu + 4x + 4 \left(2\nu + 5x + \frac{x^2}{\nu} \right) \log\left(\frac{x}{\nu+x}\right) \right\}$$

$$f_2(\nu, x) = -\frac{(\nu+x)}{6} \left\{ 5\nu + 4x + 4 \left(2\nu + 3x + \frac{x^2}{\nu} \right) \log\left(\frac{x}{\nu+x}\right) \right\}$$

Note that the leptonic vertex functions also show the scaling behaviour in the present approximation (i.e., muon mass = 0). We observe from these expressions that the dimensionality of $f_1(v,t)$ and $f_2(v,t)$ is higher than that of $W_1(u,t)$ and $W_2(u,t)$. This is due to the different dimensionality of the coupling constants in the weak interaction and in the electromagnetic and strong interactions. The weak vertex function in the Fermi theory generally grows up rapidly as the invariant mass increases.

The phase space constraint is expressed as

$$\mu^2 X^2 + 2SXY + M^2 Y^2 \leq 4t \cdot (S^2 - M^2 \mu^2) \quad (\text{IV},2)$$

where

$$X = u + t - M^2 \geq t$$

$$Y = v + t - \mu^2 \geq t$$

See also figure (39).

Before we engage in the numerical calculation, we estimate the general behaviour of the "differential" cross section $d\sigma/dt du dv$ inside the phase space. We can expect that relatively small t values will be important for our process because of the photon propagator. Under this condition the allowed phase space regions become

- (1) v large, i.e. $v \sim s$, and u small
- (2) v small and u large, i.e. $u \sim s$.
- (3) v and u are both small

The difference of the energy dependence of the leptonic and hadronic vertices described above then indicates that the region (1) is more important compared with the other two regions. This turns out to be the case

as we will see below. If we accept this reasoning we can neglect the term

$$\frac{x}{v^2} \log\left(\frac{x}{v+x}\right)$$

in $f_1(v,t)$ and $f_2(v,t)$ (in the actual numerical calculation we retained these terms and confirmed that they actually give small contributions). With this approximation we can perform another analytical integration. The final result is expressed as

$$\begin{aligned} & \frac{1}{x^2} L_{\mu\nu} H^{\mu\nu} du dv dx \\ &= \frac{F(\omega, x) d\omega x dx}{6\omega} \left\{ F_1(z) (M^2 + x\omega^2) \left(\frac{3}{1+R} - 1 \right) \right. \\ & \quad + F_2(z) \cdot (x\omega^2 - \frac{M^2 + x\omega^2}{1+R}) \\ & \quad + F_3(z) \cdot (-4S\omega) \\ & \quad \left. + F_4(z) \cdot \left(\frac{4S^2}{x} \right) \right\} \end{aligned}$$

where

$$F_1(z) = \frac{2z^2}{3} (4z+9) \log z - \frac{47}{9} z^3 + \frac{3}{2} z^2 + \frac{67}{18}$$

$$F_2(z) = \frac{2z^2}{3} (4z+3) \log z - \frac{23}{9} z^3 - \frac{1}{2} z^2 + \frac{55}{18}$$

$$F_3(z) = 4z(z+1) \log z - \left(\frac{9}{2} z^2 + 3z \right) + \frac{15}{2}$$

$$F_4(z) = (4z+1) \log z + 2(\log z)^2 - 13z + 13$$

and

$$F(\omega, x) = 2W_2(\omega, x)$$

$$z = \frac{2S}{M^2} \left\{ \sqrt{\omega^2 + M^2/x} - \omega \right\}$$

$$\frac{1}{2} \leq \omega \leq \frac{S}{2} \left(\frac{2}{x} - \frac{M^2}{2S^2} \right)$$

$$x_0 \leq x \leq (2S)^2 / (2S + M^2)$$

The momentum transfer is limited by the lower bound t_0 where the deeply inelastic process becomes presumably unimportant. We take this t_0 at $0.5(\text{BeV})^2$ rather arbitrary. The value of t_0 roughly corresponds to the cut-off mass squared of the proton elastic form factors. We assume the scaling behaviour of $W_1(\nu, t)$ and $W_2(\nu, t)$ for $t_0 \leq t$ and for all values of ω . This approximation is not good when t is small because of many resonances present for small values of ω . We separately estimated the effects of these resonances on our process.

The numerical calculation for the unpolarized incident muon shows that the contribution of the inelastic process is

$$\sigma_{\text{inelastic}} \simeq 3.5 \sigma_e, \quad \sigma_e = 3.03 \times 10^{-44} \text{ cm}^2$$

i.e. about 6% of the elastic process. This correction is very small compared with the dominant coherent elastic process and counts only 1% of it. Our numerical study also shows that the above result may contain about 50% error due to the neglect of the resonances and the mass of the muon. (See below for the error due to the nonzero muon mass in the elastic processes). The behaviour of $d\sigma/d\omega dt$ for a fixed t is shown in figure (40), and $d\sigma/dt$ in figure (41) for the incident energy of the muon at 150 BeV.

We did not calculate the effects of the deeply inelastic process on $\nu + p \rightarrow \nu + \mu + \bar{\mu} + \Gamma$. However we estimate that the effects of the deeply inelastic process are again very small. When we neglect the muon mass the energy dependence (and also invariant mass dependence) of the leptonic vertex in this process is almost identical to the process calculated above. Large leptonic invariant mass, and small momentum transfer and small hadronic invariant mass are preferred. Thus the effects of the deeply inelastic process on the neutrino induced process must also be small.

As a byproduct of the above calculation, we also derive the asymptotic formulas for the elastic processes (coherent and incoherent).

$$\bar{\mu} + (Z) \rightarrow \bar{\mu} + \nu + \bar{\nu} + (Z) \text{ at "extremely" high energies.}$$

For the elastic processes $v \gg t$ and we can approximate $f_1(v, t)$ and $f_2(v, t)$ in the leptonic vertex function as (see Eq. (IV, 1))

$$f_1(v, t) \simeq -\frac{v^2}{6} \left[8 \log\left(\frac{v}{t}\right) - 13 \right]$$

$$f_2(v, t) \simeq -\frac{v^2}{6} \left[8 \log\left(\frac{v}{t}\right) - 5 \right]$$

We also use the relation (see Eqs. (II,6) and (II,11))

$$g_1(u, t) = G_1(t) \delta(u - M^2)$$

$$g_2(u, t) = G_2(t) \delta(u - M^2)$$

Finally we get for the incoherent elastic process with unpolarized incident muons

$$\sigma = \frac{Z G^2 \alpha^2}{8 (2\pi)^3} \cdot \frac{1}{s^2} \cdot \frac{8}{9} \cdot m^4 \cdot M^2 \cdot \eta^3 \cdot$$

$$\left\{ \frac{I_1}{3} \log \eta + \frac{I_2}{3} + \frac{I_3}{72} \right.$$

$$\left. + \frac{1}{2} (1+\kappa)^2 \cdot \left(\frac{m}{M}\right)^2 \left[\frac{I_4}{3} \log \eta - \frac{59}{72} I_4 + \frac{I_5}{3} \right] \right\}$$

where

$$I_1 = \int_0^{y_0} dy \frac{1 + \kappa^2 \varepsilon^2 y^2}{(1 + y^2)^4} \cdot [3g(y) - g(y)^3 - 3(\varepsilon y)g(y)^2]$$

$$I_2 = \int_0^{y_0} dy \frac{1 + \kappa^2 \varepsilon^2 y^2}{(1 + y^2)^4} \cdot \log\left(\frac{g(y)}{y}\right) [3g(y) - g(y)^3 - 3(\varepsilon y)g(y)^2]$$

$$I_3 = \int_0^{y_0} dy \frac{1 + \kappa^2 \varepsilon^2 y^2}{(1 + y^2)^4} \cdot [-117g(y) + 81(\varepsilon y)g(y)^2 + 47g(y)^3 + 24(\varepsilon y)^2 g(y)^3]$$

$$I_4 = \int_0^{y_0} dy \frac{y^2}{(1 + y^2)^4} \cdot g(y)^3$$

$$I_5 = \int_0^{y_0} dy \frac{y^2}{(1 + y^2)^4} \cdot g(y)^3 \cdot \log\left(\frac{g(y)}{y}\right)$$

and

$$y_0 = 2S/m \cdot \sqrt{2S + M^2}$$

$$\varepsilon = \frac{m}{2M}$$

$$\eta = \frac{2S}{M \cdot m} = \frac{2E}{m}$$

$$g(y) = \sqrt{1 + \varepsilon^2 y^2} - \varepsilon y$$

m: cut-off mass of the form factors $\sim$ rho-meson mass.

