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NEW YORK, NEW YORK

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AEC Computing and Applied Mathematics Center
Institute of Mathematical Sciences
New York University

TID-4500
15th Ed.

NYO-9352
PHYSICS

A SIMPLIFIED DESCRIPTION OF SPHERICAL
AND CYLINDRICAL BLAST WAVES

by

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August 1, 1960

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Partially sponsored by Air Force Office of Scientific Research
Contract No. AF 49(638)-446

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ABSTRACT

Investigations into the behavior of the gas flow behind spherical or cylindrical blasts have shown that secondary shocks arise within the original detonation gases. The secondary shock, at first weak, is carried outward with the expanding gases. Subsequently it strengthens and bends back toward the origin, arriving there with high intensity.

By using some recently developed techniques in shock dynamics a theory is developed by which the motion of the main shock wave, as well as the formation and subsequent motion of the secondary shock, are given by explicit formulae. A formulation for giving the location of the contact surface between the detonation gases and the outside atmosphere has also been obtained. The results of a specific problem, which has been solved by numerically integrating the total equations of motion, and also checked experimentally, are compared with the results of the present theory.

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A SIMPLIFIED DESCRIPTION OF SPHERICAL AND CYLINDRICAL BLAST WAVES

1. Introduction.

The purpose of the present analysis is to demonstrate the possibility of obtaining a satisfactory description of the initial phases of spherical and cylindrical blasts, without having to go through a numerical integration of the Eulerian or Lagrangian flow equations. To accomplish this, some recent developments in shock dynamics are used and extended where necessary. In addition, a technique has been devised by which it is possible to give the motion of a contact front in terms of flow properties carried to the front by the characteristics meeting it. It is hoped that some of these results may be applicable to the numerical integration of general flow equations, whose solutions contain moving boundaries such as shocks and contact fronts.

Recent theoretical and experimental studies of spherical and cylindrical blast waves [Ref.1,2,3,7,8] enable us to give the following qualitative description of these processes. Assume at time $t = 0$ a gas sphere (or cylinder) of radius $x = x_0$ under high internal pressure, $p = p_4$, to be in a still air atmosphere at pressure $p = p_0$, $p_4 \gg p_0$. The names "gas" and "air," respectively, will henceforth be used to describe the fluids initially within and outside of this sphere. For $t > 0$ an equalization (explosion) process takes place. At the initial point $t = 0$, $x = x_0$ the region $x \geq 0$, $t > 0$ can be

separated into five domains (Fig.1): (0), undisturbed air; (1), air which has been overtaken by the main blast wave; (2), gas under the influence of the air; (3), expanding gas uninfluenced by the outside air; (4), gas not yet disturbed by the expansion. The gas in region (2) and the air in region (1) are separated by a contact front. Also, a secondary shock develops and separates regions (3) and (2).

The interesting phenomenon of a secondary shock appears only in spherical (and cylindrical) flows and does not arise in one dimensional shock tube studies. For the latter the main shock and the expansion come into an instantaneous equilibrium, being separated by a region of uniform pressure and velocity. Physical reasoning given in the literature [e.g., Ref.1] for the secondary shock formation is that the high pressure gas upon passing through a spherical rarefaction wave must expand to lower pressures than those reached through an equivalent one-dimensional expansion. This is due to the increase of volume. Because of this "over-expansion", the pressures at the tail of the rarefaction wave are lower than the pressures transmitted back by the main shock and a compression, or second shock, must be inserted to connect the two phases. By more mathematical reasoning the situation is further clarified: the centered expansion is accomplished through the negative characteristics (slope $\frac{dx}{dt} = u - a$ where u is particle velocity and a is sound speed). The characteristics at the head of the wave point in the decreasing x direction (subsonic flow). They then fan around to the increasing x direction as the velocity increases. Negative characteristics are also

reflected from the main shock, but as this shock expands it becomes weaker causing these characteristics to incline more and more toward the decreasing x direction. Hence, characteristics of the same family, but arising from two different sources, will tend to intersect. This is prevented by the secondary shock. A similar situation arises [Ref.6] in one-dimensional flows when a shock moving down a uniform tube meets an area change. The shock strength is attenuated and characteristics reflected from the shock will be bent. They will meet the characteristics of the same family which originated in the uniform section of the tube unless a secondary shock is inserted.

This reasoning with characteristics leads directly to a method for explicitly determining the path of the secondary shock. Near the initial point, the intersecting characteristics will have nearly the same slope. It is easily shown [Ref.5,p.159] that the slope of a weak shock, at each point, is equal to the average of the slopes of the incoming characteristics at that point. Using this fact, Whitham [Ref.9] developed a technique for obtaining the differential equation for the shock path. This technique is used with the present problem to get the initial motion of the secondary shock.

As the secondary shock moves outward it strengthens, and the above "weak shock" theory is no longer applicable. In order to follow further shock development a relation obtained by Chisnell [Ref.4] and amplified by Whitham [Ref.10] is introduced. Chisnell considered a shock wave moving in a channel with varying cross section area and derived a closed form expression which

gives the relation between the channel area and shock strength. By a suitable choice of channel shape the analysis may be used to describe a spherical shock. Whitham considerably simplified and extended Chisnell's analysis. He showed that the "shock-area" relation can be obtained by application of the following rule: apply the differential relation which is to be satisfied by the flow quantities along a characteristic to the flow quantities just behind the shock. Using the shock relations which express these values in terms of the shock strength, another differential equation is obtained relating shock strength change and area change. For the present analysis the channel area is identified with the shock surface area and, hence, the shock radius. We obtain, finally, a differential relation between shock radius and strength. If the flow ahead of the shock is uniform, this relation can be integrated, giving Chisnell's result. The inward facing secondary shock, however, has ahead of it the non-uniform expanding gas. Hence, the shock motion depends not only on its radius but also on the flow conditions immediately ahead of it. Whitham's derivation of the "shock-area" relation is extended to include this situation.

