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by

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DESIGN OF A FORCE-FREE INDUCTIVE STORAGE COIL

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ABSTRACT

Force-free coils have been considered for various applications as energy storage devices. A novel energy storage system consisting of both toroidal and poloidal coils has been developed. In this system, the Lorentz forces on the two coils are in opposition and can be made to cancel by having the proper ratio of ampere-turns in the two coils. A general discussion of force-free coil design is given, and the toroidal-poloidal coil system is described. The materials costs of magnets constructed with this and other geometries are compared. The conclusions are that conductor cost of this system lies between those of solenoids and toroids, and that the costs of structural material may be considerably less than either of these simpler coils.

I. INTRODUCTION

The application of inductive storage by means of superconducting coils to peak load shaving and load leveling in an electric power system has been studied by many workers in the field.¹ It has been pointed out,² however, that the mechanical structure needed to counterbalance the electromechanical forces in inductors accounts for a major fraction of the overall cost of the inductive storage installation. Hence, identifying a practical force-free configuration may lead to a substantial saving in cost.

Although the theory of force-free coils, coils in which the current and field are parallel, $\underline{J} = \alpha \underline{H}$ has been known for some time,³ there has been some confusion surrounding practical implications of these coils. It is important to realize, for instance, that a coil of finite size cannot be strictly force-free. This fact, which was pointed out for the first time by Rosenbluth,⁴ is described in detail in Longmire's text, "Elementary Plasma Physics."⁵ The proof makes use of well-known arguments based on Maxwell's development of the stress tensor associated with the electromagnetic field and on an extension of the virial theorem of Clausius.

For convenience, a brief outline of Longmire's proof has been duplicated in the appendix. The

essential conclusions are given below.

- 1) When a system consisting of a coil and its associated magnetic field is in equilibrium, then the mechanical energy due to the electromechanical stresses in the coil cannot be less than the magnetic energy of the field.
- 2) The mechanical energy in the coil is a minimum when the coil is force-free.
- 3) The stresses in a force-free coil are tensile in character. In a conventional (nonforce-free) coil, the forces are both tensile and compressive.
- 4) The mass of a coil - designed so that the material of the coil is subjected to the maximum allowable working stresses - is directly proportional to the stress energy in the coil. An immediate corollary of this conclusion is that a force-free coil will lead to a structure with the minimum mass. In order to illustrate this statement, Stekly² calculated the mass of a structure of a conventional toroidal coil in which the winding is purely poloidal. He subsequently pointed out that such a coil has a mass three times that of a force-free toroidal coil associated with the same magnetic energy.

The standard procedure of estimating force-free fields is to look for current distributions in which

the current is everywhere parallel to the magnetic field. This is the approach that has been followed here. However, an additional requirement has been introduced, viz., that only solutions leading to surface currents which can be easily produced in practice are selected.

II. FORMULATION OF DESIGN PROCEDURE

The starting point of the calculations is the basic equation for force-free criteria,

$$\underline{F} = \underline{J} \times \underline{H} = (\nabla \times \underline{H}) \times \underline{H} = 0. \quad (1)$$

As is well known, the above expression implies that the current is parallel to the field,

$$(\nabla \times \underline{H}) = \alpha \underline{H} = \underline{J}, \quad (2)$$

where α is a function of position which needs to be evaluated.

It is convenient to introduce a vector potential $\underline{A}$, defined by

$$\underline{H} = (\nabla \times \underline{A}). \quad (3)$$

Combining the above expression with Ampère's equation,

$$\begin{aligned} \underline{J} &= (\nabla \times \underline{H}) \\ &= (\nabla \times (\nabla \times \underline{A})), \end{aligned} \quad (4)$$

then the set of Eqs. 2, 3, and 4 completely defines the problem, once the shape of the coil is decided upon.

It is proposed to discuss the toroidal configuration. Conventional toroidal coils for which the field is completely inside the torus have the disadvantage of displaying a poor figure of merit,⁶ defined as the magnetic energy stored per unit mass of conductors. This statement is not for force-free coils in which the field produced by the toroidal coil extends to all space. The stored energy increases appreciably and it will be shown that the figure of merit improves a great deal.