Numerical calculation for the proton target gives ⁴³⁾

$$\sigma = \frac{G^2 \alpha^2 m E}{9 \pi^3} \left\{ [0.28 + 0.0071 \kappa^2 + 0.0047(1+\kappa)^2] \log \eta \right. \\ \left. - [0.013 + 0.010 \kappa^2 + 0.0097(1+\kappa)^2] \right\}$$

This formula gives $\sigma = 70 \sigma_c$ at $E_{inc} = 150$ BeV. Our "exact" numerical integration gives $\sigma = 50 \sigma_c$. The muon mass dependent term still gives a considerable contribution for the incident energy of the muon at 150 BeV. The comparison of our formula with the asymptotic formula

for the process $\nu + p \rightarrow \nu + \mu + \bar{\mu} + p$ given in Ref. (4) shows that the ratio of the cross sections at extremely high energies is

$$\sigma_{\mu} : \sigma_{\nu} \approx 1 : 7$$

For the coherent process, $\epsilon = \frac{m}{2M} \sim 0.1$, and we can approximate $g(y) \approx g(0) = 1$, $\epsilon = 0$ (and of course $K = 0$). We get the formula for the coherent process

$$\sigma = \frac{4Z^2 G^2 \alpha^2}{9\pi^3} (mE) \cdot \left(\frac{5\pi}{32}\right) \cdot \frac{1}{6} \cdot \left[\log \eta + \frac{3}{4}\right]$$

where m is the cut-off mass of the nuclear form factor.

With $E = 150$ BeV and for the Fe^{56} target, this formula gives

$$\sigma = 22 Z^2 \sigma_c$$

Our "exact" numerical calculation gives

$$\sigma = 12 Z^2 \sigma_c$$

i.e., the muon mass correction is not negligible at all for the muon energy at 150 BeV.

This poor approximation can be expected because the muon mass is comparable with the cut-off mass of the Fe^{56} form factor; we can not neglect the mass of the muon until we come up to the truly asymptotic region where the first term of the asymptotic formula dominates and the effect of the muon mass is small. For the electron induced neutrino pair production process at 150 BeV (note that our formula above is lepton mass independent), our numerical integration shows

$$\sigma = 20 Z^2 \sigma_c$$

The agreement with the above asymptotic formula is reasonable.

Again the comparison of our formula with the asymptotic formula for

$\nu + Z \rightarrow \nu + \mu + \bar{\mu} + Z$ given in Ref. (4) shows

$$\sigma_{\mu} : \sigma_{\nu} = 1 : 8$$

at the truly asymptotic region. At 150 BeV where our formula is not yet applicable, the actual ratio given by the numerical integration is

$$\sigma_{\mu} : \sigma_{\nu} = 1 : 6$$

The asymptotic formula above becomes useful when we modify it as

$$\sigma = \frac{4Z^2 G^2 \alpha^2}{9\pi^3} (mE) \left(\frac{5\pi}{32}\right) \cdot \frac{1}{6} \cdot \left[\log \eta + \frac{3}{4} + C \right]$$

and adjust C at some fixed energy E by comparing it with an "exact" numerical integration; C corresponds to the muon mass corrections. For example,

$$\sigma = \frac{4Z^2 G^2 \alpha^2}{9\pi^3} (mE) \left(\frac{5\pi}{32}\right) \cdot \frac{1}{6} \cdot \left[\log \eta + \frac{3}{4} - 3.45 \right]$$

fits our numerical values for the coherent process well.

V. DISCUSSION AND CONCLUSION

The major background process to the muon pair creation by neutrinos in a nuclear Coulomb field may be the inelastic process⁴⁴⁾,

$$\nu + N \rightarrow \mu^- + N^* \rightarrow N + \pi \rightarrow N + \mu^+ + \nu$$

where N denotes the nucleon. This process contains no electromagnetic interaction. Thus the cross section will be greater than our process by a factor of $A/(Z^2 \cdot \alpha^2) \sim 10^3$. The forward energy spectrum of the muon can be estimated by the PCAC hypothesis⁴⁵⁾; it depends, however, on the mass spectrum of N^* which is unknown. If one assumes that the 3-3 resonance region gives a dominant contribution even for the incident neutrinos of $10 \sim 40$ BeV, the energy spectrum of the negative muon has a sharp peak at high energies. If the large N^* mass contributes, the energy distribution becomes more uniform over the phase space. Still we have no way to estimate the angular distribution of the muon, and the energy and angular distribution of the pion. The low energy data at CERN⁴⁶⁾ shows that wide angle production of the pion is dominant. This property could be greatly modified as we go to higher energies.

The radiative corrections to the muon pair creation are estimated from the calculation of the radiative corrections to the muon pair production by energetic photons in a nuclear Coulomb field (Bethe-Heitler process). The radiative corrections are about 5 percent for photon energies of ~ 15 BeV and an energy resolution of ~ 200 MeV⁴⁷⁾. If we apply this result to our process, it means that we can neglect the radiative corrections. The

ambiguity of the nuclear (not nucleon) form factor and the numerical integration can easily produce ~ 10 percent error in our calculation. The effect of the distorted Coulomb wave functions of muons will be also small for energetic muons. The low energy muons may be influenced by the nuclear Coulomb field⁴⁸⁾; the detection of such slow muons will in any case be difficult in the actual experiment.

The major background process to the neutrino radiation from the muon will be the bremsstrahlung process. We already discussed this problem in Subsection 3 of Section III. The bremsstrahlung process is about 10^9 times more frequent than our process even at the most favourable configurations.

Another background process to the neutrino radiation will be the "deeply inelastic process,"

$$\mu + N \longrightarrow \mu + \Gamma$$

where Γ is some system of hadrons. The energy and angular distribution of the muon depends on the invariant mass of Γ and the form factors. In any case it is expected that one will measure the deeply inelastic process before one tries to perform the neutrino radiation experiment. We can, therefore, determine the background rate based on these experiments.

The radiative corrections in this case are also expected to be ~ 5 percent. Thus we can neglect them, although the detailed calculation of the radiative corrections is desirable and may be useful in connection with the analysis of the deeply inelastic process.

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Appendix A. Trace Calculation.

(a) Muon pair radiation from neutrinos.