The secondary shock is therefore obtained in two sections which take into account its changing nature from very weak, initially, to quite strong when it finally implodes at the origin. When it is weak, the average slope technique will be used to describe its motion until the point where it either starts bending back toward the origin, or where the shock strength becomes too large. From this point on, the extended "shock-area" rule is used.

The determination of the main shock wave separating regions (0) and (1) is much simpler than that of the secondary shock. This is because the flow ahead of the main shock in region (0) is uniform, and direct application of the "shock-area" relation, without extensions, suffices. As a consequence of the form of the "shock-area" relation the x and t coordinates of the main shock are determined parametrically as functions of shock Mach number. Similarly, since the contact front and secondary shock are determined by extrapolation of the known flow conditions at the main shock, their coordinates are also determined as parametric functions of the main shock Mach number. Therefore, starting at the initial point with the known main shock Mach number, small incremental steps in this Mach number are made. Corresponding to each increment, a point on the main shock, contact front, and secondary shock are determined.

Throughout the present analysis, the effects of entropy changes are neglected. This does not affect the calculations in the expansion region (3) since the expansion is isentropic. As the secondary shock starts, it is weak, and entropy changes in region (2) are small near the origin. In the application of the shock-area rule, it is seen, using Whitham's derivation, that the characteristic differential relation used does not contain entropy terms explicitly. Also, the flow quantities appearing there are evaluated immediately behind the shock by use of the Hugoniot shock relations. Hence, if the flow ahead of the shock is isentropic, entropy considerations will not affect the shock motion; this is the case for both the main and secondary shock.

The contact front, being a particle path, carries a constant entropy, or, as in the present problem, the change in entropy across the front is constant. It is seen, then, that the only place where neglect of entropy changes could affect the calculations is in the extrapolation of flow properties from the main shock through the small region (1) to the contact front.

In Section 2 an approximate solution to the flow in the expansion region (3) is determined. Since the flow near the initial point can be accurately described by one dimensional shock tube theory, the full solution is approximated by perturbing about the known one dimensional solution. Equations describing the main shock and negative characteristics in region (1) are derived in Section 3. In Section 4 the contact front and negative characteristics in region (2) are determined by the utilization of known fluid dynamical properties of contact discontinuities. The secondary shock equations are obtained in Section 5. Here, both the "weak-shock" theory and the extended "shock-area" rule are presented. In Section 6 a specific problem is treated, and results are compared with other solutions to this problem.

The present analysis will describe the motion of the secondary shock as it initially moves outward and until it finally implodes at the origin. Beyond that time, its motion cannot be given by this theory. Similarly, this time will roughly delineate the region of validity for the main shock and contact front approximations.

2. Expansion Region.

Immediately after detonation, the high pressure gas in region (4) of Figure 1 is penetrated by a centered expansion fan and moves rapidly outward. This expansion, region (3), is isentropic and satisfies the flow equations (1) and (2), [Ref.5], written in characteristic form

$$(1) \quad \left(\frac{1}{\gamma-1} a_t + \frac{1}{2} u_t\right) + (u+a) \left(\frac{1}{\gamma-1} a_x + \frac{1}{2} u_x\right) + \frac{nua}{2x} = 0$$

$$(2) \quad \left(\frac{1}{\gamma-1} a_t - \frac{1}{2} u_t\right) + (u-a) \left(\frac{1}{\gamma-1} a_x - \frac{1}{2} u_x\right) + \frac{nua}{2x} = 0$$

$n = 1$ for cylindrically symmetric flows,

$n = 2$ for spherically symmetric flows,

and γ is the ratio of specific heats.

The flow variables a and u , sound speed and particle velocity, as well as the independent variables x and t , radial distance and time, are all assumed to have been made dimensionless as follows:

$$a = \bar{a}/a_0, \quad u = \bar{u}/a_0, \quad x = \bar{x}/x_0, \quad t = \bar{t}a_0/x_0.$$

Barred quantities are true physical variables, a_0 is the sound speed outside the blast wave in region (0) of Figure 1 and x_0 is the initial blast radius.

An approximate solution to equations (1) and (2) is obtained by perturbing about a one dimensional centered expansion wave, whose solution can be given explicitly [Ref.5,p.104]. This

approximation holds exactly as the initial point $x = 1, t = 0$ is approached. To obtain this approximation we first make the following definitions:

$$(3) \quad \left\{ \begin{array}{l} u = u_1 + u_2 \\ a = a_1 + a_2 \\ r = \frac{1}{\gamma-1} a_1 + \frac{1}{2} u_1, \quad s = \frac{1}{\gamma-1} a_1 - \frac{1}{2} u_1 \\ R = \frac{1}{\gamma-1} a_2 + \frac{1}{2} u_2, \quad S = \frac{1}{\gamma-1} a_2 - \frac{1}{2} u_2. \end{array} \right.$$

The variables with subscript 1 are solutions to the one dimensional centered expansion problem (equations (1) and (2) with $n = 0$):

$$(4) \quad \left\{ \begin{array}{l} u_1 = 2\mu r + (1-\mu) \frac{x-1}{t} \\ a_1 = \mu(2r - \frac{x-1}{t}) ; \quad \mu = \frac{\gamma-1}{\gamma+1} \\ r = \text{constant}, \quad s = (1-2\mu)r - (1-\mu) \frac{x-1}{t}. \end{array} \right.$$

Substituting (3) and (4) into (1), retaining lowest order terms, we obtain an equation for R :

$$R_t + (u_1 + a_1)R_x + \frac{nu_1 a_1}{2x} = 0$$

or

$$(5) \quad \frac{dt}{t} = \frac{dx}{4\mu r t + (1-2\mu)(x-1)} = \frac{-2xdR}{ntu_1 a_1}$$

The first relation in (5) gives for the curvilinear positive

characteristics

$$(6) \quad \frac{x-1}{t} - 2r = -Kt^{-2\mu}, \quad \text{or} \quad a_1 t^{2\mu} = K\mu$$

where K is a constant for each characteristic.

