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MASTER

ZERO-MASS STATES IN NON-FREDHOLM

BETHE-SALPETER $q\bar{q}$ MODELS*

by

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ABSTRACT

We consider a $q\bar{q}$ interaction given by a (non-Fredholm) Bethe-Salpeter equation whose kernel is a sum of S, V, T, A and P single boson exchange. With the heavy-quark model of mesons in mind, we determine when massless bound states of given tensor character (S, V, T, A or P) will occur, and what exchanged-particle coupling constants will produce them. The technique used is gradually to increase the magnitude of the coupling constants from zero, and determine when the lowest bound state mass descends to zero. This point is detected by the development of a singularity in the previously regular $s=0$ scattering amplitude. The singular, non-Fredholm property of the Bethe-Salpeter equation allows considerable simplifications which make the calculation possible.

I. INTRODUCTION

It is desirable to make a relativistic quark model of mesons. Attempts to describe mesons as $q\bar{q}$ bound states by solving a Bethe-Salpeter (BS) equation have been frustrated by the fact that, among other difficulties, the BS equations for spin- $\frac{1}{2}$ particles do not obey the Fredholm conditions.¹ (A typical example of such an equation is the QED equation for electron-positron scattering in the ladder approximation.²)

Current ways³ of looking at the strong interactions imagine there to be a great multiplicity of mesons in nature, each of which is, in the simplest point of view, a quark-antiquark bound state. Quarks, when formulated as entities, presumably have very large mass compared with the masses of the mesons. If any $q\bar{q}$ Bethe-Salpeter equation⁴ contains low-mass bound states in the multiplicity envisioned nowadays for mesons, a little thought^{4a} will persuade the reader that the equation is very unlikely to be Fredholm. Fredholm models which have been solved have each yielded only a small number of low-mass bound states.⁵ Thus there would be great interest in working out the bound-state masses of any non-Fredholm model, however simple. Unfortunately, this seems unlikely to be done soon. In fact -- surprisingly -- nothing about

the bound states of any such models is known so far, except a simple inequality for the bound state masses in ladder approximation models.⁶ For any bound state mass M_B , the inequality is $M_B > (2-g/\pi)m$, where m is the q mass and g is a constant proportional to the coupling constant(s) of the exchanged particles. All that this inequality does is to suggest that some bound state masses may descend near to zero as the coupling constants of the model increase to values of order unity.

In this article we will carry the last result slightly farther (while eagerly looking forward to the time when somebody will make all these results obsolete, by actually finding the bound state masses of a non-Fredholm model). The heavier that quarks are assumed to be, the lighter the known mesons are by comparison. In this article we will consider, as a convenient limit of successively smaller bound state masses M_B (analogous to fixed meson mass and greater and greater quark mass), a bound state mass of zero.^{6a} In a class of simple ladder-approximation models, we will determine what values the coupling constants must be increased to so that the lowest bound state's mass descends to zero. We will also determine the tensor character of the resulting massless bound states.

We will consider the $q\bar{q}$ equation in the ladder approximation where the kernel is given by an arbitrary sum of single particle S , V , T , A and P exchange, whose

details are below. (The QED equation is V exchange alone.) We will consider the interaction as being gradually made stronger and stronger by increasing the exchanged particle coupling constants, starting from zero. As the attractive coupling constants increase, any bound states will become deeper and deeper, and eventually the lowest bound state's mass will decrease to zero. We will determine the answers to the following two questions. (i) What values of the exchanged-particle coupling constants will produce zero-mass bound states? (ii) What tensor character (S, V, T, A, P) have the bound states themselves in this case?

As discussed in Ref. 6, it is necessary to detect bound-state solutions of a non-Fredholm equation by looking for poles in s in the scattering Green's function solution $T(s)$. This is because when the Fredholm conditions are not obeyed, the existence of a solution to the homogeneous equation at some value of s does not mean that there is necessarily a bound state there. The derivation in this paper starts by considering the $q\bar{q}$ scattering Green's function $T(s)$ with s set equal to zero. This function, $T(0)$, is known⁶ to exist until the interaction coupling constants reach a certain value. When the coupling constants reach a value such that $T(0)$ develops a singularity, this is interpreted to mean that the mass of the lowest bound state has descended to zero.⁷ We will determine for what coupling strength of the $S, V, \dots$ exchanged particles to the

fermions the singularity in $T(0)$ appears, and the nature (S,V,...) of the singularity.

The results are given in a form from which the consequences of any combination of S,V,T, A and P exchange can be read off (Section V). Some of the results are known already. Typical results are these.