We consider Eq. (II,12),

$$\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4 G^2}{q^4} \cdot M^{\mu\nu} \cdot J_{\mu\nu}$$

where $J_{\mu\nu}$ is defined in Eq. (II,11).

$$\begin{aligned} M^{\mu\nu} = & 4 \cdot \text{Tr} [K \gamma^\alpha K \gamma^\alpha (1 - \gamma_5)] \\ & \cdot \left\{ \frac{1}{A^2} \text{Tr} [\not{p} (2 p^\mu + \gamma^\mu \not{p}) \not{q}_\alpha \not{p} \not{q}_\beta (\not{q} \gamma^\nu + 2 p^\nu) (1 - \gamma_5)] \right. \\ & + \frac{1}{B^2} \text{Tr} [\not{p} \gamma_\alpha (2 \bar{p}^\mu + \not{q} \gamma^\mu) \not{p} (\gamma^\nu \not{q} + 2 \bar{p}^\nu) \not{q}_\beta (1 - \gamma_5)] \\ & \left. - \frac{2}{AB} \text{Tr} [\not{p} \gamma_\alpha (2 \bar{p}^\mu + \not{q} \gamma^\mu) \not{p} \not{q}_\beta (\not{q} \gamma^\nu + 2 p^\nu) (1 - \gamma_5)] \right\} \end{aligned}$$

where $A = q^2 + 2(q \cdot p)$ and $B = q^2 + 2(q \cdot \bar{p})$. See Eq. (II,12). Note that $J_{\mu\nu}$ is symmetric under the exchange $\mu \leftrightarrow \nu$. Thus it is enough to calculate the symmetric part of $M^{\mu\nu}$. An explicit expression of $M^{\mu\nu}$ is given by

$$M^{\mu\nu} = (-256) \left[\frac{1}{A^2} M_1^{\mu\nu} + \frac{1}{B^2} M_2^{\mu\nu} + \frac{1}{AB} M_3^{\mu\nu} \right]$$

where

$$M_1^{\mu\nu} = \left\{ g^{\mu\nu} [2(q \cdot k')(q \cdot p) - (q \cdot q)(k' \cdot p)] \right. \\ \left. + (p^\mu k'^\nu + p^\nu k'^\mu) [(p+q)^2 - \mu^2] \right. \\ \left. - 2[(p^\mu q^\nu + p^\nu q^\mu) + 2p^\mu p^\nu] [k' \cdot (p+q)] \right\} (k \bar{p})$$

$$M_2^{\mu\nu} = \left\{ g^{\mu\nu} [2(q \cdot k)(q \cdot \bar{p}) - (q \cdot q)(k \cdot \bar{p})] \right. \\ \left. + (\bar{p}^\mu k'^\nu + \bar{p}^\nu k'^\mu) [(\bar{p}+q)^2 - \mu^2] \right. \\ \left. - 2[(\bar{p}^\mu q^\nu + \bar{p}^\nu q^\mu) + 2\bar{p}^\mu \bar{p}^\nu] [k(\bar{p}+q)] \right\} (k' \cdot p)$$

$$M_3^{\mu\nu} = 2 g^{\mu\nu} \left\{ (p \cdot q) [(k' \cdot \bar{p})(q \cdot k) - (q \cdot \bar{p})(k \cdot k')] \right. \\ \left. + (q \cdot k') [(q \cdot \bar{p})(k \cdot p) - (p \cdot \bar{p})(q \cdot k)] \right. \\ \left. + (q \cdot q) [(p \cdot \bar{p})(k \cdot k') - (p \cdot k)(\bar{p} \cdot k')] \right\} \\ + (p^\mu \bar{p}^\nu + p^\nu \bar{p}^\mu) [4(k \cdot \bar{p})(k' \cdot p) + 2(k \cdot q)(k' \cdot q) - (q \cdot q)(k \cdot k') \\ + 2(k \cdot \bar{p})(q \cdot k') + 2(k' \cdot p)(q \cdot k)] \\ + (p^\mu q^\nu + p^\nu q^\mu) [(k \cdot \bar{p})(q \cdot k') + (q \cdot \bar{p})(k \cdot k') - (q \cdot k)(k' \cdot \bar{p}) + 2(k' \cdot p)(k \cdot \bar{p})] \\ + (\bar{p}^\mu q^\nu + \bar{p}^\nu q^\mu) [(k' \cdot p)(q \cdot k) + (q \cdot p)(k \cdot k') - (q \cdot k')(k \cdot p) + 2(k \cdot \bar{p})(k' \cdot p)] \\ - (p^\mu k'^\nu + p^\nu k'^\mu) [2(k' \cdot p)(q \cdot \bar{p}) + 2(q \cdot \bar{p})(q \cdot k') - (q \cdot q)(k' \cdot \bar{p})] \\ - (\bar{p}^\mu k'^\nu + \bar{p}^\nu k'^\mu) [2(k \cdot \bar{p})(q \cdot p) + 2(q \cdot p)(q \cdot k) - (q \cdot q)(k \cdot p)]$$

$$\begin{aligned}
 & + (g^\mu k^\nu + g^\nu k^\mu) \cdot [(p\bar{p})(gk') - (k'p)(g\bar{p}) - (k'\bar{p})(gp)] \\
 & + (g^\mu k'^\nu + g^\nu k'^\mu) \cdot [(p\bar{p})(gk) - (k\bar{p})(gp) - (kp)(g\bar{p})] \\
 & + (k^\mu k'^\nu + k'^\mu k^\nu) \cdot [2(gp)(g\bar{p}) - (gg)(p\bar{p})] \\
 & + (g^\mu g^\nu + g^\nu g^\mu) \cdot [(k'\bar{p})(kp) - (kk')(p\bar{p}) + (k'p)(k\bar{p})]
 \end{aligned}$$

This $M^{\mu\nu}$ must satisfy the positivity condition

$$a_\mu a_\nu M^{\mu\nu} \geq 0 \text{ for arbitrary 4-vector } a_\mu.$$

It is difficult to show this for the general case. We checked this condition for the following simplest case:

We consider the configuration where all particles appear in the forward direction (2-dimensional problem). Upon this assumption we can write

$$p = (E, p_1, 0, 0), \quad -\infty < p_1 < +\infty$$

$$\bar{p} = (\bar{E}, \bar{p}_1, 0, 0), \quad -\infty < \bar{p}_1 < +\infty$$

$$k = (k_0, \alpha k_0, 0, 0), \quad \alpha = \pm 1, \quad 0 \leq k_0$$

$$k' = (k'_0, \beta k'_0, 0, 0), \quad \beta = \pm 1, \quad 0 \leq k'_0$$

and the momentum transfer (which is space-like)

$$q = (0, q, 0, 0), \quad -\infty < q < +\infty$$

where p , $\bar{p}$, k and k' correspond to the 4-momenta of the muon, antimuon, incident neutrino and outgoing neutrino, respectively.

Using the gauge condition

$$q_\mu M^{\mu\nu} = 0, \text{ and } M^{\mu\nu} = M^{\nu\mu},$$

$M^{\mu\nu}$ can be written as

$$M^{\mu\nu} = \begin{bmatrix} M^{00} & 0 & 0 \\ 0 & g^{22} U & \\ 0 & & g^{33} V \end{bmatrix}$$

where U and V are scalar quantities.

Thus it is enough to show that

$$M^{00} \geq 0$$

$$U \leq 0$$

$$V \leq 0$$

Our result for $M^{\mu\nu}$ gives

$$M^{00} = k_0 k'_0 (\bar{E} - \alpha \bar{p}_1)(E - \beta p_1) \left[\frac{2E - \beta q}{q^2 + 2q p_1} - \frac{2\bar{E} - \alpha q}{q^2 + 2q \bar{p}_1} \right]^2 \geq 0$$

$$U = V = -k_0 k'_0 \cdot q^2 \left[\frac{(\bar{E} - \alpha \bar{p}_1)(E + \beta p_1)}{(2q p_1 + q^2)^2} + \frac{(E - \beta p_1)(\bar{E} + \alpha \bar{p}_1)}{(2q \bar{p}_1 + q^2)^2} \right] \leq 0$$

Note that we use the on-the-mass-shell condition on 4-vectors, k , k' , p and $\bar{p}$, to prove the positivity condition. This is due to the fact that we use projection operators, which are valid only for on-the-mass-shell particles, during the trace calculation.