Using (6) to eliminate x , we obtain the differential equation for R along the positive characteristics.

$$(7) \quad \frac{dR}{dt} = \frac{-n\mu K[2r - (1-\mu)Kt^{-2\mu}]t^{-2\mu}}{2[1+t(2r-Kt^{-2\mu})]}$$

It does not seem possible to integrate this equation for arbitrary μ . However, for certain specific values a closed form integration is possible. The integration for $\mu = \frac{1}{6}$ and $\frac{1}{4}$, corresponding to $\gamma = \frac{7}{5}$ and $\frac{5}{3}$, can be carried out, but with the present approximate theory the added complexity of the exact integral does not seem justified. The expression on the right hand side of (7) is therefore simplified by assuming the bracketed term in the denominator equal to 1. This implies we remain close to the line $x = 1$, which is roughly correct except for very high intensity blast waves. With this assumption we have

$$\frac{dR}{dt} = -\frac{n\mu K}{2} [2r - (1-\mu)Kt^{-2\mu}]t^{-2\mu}.$$

This is integrated to give

$$R = -\frac{nK\mu t}{2} \left[\frac{2r}{1-2\mu} t^{-2\mu} - \frac{(1-\mu)K}{1-4\mu} t^{-4\mu} \right] + f(a_1 t^{2\mu}).$$

Eliminating K by means of (6)

$$(8) \quad R = \frac{nt}{2} \left[\frac{(1-\mu)}{\mu(1-4\mu)} a_1^2 - \frac{2r}{1-2\mu} a_1 \right] + f(a_1 t^{2\mu}).$$

The arbitrary function $f(a_1 t^{2\mu})$, constant along each positive characteristic, and the constant r are determined by continuously connecting $\frac{1}{\gamma-1} a + \frac{1}{2} u$ across the boundary characteristic between regions (3) and (4). If (4) were a region of uniform flow with u_4 and a_4 constant, the boundary characteristic would be

$$(9) \quad x-1 = (u_4 - a_4)t.$$

For this case we would have

$$(10) \quad \begin{cases} r = \frac{1}{\gamma-1} a_4 + \frac{1}{2} u_4 \\ f(a_1 t^{2\mu}) = - \frac{tn}{2} \left(\frac{a_1}{a_4} \right)^{\frac{1}{2\mu}} \left[\frac{1-\mu}{\mu(1-4\mu)} a_4^2 - \frac{2r}{1-2\mu} a_4 \right]. \end{cases}$$

The second equation in (10) is obtained by requiring R to vanish at the boundary between regions (3) and (4). We shall assume for definiteness in later calculations (and simplicity) that region (4) is uniform and that equations (8) and (10) hold in region (3). It is clear that by properly choosing the arbitrary characteristic function $f(a_1 t^{2\mu})$ appearing in equation (8), a more general boundary condition could be satisfied. Combining (8) and (10) we have, for region (3)

$$(11) \quad \frac{1}{\gamma-1} a_3 + \frac{1}{2} u_3 = \frac{1}{\gamma-1} a_4 + \frac{1}{2} u_4 + tH(a_1).$$

with

$$H(a_1) = \left\{ \frac{n}{2} \frac{1-\mu}{\mu(1-4\mu)} \left[a_1^2 - \left(\frac{a_1}{a_4} \right)^{\frac{1}{2\mu}} a_4^2 \right] - \frac{2r}{1-2\mu} \left[a_1 - \left(\frac{a_1}{a_4} \right)^{\frac{1}{2\mu}} a_4 \right] \right\}$$

$$a_1 = \mu(2r - \frac{x-1}{t}).$$

Substituting the relations (3) and (4) into the negative characteristic equation (2), and retaining lowest order terms an equation for S is obtained:

$$(12) \quad tS_t + (x-1)S_x + [(2\mu-1)R+S] + \frac{nu_1 a_1 t}{2x} = 0,$$

where $R = tH(a_1)$, and H is defined in (11). Writing (12) in characteristic form,

$$(13) \quad \frac{dt}{t} = \frac{dx}{x-1} = \frac{dS}{-[(2\mu-1)R+S] - \frac{nu_1 a_1 t}{2x}}$$

The first of equations (13) give for the characteristics the lines $x-1 = Lt$, where L is constant along each characteristic. Along the characteristics, (13) also gives

$$(14) \quad \begin{aligned} \frac{d}{dt}(tS) &= (1-2\mu)R - \frac{nu_1 a_1 t}{2(1+Lt)} \\ &= (1-2\mu)tH(a_1) - \frac{nu_1 a_1 t}{2(1+Lt)} \end{aligned}$$

Since u_1 and a_1 are constant along the characteristic lines, (14) can be integrated immediately to give

$$S = (1-2\mu) \frac{t}{2} H(a_1) - \frac{nu_1 a_1}{2L} \left[1 - \frac{\log(1+Lt)}{Lt} \right]$$