- (a) For pure S exchange, with coupling constant f_S , the massless bound state is pure P and occurs when the coupling constant reaches the value given by $f_S^2/4\pi = \pi$.
- (b) For pure P exchange, the massless bound state is pure S, and occurs when the coupling constant f_P reaches the value given by $f_P^2/4\pi = \pi$.⁸
- (c) For pure V exchange, as in the QED case, massless S and P bound states occur together, at $e^2/4\pi = \pi/4$.⁹
- (d) For V and A exchange with equal coupling constant f , massless V and A bound states occur together at a value $f^2/4\pi = (\sqrt{13} - 2)\pi/4 \approx 1.26$. The possible physical interest of this case is obvious.
- (e) There cannot be any tensor (T) massless bound states.

These coupling constants are independent of the q mass, so that no matter how heavy the fermions are, zero- or almost zero-mass bound states can be achieved with very moderate, dimensionless coupling constants. In cut-off models, on the other hand, the needed coupling constants are very dependent on the cutoff.¹⁰ In non-covariant, potential-theory quark models they are sometimes ridiculously large.

The rest of this article is arranged as follows. In Section II the BS equations considered are introduced, and are decomposed into S,V,T,A and P components in a manner to make calculation easy. For technical reasons the exchanged bosons are taken to be massless, but it is thought that the results are independent of this assumption. The crucial observation which makes the derivation of the present article's results feasible is presented here: the scale invariance in mass of the equation allows the singularities to be derived from the asymptotic behavior alone, independently of the quark mass. (This great simplification is no longer possible when a cutoff has been introduced.) In Section III a simple case which has already appeared in the literature -- the pseudoscalar (P) component of the $s=0$ scattering Green's function -- is gone through in detail in order to describe the method used for the other (S,V,T,A), more complicated components. These components are reported on, with the calculations' details omitted, in Section IV. In Section V the results are collected and summarised, with a table, so it is possible to read off what combinations of exchanged particles, with what coupling constants, will give a certain type of zero-mass bound state. The reader may pass from Section II directly to Section V if he wishes. The conclusions are given in Section VI.

II. BETHE-SALPETER EQUATION

The interaction of the quark and antiquark is considered to be described by a BS equation with a kernel given by a sum of S, V, T, A and P massless single particle exchange.

The $q\bar{q}$ scattering Green's function will be denoted by $T_{\alpha\beta;\gamma\sigma}(p, q; K)$ according to Fig. 1. It will obey a BS equation

$$T_{\alpha\beta;\gamma\sigma}(p, q; K) = I_{\alpha\beta;\gamma\sigma}(p, q; K) + i \int \frac{d^4 k}{(2\pi)^4} \times I_{\alpha\delta;\gamma\theta}(p, k; K) G_{\delta\epsilon}(k + \frac{1}{2}K) G_{\lambda\theta}(k - \frac{1}{2}K) T_{\epsilon\beta;\lambda\sigma}(k, q; K) \quad (2-1)$$

Here K denotes the arbitrary, constant, complex total four-momentum; $s = -K^2$.

In Eq. (2-1) Wick rotation has been performed on all the relative energy variables: $p^0 = ip_4$, etc. The metric is given by $a \cdot b = \underline{a} \cdot \underline{b} - a^0 b^0 = \underline{a} \cdot \underline{b} + a_4 b_4$. $d^4 k$ means $d\underline{k} dk_4$. The propagators $G(p)$ will be taken to be $G(p) = 1/(\gamma \cdot p + m)$, where m is the q mass. Using $A \otimes B$ as an abbreviation for $A_{\alpha\beta} B_{\sigma\gamma}$, the kernel will be taken to be this sum of single particle exchanges:

$$\begin{aligned}
I(p, q; K) = & \frac{1}{i} \frac{1}{(p-q)^2} \left\{ f_s^2 1 \otimes 1 + f_v^2 \gamma^l \otimes \gamma^l \right. \\
& \left. + f_T^2 \sigma^{lm} \otimes \sigma^{lm} + f_A^2 \gamma^l \gamma_5 \otimes \gamma^l \gamma_5 + f_P^2 i \gamma_5 \otimes i \gamma_5 \right\}
\end{aligned}
\tag{2-2}$$

(It will be explained below why the exchanged particles are chosen to be massless and why it is not expected to affect the results.) The Dirac matrices have the following properties. $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$; $\gamma_4 = -i\gamma^0$;

Latin indices run from 1 to 4; $\sigma^{lm} = (i/2)[\gamma^l, \gamma^m]$; $\gamma^{l\dagger} = -\gamma^l$; $\gamma_5^2 = 1$.