(b) Neutrino pair radiation from leptons

We have two trace calculations,

$$\text{Tr}[\not{p} W_\alpha^\mu \not{p}' V_\beta^\nu] \quad \text{and} \quad \text{Tr}[\not{p} W_\alpha^\mu \not{p}' V_\beta^\nu]$$

where

$$W_\alpha{}^\mu = \frac{-2p^\mu + \not{q} \gamma^\mu}{q^2 - 2(qp)} \gamma_\alpha - \gamma_\alpha \frac{2\bar{p}^\mu + \not{q} \gamma^\mu}{q^2 + 2(q\bar{p})}$$

$$V_\beta{}^\nu = \gamma_\beta \frac{-2p^\nu + \not{q} \gamma^\nu}{q^2 - 2(qp)} - \frac{2\bar{p}^\nu + \not{q} \gamma^\nu}{q^2 + 2(q\bar{p})} \gamma_\beta$$

See Eqs. (III,3), (III,5) and (III,16). Note that $W_\alpha{}^\mu$ and $V_\beta{}^\nu$ satisfy the gauge condition,

$$q_\mu W_\alpha{}^\mu = 0$$

$$q_\nu V_\beta{}^\nu = 0$$

If we know the second trace, we can get the first one by replacing Q by p . However we calculate these two traces independently in the following.

From Eq. (III,13), the first trace appears as the following combination in the matrix element (the first factor is from the neutrino pair),

$$N^{\mu\nu} \equiv \left[\frac{g_1^* g_1^a}{m^2} - g^{ab} \right] \cdot \text{Tr}[\not{X} W_\alpha{}^\mu \not{X} V_\beta{}^\nu]$$

We rewrite this slightly as

$$N^{\mu\nu} = \left[\frac{g_1^* g_1^a}{m^2} - g^{ab} \right] \cdot \left\{ \text{Tr}[(\not{p} - \not{\mu}) W_\alpha{}^\mu (\not{p} - \not{\mu}) V_\beta{}^\nu] - \mu^2 \text{Tr}[W_\alpha{}^\mu V_\beta{}^\nu] \right\}$$

Note that the first trace in $N^{\mu\nu}$ expresses the same result as we would get from the vector current alone. This means the current is conserved, i.e.,

$$g_1^a \text{Tr}[(\not{p} - \not{\mu}) W_\alpha{}^\mu (\not{p} - \not{\mu}) V_\beta{}^\nu] = 0$$

Thus

$$N^{\mu\nu} = -\text{Tr}[\not{X} W_\alpha^\mu \not{X} V^{\nu\alpha}] - \frac{\mu^2}{m^2} g_\alpha^\alpha g_\beta^\beta \text{Tr}[W_\alpha^\mu V_\beta^\nu]$$

With this form the trace calculation is almost trivial. The result is

$$\begin{aligned} & -\text{Tr}[\not{X} W_\alpha^\mu \not{X} V^{\nu\alpha}] \\ &= \frac{\not{p}}{A^2} \left\{ g^{\mu\nu} [(p\bar{p}) q^2 - 2(p\cdot q)(\bar{p}\cdot q)] - (\bar{p}^\mu p^\nu + \bar{p}^\nu p^\mu) \cdot A \right. \\ & \quad \left. + 2 [2 p^\mu p^\nu - (p^\mu q^\nu + p^\nu q^\mu)] [\bar{p} \cdot (p - q)] \right\} \\ &+ \frac{\not{p}}{B^2} \left\{ g^{\mu\nu} [(p\bar{p}) q^2 - 2(p\cdot q)(\bar{p}\cdot q)] - (\bar{p}^\mu p^\nu + \bar{p}^\nu p^\mu) \cdot B \right. \\ & \quad \left. + 2 [2 \bar{p}^\mu \bar{p}^\nu + (\bar{p}^\mu q^\nu + \bar{p}^\nu q^\mu)] [p \cdot (\bar{p} + q)] \right\} \\ &+ \frac{16}{AB} \left\{ -g^{\mu\nu} (\bar{p}p) q^2 + (\bar{p}^\mu p^\nu + \bar{p}^\nu p^\mu) [2(p\bar{p}) + q^2] \right. \\ & \quad - \frac{1}{2} (\bar{p}^\mu p^\nu + \bar{p}^\nu p^\mu) (A+B) - \bar{p}^\mu \bar{p}^\nu A - p^\mu p^\nu B \\ & \quad + (\bar{p}^\mu \bar{p}^\nu + p^\mu p^\nu) \cdot q^2 \\ & \quad \left. + [q^\mu (p^\nu - \bar{p}^\nu) + q^\nu (p^\mu - \bar{p}^\mu)] \cdot (p\bar{p}) \right\} \end{aligned}$$

$$\begin{aligned} & -\frac{\mu^2}{m^2} g_\alpha^\alpha g_\beta^\beta \text{Tr}[W_\alpha^\mu V_\beta^\nu] \\ &= -4\mu^2 \left\{ \frac{1}{A^2} [4 p^\mu p^\nu - 2(p^\mu q^\nu + p^\nu q^\mu) + q^2 g^{\mu\nu}] \right. \\ & \quad + \frac{1}{B^2} [4 \bar{p}^\mu \bar{p}^\nu + 2(\bar{p}^\mu q^\nu + \bar{p}^\nu q^\mu) + q^2 g^{\mu\nu}] \\ & \quad \left. + \frac{2}{AB} [4 \bar{p}^\mu p^\nu + 2(p^\nu q^\mu - \bar{p}^\mu q^\nu) + q^2 g^{\mu\nu} - 2 q^\mu q^\nu] \right\} \\ & - \frac{\mu^2}{m^2} \frac{16}{AB} \left\{ 2(g\cdot g) g_\beta^\beta g_\alpha^\alpha - g^2 g_\beta^\beta g_\alpha^\alpha - (g\cdot g)(g\cdot g) g^{\mu\nu} \right\} \end{aligned}$$

where

$$A = q^2 - 2(q \cdot p)$$

$$B = q^2 + 2(q \cdot \bar{p})$$

Next we calculate the second trace which is needed in order to calculate the matrix element for a polarized lepton beam. Inspecting the structure of the trace, we find the following result (we use the usual summation convention over the repeated indices),

$$\begin{aligned} \frac{1}{8} \text{Tr}[Q W_{\alpha}^{\mu} \not{V}_{\beta}^{\nu}] &= \frac{1}{A^2} [M^{\mu\nu}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} M^{\mu\nu}_{\gamma\gamma}] \\ &+ \frac{1}{B^2} [N^{\mu\nu}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} N^{\mu\nu}_{\gamma\gamma}] \\ &+ \frac{2}{AB} [T^{\mu\nu}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T^{\mu\nu}_{\gamma\gamma} \\ &+ S^{\mu\nu}_{\alpha\beta} - \frac{1}{2} g^{\mu\nu} S_{\lambda\lambda}^{\lambda\lambda}] \end{aligned}$$

where $A = q^2 - 2(q \cdot p)$, $B = q^2 + 2(q \cdot \bar{p})$ and

$$\begin{aligned} M^{\mu\nu}_{\alpha\beta} &= \bar{p}_{\alpha} \{ g^{\mu\nu} [q^2 Q_{\beta} - 2(q \cdot q) q_{\beta}] \\ &+ 4 p^{\mu} p^{\nu} Q_{\beta} - 2 Q_{\beta} [p^{\mu} q^{\nu} + p^{\nu} q^{\mu}] \\ &- 2 q_{\beta} [p^{\mu} q^{\nu} + p^{\nu} q^{\mu}] + 2(q \cdot q) [p^{\mu} q_{\beta}^{\nu} + p^{\nu} q_{\beta}^{\mu}] \\ &+ 2 q_{\beta} [q^{\mu} q^{\nu} + q^{\nu} q^{\mu}] - q^2 [q^{\mu} q_{\beta}^{\nu} + q^{\nu} q_{\beta}^{\mu}] \} \end{aligned}$$