Letting $L = (x-1)/t$ and using equation (4) we have

$$(15) \quad \frac{1}{\gamma-1} a_3 - \frac{1}{2} u_3 = (1-2\mu)r - (1-\mu) \frac{x-1}{t} + \frac{(1-2\mu)}{2} tH(a_1) - \frac{nu_1 a_1 t}{2(x-1)^2} [x-1 - \log x]$$

For determination of the secondary shock it is necessary to have the equation for the negative characteristics. The lowest order approximation for this equation is $x = 1 + Lt$. However, by using the above results, a better approximation can be obtained. To get this we first obtain $u - a$ by taking a proper linear combination of (11) and (15)

$$(16) \quad u_3 - a_3 = \frac{x-1}{t} + \frac{1-2\mu}{2-2\mu} tH(a_1) + \frac{nu_1 a_1 t}{2(1-\mu)(x-1)^2} [x-1 - \log x]$$

Along the negative characteristics we have

$$\frac{dx}{dt} = u - a.$$

To facilitate integration of the above, the right side of (16) is simplified by assuming $x = 1 + Lt$, and u_1 , a_1 , and L

are constant along the negative characteristics, giving

$$\frac{dx}{dt} = L + \frac{1-2\mu}{2-2\mu} tH(a_1) + \frac{nu_1 a_1}{2(1-\mu)L} \left[1 - \frac{\log(1+Lt)}{Lt} \right]$$

$$x \cong 1 + Lt + \frac{(1-2\mu)t^2}{4(1-\mu)} H(a_1) + \frac{nu_1 a_1}{2(1-\mu)L} \left[1 - \frac{\log(1+Lt)}{Lt} \right] \frac{t}{2}$$

or, letting $L = (x-1)/t$ in the last term

$$(17) \quad x = 1 + Lt + \frac{(1-2\mu)t^2}{4(1-\mu)} H(a_1) + \frac{nu_1 a_1 t^2}{4(1-\mu)(x-1)} \left[1 - \frac{\log x}{x-1} \right]$$

where $H(a_1)$ is given in equation (11). Since the logarithmic term in (16) does not vary much, an approximate integration was used to obtain (17).

Equations (11), (15), (16), and (17) will be used for describing the flow in region (3).

3. Main Shock and Region (1).

The characteristic equations (1) and (2), written with pressure and velocity as dependent variables, are

$$(18) \quad p_t + \rho a u_t + (u+a)(p_x + \rho a u_x) + \frac{\rho a^2 u n}{x} = 0$$

$$(19) \quad p_t - \rho a u_t + (u-a)(p_x - \rho a u_x) + \frac{\rho a^2 u n}{x} = 0,$$

with p pressure and ρ density.

These equations, when written in this form, hold whether or not the flow is isentropic.

Since the positive characteristics are the ones coming into the main shock, equation (18) is used for application of Whitham's rule (given in the introduction) for determination of the main shock motion. We write (18) in differential form

$$(20) \quad dp + \rho a du = - \frac{\rho a^2 u n}{u+a} \frac{dx}{x}.$$

At the shock the Hugoniot relations are

$$(21) \quad \begin{cases} u = \frac{2a_0}{\gamma+1} (M-M^{-1}); & p = \frac{p_0}{\gamma+1} (2\gamma M^2 - \gamma + 1) \\ \rho = \frac{\rho_0(\gamma+1)M^2}{(\gamma-1)M^2+2}; & a^2 = \frac{a_0^2(2\gamma M^2 - \gamma + 1)(M^2(\gamma-1)+2)}{(\gamma+1)^2 M^2} \end{cases}$$

where $M = U/a_0$, U is the shock velocity and subscript 0 refers to values ahead of the shock (region (0) Figure 1).

Substituting these relations into (20) we obtain a differential relation between shock Mach number and its radius, x_m :

$$(22) \quad -n \frac{dx_m}{x_m} = dM \left\{ \frac{4M}{2\gamma M^2 - \gamma + 1} + \frac{2(M^2+1)}{M \sqrt{(2\gamma M^2 - \gamma + 1)(M^2(\gamma-1)+2)}} + \frac{2M}{M^2-1} \sqrt{\frac{(\gamma-1)M^2+2}{2\gamma M^2 - \gamma + 1}} + \frac{M^2+1}{M(M^2-1)} \right\}.$$

This can be integrated to give

$$(23) \quad (x_m)^n (2\gamma M^2 - \gamma + 1)^{1/\gamma} \cdot \left(\sqrt{(\gamma-1)(2\gamma M^2 - \gamma + 1)} + \sqrt{2\gamma(M^2(\gamma-1) + 2)} \right)^{\sqrt{\frac{2\gamma}{\gamma-1}}} \cdot \frac{(\sqrt{2\gamma M^2 - \gamma + 1} - \sqrt{(\gamma-1)M^2 + 2})^2}{M} \cdot \exp\left[\frac{1}{\sqrt{2(\gamma-1)}} \arcsin\left\{\frac{2(2\gamma M^2 - \gamma + 1) - (\gamma-1)(M^2(\gamma-1) + 2)}{M^2(\gamma+1)^2}\right\}\right] = \text{constant}.$$

Using the dimensionless coordinates introduced in (3) we have the relation between t_m and M , along the shock,

$$(24) \quad \frac{dt_m}{dM} = \frac{dt_m}{dx_m} \frac{dx_m}{dM} = \frac{1}{M} \frac{dx_m}{dM}.$$

Where $\frac{dx_m}{dM}$ is defined in equation (22).