Eq. (2-1) has 16 independent components corresponding to the 16 pairs of free indices (β, σ) . Using a natural tensor basis instead, eq. (2-1) will be put in a form easier to calculate with in this article. Choose a basis of 16 linearly independent Dirac matrices X^j as shown in Table 1.

TABLE I

<u>X</u>	<u>X^j</u>	<u>j_X</u>
S	1	1
V	$i\gamma^l$	4
T	σ^{lm}	6
A	$\gamma^l \gamma_5$	4
P	γ_5	1

Then the sets of components are given in terms of each other using the relations

$$\frac{1}{4} k_{\mu} (X^i X^j) = \delta_{ij}, \quad \delta_{\alpha\beta} \delta_{\rho\sigma} = \frac{1}{4} \sum_{X=S}^P \sum_{j=1}^J X^j_{\alpha\beta} X^j_{\rho\sigma}, \quad (2-3)$$

$$T_{\alpha\beta;\rho\sigma} = \sum_{X=S}^P \sum_{j=1}^J T_{(X^j)_{\alpha\beta}} X^j_{\rho\sigma}, \quad T_{(X^j)_{\alpha\beta}} = \frac{1}{4} T_{\alpha\beta;\rho\sigma} X^j_{\rho\sigma} \quad (2-4)$$

That is, the Green's function $T_{\alpha\beta;\rho\sigma}$ may be written according to (2-4) as a sum over basis matrices X^j , with components $T_{(X^j)}$. The BS equations (2-1) can now be written as 16 other independent equations, one for each of the 16 components $T_{(X^j)}(p,q;K)$. These equations will be given below, after further simplifications.

As Ref. 6 has shown, the solution T to (2-1) is the solution by successive iterations, and it is analytic in s in a domain of the s -plane which shrinks in size around the point $s = 0$ as the attractive interaction coupling constants increase. Here, as explained in Section I, we wish to find out at what values of the coupling constants f_Y ($Y = S, V, T, A, P$) the solution T ceases to exist (begins to be singular) even at $s = 0$. Therefore we must find for what values of f_Y the solution by successive iterations to at least one of the equations for the components $T_{(X^j)}$ of the scattering Green's function just begins to diverge (or to be singular) at $s = 0$, while all the other components remain convergent. This is what is calculated in the present paper.

We will put $K = 0$ in order to get $s = 0$,
 even though $s = 0$ only means $K^2 = 0$. This is justified
 because, as is clear from Section I, we are interested in
 s near to zero as much as in the particular value $s = 0$.
 However it is known⁶ that for any $s \neq 0$, there exists a
 Lorentz frame in which each component K^μ of K is of order
 $s^{\frac{1}{2}}$.¹²

Now that $K = 0$, an important simplification is possible.
 It is clear from the substitution⁶ $T(p, q; 0) = m^{-2} T'(p/m, q/m; 0)$
 that Eq. (2-1) with the kernel (2-2) gives an equation
 for the function T' which does not contain the mass m . Then
 the convergence of the solution by successive iterations to
 the equation for T' is independent of m , so clearly this
 must also be true for T . The convergence therefore can
 equally well be investigated in the case $m = 0$; singularities
 in T will develop at values of the coupling constants f_Y
 which are independent of m .⁶ Thus to investigate conver-
 gence, we may put $K = 0$ and $m = 0$ in the BS equation and
 write $1/(\gamma \cdot k)$ for $G(k)$ in it.

The case $m = 0$ corresponds to the asymptotic case $m \neq 0$,
 $p^2 \rightarrow \infty$, $q^2 \rightarrow \infty$. Therefore it is plausible that the
 addition of a mass term to the denominators of the exchanged
 particle propagators in the kernel (2-2) would make no
 difference to the results. Without strict proof, we believe
 this to be the case. The absence of mass in the exchanged
 particle propagators allows the calculations to be made far
 easier by use of the Laplacian relation (2-7) below which

turns the integral equation into a differential equation.

When $m = 0$, the solution of the BS equation with the kernel (2-2) simplifies a great deal. Inspection shows that the iteration solution for each component $T_{(X^j)}$ is always a sum of matrices of the same tensor class X as the basis matrix X^j . (That is, $T_{(\gamma^5)} = f \gamma^5$, $T_{(\gamma^k)} = f_{ki} \gamma^i$, etc.) The effect of the kernel (2-2) also simplifies, the coupling constants f_Y^2 combining into a single constant g_X^2 according to the definition

$$g_X^2 = \sum_{Y=S}^P a_{XY} f_Y^2 \quad (2-5)$$

where the numbers a_{XY} are given in Table 2 (in Section V, below). The equation for $T_{(X^j)}(p,q;0)$ with $m = 0$, denoted by $T_{(X^j)}(p,q)$, is

$$T_{(X^j)}(p,q) = \frac{1}{i} \frac{1}{4} \frac{g_X^2}{(p-q)^2} X^j + g_X^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(p-k)^2} \frac{\gamma \cdot k}{k^2} T_{(X^j)}(k,q) \frac{\gamma \cdot k}{k^2}. \quad (2-6)$$