$$\{ \psi_b(\underline{d}) - \psi(\underline{d}) + (\underline{b})\underline{d} \} [{}^{\psi}\underline{b}_nd + {}^{\psi}\underline{b}_nd] +$$

$$\{ \psi(\underline{b}_nd + {}^{\psi}\underline{b}_nd) + \underline{d} [{}^{\psi}\underline{b}_nd + {}^{\psi}\underline{b}_nd] \} \psi \mp = {}^{\psi}_n S$$

$$\begin{aligned} & (\underline{b})[\underline{d} + \psi_b] [{}^{\psi}\underline{b}_nd + {}^{\psi}\underline{b}_nd] + \\ & (\underline{d})\{ \psi(\underline{b}_nd + {}^{\psi}\underline{b}_nd) - \psi_b [{}^{\psi}\underline{b}_nd + {}^{\psi}\underline{b}_nd] \} + \\ & [(\underline{d})(\underline{b})\psi - (\underline{d})\psi_b] {}^{\psi}\underline{b}_nd + \\ & \psi \psi \underline{d} {}^{\psi}\underline{b}_nd \psi - \end{aligned}$$

$$\begin{aligned} & \psi \psi \{ [{}^{\psi}\underline{b}_nd + {}^{\psi}\underline{b}_nd] - [{}^{\psi}\underline{b}_nd + {}^{\psi}\underline{b}_nd] \} + \\ & [\psi \psi + \psi \psi \underline{d}] [{}^{\psi}\underline{d}_nd + {}^{\psi}\underline{d}_nd] - \\ & [\psi \psi \psi + \psi \psi \psi \underline{d}] [{}^{\psi}\underline{d}_nd + {}^{\psi}\underline{d}_nd] = {}^{\psi}_n T \end{aligned}$$

$$\begin{aligned} & \{ [{}^{\psi}\underline{b}_nd + {}^{\psi}\underline{b}_nd] \psi - [{}^{\psi}\underline{b}_nd + {}^{\psi}\underline{b}_nd] \psi \psi + \\ & [{}^{\psi}\underline{b}_nd + {}^{\psi}\underline{b}_nd] (\underline{d})\psi - 4 \psi \psi \underline{d} + \\ & [{}^{\psi}\underline{b}_nd + {}^{\psi}\underline{b}_nd] \psi \psi + 4 \psi \psi \underline{d} + \\ & [\psi \psi (\underline{b})\psi - \psi \psi \psi] {}^{\psi}\underline{b} \} \psi = {}^{\psi}_n N \end{aligned}$$

During the trace calculation, we use the fact that only the part symmetric under the exchange $\mu \leftrightarrow \nu$, or, $\alpha \leftrightarrow \beta$ contributes to the final matrix element (although the above expression is not yet fully symmetrized). The final matrix element is obtained when we combine this expression with the "boson projection operator," $(q_1^\alpha q_1^\beta / m^2 - g^{\alpha\beta})$ and the nucleon current part $J_{\mu\nu}$.

One can easily check that all the above results satisfy the gauge condition.

Appendix B. Analytic Integration

Our general integration procedure is as follows:

We first perform the integration over the internal angular variables $d\Omega_{q_2}$ in the frame where $\vec{q}_1 = 0$. See, e.g., Eq. (II,14). The result can be written as "manifestly" Lorentz covariant. Next we go to the lab. frame where we perform the integration over the variables of the target nucleus (or nucleon). When we encounter an integration of the form

$$\int \frac{\log x}{1+x} dx$$

it is time to quit the analytical approach. We can generally reduce the phase space integration to a 4-dimensional one. We then use the computer to do the remaining integration.

We discuss an example in the following.

Example: Muon pair radiation from neutrinos

We consider the integration

$$\frac{1}{4\pi} \int \frac{(a \cdot q_2)(b \cdot q_2)}{[q^2 + (q \cdot q_1) + (q \cdot q_2)]^2} d\Omega_{q_2}$$

or, equally the following integration,

$$I(a,b) = \frac{1}{4\pi} \int \frac{4(a \cdot \bar{p})(b \cdot k')}{[q^2 + (q \cdot q_1) + (q \cdot q_2)]^2} d\Omega_{q_2}$$

where

$q_1 = \bar{p} + k'$, $\bar{p}$ = 4-momentum of the antimuon.

$q_2 = \bar{p} - k'$, k' = 4-momentum of the outgoing neutrino.

q = momentum transfer to the nucleon system (which is space-like)

and a and b are arbitrary 4-vectors.

We introduce several auxiliary functions to get a "compact" expression for the result of the integration.

$$B(a, b) = (aq_1)(bq_1) - m^2(ab)$$

$$C(a) = B(a, q)$$

$$A(a) = (aq_1)(1 + \mu^2/m^2) - H \cdot C(a)/D^2$$

$$A(a, b) = B(a, b) - C(a)C(b)/D^2$$

$$H = q^2 + R \cdot (qq_1)$$

$$D = (qq_1)^2 - m^2 q^2$$

$$m^2 = q_1^2$$

$$R = (1 + \mu^2/m^2), P = (1 - \mu^2/m^2)$$

$$G = \log[(H + D \cdot P)/(H - D \cdot P)]$$

$$W(H) = (1/2)(P^2 - H^2/D^2)$$

Then

$$\begin{aligned} I(a, b) &= \frac{4}{4\pi} \int \frac{(a\bar{p})(b\bar{k}')}{[q^2 + (q \cdot q_1) + (q \cdot q_2)]^2} d\Omega_{q_2} \\ &= \frac{1}{H^2 - D^2 P^2} [A(a)B(b) - W(H) \cdot A(a, b)] \\ &+ \frac{G}{2 D^3 P} [C(a)B(b) - C(b)A(a) - H \cdot A(a, b)] \\ &+ \frac{1}{2 D^2} [A(a, b) - \frac{2}{D^2} C(a)C(b)] \end{aligned}$$

The integral, e.g.,

$$I(a, b, c, d) = \frac{1}{4\pi} \int \frac{16 (a\bar{p})(b\bar{p})(c\bar{p})(d\bar{k}')}{[q^2 + (q \cdot q_1) + (q \cdot q_2)]^2} d\Omega_{q_2}$$

is also obtained by a similar method although the result is more

complicated. Our matrix element contains about 50 of these "basic" integrations.

Our next task is to integrate over $d\Omega_q$. See Eq. (II,17).

$d\Omega_q$ is the solid angle of the momentum transfer q in the lab. frame.