An axis-symmetrical magnetic field which is independent of the azimuthal component can be expressed by:⁷

$$\underline{H} = \underline{i}_z \times rT + \nabla \times (\underline{i}_z \times rP), \quad (5)$$

where T and P are toroidal and poloidal components, respectively, and $\underline{i}_z$ is a unit vector along z , the axis of symmetry. It is convenient to express the magnetic field in cylindrical coordinates: z , $\tilde{\omega}$, and ϕ , so that

$$\underline{H} = -\tilde{\omega} \frac{\partial P}{\partial z} \underline{i}_{\tilde{\omega}} + \tilde{\omega} T \underline{i}_{\phi} + \frac{1}{\tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} (\tilde{\omega}^2 P) \underline{i}_z, \quad (6)$$

where $\underline{i}_{\tilde{\omega}}$, $\underline{i}_{\phi}$ are unit vectors along the radial and azimuthal directions.

In a similar manner, we define the vector potential in terms of toroidal and poloidal components T^* and P^* respectively. Therefore one finds:

$$\underline{A} = \underline{i}_z \times r T^* + \nabla \times (\underline{i}_z \times r P^*). \quad (7)$$

The components of the magnetic field are found from

$$\underline{H} = (\nabla \times \underline{A}) = \tilde{\omega} \frac{\partial T^*}{\partial z} \underline{i}_{\tilde{\omega}} - \tilde{\omega} \Delta_5 P^* \underline{i}_{\phi} + \frac{1}{\tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} (\tilde{\omega}^2 T^*) \underline{i}_z, \quad (8)$$

where

$$\Delta_5 \equiv \frac{\partial^2}{\partial \tilde{\omega}^2} + \frac{3}{\tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} + \frac{\partial^2}{\partial z^2}.$$

The requirement of force-free fields together with Eq. 8 yields

$$j_{\tilde{\omega}} = \alpha H_{\tilde{\omega}} = \frac{\partial}{\partial z} (\tilde{\omega} \Delta_5 P^*) = -\alpha \tilde{\omega} \frac{\partial T^*}{\partial z}, \quad (9)$$

$$\begin{aligned} j_{\phi} &= \alpha H_{\phi} = -\left(\frac{1}{\tilde{\omega}} \frac{\partial^2 (\tilde{\omega}^2 T^*)}{\partial z^2} + \frac{\partial}{\partial \tilde{\omega}} \frac{1}{\tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} (\tilde{\omega}^2 T^*) \right) = \\ &= -\alpha (\tilde{\omega} \Delta_5 P^*), \end{aligned} \quad (10)$$

and

$$j_z = \alpha H_z = -\frac{1}{\tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} (\tilde{\omega}^2 \Delta_5 P^*) = \alpha \frac{1}{\tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} (\tilde{\omega}^2 T^*). \quad (11)$$

An immediate consequence of Eq. 10, which we rewrite as

$$\frac{\partial^2}{\partial z^2} (\tilde{\omega}^2 T^*) + \tilde{\omega} \frac{\partial}{\partial \tilde{\omega}} \frac{1}{\tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} (\tilde{\omega}^2 T^*) = -\tilde{\omega} j_{\phi}, \quad (10')$$

is that any $(\tilde{\omega} j_{\phi}) = f_n(\tilde{\omega}^2 T^*)$ which satisfies Eq. 10' describes a force-free configuration. This important

Fig. 1. Coordinate system used for calculations of toroidal coil geometries. $\tilde{\omega}$, d , and z are the usual polar coordinates. ρ and θ are the poloidal coordinates.