It now only remains to determine, for each X , what value of g_X^2 will cause the iteration solution of the simplified equation (2-6) to diverge. The solution to (2-6) is most easily found by turning (2-6) into a differential equation, resolving into 0(4) partial waves, and using radial-variable Wronskians to obtain the Green's function solution. Starting with the Laplace Green's

function equation in four dimensions, with Δ referring to the p^i variables,

$$\Delta \frac{1}{(p-q)^2} = -4\pi^2 \delta^4(p-q), \quad (2-7)$$

the inhomogeneous and homogeneous differential versions of (2-6) are

$$\begin{aligned} \Delta T_{(xj)}(p,q) + \beta_X (1/u)(\gamma \cdot \hat{p}) T_{(xj)}(p,q)(\gamma \cdot \hat{p}) \\ = i g_X^2 (1/v) \pi^2 \delta^4(p-q) X^j, \end{aligned} \quad (2-8)$$

$$\Delta Y + \beta_X (1/u)(\gamma \cdot \hat{p}) Y(\gamma \cdot \hat{p}) = 0, \quad (2-9)$$

where the abbreviations

$$u = p^2, \quad v = q^2, \quad \hat{p} = p/u^{1/2}, \quad \beta_X = g_X^2/4\pi^2$$

have been made. In the following two Sections we will use Eqs. (2-8) and (2-9) to determine when the iteration solutions to the various tensor components of the BS equation (2-6) converge. The results are collected in Section V.

III. EXAMPLE OF P COMPONENT

In this Section the $X = P$ component of Eq. (2-6) will be treated. The solution is known.¹³ The method of finding it will be expounded as an example of the method we use on the other, more complicated S, V, T, A components. We are thereby enabled to spare the reader from a description of the rather long calculations of these latter cases.

Inspection of (2-6) ($X = P$) shows that the solution by successive iterations is of the form

$$T_{(P)}(p, q) = F(p, q) \gamma_5,$$

F a scalar. In fact the solution is^{13, 6}

$$T_{(P)}(p, q) = \gamma_5 \frac{1}{i} \frac{1}{4} q_P^2 \sum_{n=0}^{\infty} \frac{n+1}{\epsilon_n} u_{<}^{(\epsilon_n-1)/2} u_{>}^{(\epsilon_n+1)/2} C_n(z) \quad (3-2)$$

where

$$u_{<} = \min(u, v), \quad u_{>} = \max(u, v), \quad z = \hat{p} \cdot \hat{q}, \quad C_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta};$$

$$\epsilon_n = [(n+1)^2 - \beta_P]^{\frac{1}{2}}. \quad (3-3)$$

(Recall $\beta_P = q_P^2 / 4\pi^2$.)

Solution (3-2), whose sum over n converges,¹⁴ is real and finite as long as ϵ_n is real for all n ; that is, as long as $(n+1)^2 - \beta_P > 0$ for all n . When β_P reaches the value 1, the $n = 0$ partial wave component of solution (3-2) becomes singular. At this value of β_P the solution by successive iterations of Eq. (2-6)

($X = P$) no longer converges. The scattering amplitude (with $m \neq 0$ in general) develops a singularity in the coupling constant.¹⁵ This is the behaviour that, as explained in Section I, we interpret as meaning that the mass of a pseudoscalar bound state has descended to zero.

The result of this Section, then, is that the $X = P$ component of Eq. (2-8) will converge when

$$\beta_P = g_P^2 / 4\pi^2 < 1 \quad (3-4)$$

and diverges when the extreme of (3-4) is reached.

We will now show the relevant steps of the derivation of the result (3-4).