We have an integration of the form

$$J = \int g((k-p) \cdot q, (p \cdot q)) \cdot \delta[m^2 - (k-p-q)^2] d\Omega_q$$

where g is a certain function of $(k-p) \cdot q$ and $(p \cdot q)$. We take the direction $\vec{k} - \vec{p}$ as the z -axis. The integration over $d\cos\theta_q$ is done using the δ -function $\delta(m^2 - (k-p-q)^2)$. We get a kinematical constraint from this integration. See Eq. (II, 18). The integration over $d\varphi_q$ can be done analytically as we discussed in Section II. Only $(p \cdot q)$ depends on φ_q . However, as one can guess from the expression for $I(a,b)$ above, the function $g((k-p) \cdot q, (p \cdot q))$ is very lengthy and complicated. We have to first reexpress this $g((k-p) \cdot q, (p \cdot q))$ as a rational function of $\phi = 2(p \cdot q) + q^2$. The reason why we choose this special combination instead of $(p \cdot q)$ lies in the fact that $g((k-p) \cdot q, (p \cdot q))$ contains ϕ in the denominator. See Eq. (II,12). After this rearrangement, we have a term like

$$\int \frac{X_2 \phi^2 + X_1 \phi + X_0}{\phi^2} d\varphi_q = X_2 \int d\varphi_q + X_1 \int \frac{1}{\phi} d\varphi_q + X_0 \int \frac{1}{\phi^2} d\varphi_q$$

where X_2 , X_1 and X_0 are independent of φ_q . We can perform this integration using the relations

$$\frac{1}{2\pi} \int \frac{d\varphi_q}{\phi} = - \frac{1}{\sqrt{A^2 - B^2}}$$

$$\frac{1}{2\pi} \int \frac{d\varphi_q}{\phi^2} = - \frac{A}{[A^2 - B^2]^{3/2}}$$

where

$$A = q^2 + 2 q^0 p^0 - 2 |\vec{q}| |\vec{p}| \cos \theta_q \cos \theta_p$$

$$B = 4 |\vec{q}|^2 |\vec{p}|^2 \sin^2 \theta_q \sin^2 \theta_p$$

Angles θ_q and θ_p are defined with respect to the z-axis, $\vec{k} - \vec{p}$.

If we define

$$s = (k \cdot Q)$$

$$x = (p \cdot Q)$$

$$y = (p \cdot k)$$

$$t = -q^2$$

$$p = \sqrt{(s - x)^2 + 2M^2 y - M^2 \mu^2}$$

where

Q : 4-momentum of the target nucleus (or nucleon)

p : 4-momentum of the outgoing muon

k : 4-momentum of the incident neutrino

q : 4-momentum transfer to the nucleon system

and $Q^2 = M^2$, $p^2 = \mu^2$

A and B can be also expressed as,

$$A = - \left(1 - \frac{x}{M^2} \right) \cdot x - \frac{1}{M^2} \sqrt{x^2 + 4M^2 x} \sqrt{x^2 - M^2 \mu^2} \cdot z \cdot \cos \theta_p$$

$$B^2 = \frac{1}{M^4} (x^2 + 4M^2 x)(x^2 - M^2 \mu^2)(1 - z^2)(1 - \cos^2 \theta_p)$$

where

$$\cos \theta = \frac{1}{p \cdot \sqrt{x^2 - M^2 \mu^2}} \cdot [x(s - x) - M^2 y + M^2 \mu^2]$$

$$z = \frac{M^2}{p \cdot \sqrt{x^2 + 4M^2 x}} \cdot [m^2 + \left(1 + \frac{s - x}{M^2} \right) \cdot x + 2y - \mu^2]$$

Summary:

After the integration over $d\Omega_{g_2}$ and $d\Omega_g$, we can express the cross section in terms of $(Q \cdot k)$, $(Q \cdot p)$, $(k \cdot p)$, m^2 and t , or equivalently, in terms of the energy of the incident neutrino, the energy and angle of the outgoing muon, invariant mass of the antimuon-outgoing neutrino system and the square of the momentum transfer to the nucleon system.

This is the result (II,20) in Section II.

The above integration procedure looks very tedious. However, one can not count out the computer which is designed to do complicated book-keeping. All we have to do is to write the initial mathematical expression in terms of the computer language instead of the familiar algebraic symbols. In this sense, one may call the above integration procedure as "semi-analytical"; we do not know the explicit analytical expression for the final result. We just tell the computer how to rearrange the various terms and how to replace the various integrals by the exact analytic results during the numerical calculation.

We repeat the same procedure to get the antimuon distributions, and also the invariant mass distribution of the muon-pair. We can also apply the similar procedure to the calculation of neutrino radiation from muons; it is actually much easier.

Appendix C. General Procedure of the Numerical Integration

For the muon pair creation process, we used the computer to do the formidable bookkeeping as well as numerical integration, as we discussed in Appendix B.

In this appendix we discuss the general integration procedure which applies to both processes. We used the Simpson method throughout for the numerical integration in this work. The integrand is, however, often not well behaved. To cope with this difficulty, we applied the most basic idea of the Simpson method; if we have only two divisions, the Simpson method is one of the most "reasonable" approximations to the integral. Thus we divided the integration region into many sub-intervals. Each sub-interval was divided into 2 or 4, and we applied the Simpson method for this sub-interval. (There are, of course, many other more sophisticated integration methods). The integration over the momentum transfer squared is most difficult. We discuss this integration in detail in the following.

Let us consider an integral

$$\int_{t_{\min}}^{t_{\max}} f(t) dt$$

where the integrand $f(t)$ is semi-singular at small values of t , and $t_{\min}$ is very small compared with $t_{\max}$, $0 < t_{\min}/t_{\max} \ll 1$. We divide this interval $[t_{\min}, t_{\max}]$ by "exponential" progress. We first fix two numbers N and n which satisfy the condition,

$$t_{\max}/t_{\min} \leq N^n$$

where n is an integer. Then we integrate from

$$t_0 = t_{\min} \text{ to } t_1 = (t_{\max} - t_{\min})/N^n + t_0$$

by the Simpson method, and next from

$$t_1 \text{ to } t_2 = (t_{\max} - t_1)/N^{n-1} + t_1$$

then from

$$t_2 \text{ to } t_3 = (t_{\max} - t_2)/N^{n-2} + t_2$$

and so on.

We adjust N and n such that

(i) the first interval, i.e., $\int_{t_0}^{t_1} f(t) dt$, or the last interval, $\int_{t_{n-1}}^{t_n} f(t) dt$, does not give the largest contribution to the total integral,

(ii) each sub-interval gives roughly the same contribution to the total integral.

We also print out the values of $f(t_i)$, $i=1 \sim n$, i.e., the values of the integrand at each end of the sub-interval, and also the maximum value of $f(t)$. We thus check that we do not miss the maximum point of $f(t)$. Usually $N \sim 10$ worked well within a reasonable computer time and with an error of $\sim 5\%$.

The Monte Carlo method can give a very good result for the total cross section. Actually, about 200 points in the 4-dimensional phase space $(E, \cos \theta, m, t)$ can give the total cross section with an error of ~ 20 percent, if we limit the variation of the momentum transfer t within the interval $[t_{\min}, t_{\max}/10]$. Larger momentum transfers do not give any substantial contribution. This is amazing when one recalls that we calculated about 150,000 points inside the phase space. However, the Monte Carlo

method is not good when one wants to know the detailed behaviour of the differential cross section inside the phase space volume, which we try to understand in this work. We can also find mistakes (at least "apparent" mistakes) in the calculation by looking at the detailed behaviour of the integrand.

The typical computer time of our calculation was ~ 5 minutes for a fixed energy of the incident particle and for a fixed target. The major part of the computer time was spent on the complicated bookkeeping. Actually, the neutrino radiation process in Section III, which does not require any bookkeeping, took only $10 \sim 20$ seconds for a similar calculation, i.e., the entire distribution for a fixed incident muon energy, say at 150 BeV, and a proton target.

References and Footnotes

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- 6) L. Lederman and C. Baltay, and also, D. Cline and A. Mann, Private communication.
- 7) J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics (MacGraw-Hill, New York, 1964). However we use the invariant normalization of states, $\langle p' | p \rangle = (2\pi)^3 \cdot 2p^0 \cdot \delta(\vec{p} - \vec{p}')$, and the normalization of spinors, $U(s,p)\bar{U}(s,p) = (\not{p} + \mu) \left(\frac{1 + \gamma_5 \not{s}}{2} \right)$.
- 8) We consider only spin-0 nuclei, and nucleons. Higher spin particles will introduce more form factors.
- 9) This W_{α}^{μ} is also vanishingly small when the momentum transfer q is small and the two muon momenta p and $\bar{p}$ are parallel. This property of W_{α}^{μ} is responsible for the cancellation among the leading terms in the matrix element. It is well known that the gauge condition generally causes a large cancellation among leading terms at small momentum transfers.
- 10) The first half of this statement is essentially the result of CP invariance, and the latter half is the reminiscence of the P (or C) violation in the weak current. However, note that P (or C) violation does not necessarily give the asymmetry of the energy and angular distributions

References and Footnotes (continued)

between the muon and the antimuon.