result was pointed out by Furth et al (Ref. 4). To completely define the problem we need to find the component P^* and the function α . We will find it very useful to introduce polar coordinates ρ , θ , as shown in Fig. 1, referred to a plane which includes the axis of symmetry of the toroid. If we introduce a poloidal current j_θ , then

$$\tilde{j}_\omega = -\sin \theta j_\theta, \quad (12)$$

and

$$j_z = \cos \theta j_\theta. \quad (13)$$

Multiplying Eq. 9 by $\sin \theta$, Eq. 10 by $\cos \theta$, and subtracting one from other, we obtain

$$\begin{aligned} \sin \theta \frac{\partial}{\partial z} (\tilde{\omega}^2 \Delta_5 P^*) + \cos \theta \frac{\partial}{\partial \tilde{\omega}} (\tilde{\omega}^2 \Delta_5 P^*) = \\ \frac{\partial}{\partial \rho} (\tilde{\omega}^2 \Delta_5 P^*) = -j_\theta \tilde{\omega}, \end{aligned} \quad (14)$$

and

$$\frac{\partial}{\partial \rho} (\tilde{\omega}^2 \Delta_5 P^*) = -\alpha \frac{\partial}{\partial \rho} (\tilde{\omega}^2 T^*). \quad (15)$$

If one writes $\tilde{\omega}^2 \Delta_5 P^* = \Omega$, then Eq. 14 can be integrated at once,

$$\Omega = -\int j_\theta \tilde{\omega} d\rho + f(\theta), \quad (16)$$

where $f(\theta)$ is an arbitrary function of θ . It is readily checked that $f(\theta)$ is a constant or zero. Thus if j_θ is prescribed, Ω can be found by quadrature.

Combining Eqs. 9, 10, 11, and 14, a relation defining the unknown function is found. This is

$$\frac{1}{\alpha^2} \left(\frac{\partial^2 \Omega}{\partial z^2} + \tilde{\omega} \frac{\partial}{\partial \tilde{\omega}} \left(\frac{1}{\tilde{\omega}} \frac{\partial \Omega}{\partial \tilde{\omega}} \right) \right) + \frac{1}{2} \left(\frac{\partial \Omega}{\partial z} \frac{\partial}{\partial z} \left(\frac{1}{\alpha^2} \right) + \frac{\partial \Omega}{\partial \tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} \left(\frac{1}{\alpha^2} \right) \right) = -\tilde{\omega} \Omega. \quad (17)$$

If Ω is known from Eq. 16, then α may be found by solving Eq. 17. This in turn defines the toroidal component of the current, J_ϕ , by means of Eq. 10, viz.,

$$j_\phi = -\frac{\alpha \Omega}{\tilde{\omega}}. \quad (18)$$

III. A REDUCED FORCE CURRENT DISTRIBUTION

The first two conclusions derived from the virial theorem were stated in the introduction:

1) When a system consisting of a coil and its associated magnetic field is in equilibrium, then the mechanical energy due to the electromechanical stresses in the coil cannot be less than the magnetic energy of the field, and

2) The mechanical energy in the coil is a minimum when the coil is force-free. These conclusions lead one to intuitively guess the current distribution which will yield a force-free coil. In the past the solutions have been coils in which the field and the current slowly change from toroidal to poloidal across the windings thickness. Solutions of this nature have been investigated by Mills et al.⁸ and Wakefield.⁹ However, they lead to complicated shapes which are hard to fabricate in practice.

A poloidal surface current on a form produces a toroidal magnetic field which is completely contained inside the torus. This type of coil is subjected to an unbalanced mechanical force in the sense that the magnetic pressure acts on one side, viz., from the interior of the torus. For a purely toroidal surface current, the field is poloidal and is localized outside the torus. The direction of the mechanical forces on the torus are compressive for the toroidal current sheet and expansive for the poloidal current sheet. Hence, a surface current

Fig. 2. Reduced force energy storage concept. The poloidal coils produce outward forces which are in opposition to the inward forces of the toroidal coils.

which has both a toroidal and a poloidal component can lead to a force-free solution, since it appears that the forces are such that they will cancel for a proper choice of current.

We therefore separate the two components of the current (toroidal and poloidal) into two distinct coils: a coil carrying a poloidal current located inside a toroidal shell, and a coil carrying a toroidal current outside the shell. A system such as this can be manufactured from circular coils. Thus, it is a practical solution, as shown in Fig. 2.