Consider solutions of the homogeneous equation (2-9) whose matrix part will span the solution $T_{(P)}$, Eq. (3-1). In this case, there is only one, of the form

$$Y = Y_1(u, z) \gamma_5, \quad (3-5)$$

Y_1 a scalar. Substitution of (3-5) into (2-9) allows the γ_5 to be cancelled throughout and, after multiplication across by u , gives

$$\{D[u] + D_{11}[z]\} Y_1 = 0 \quad (3-6)$$

with

$$D[u] = 4u^2(\partial/\partial u)^2 + 8u(\partial/\partial u) + 1 \quad (3-7)$$

$$D_{11}[z] = (1-z)^2(\partial/\partial z)^2 - 3z(\partial/\partial z) - 1 + \beta_P. \quad (3-8)$$

Separation of variables, with separation constant λ , gives

$$Y_1(u, z) = U(u) Z_1(z), \quad (3-9)$$

$$D[u] U(u) = \lambda U(u) \quad (3-10)$$

$$\{D_{11}[z] + \lambda\} Z_1(z) = 0 \quad (3-11)$$

Here we know $Z_1(z) = C_n(z)$, but if we did not (as for the other components $X \neq P$) we could find it as follows. Let

$$Z_1(z) = \sum_{j=0}^{\infty} d_{1,j} z^j \quad (3-12)$$

Eq. (3-11) now gives the recurrence relation

$$(j+2)(j+1) d_{1,j+2} = [(j+1)^2 - \beta_P - \lambda] d_{1,j} \quad (3-13)$$

In order to obtain a complete set of angular solutions it is required that the series (3-12) terminate at z^n , which by (3-13) requires that

$$\lambda = (n+1)^2 - \beta_P \quad (3-14)$$

Then $Z_1(z) = C_n(z)$.

The expansions¹⁶

$$\delta^4(p-q) = (1/2\pi) \delta(u-v) \delta(\hat{p}-\hat{q}) \quad (3-15)$$

$$\delta(\hat{p}-\hat{q}) = (1/2\pi^2) \sum_{n=0}^{\infty} (n+1) C_n(z) \quad (3-16)$$

now show that we can write

$$T_{(P)}(p, q) = \gamma_5 \sum_{n=0}^{\infty} F_n(u, v) C_n(z) \quad (3-17)$$

and then for each n , Eq. (2-8) with $X=P$ has a separate component, which after cancelling through $C_n(z)$ leaves the radial equation

$$\{D[u] - \lambda\} F_n(u, v) = i g_p^2 \frac{1}{2} (n+1) \delta(u-v). \quad (3-18)$$

Eq. (3-18) is a simple Green's function problem, solved by the Wronskian method, using the solutions of the homogeneous equation (3-10). Eq. (3-7) shows that the solutions of (3-10) are given by

$$U(u) = u^{\alpha_1}, u^{\alpha_2}, \quad \left[(2\alpha_i + 1)^2 = \lambda \right], \quad (3-19)$$

their Wronskian being

$$W(u) = u^{\alpha_1} \partial u^{\alpha_2} / \partial u - u^{\alpha_2} \partial u^{\alpha_1} / \partial u = \lambda^{\frac{1}{2}} u^{-2}. \quad (3-20)$$

The solution to (3-18) for each n is shown in (3-2); as (3-14) shows, the denominator ϵ_n is the Wronskian term $\lambda^{\frac{1}{2}}$. It is now clear that the solution (3-2) diverges when the effective coupling constant g_p^2 (defined by (2-5)) reaches a value such that the Wronskian of any one of the pairs of radial solutions becomes zero. By (3-20), this happens when the constant λ reaches zero for some n , that is, when condition (3-4) is no longer obeyed.

Thus the following sequence of steps is possible for each component X of the scattering Green's function.

(i) Find a separation of the homogeneous equation (2-9) into angular and radial components. (ii) Find the separation constant λ needed to make each angular part terminate. (iii) Find when the coupling constants give any λ such a value that the radial solutions become degenerate. At this point the coupling constants have reached a value such that the solution by successive iterations no longer converges at $K=0$, that is, the coupling constant gives a singularity in the scattering amplitude which is identified with a zero-mass bound state.

The next Section will outline the results for the components $X = S, V, A, T$. We hope that this Section's derivation of a simple known result will justify the omission of the far more complicated details of the other cases.

IV. OTHER COMPONENTS

(a) X=S

From Table I, $X^j=1$ when $X=S$. Inspection shows that the iteration solution to (2-6) is a multiple of the unit matrix, of the form

$$T_{(S)}(p, q) = F(p, q), \quad (4a-1)$$

F a scalar. The homogeneous solution to (2-9) must have the form

$$Y = Y_1(u, z)$$

Y_1 a scalar. Since $(\gamma \cdot \hat{p}) \gamma_5 (\gamma \cdot \hat{p}) = +\gamma_5$ but $(\gamma \cdot \hat{p}) 1 (\gamma \cdot \hat{p}) = -1$, the homogeneous equation (2-9) gives the same form as (3-6) - (3-8) with β_p replaced by $-\beta_s$. The same arguments show that convergence of the iteration solution to (2-6) needs

$$\beta_s = g_s^2 / 4\pi^2 > -1, \quad (4a-2)$$

and the solution will diverge when the extreme of (4a-2) is reached.