- 11) The result is also applicable to the zero spin nuclear target after this spin average. It is enough just to put the anomalous magnetic moment κ at 0, and $G_1(q^2) \rightarrow 0$.
- 12) When we perform this integration analytically, we encounter extremely tedious bookkeeping. This bookkeeping has been taken care of by the computer in our calculation.
- 13) Reference (4) above also gives a discussion on the form factor dependence of the total cross section.
- 14) We discuss only the muon distribution. The antimuon distribution is readily obtained by replacing the muon variables by the antimuon ones, and vice versa. Note that kinematical structure is identical for these two particles.
- 15) $s_0 = (Q+k)^2 = M^2 + 2Qk = M^2 + 2s$ is the usual Mandelstam invariant.
- 16) This result is for energetic neutrinos. The phase space is slightly more complicated for low energy neutrinos with energy less than $2M_\mu/(M-\mu)$.
- 17) See Appendix C for the details of the numerical integration.
- 18) If S-P combination were dominant at high energies (this combination contradicts with low energy data), this asymmetry is reversed.
- 19) For the proton target, the narrow phase space determines this behaviour. For the heavier nuclei, the small momentum transfer due to the form factors is responsible for this behaviour.

Refs. and footnotes (cont.)

- 20) We first calculated $d\sigma/d \cos\theta dE$ with fixed $\cos\theta$ and varying E . The angular distribution at fixed energy was obtained from this by a simple quadratic interpolation formula. Therefore the distribution $d\sigma/d \cos\theta dE$ with fixed E and also $d\sigma/dE$ are less accurate compared with $d\sigma/d \cos\theta dE$ with fixed $\cos\theta$ and $d\sigma/d \cos\theta$.
- 21) All these characteristic features (i)~(v) remain almost unchanged when one replaces the Fe^{56} target by the Pb^{208} target.
- 22) The discrepancy between $\sigma(+)$ and $\sigma(-)$ can be made less than 2 percent if we take a fine enough mesh for $\cos\theta$ integration as one can see from the total cross section on the proton target at 10 BeV and the neutron target at 40 BeV. We used finer meshes for these two cases to test the convergence of the $(\cos\theta, E)$ integration. The major error we might have is the systematic error in the (t, m^2) integration before we get the energy and angular distribution. To estimate the error due to this (t, m^2) integration, we changed the integration mesh. We also printed out the momentum dependence of our cross section for every calculation we did. Based on these checks, we estimate that over all error in the numerical integration is less than 5%.
- 23) The derivation of this condition is identical to that of the condition (II,24).
- 24) The program to do this calculation turned out to be very time consuming. We had to be satisfied with only these two calculations.

Refs. & Footnotes (cont.)

- 25) This behaviour is expected because the momentum transfer to the individual nucleon can be much larger than that to the nucleus as a whole due to the form factors. The upper limit of the invariant mass is estimated by $\sim \sqrt{2E_\nu \cdot m^*}$, where m^* is about the same as the cut-off mass of the form factor. This estimate of the upper limit gives ~ 3 BeV for the proton target and ~ 1.5 BeV for the Te^{56} target with incident energy of neutrinos at 10 BeV.
- 26) V.M. Shekhter, Soviet Physics - JETP, 7, 179 (1958).
- 27) B. Pontecorvo, Soviet Physics - JETP, 9, 1148 (1959).
- 28) G.M. Gandel'man and V.S. Pinaev, Soviet Physics - JETP, 10, 764 (1960).
- 29) M.E. Ebel and W.D. Walker, Phys. Rev. 122, 1639 (1961).
- 30) L. Lederman, Private communication.
- 31) The result for the spin zero nucleus can be readily obtained from this result as in Section II.
- 32) The term which contains γ_5 in the lepton current squared gives no contribution to the result.
- 33) If we use the normal lepton instead of the antilepton as the incident particle, the sign in front of P is reversed.
- 34) However note that Eq. (III, 4) cannot describe a general superposition of positive and negative helicity states (so called pure states).

Refs. etc. (cont.)

- 35) The integration over m is analytically doable.
- 36) We neglected the polarization term due to the incoherent process and also the incoherent contribution from neutrons. These contributions must be small. Strictly speaking, the total cross section is not the simple sum of the coherent and incoherent contributions. The incoherent contribution is slightly suppressed due to the Fermi statistics. Nucleon motions inside a nucleus also modify the incoherent cross section substantially.³⁷⁾ However the incoherent contribution is small compared with coherent one in our problem. Thus we can safely neglect these corrections.
- 37) See, e.g., G. von Gehlen, Nuovo Cimento, 30, 859 (1963).
- 38) Note that the kinematical limit of the invariant mass for the Fe^{56} target is much greater than that for the proton target.
- 39) P.N. Kirk et al., SLAC preprint, SLAC-PUB-656 (1969) reports that the cut-off mass of the proton magnetic form factor at 0.71 (BeV)^2 reproduces the experimental result well at large momentum transfers. This change of the cut-off mass from the rho-mass to 0.71 (BeV)^2 increases the incoherent cross section by ~ 10 percent.
- 40) If one neglects the lepton mass, the axial vector current is also conserved and gives the same contribution as the vector current after an integration over the neutrino variables.

Refs. (cont.)

However, we keep these small "mass terms" during the numerical calculation.

- 41) E.D. Bloom et al., A Report Presented to the XV International Conference on High Energy Physics, Kiev, U.S.S.R., SLAC-PUB-796, August, 1970. Our notation is slightly different from that of this reference.
- 42) The notation in this section generally follows that in Section III. In particular $s = E_{\mu} \cdot M$ = incident energy of the muon times proton mass. σ_S and σ_T denote the scalar and transversal "photon"-proton total cross sections, respectively.
- 43) This formula is also applicable to the neutron target when one puts every κ -independent term at zero and put the anomalous magnetic moment κ at -1.91. $\kappa = 1.79$ for the proton target.
- 44) C. Baltay and H. Wachsmuth, 1969 Aspen Summer Study, SS-133 (1969).
- 45) S.L. Adler, Phys. Rev., 135, B936 (1964); S.L. Adler, Ann. of Phys. 50, 189 (1958).
- 46) See Ref. (44) above.
- 47) R. G. Parsons, Phys. Rev., 150, 1165 (1966).
- 48) M. Veltman, Physica, 29, 161 (1963).

Figure Captions

1. Feynman diagrams for the process $\nu + (z) \rightarrow \nu + \mu + \bar{\mu} + (z)'$.
2. Phase space for the energy and angle of the produced muon or antimuon. Note that the phase space is identical for these two particles.
3. Phase space for the momentum transfer t and the invariant mass of the outgoing neutrino-antimuon pair m .
4. Phase space for the energy and angle of the scattered neutrino. See Eq. (II,27).
- 5.&6. Lab. energy spectrum of μ^+ and μ^- from a proton target at $\cos\theta = 1.0$ (Fig. 5) and at $\cos\theta = 0.997$ (Fig. 6). Incident energy of the neutrino at 10 BeV. $\sigma_0 = 3.03 \times 10^{-44} \text{ cm}^2$.
- 7.& 8. Lab. energy spectrum of μ^+ and μ^- from an Fe^{56} target at $\cos\theta = 1.0$ (Fig. 7) and at $\cos\theta = 0.9997$ (Fig. 8). Incident energy of the neutrino at 40 BeV. $z = \text{charge number of } \text{Fe}^{56} = 26$. $\sigma_0 = 3.03 \times 10^{-44} \text{ cm}^2$. For an Fe^{56} target we always show only the coherent contribution.
9. Angular distribution of μ^+ and μ^- from an Fe^{56} target. Incident energy of the neutrino at 10 BeV and 40 BeV. Note that the long tail of the angular distribution gives an important contribution to the total cross section. $z = 26$ and $\sigma_0 = 3.03 \times 10^{-44} \text{ cm}^2$.