The magnitude of the currents required for the forces to nearly cancel may be obtained as follows. The poloidal current, j_θ , from

$$\nabla \cdot j_\theta = 0, \quad (19)$$

must satisfy the condition

$$j_\theta \tilde{\omega} = j_\theta^0 (R + a), \quad (20)$$

where j_θ^0 is the current per unit length at the rim of the torus. But, the current is a surface current, i.e.,

$$j_\theta^0 = j_\theta^0 \delta(\rho - a) (R + a), \quad (21)$$

and

$$j_\theta = j_\theta^0 \delta(\rho - a) \frac{(R + a)}{(R + \rho \cos \theta)}. \quad (22)$$

Substitution of Eq. 22 into Eq. 16 indicates that

$$\Omega = -j_\theta^0 (R + a), \quad (23)$$

inside the torus and is zero elsewhere. This property follows from the manner of taking the limit of integration.

The discontinuous behavior of Ω causes the differential Equation 17 to become

$$\left(\frac{1}{\alpha}\right) \frac{R}{\rho(R + \rho \cos \theta)} + \frac{1}{2} \frac{\partial}{\partial \rho} \left(\frac{1}{\alpha}\right) = R + \rho \cos \theta. \quad (24)$$

The general solution of Eq. 24 is

$$\alpha = \frac{(R + a \cos \theta)}{a} \sqrt{C + a^2}, \quad (25)$$

where C is an arbitrary constant.

The toroidal current from Eq. 18 is

$$j_\theta = j_\theta^0 (R + a) \frac{\sqrt{C + a^2}}{a} \delta(\rho - a). \quad (26)$$

We now select the value of C so that the magnetic forces cancel on both sides of the toroidal shell. This is equivalent to

$$H_\omega^2 + H_z^2 = H_\phi^2, \quad (27)$$

or from Eqs. 9, 10, and 11,

$$j_\theta^2 = j_\phi^2,$$

which yields $C = 0$, and

$$j_\theta = j_\phi = j_\theta^0 (R + a) \delta(\rho - a).$$

IV. FIGURE OF MERIT OF WINDING CONFIGURATION

The effective utilization of the superconducting material in the coil is expressed by the magnetic energy stored by the coil per unit mass of the conductor. To obtain a realistic, relative figure of merit for the material utilization in different coils, it will be useful to compare the coils on the basis of the magnetic energy stored.

The magnetic energy stored, E_m , is given by

$$E_m = \frac{1}{2} L I^2, \quad (28)$$

where L is the inductance of the coil and I is the current. But the current is related to the current density j by

$$I = j A, \quad (29)$$

where A is the cross-sectional area of the conductor. Combining Eqs. 28 and 29, the figure of merit for the winding is

$$fm = \frac{j}{2} \frac{LI^2}{A\ell\rho_s}, \quad (30)$$

where ℓ is the length of the superconducting winding and ρ_s is the density of the conductor. In order to show the dependence of the figure of merit on the energy, we manipulate Eqs. 28, 29, and 30 so that

$$fm = \frac{j}{20} \frac{1}{\ell} \sqrt{2E_m L}. \quad (31)$$

Since the current density is normally close to the maximum allowable value, $\frac{j}{\rho_s}$ is a constant for a given material, so that the meaningful parameter is $\sqrt{\frac{E}{\ell^2}}$.

In our case, since we have two types of coils, the figure of merit has to take into account the contribution of the toroidal and poloidal current elements, i.e.,

$$fm = \frac{\Sigma(LI^2)}{2E(A\ell\rho_s)}. \quad (32)$$

The inductance of a single-turn poloidal current sheet on a torus of minor radius a and major radius R is given by²

$$L = \mu_0 R \left(1 - \sqrt{1 - \frac{a^2}{R^2}} \right). \quad (33)$$

However, it is known that the maximum magnetic energy stored in a torus for a given quantity of superconductor is obtained for an aspect ratio $\frac{a}{R} \sim \frac{1}{3}$. Thus

$$L_{\text{poloidal}} = 0.0007183 R \text{ (}\mu\text{H)}, \quad (34)$$

where R is expressed in centimeters.

But the value of ℓ for the poloidal current sheet is $2\pi a$. Thus $\frac{L}{\ell^2}_{\text{poloidal}}$ is $\frac{0.0064647}{4\pi^2 R}$.