The motion of the main shock as a function of its Mach number is given by equations (23) and (24) in conjunction with an initial condition specifying its position and Mach number at a certain time. Ordinarily, the conditions are prescribed at the instant of detonation. If, for example, we were considering the bursting of a gas bubble at high pressure, the initial shock Mach number could be determined by use of one dimensional shock tube theory. Specifically, if it were assumed that the gas inside the bubble were steady with pressure and sound speed p_4 and a_4 , and that the outside atmosphere were steady at p_0 and a_0 , the one dimensional theory would give the following relation which the

initial shock Mach number must satisfy.

$$(25) \quad \frac{p_4}{p_0} \left\{ 1 - \frac{\mu a_0}{a_4} (M - M^{-1}) \right\}^{\frac{\mu+1}{\mu}} = (1+\mu)M^2 - \mu.$$

This result is derived in Appendix A.

It will be seen that the only flow property required to be known in region (1) is the position of the negative characteristics reflected from the main shock. These characteristics are defined by the relation

$$\frac{dx}{dt} = u - a.$$

Since the contact front follows very closely behind the main shock, region (1) is very small and it suffices to approximate the negative characteristics there by straight lines. Effects of the changing shock strength, due to expansion, are accounted for by adjusting the slope of each characteristic to be that value of $u - a$ where the characteristic meets the shock. We have, therefore,

$$(26) \quad x = x_m + w_1(M)(t - t_m)$$

with $w_1(M) = u - a$ evaluated at the point x_m, t_m on the main shock.

Equations (23), (24) and (26) define the main shock and negative characteristics behind the shock as functions of the

parameter M , which is the shock Mach number.

4. Contact Front.

The contact front separates the gas in region (2) from the air in region (1). The information about the front required for the present analysis is the location of the front, and the change in slope of the negative characteristic upon crossing the front. It will be seen in the next section that it is necessary for the determination of the secondary shock to know the slope of the negative characteristics in region (2). However, as will be shown below, in order to determine this slope one must know not only the characteristic's slope in region (1) but also the location of the point where the characteristic crosses the contact front.

In general, pressure and normal velocity are continuous, and density and temperature may undergo jump discontinuities across contact fronts. Since the present analysis is concerned with flows which are purely spherical or cylindrical; i.e., with one space dimension, contact fronts are always normal to the flow direction, and velocity is continuous across the front.

As the contact front moves with the local particle velocity, u , it will be met by positive characteristics, slope $u + a$, from region (2) and negative characteristics, slope $u - a$, from region (1). In order to determine the positive characteristic values we shall assume equation (11), which gives $\frac{1}{\gamma-1} a + \frac{1}{2} u$ in region (3), to be applicable in region (2); this due to the fact that the secondary shock, separating regions (2) and (3), is initially very weak. Letting Q_2 , Q_1 and w_2 , w_1 denote

$\frac{1}{\gamma-1} a + \frac{1}{2} u$ and $u - a$ in regions (2) and (1) respectively.

We have, due to continuous velocity at the contact front

$$(27) \quad \left\{ \begin{array}{l} Q_2 + \frac{1}{\gamma-1} w_2 = Q_1 + \frac{1}{\gamma-1} w_1 \\ \text{or} \\ w_2 - w_1 = -(\gamma-1)(Q_2 - Q_1) . \end{array} \right.$$

For isentropic flows we have the relation between pressure and sound speed [Ref.5,p.95],

$$\frac{p}{p_1} = \left(\frac{a}{a_1} \right)^{\frac{2\gamma}{\gamma-1}}$$

where subscript 1 stands for some initial state. Consequently, due to continuous pressure at the contact front

$$(28) \quad \left\{ \begin{array}{l} \left(\frac{a}{a_1} \right)_2 = \left(\frac{a}{a_1} \right)_1 \\ \text{or} \\ 2Q_2 - w_2 = \frac{a_{12}}{a_{11}}(2Q_1 - w_1) . \end{array} \right.$$

(It should be noted that in the derivation of equations (27) and (28) the assumption has been made that γ , the ratio of specific heats, has the same magnitude in both regions (1) and (2). The extension to different γ 's is direct but not too simple.) We

obtain the slope of the negative characteristics in region (2) in terms of known quantities by eliminating Q_1 from equations (27) and (28).

$$(29) \quad w_2 = \frac{\frac{\gamma+1}{\gamma-1} \frac{a_{12}}{a_{11}} w_1 + 2(1 - \frac{a_{12}}{a_{11}}) Q_2}{1 + \frac{2}{\gamma-1} \frac{a_{12}}{a_{11}}}$$

Evaluation of (29) is not direct, since w_1 is known only as a function of main shock Mach number (eqn.(26)) and Q_2 is known only as a point function of x and t (eqn.(11)). We must determine, therefore, the (x,t) coordinates of the contact front in terms of the parameter M .

First we express a_1 in terms of w_1 and Q_2 : since velocity is continuous across the contact front

$$2Q_2 - w_1 = \frac{2}{\gamma-1} a_2 + a_1,$$

combining this with the first of equations (28)

$$(30) \quad a_1 = \frac{2Q_2 - w_1}{1 + \frac{2}{\gamma-1} \frac{a_{21}}{a_{11}}}.$$

Next, let $x_c = c(t_c)$ be the equation of the contact front. The negative characteristic given in equation (26) meets the contact front at the point (x_c, t_c) . Hence, at this point

$$c(t_c) = x_m + w_1(M)(t_c - t_m).$$

Differentiating with respect to t_c we have

$$\frac{dc}{dt_c} = w_1 + \frac{dM}{dt_c} \left\{ x'_m + w'_1(t_c - t_m) - w_1 t'_m \right\}.$$