(b) $X=V$

Here $X^{\hat{l}} = i\gamma^{\hat{l}}$, from Table 1.

Consideration of the solutions to the homogeneous equation (2-9) shows that they must be of a form more complicated than (3-5). In fact

$$Y = \sum_{A=1}^5 Y_A(u, z) E_A \quad (4b-1)$$

where

$$\begin{aligned} E_1 &= \gamma^{\hat{l}} \\ E_2 &= \hat{p}^{\hat{l}}(\gamma \cdot \hat{p}) \\ E_3 &= \hat{p}^{\hat{l}}(\gamma \cdot \hat{q}) \\ E_4 &= \hat{q}^{\hat{l}}(\gamma \cdot \hat{p}) \\ E_5 &= \hat{q}^{\hat{l}}(\gamma \cdot \hat{q}) \end{aligned} \quad (4b-2)$$

and Y_A are scalars.

Substitution of (4b-1) into (2-9) gives, as the analogue of (3-6), five equations of the form

$$D[u] Y_A + \sum_{B=1}^5 D_{AB}[z] Y_B = 0 \quad (4b-3)$$

where $D[u]$ is still given by (3-7) but $D_{AB}[z]$ is now a 5×5 matrix of operators analogous to (3-8). Eq. (4b-3) separates as (3-9) does, according to

$$Y_A(u, z) = U(u) Z_A(z) \quad (4b-4)$$

where, as in (3-10) and (3-11),

$$D[u] U(u) = \lambda U(u) \quad (4b-5)$$

$$\sum_B \{ D_{AB}[z] + \lambda \delta_{AB} \} Z_B(z) = 0 \quad (4b-6)$$

The radial solution to (4b-5) is identical to that in Section III; it simply remains to find when λ will arrive at the value 0, as the effective coupling parameter $\beta_V = g_V^2/4\pi^2$ moves away from 0. To determine

λ , it is required as before that $Z_A(z)$ be a terminating series in z (for all A) in order to define a complete set of angular solutions. Counting powers of $\hat{q}$ in the angular matrices E_A suggests that the analogue to (3-12) is

$$Z_A(z) = \sum_{j=C_A}^{\infty} d_{A,j} z^{j-C_A}, \quad (4b-7)$$

$$C_A = \begin{cases} 0, & A=1, 2 \\ 1, & A=3, 4 \\ 2, & A=5 \end{cases}.$$

Substitution of (4b-7) into (4b-6) shows that the form (4b-7) is appropriate, since it gives an equation which is recurrent as (3-13) is:

$$(j-C_A+2)(j-C_A+1) d_{A,(j+2)} = \sum_B M_{AB} \{ j, \lambda, \beta_V \} d_{B,j} \quad (4b-8)$$

where now M_{AB} is a 5x5 matrix in general (but only 2x2 for $j=0$, 4x4 for $j=1$). The requirement that the series (4b-7) terminate at $j=n$ is the requirement

that $d_{A,(n+2)}=0$ (all A), which requires

$$\det(M_{AB}\{n, \lambda, \beta_V\}) = 0. \quad (4b-9)$$

This condition, analagous to (3-14), gives λ as a function of n and β_V . For $n=0$ there are two eigenvalues

λ ; for $n=1$, four; for each $n \geq 2$, five. It turns out that there are two equal eigenvalues, $\lambda = 5 - (\beta_V^2 + 4\beta_V + 16)^{\frac{1}{2}}$, such that these λ reach 0 before any other as β_V increases away from zero. (One of these values of λ occurs

for $n=0$ and the other for $n=2$. The occurrence of two values is connected with the fact that the solution by successive iterations to the BS equation (2-1) has symmetry in p and q when $K=0$.) On the other hand when

β_V decreases from zero, the eigenvalue λ which reaches zero first turns out to be $\lambda = \beta_V + 4$, occuring for $n=1$.

Thus λ reaches zero when β_V reaches the extremes of these conditions:

$$-4 < \beta_V < \sqrt{13} - 2. \quad (4b-10)$$

Accordingly, the V component of the BS equation at $K=0$ will converge when (4b-10) is satisfied, and will diverge when the extremes of (4b-10) are reached.

(c) $X=A$

Here $X^i = \gamma^i \gamma_5$, from Table 1.

This case is the same as $X=V$ except that the angular matrices E_A are replaced by $E_A \gamma_5$, and β_V is replaced

by $-\beta_A$ for the same reason as in the case $X=S$. The result then is that the convergence of the solution of the A component of the BS equation at $K=0$ requires

$$-\sqrt{13} + 2 < \beta_A < 4 \quad (4c-1)$$

with singularities occurring when the extremities of (4c-1) are reached.