When we include the contribution of the toroidal current sheet in the force-free coil, we find

$$\begin{aligned} L_{\text{toroidal}} &= \mu_0 R \left\{ \left(1 + \frac{a^2}{8R^2} \right) \log\left(\frac{8R}{a}\right) - \frac{7}{4} + \frac{a^2}{24R^2} \right\} \\ &= 0.01849 R \quad (\mu\text{H}), \end{aligned} \quad (35)$$

where we have assumed $\frac{a}{R} = 1/3$. Since the appropriate value of ℓ for the toroidal current sheet is $2\pi R$, the quantity $\frac{L}{\ell^2}_{\text{toroidal}}$ is $\frac{0.01849}{4\pi^2 R}$.

Now we note that

$$E_m^{\text{toroidal}}/E_m^{\text{poloidal}} = LI^2_{\text{toroidal}}/LI^2_{\text{poloidal}}. \quad (36)$$

Since the calculations of Section III indicated that the current densities in the toroidal and poloidal coils must be equal in order to satisfy the force-free condition, we find

$$\begin{aligned} \frac{E_m^{\text{toroidal}}}{E_m^{\text{poloidal}}} &= \frac{(0.01849 R)}{(0.000718 R)} \frac{a^2}{R^2} \\ &= 2.8613, \end{aligned}$$

or

$$\left. \begin{aligned} E_m^{\text{toroidal}} &= 0.741 E_m \\ E_m^{\text{poloidal}} &= 0.259 E_m \end{aligned} \right\} \quad (37)$$

where $E_m = E_m^{\text{toroidal}} + E_m^{\text{poloidal}}$ is the total energy stored in the coil. The expression for the figure of merit for the force-free coil becomes

$$\begin{aligned} fm &= \frac{j}{\rho_s} \sqrt{\frac{E_m}{8\pi^2 R}} \frac{\sqrt{0.741 L_{\text{toroidal}} + \frac{a}{R}} \sqrt{0.259 L_{\text{poloidal}}}}{1 + \left(\frac{a}{R}\right)^2} \\ &= \frac{0.111 j}{\rho_s} \sqrt{\frac{E_m}{8\pi^2 R}}. \end{aligned} \quad (38)$$

We now compare the fm of other coils with that of the force-free coil discussed here.

The first coil we consider is a cylindrical coil of radius a_0 and length ℓ_0 . Nagaoka¹⁰ evaluated

the inductance of a cylindrical current sheet which is given by

$$L = 0.002\pi^2 a_0 \left(\frac{2a_0}{\ell_0} \right) K(\mu H) , \quad (59)$$

where the Nagaoka factor, K , is a function of the ratio $\frac{2a_0}{\ell_0}$, and all dimensions are expressed in centimeters.

Because of the idealized situation assumed here, viz., that of a very thin coil, the maximum inductance is not that for a Brooks coil¹¹, but instead¹² it is for $\frac{2a_0}{\ell_0} = 2.4510$. The corresponding inductance is

$$L = 0.02305 a_0 (\mu H) . \quad (40)$$

The figure of merit for the cylindrical current sheet is readily found from Eqs. 31 and 40 to be

$$fm = 0.152 \frac{j}{c_s} \frac{E_m}{8\pi^2 a_0} , \quad (41)$$

Had we considered the poloidal element alone, as is the case for the conventional toroidal coil, then the figure of merit would be

$$fm = 0.011 \frac{j}{c_s} \frac{E_m}{8\pi^2 R} , \quad (42)$$

The above calculations show clearly that in the force-free coil the material is used more efficiently than in a conventional toroidal winding. The figure of merit for a force-free coil is intermediate between that of a coil with pure poloidal current and that of a coil with pure toroidal current. The use of a superconductor is thus slightly less efficient than in a solenoid; however, this is more than compensated for by the economy of the mechanical structure.

V. COIL REINFORCING STRUCTURE

The above calculations for the current distributions needed to obtain a force-free coil were performed on the basis of the current being concentrated

on a surface (sheet). Actually, a toroidal form is needed to support the two sets of coils which carry the toroidal and poloidal currents. The two coils are therefore separated by a distance equal to the thickness of the form. As a result the forces are almost but not quite cancelled.