Primes indicate differentiation with respect to M . As the contact front moves with local particle velocity,

$$\frac{dc}{dt_c} = \frac{dx_c}{dt_c} = u.$$

Combining the two above equations, using $w = u - a$

$$(31) \quad a_1 = \frac{dM}{dt_c} \left\{ x'_m + w'_1(t_c - t_m) - w_1 t'_m \right\}.$$

Eliminating a_1 from (30) and (31) we obtain a differential equation relating t_c and M

$$(32) \quad \frac{dt_c}{dM} = \frac{\left\{ 1 + \frac{2}{\gamma-1} \frac{a_{21}}{a_{11}} \right\} \left\{ x'_m + w'_1(t_c - t_m) - w_1 t'_m \right\}}{2Q_2 - w_1}.$$

Equation (32) together with

$$(33) \quad x_c = x_m + w_1(M)(t_c - t_m)$$

define the coordinates of the contact front (x_c, t_c) as functions of the parameter M . This result enables us to evaluate the change in slope of the negative characteristic upon crossing the contact front as given in (29). In fact, we have directly the negative characteristics in region (2)

$$(34) \quad x = x_c + (t - t_c)w_2 ,$$

where t_c and x_c are given by (32) and (33), and w_2 is given by (29).

5. Secondary Shock.

We now have enough information about the flow to fit the secondary shock between regions (3) and (2). It is seen that those negative characteristics arising in the expansion region (3) tend to fan into the increasing x direction. Those which arise at the main shock will tend to point more and more toward the decreasing x direction. This is due to the fact that the shock weakens as it expands, as can be inferred from equations (22) or (23). Thus, as the shock weakens the quantity $u_1 - a_1$, which is the characteristic slope, decreases.

The secondary shock must be inserted so that the shock relations are satisfied. For a weak shock the shock conditions show that its slope at each point in the (x, t) plane is the average of the slopes of the characteristics meeting it. Since we have the equations, (17) and (34), describing the characteristics which meet the shock, we can, using a technique developed by

Whitham [Ref.9], obtain the shock motion.

The characteristics which arise in the expansion fan and at the main shock are respectively

$$(35) \quad \begin{cases} x = 1 + Lt + f(x,t) \\ x = x_c + w_2(t-t_c) \end{cases}$$

where $f(x,t)$ represents terms appearing in equation (17), and x_c , t_c , and w_2 are functions of main shock Mach number, M , as given in equations (33), (32) and (29). Assuming the secondary shock path to be given by

$$x_s = 1 + S(t_s)$$

we have along the shock path

$$(36) \quad \begin{aligned} S(t_s) &= w_2(t_s - t_c) + x_c - 1 \\ &= Lt_s + f(x_c + (t_s - t_c)w_2, t_s) \end{aligned}$$

Expressing the shock slope as the average of the characteristic slopes as obtained from (35),

$$\frac{dS}{dt_s} = \frac{1}{2} \left(w_2 + \frac{L + f_t}{1 - f_x} \right)$$

Letting the parameters M and L and the variable x_s all be

dependent on t_s , we can obtain another representation for

$\frac{dS}{dt_s}$, using (35) and (36); letting primes represent differentiation with respect to M :

$$\frac{dS}{dt_s} = \frac{1}{2} (w_2 + \frac{dM}{dt_s} [x'_c + w'_2(t_s - t_c) - w_2 t'_c] + \frac{L + f_t}{1 - f_x} + \frac{t \frac{dL}{dt}}{1 - f_x})$$

This equation, when combined with the previous expression, yields

$$(37) \quad t \frac{dL}{dM} = - (1 - f_x) (x'_c + w'_2(t_s - t_c) - w_2 t'_c).$$

Eliminating L from (37) by use of (36) we obtain, after some algebra, a relation between t_s and M :

$$(38) \quad \frac{d}{dM} \left(\frac{[x_c - 1 - t_c w_2]^2}{t_s} \right) = [x_c - 1 - t_c w_2] \cdot [-2w'_2 + \frac{d}{dM} \left(\frac{f}{t_s} \right) + \frac{f_x}{t_s} (x'_c + w'_2(t_s - t_c) - w_2 t'_c)].$$

Here, f and f_x are evaluated at $x = x_c + (t_s - t_c)w_2$, $t = t_s$.

Integrating both sides of (38) with respect to M we have finally

$$(39) \quad [x_c - 1 - t_c w_2]^2 = t_s \int_{M_1}^M [x_c - 1 - t_c w_2] \cdot [-2w'_2 + \frac{d}{dM} \left(\frac{f}{t_s} \right) + \frac{f_x}{t_s} (x'_c + w'_2(t_s - t_c) - w_2 t'_c)] dM$$

where M_1 is the initial Mach number of the main shock wave. Equation (39) is rather messy and must be solved by an iterative numerical technique. Since for a fixed M the right hand side of (39) increases monotonically with t_s and the left hand side is a function of M alone, it is possible, for each increment in M , to solve for the corresponding t_s by a simple "trial and error" scheme. Once t_s is determined, x_s is given by

$$(40) \quad x_s = x_c + w_2(t_s - t_c).$$

The technique described above, while useful to describe the initial motion of the secondary shock, is valid only when this shock is still weak. In the present situation, however, this shock strengthens as it is carried outward by the expanding gases and another method must be used to describe its subsequent motion. (The change-over point for a specific problem was arbitrarily chosen to be either when the shock begins to turn back toward the origin or where the shock strength becomes 1 in magnitude, whichever comes first. Shock strength is defined in this case as $M_s - 1$, where M_s is the shock Mach number.)