(d) $X=T$.

Here $X^j = \sigma^{lm}$, from Table 1.

This case is the most complicated of all. The procedure is the same as in part (b), $X=V$. With [] meaning "antisymmetrised with respect to the indices l and m ," the angular matrices needed are

$$\begin{aligned} E_1 &= [\sigma^{lm}] \\ E_2 &= [\hat{p}^l \hat{p}^k \sigma^{km}] \\ E_3 &= [\hat{p}^l \hat{q}^k \sigma^{km}] \\ E_4 &= [\hat{q}^l \hat{p}^k \sigma^{km}] \\ E_5 &= [\hat{q}^l \hat{q}^k \sigma^{km}] \\ E_6 &= [\hat{q}^l \hat{p}^m \hat{p}^j \hat{q}^k \sigma^{jk}] \end{aligned} \quad (4d-1)$$

The procedure of Section IV(b), but now involving 6x6 matrices, shows that there is a pair of eigenfunctions having the property that their λ reaches 0 before any other as β_T moves away from zero. These λ are $\lambda = 5 - (\beta_T^2 + 16)^{1/2}$, giving the condition

$$-3 < \beta_T < 3 \quad (4d-2)$$

for convergence of the solution to (2-8); divergence in the T component occurs when the extremes of (4d-2) are reached.

V. COLLECTION OF RESULTS

We are looking for values of the exchanged-particle coupling constants f_Y which just cause a singularity in the $K=0$ scattering Green's function solution $T(p,q;0)$ to the BS equation (2-1). We are interpreting the onset of such a singularity as the point where the lowest bound state mass has descended to zero.

We have found conditions on linear combinations of f_Y^2 such that $T(p,q;0)$ is regular. With the help of the numbers a_{XY} given in Table 2, these conditions on f_Y^2 can be quite simply given as the following summary of results from the last two Sections.

$$\beta_X = \sum_{Y=S}^P a_{XY} f_Y^2 / 4\pi^2 ; \quad (5-1)$$

$$-1 < \beta_S$$

$$-4 < \beta_V < \sqrt{13} - 2$$

$$-3 < \beta_T < 3 \quad (5-2)$$

$$-(\sqrt{13} - 2) < \beta_A < 4$$

$$\beta_P < 1.$$

Table 2. Values of a_{XY} .

		Y (EXCHANGED PARTICLES)				
		S	V	T	A	P
X (COM- PONENT)	S	1	-4	6	4	-1
	V	1	2	0	2	1
	T	1	0	-4	0	-1
	A	1	-2	0	-2	1
	P	1	4	6	-4	-1

When the coupling constants f_Y are such that all the conditions (5-2) are obeyed, then $T(p,q;0)$ is regular and there are no zero-mass bound states. If all of (5-2) are obeyed except that certain β_X have just reached the limit of their allowed values, then that X component of $T(p,q;0)$ (as given by (2-4)) will just begin to diverge, and this is interpreted as a bound state having just descended down to zero mass with the tensor character X.

The examples listed in Section I are obtained as follows.

(a) Pure S exchange. Here $f_Y=0$ except for $Y=S$. Equation (5-1) and use of Table 2 shows that $\beta_S = \beta_V = \beta_T = \beta_A = \beta_P = f_S^2/4\pi^2$. As f_S^2 increases from 0, (5-2) shows that β_P reaches its limit first, and then $f_S^2/4\pi = \pi$.

At this value the P component of the K=0 scattering Green's function acquires a singularity, which is interpreted as a massless pseudoscalar bound state.

(b) Pure P exchange. Here $f_Y=0$ except for $Y=P$. Eq. (5-1) and Table 2 show that $\beta_S = \beta_T = \beta_P = -f_P^2/4\pi^2$, $\beta_V = \beta_A = f_P^2/4\pi^2$. Eq. (5-2) shows that as f_P^2 increases from zero, β_S reaches its limit first; the zero-four-momentum scattering Green's function begins to have a scalar singularity,⁸ interpreted as a massless scalar bound state occurring when $f_P^2/4\pi = \pi$.

(c) Pure V exchange (as in QED). Here $f_Y = 0$ unless $Y=V$. Eq. (5-1) and Table 2 show that $\beta_S = -f_V^2/\pi^2$, $\beta_V = f_V^2/2\pi^2$, $\beta_T = 0$, $\beta_A = -f_V^2/2\pi^2$, $\beta_P = f_V^2/\pi^2$. As f_V^2 increases from zero, (5-2) shows that β_S and β_P reach their limits first together, which is interpreted as simultaneous pseudoscalar and scalar massless bound states⁹ arrived at when $f_V^2/4\pi = \pi/4 \approx 0.8$.