Figure Captions (continued)

10. & 11. Angular distribution of μ^+ and μ^- at fixed outgoing energy, namely, at 16 BeV (Fig. 10) and at 2.1 BeV (Fig. 11). Incident energy of the neutrino at 40 BeV. Fe^{56} target.
12. Angular distribution of μ^+ and μ^- from a proton target. Incident energy of the neutrino at 10 BeV and 40 BeV.
 $\sigma_0 = 3.03 \times 10^{-44} \text{ cm}^2$.
13. & 14. Angular distribution of μ^+ and μ^- at fixed outgoing energy, namely at 4 BeV (Fig. 13) and at 0.59 BeV (Fig. 14). Incident energy of the neutrino at 10 BeV, proton target.
15. Energy spectrum of μ^+ and μ^- from an Fe^{56} target. Incident energy of the neutrino at 10 BeV. $\sigma_0 = 3.03 \times 10^{-44} \text{ cm}^2$.
 $z = 26$. Only the coherent contribution is shown, as usual for an Fe^{56} target.
- 16 (a). Energy spectrum of μ^+ and μ^- from an Fe^{56} target. Incident energy of the neutrino at 40 BeV.
 (b). Low energy part of (a) is expanded.
17. Energy spectrum of μ^+ and μ^- from a proton target for the incident energy of the neutrino at 10 BeV.
18. Momentum transfer dependence of the muon (not antimuon) distribution at fixed outgoing energy (=4 BeV) and angle ($\cos\theta = 1.0$). Incident energy of the neutrino at 10 BeV. Proton target. The level line shows the relative importance of the interval it covers when one integrates the distribution over the interval.

Figure Captions (cont.)

19. Momentum transfer dependence of the muon distribution at fixed outgoing energy ($= 8 \text{ BeV}$) and angle ($\cos\theta = 0.99985$). Incident energy of the neutrino at 40 BeV . Fe^{56} target. The level line shows the relative importance of the interval it covers.

20. Invariant mass distribution of the outgoing neutrino- antimuon pair from an Fe^{56} target. This corresponds to the mass of the virtual weak boson. The energy and angle of the muon (not antimuon) are fixed at 8 BeV and $\cos\theta = 0.99985$, respectively. The momentum transfer to the hadronic system is also fixed at 0.01 and $0.001 (\text{BeV}/c)^2$. Incident energy of the neutrino at 40 BeV . $\sigma_0 = 3.03 \times 10^{-44} \text{ cm}^2$. $z = 26$.

21. General qualitative tendency of the energy and angular distribution of μ^+ and μ^- . The muon is more energetic and appears more in the forward direction compared with the antimuon. The long tail of the distribution at large angles essentially determines the energy spectrum when one integrates over $\cos \theta$.

22. Invariant mass distribution of the μ^+ and μ^- pair from an Fe^{56} target for the incident energy of the neutrino at 10 BeV . Only the coherent contribution is shown. $\sigma_0 = 3.03 \times 10^{-44} \text{ cm}^2$.

23. Invariant mass distribution of the μ^+ and μ^- pair from a proton target for the incident energy of the neutrino at 10 BeV .

Fig. Captions (cont.)

24. & 25. Energy spectrum of the dimuon system from an Fe^{56} target.
Invariant mass of the dimuon is fixed at 0.3 BeV (Fig. 24)
and at 0.93 BeV (Fig. 25). Incident energy of the neutrino
at 10 BeV. $\sigma_0 = 3.03 \times 10^{-44} \text{ cm}^2$.
26. Feynman diagrams for the process $\mu^+ + (z) \rightarrow \mu^+ + \nu + \bar{\nu} + (z)$.
27. Phase space for the energy and angle of the scattered antimuon.
28. Phase space for the invariant mass of the radiated neutrino
pair and the momentum transfer to the hadronic system.
29. Forward energy spectrum of the scattered muon from an Fe^{56}
target. Incident energy of the antimuon at 120 BeV.
 $\sigma_0 = 3.03 \times 10^{-44} \text{ cm}^2$, $z = 26$. For an Fe^{56} target we show
only the coherent contribution.
- 30 (a) Energy spectrum of the scattered antimuon from an Fe^{56} target
for 150 BeV incident energy.
(b) Low energy part of (a) is expanded.
31. Same as in Fig. 30 except that the incident energy of the antimuon
is now at 120 BeV.
- 32 (a) Energy spectrum of the scattered antimuon from a proton target
for 150 BeV incident energy.
(b) Low energy part of (a) is expanded.
33. & 34. Angular distribution of the scattered antimuon at fixed outgoing
energy; 5.7 BeV in Fig. 33 and 30 BeV in Fig. 34. Incident
energy of the antimuon at 120 BeV. Fe^{56} target. $\sigma_0 = 3.03 \times 10^{-44}$

Fig. Caps. (cont.)

cm^2 and $z = 26$. The qualitative feature of this distribution remains unchanged when one increases the incident energy up to 150 BeV.

35. Invariant mass of the radiated neutrino pair from a proton target for the fixed energy ($= 9.45$ BeV) and angle ($\cos\theta = 0.99963$) of the outgoing antimuon. Incident energy of the antimuon at 150 BeV.
36. Invariant mass of the radiated neutrino pair from an Fe^{56} target for the fixed energy ($= 5.7$ BeV) and angle ($\cos\theta = 0.9998$) of the outgoing antimuon. Incident energy of the antimuon at 120 BeV.
37. Momentum transfer dependence of the antimuon distribution at fixed energy ($= 9.47$ BeV) and angle ($\cos\theta = 0.9996$) of the outgoing muon. Invariant mass of the radiated neutrino mass is also fixed at 1.45 BeV. The level line shows the relative importance of the interval it covers when one integrates over the interval.
38. Feynman digrams for the process $\mu^+ + p \rightarrow \mu^+ + \nu + \bar{\nu} + \Gamma$.
39. Phase space for the variable (X, Y) . See Eq. (IV,2) for the definition of X and Y .
40. Dependence of the inelastic cross section on the scaling variable $\omega = \nu/t \geq 1/2$ for momentum transfer t fixed at $1(\text{BeV}/c)^2$. Incident energy of the muon at 150 BeV.
 $\sigma_0 = 3.03 \times 10^{-44} \text{cm}^2$.

Fig. Caps. (cont.)

41. Momentum transfer dependence of the inelastic cross section.
Incident energy of the muon at 150 BeV.

(a)

(b)

Fig.1

Fig.2

Fig.3

Fig.4

Fig.5

Fig.6

Fig.7

Fig.8

Fig. 9

Fig.10

Fig.II

Fig.12

Fig.13

Fig.14

Fig.17

Fig.18

Fig.20

Fig. 21

Fig. 24

$$\Delta k \approx E_{\mu\bar{\mu}} \text{ (BeV)}$$

Fig.25 $\Delta k \approx E_{\mu\bar{\mu}} \text{ (BeV)}$

Fig. 26

Fig. 27

Fig.28

Fig.29

Fig.30(a)

Fig.30(b)

Fig.31

Fig.32(a)

Fig.32(b)

Fig.33

Fig.34

Fig.35

Fig.36

Fig.37

(a)

(b)

Fig.38

Fig.39

Fig.40

Fig.4I