For moderate to strong shocks, the technique derived in Section 3 for describing the main blast wave is applicable. It must be altered in the present case, since the flow coming into the secondary shock is non-uniform and terms describing this incoming flow variation along the shock must be included. To do this we start with the characteristic differential relation

corresponding to (20), but this time, since the negative characteristics are the ones coming into the shock, we use

$$(41) \quad dp - \rho a du = - \frac{\rho a^2 u n}{u-a} \frac{dx}{x}.$$

As the expansion region (3) is one of isentropic flow,

$$\left(\frac{p_3}{p_1}\right) = \left(\frac{a_3}{a_1}\right)^{\frac{2\gamma}{\gamma-1}} = \left(\frac{\rho_3}{\rho_1}\right)^\gamma.$$

The Hugoniot relations given in (21) are used with the variables a_0, p_0, ρ_0 replaced by a_3, p_3, ρ_3 . Also the velocity relation is altered to account for the fact that the shock is backward facing and the flow in region (3) is non-stationary, giving

$$u = u_3 - \frac{2a_3}{\gamma+1} (M-M^{-1})$$

where $M = (u_3 - U_s)/a_3$, and U_s is the secondary shock velocity. Substituting these relations into (41) we obtain equations which hold along the secondary shock

$$\begin{aligned}
 (42) \quad \left\{ \frac{dM}{dx} = \left\{ \frac{\gamma H}{2} \frac{M}{R} \left(\frac{u_3 t / U_s + u_3 x}{a_3} \right) - \left(\frac{M^2 - 1}{R} + \frac{1}{\gamma - 1} \right) \left(\frac{a_3 t / U_s + a_3 x}{a_3} \right) \right. \right. \\
 \left. \left. + \left(\frac{2a_3(M^2 - 1) - (\gamma + 1)Mu_3}{(\gamma + 1)Mu_3 - 2a_3(M^2 - 1) - R} \right) \frac{n}{2x} \right\} \div \left\{ \frac{2M}{2\gamma M - \gamma + 1} + \frac{M^2 + 1}{MR} \right\} \right. \\
 \left. \frac{dt}{dx} = U_s^{-1} \right. \\
 \left. U_s = u_3 - Ma_3 \right\}
 \end{aligned}$$

with $R = \sqrt{(2\gamma M^2 - \gamma + 1)(M^2(\gamma - 1) + 2)}$.

The quantities u_3 , C_3 , and their derivatives are obtained from equations (11) and (16). Equations (42) are solved stepwise from a given point where x , t , and U_s or M are prescribed.

6. Application to a Specific Problem.

We shall describe here the application of the present theory to a specific problem which has been solved theoretically by Brode [Ref.2], and carried out experimentally by Boyer [Ref.1]. A sphere of radius 1, containing a gas at 22 atm. and at a temperature of 299°K, is assumed to be surrounded by air at 1 atm. and the same temperature. The compressed gas is also air, and the ratio of specific heats, γ , is assumed to equal 1.4 everywhere. The initial Mach number of the main shock, as determined by one

dimensional shock tube theory (Appendix A), is 1.846. With this, the constant on the right hand side of the main shock equation (23) is 26.1. For this case the sound speed and particle velocity, made dimensionless with respect to the flow in region (0), are as follows, at the initial point $x = 1, t = 0$:

$$\begin{array}{ll} u_0 = 0 & a_0 = 1 \\ u_1 = 1.087 & a_1 = 1.252 \\ u_2 = 1.087 & a_2 = .729 \\ u_3 = \frac{5}{6}(1 + \frac{x-1}{t}) & a_3 = \frac{1}{6}(5 - \frac{x-1}{t}) \\ u_4 = 0 & a_4 = 1. \end{array}$$

The leading characteristic of the expansion fan in region (3) is $(x-1)/t = -1$.

For region (3) we have, corresponding to equations (11), (16) and (17)

$$(6.11) \quad 2.5a_3 + .5u_3 = 2.5 - \frac{t}{28.8}(5 - \frac{x-1}{t})(1 + \frac{x-1}{t})^2$$

$$(6.16) \quad u_3 - a_3 = \frac{x-1}{t} - \frac{(5 - \frac{x-1}{t})(1 + \frac{x-1}{t})t}{72}[(1 + \frac{x-1}{t}) - \frac{12}{x-1}(1 - \frac{\log x}{x-1})]$$

$$(6.17) \quad x = 1 + Lt - \frac{(5 - \frac{x-1}{t})(1 + \frac{x-1}{t})t^2}{144}[(1 + \frac{x-1}{t}) - \frac{12}{x-1}(1 - \frac{\log x}{x-1})].$$

The contact front is

$$(6.32) \quad \frac{dt_c}{dM} = \frac{4.127(x'_m + w'_1(t_c - t_m) - w_1 t'_m)}{5a_3 + u_3 - w_1}$$

$$(6.33) \quad x_c = x_m + w_1(M)(t_c - t_m),$$

$w_1(M)$ is the negative characteristic slope, $u - a$, in region (1), it is evaluated, as a function of the main shock Mach number M , by using the Hugoniot relations in equation (21). In region (2) the negative characteristic slope, w_2 , is

$$(6.29) \quad w_2 = .909w_1 + .182(2.5a_3 + .5u_3)$$

In Figure 2 the present theory is compared with that of Boyer and Brode. The results of several experimental explosions carried out by Boyer are indicated by dashed lines. Brode's work, numerically integrating the Lagrangian equations of motion, is indicated by solid lines. The present theory is given by the curves composed of heavy dots. A simplified determination of the secondary shock is indicated by x 's. For this the one-dimensional solution was used in region (3). This is given by omitting the last term in equations (6.11), (6.16) and (6.17), and also setting f and f_x equal to zero in the secondary shock equation (39). The time to compute the two shocks and the contact front on an IBM 704 was about 3 minutes.