(d) Equal V and A exchange. Here we put $f_V = f_A = f$, $f_S = f_T = f_P = 0$. Eq. (5-1) and Table 2 show that $\beta_S = \beta_T = \beta_P = 0$, $\beta_V = f^2/\pi^2$, $\beta_A = -f^2/\pi^2$. As f^2 increases from 0, (5-2) shows that β_V and β_A reach their limits together, when $f^2/4\pi = (\sqrt{13} - 2)\pi/4 \approx 1.26$. This is interpreted as the simultaneous occurrence of massless V and A bound states (i.e. states with $J^P = 1^-, 0^+, 1^+, 0^-$). This is physically interesting as showing that a chirally symmetric Bethe-Salpeter kernel

can produce a scattering Green's function with some chiral symmetry properties.

(e) Tensor states. It can be seen from Eqs. (5-1), (5-2) and Table 2 that there is no combination of particle exchange which can produce a massless T bound state.

Any other combination of interactions may be tested similarly to the above examples.

VI. CONCLUSIONS

We have considered the development of singularities in the scattering solution $T(p,q;K)$ (where K =total four-momentum) to a non-Fredholm Bethe-Salpeter equation for the interaction of two spin- $\frac{1}{2}$ particles. The kernel of the equation is given by a sum of S, V, T, A and P massless particle exchange, with dimensionless coupling constants f_S , f_V , f_T , f_A and f_P .

Under variation of these coupling constants, we have regarded the onset of a singularity in a previously regular zero-four-momentum scattering amplitude $T(p,q;0)$ as the descent of the mass of the lowest bound state to zero. We have derived a simple formula (Section V) which determines in the most general case when such singularities occur in this model, and what their tensor classification is.

The physical interest of these questions is that if mesons are made of heavy quarks, the mesons are relatively almost massless. The interest in considering the kind of non-Fredholm model discussed here is at least twofold: (i) the dimensions of such models are those of natural field theories, such as QED; (ii) Fredholm models, such as spin- $\frac{1}{2}$ equations with a cut-off, or scalar equations, have been tried⁵ and found not to contain low-mass bound states in the multiplicity desired.

In particular it is found that if the interaction kernel is given by a (chirally symmetric) V and A exchange of equal strength, the scattering Green's function $T(p,q;0)$ develops a V and A singularity simultaneously, suggesting the possibility of a corresponding chiral symmetry in the lowest bound states of a heavy quark model of mesons.

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⁴ In the work of the present article q denotes any massive spin-1/2 particle, with no internal quantum numbers explicitly ascribed to it.

^{4a} About the probable distribution of exchanged-particle coupling-constant eigenvalues for fixed bound-state mass.

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⁶ J. Connell, Phys. Rev. 179, #### (1969).

^{6a} We are regarding this value as a limit point only. There are difficulties with the concept of a BS bound state of actually zero mass, as is shown, for example, by both Ref. 6 and the present work.

7 It is an assumption. However we do know (see Ref.6) that $T(s)$ has a domain of analyticity in the s -plane which shrinks as the (suppose only one) exchanged-particle coupling constant increases. The inequality cited above is a consequence of this result. It certainly appears as if the bound states may be "pushing" the domain of analyticity down the real s -axis towards zero as the coupling constant increases. If on the other hand the lowest bound-state mass were still a finite distance from zero when the coupling constant reached its limiting value (such that $T(0)$ becomes singular), it is difficult to imagine what the subsequent variation of the coupling constant would have to be to bring the bound state mass down to zero, and -- more important -- what the physical significance of the model would be at all.

8 First derived by Lee and Swift, Ref. 1.

9 G. Domokos, personal remark.

10 See Narayanaswamy and Pagnamenta, Ref. 5.

11 See report of R. Dalitz in Proceedings of the Thirteenth International Conference on High Energy Physics, Berkeley, 1966 (U. of Cal. Press, 1967).

12 Of course the Lorentz frame has to be chosen before performing the Wick rotation.

13 R.S. Willey, Phys. Rev. 153, 1364 (1967).

14 See Ref. 6 for a discussion of exceptional points.

15 See the latter part of Ref. 13 for an illustration of this.

16 G. Domokos and P. Suranyi, Nuclear Phys. 54, 529 (1964).

FIGURE CAPTION

Fig. 1. The Feynman diagram for the Green's function

$$i^{-2} T_{\alpha\beta;\gamma\sigma}(p,q;K).$$