Before we estimate this imbalance, we first need to evaluate the thickness of the toroidal shell required to sustain the forces. From the virial theorem, one obtains⁵

$$W = \frac{1}{2} \frac{B^2}{\mu_0} E_m , \quad (43)$$

where σ_w is the working elastic stress in the medium, and W the mass of the shell. Introducing the thickness t of the shell, one can readily show that¹³

$$t = \left(\frac{B^2}{2\mu_0 \sigma_w} \right) \frac{a}{2R} R . \quad (44)$$

Since $\frac{a}{R} = \frac{1}{3}$, and for typical values $B \approx 0.005$ T and $\sigma_w \approx 1.1$ GPa, one finds that

$$\frac{t}{R} \approx \frac{1}{600} , \quad (45)$$

which indicates that the thin-shell theory is valid. An immediate consequence of this theory is that one can estimate the effect of imbalance in the magnetic forces.

In the force-free situation, the shell is under pure compression (from the inside and from the outside). If there is an imbalance due to the finite thickness of the shell, we expect to have a difference in the magnetic field, ΔB_z of the order t/R .

There are therefore additional stresses, in the form of hoop stresses caused by this net force. The net force is $\frac{(\Delta B)^2}{2\mu_0} \sim \frac{((t/R)B)^2}{2\mu_0} \sim \left(\frac{t}{R}\right)^2 \frac{B^2}{2\mu_0}$.

The hoop stresses (in direction tangential to the minor and major circles) are,

$$\begin{aligned} \sigma &= \frac{2\pi}{3} \left(\frac{\Delta B^2}{2\mu_0} \right) \frac{R}{t} (1 + a/R) \\ &= 8.377 \frac{t}{R} \left(\frac{B^2}{2\mu_0} \right) . \end{aligned} \quad (46)$$

The stresses are increased by approximately 1/75 of the normal working stresses.

An essential conclusion is that for the force-free case, the mass defined by Eq. 43 is about 1/3 that for the mass needed in a conventional coil. Indeed Stekly² has shown that for a regular coil

$$M \approx 3.1 \frac{B_{\max}^2}{\sigma_w} E_m \quad (47)$$

VI. SYSTEM COSTS

The cost of the Superconducting Magnetic Energy Storage system is obtained by the procedure outlined by Hassenzahl et al.¹⁴ The cost of the system will depend on the costs of

- 1) the superconductor material
- 2) the dewar
- 3) the structure
- 4) the interfacing equipment (ac-dc converters)
- 5) miscellaneous.

Superconductor Material

Since two sets of coils are used, one would be tempted to state that the cost of the material should be doubled. Actually, the increase is less because the inductance of the force-free coil is greater than that of a conventional toroidal coil.

If the inductance of a solenoidal coil is L_1 and that of the additional toroidal winding is L_2 , then the energy stored in a conventional coil is

$$E_{mi} = \frac{1}{2} I_i^2 L_i \quad ; \quad i = 1, 2. \quad (48)$$

On the other hand, for the force-free coil consisting of both toroidal and solenoidal coils,

$$E_m = \frac{1}{2} I^2 (L_1 + L_2). \quad (49)$$

To determine the inductance of a force-free coil, which will store the same energy as that stored in a conventional solenoidal coil, we must have

$$I_1^2 L_1 = I_2^2 (L_1 + L_2). \quad (50)$$

The current, therefore, must be reduced by the ratio

$$\frac{I_1}{I_2} = \left(\frac{L_1 + L_2}{L_1} \right)^{1/2} = \left(1 + \frac{L_2}{L_1} \right)^{1/2}. \quad (51)$$

If we keep the current the same, the inductance of the comparison coil must be increased by the factor

$$\left(1 + \frac{L_2}{L_1} \right).$$

Since the inductance is proportional to the square of the number of turns, the mass of the superconductor would vary like the square root of the inductance.

Therefore, we conclude that instead of doubling the mass of superconductor, we have indeed increased

it by $1 + \left(\frac{L_1}{L_2} \right)^{1/2}$. Using the values of Eqs. 36

and 40, we find that the factor is 1.18 (i.e., an increase of 18%).