Since Boyer's experimental work involved the bursting of a glass sphere the differences between his result and Brode's

may be due to the presence of glass particles in the flow field. As the present model, a hypothetical sudden expansion of a gas sphere, is the same as Brode's it is expected that the results be comparable with his. The agreement as to the initial motion of the main shock and contact front is quite good. Brode shows the secondary shock expanding slightly more and imploding sooner than both the present theory and Boyer's experiments. The differences in the initial motion could be due to the fact that the shock is initially weak. Brode, using the von Neumann-Richtmyer pseudo viscosity technique, which spreads out the shock, might have difficulty in determining the true location of a weak shock. Boyer's experimental technique which used the schlieren shadowgraph to determine the secondary shock, has difficulty in locating weak shocks which give correspondingly weak shadows. As to the determination of the time of implosion Brode's use of Lagrangian mass coordinates may lead to inaccuracies near the flow center as the density there is quite low. Most of the mass is located near the contact front and main shock. However, since the present theory becomes less valid as the distance to the initial point, $x = 1, t = 0$, increases; it is rather difficult to precisely determine the causes of the small discrepancies in the different results.

The approximation using the one-dimensional flow solution to determine the secondary shock agrees remarkably well with Boyer's experimental work. This approximation may be useful as it gives a qualitative description of the flow and affords a marked simplification of some of the equations. Overall, however, the agreement between the different theories is quite good.

Appendix A. One Dimensional Shock Tube Theory.

Consider a tube separated into two regions (Figure 3), each region containing a uniform stationary gas but at different pressures. The separation point is assumed to be at $x = 0$, of a coordinate along the tube, and the high pressure region is in $x < 0$. At time $t = 0$ the separation is removed and a shock moves into the uniform region (0). Regions (1) and (2), separated by a contact front, are both uniform and have the same pressure and particle velocity. Region (3) is a centered expansion separating the uniform regions (4) and (2). Knowing the flow properties in regions (0) and (4) it is desired to determine the shock strength and the slope of the last characteristic of region (3); i.e., the characteristic separating regions (2) and (3).

In the expansion region the solution is [Ref.5,p.104]

$$(A.1) \quad \left\{ \begin{array}{l} a_3 = \mu \left(\frac{1-\mu}{\mu} a_4 - \frac{x}{t} \right) \quad ; \quad \frac{p_3}{p_4} = \left(\frac{a_3}{a_4} \right)^{\frac{1+\mu}{\mu}} \\ u_3 = (1-\mu) \left(a_4 + \frac{x}{t} \right) \quad ; \quad \frac{\rho_3}{\rho_4} = \left(\frac{a_3}{a_4} \right)^{\frac{1-\mu}{\mu}} \end{array} \right.$$

where $\mu = \frac{\gamma-1}{\gamma+1}$. The Hugoniot shock relations giving u_1 and p_1 are

$$(A.2) \quad u_1 = (1-\mu)a_0(M-M^{-1}), \quad p_1 = p_0((1+\mu)M^2-\mu).$$

Since the velocity and pressure are continuous across the contact

front, u_2 and p_2 are also given by (A.2). Letting S denote the slope of the last characteristic of the expansion region (3), we have at this characteristic due to continuous velocity and pressure

$$(A.3) \quad \begin{cases} a_4 + S = a_0(M-M^{-1}) \\ p_4 \left(\frac{\mu}{a_4} \left(\frac{1-\mu}{\mu} a_4 - S \right) \right)^{\frac{1+\mu}{\mu}} = p_0((1+\mu)M^{2-\mu}) \end{cases}$$

Eliminating S in (A.3) an equation for M is obtained

$$(A.4) \quad \frac{p_4}{p_0} \left(1 - \frac{\mu a_0}{a_4} (M-M^{-1}) \right)^{\frac{1+\mu}{\mu}} = (1+\mu)M^{2-\mu}.$$

This gives the required shock Mach number in terms of the initial conditions.

BIBLIOGRAPHY

- [1] Boyer, D. W., Spherical Explosions and Implosions, UTIA Report No. 58, Institute of Aerophysics, University of Toronto, Nov. 1959.
- [2] Brode, H. L., Theoretical Solutions of Spherical Shock Tube Blasts, RM-1974, The Rand Corporation, Sept. 1957.
- [3] Brode, H. L., Blast Wave from a Spherical Charge, The Phys. of Fluids, 2(1959), p. 217.
- [4] Chisnell, R. F., The Motion of a Shock Wave in a Channel with Applications to Cylindrical and Spherical Shock Waves, J. Fluid Mech., 2(1957), p. 286.
- [5] Courant, R., and Friedrichs, K. O., Supersonic Flow and Shock Waves, Interscience, New York, (1948).
- [6] Friedman, M. P., An Improved Perturbation Theory for Shock Waves Propagating Through Non-Uniform Regions, J. Fluid Mech., 8(1960), p. 193.
- [7] Shardin, H., Measurement of Spherical Shock Waves, Comm. Pure and Appl. Math., 7(1954), p. 223.
- [8] Wecken, F., Expansion einer Gaskugel hohen Druckes, Z. Agnew. Math. Mech., Aug./Sept. 1950, p. 271.
- [9] Whitham, G. B., The Flow Pattern of a Supersonic Projectile, Comm. Pure and Appl. Math., 5(1952), p. 301.
- [10] Whitham, G. B., On the Propagation of Shock Waves Through Regions of Non-Uniform Area or Flow, J. Fluid Mech., 4(1958), p. 337.

Figure 1. Explosion Flow Diagram

Figure 2. Experimental and Theoretical Spherical Blast Results

Figure 3. Shock Tube Flow Diagram