The cost of the dewar is the same as for a simple toroidal coil.

The cost of the support structure, however, is reduced appreciably. We had mentioned above that Stekly predicted the mass of a conventional coil wound on a torus to be 3.1 times bigger than that of the minimum mass. This factor comes close to the value of 3.0 found by Hassenzahl.¹⁴ For the force-free coil, it is taken to be equal to the minimum value. The cost estimates given for a force-free coil in Table I are based on $B_{\max} = 8.0$ T and $a/R = 1/3$. These costs are compared to the costs of some conventional coils in Table II.

TABLE I

COST ESTIMATES FOR FORCE-FREE COIL

	3×10^4	10^6	3.9×10^7	10^8
Energy Stored (MJ)				
Cost of superconductor (\$ x 10^6)	2.24	22.77	261.96	489.7
Dewar cost (\$x 10^6)	2.0	24.6	300	581
Structure (\$x 10^6)	0.267	8.67	338	866.67
ac-dc converter (\$x 10^6)	0.04	1.4	54	140
Miscellaneous (\$x 10^6)	.47	2.1	51	128
TOTAL	5.02	59.54	1004.96	2205.37

TABLE II

COMPARISON OF CONSTRUCTION COSTS OF FORCE-FREE COIL WITH COSTS OF CYLINDRICAL COIL

	3×10^4	10^6	3.9×10^7	10^8
Energy (MJ)				
Force-free coil (\$ 10^6)	5.02	59.54	1004.96	2205.37
Conventional coil (\$ 10^6)	5.2	73.4	1641	3864
This table clearly demonstrates the economic advantage of the force-free coil.				

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APPENDIX

PROOF OF VIRIAL THEOREM

The virial theorem of Clausius can be extended by analogy to the case of a coil and its magnetic field. It is known from classical electromagnetic theory that Maxwell's stress tensor $\underline{T}$ satisfies a conservation relation of the form

$$\frac{\partial}{\partial x_j} (T_{ij}) = 0. \quad (A-1)$$

When the electromagnetic forces are in equilibrium with elastic forces in a medium that contains the electromagnetic field, then through the introduction of an elastic stress tensor $\underline{S}$, the above equation is modified to become

$$\frac{\partial}{\partial x_j} (T_{ij} + S_{ij}) = 0. \quad (A-2)$$

If we multiply the above expression by the position vector $\underline{r}$ and integrate over all space

$$\begin{aligned} 0 &= \int_V x_i \frac{\partial}{\partial x_j} (T_{ij} + S_{ij}) d\tau \\ &= \int_S x_i (T_{ii} + S_{ii}) n_i d\sigma - \int_V (T_{ii} + S_{ii}) d\tau \end{aligned} \quad (A-3)$$

In the above equation, $d\tau$ is an element of volume, $d\sigma$ an element of surface, the n_i 's are the direction cosines of the outward normal to the surface S and the transition from the first to the second line of Eq. A-3 has made use of Gauss's theorem. Since the trace of the tensor $\underline{T}$ is the magnetic energy, while the trace of $\underline{S}$ is the elastic energy, one finds at once

$$\int \left(\frac{B^2}{2\mu_0} + \frac{1}{2} \frac{\sigma^2}{\epsilon} \right) d\tau = \int_S x_i (T_{ii} + S_{ii}) n_i d\sigma. \quad (A-4)$$

The above relation is the virial theorem.

If the coil is of infinite dimension, the surface integral vanishes, and the mechanical energy is equal to the magnetic energy except for a negative sign. The negative sign indicates that the elastic forces have to be tensile in character.

When the coil is of finite dimensions, the surface integral need not vanish. But because of the positive definite character of the integrand

$$\int \frac{B^2}{2\mu_0} d\tau \leq \left(-\int \frac{1}{2} \frac{\sigma^2}{\epsilon} d\tau \right) . \quad (A-5)$$

It is thus clear that the elastic energy stored in the coil is at least as small as the magnetic energy but is usually larger. In a force-free coil the mechanical energy is a minimum, i.e.,

E_m = Mechanical energy.