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LINE AND AREA ORTHOGONALITY OF JACOBI POLYNOMIALS

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DEPARTMENT OF COMPUTER SCIENCE
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LINE AND AREA ORTHOGONALITY OF JACOBI POLYNOMIALS

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1. Introduction

Let $P_n^{(\alpha,\beta)}(x)$, $n=0,1,2,\ldots$, be the set of Jacobi polynomials which form an orthogonal system on $[-1,1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, $\alpha > -1$, $\beta > -1$; that is,

(1) $$\left(P_n^{(\alpha,\beta)}, P_m^{(\alpha,\beta)}\right) = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x)\, P_m^{(\alpha,\beta)}(x)\, dx = 0, \quad m \neq n$$

These polynomials are standardized, as usual, such that $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. Let $p_n^*(x)$ designate the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ normalized according to (1) so that $p_n^*(x) = P_n^{(\alpha,\beta)}(x)/\|P_n^{(\alpha,\beta)}\|$, where $\|P_n^{(\alpha,\beta)}\| = (P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)})^{\frac{1}{2}}$.

Let $\mathcal{E}_\rho$ designate the ellipse $z=\frac{1}{2}(\xi+\xi^{-1})$, $\xi=\rho e^{i\theta}$, $0 \leq \theta \leq 2\pi$, $\rho > 1$, in the z-plane $(x=\mathrm{Re}(z))$ with foci at $z = \pm 1$ and semiaxes $\frac{1}{2}(\rho+\rho^{-1})$ and $\frac{1}{2}(\rho-\rho^{-1})$. By $A(\mathcal{E}_\rho)$ we shall designate the class of functions analytic on the closed interval $[-1,1]$ $(A[-1,1])$ and which can be continued analytically into the z-plane so as to be single valued and regular in $\hat{\mathcal{E}}_\rho$, the interior of $\mathcal{E}_\rho$, and continuous in $\bar{\mathcal{E}}_\rho$, the closed ellipse $\mathcal{E}_\rho$.

For $f,g \in A(\mathcal{E}_\rho)$ we introduce the line integral inner product

(2) $$(f,g) = \int_{\mathcal{E}_\rho} |w(z)|\, f(z)\, \overline{g(z)}\, ds \qquad (ds = |dz|)$$

which induces the line integral norm

(3) $$\|f\|_{\mathcal{E}_\rho} = \int_{\mathcal{E}_\rho} |w(z)|\, |f(z)|^2\, ds$$

and the area integral inner product

$$(4) \qquad (f, g) = \iint_{\hat{\mathcal{E}}_\rho} |W(z)|\, f(z)\, \overline{g(z)}\, dx\, dy$$

which induces the area integral norm

$$(5) \qquad \|f\|_D^2 = \iint_{\hat{\mathcal{E}}_\rho} |W(z)|\, |f(z)|^2\, dx\, dy$$

In (2) and (4), $|w(z)|$ and $|W(z)|$ are weight functions such that $\|1\|_{\mathcal{E}_\rho} < \infty$ and $\|1\|_D < \infty$.

It has been shown by Walsh [1] that the Chebyshev polynomials of the first kind, $T_n(z) = \cos(n \arccos z) = P_n^{(-\frac{1}{2},-\frac{1}{2})}(z)$, form an orthogonal system on $\mathcal{E}_\rho$ with respect to the inner product (2) with weight $|w(z)| = |1-z^2|^{-\frac{1}{2}}$, and that these polynomials are also orthogonal on $\hat{\mathcal{E}}_\rho$ with respect to the inner product (4) with weight $|W(z)| = |1-z^2|^{-1}$. Szego [2] has shown that the Chebyshev polynomials of the second kind, $U_n(z) = \sin[(n+1) \arccos z]/(1-z^2)^{\frac{1}{2}} = P_n^{(\frac{1}{2},\frac{1}{2})}(z)$, form an orthogonal system with respect to (2) with weight $|1-z^2|^{\frac{1}{2}}$; it is also known that these polynomials are orthogonal on $\hat{\mathcal{E}}_\rho$ with respect to (4) with weight unity (See, e.g., Davis [3]).

The purpose of this paper is to show that the (complex) Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ form an orthogonal system on $\mathcal{E}_\rho$ with respect to the line integral inner product (2) with weight $|w(z)| = |1-z|^\alpha |1+z|^\beta$, $\alpha > -1$, $\beta > -1$; further, for $\alpha > -1$, $\beta > -1$, these polynomials also form an orthogonal system on $\hat{\mathcal{E}}_\rho$ with respect to the area integral inner product (4) with weight $|W(z)| = |1-z|^{\alpha-\frac{1}{2}} |1+z|^{\beta-\frac{1}{2}}$.

Let $L^2(\hat{\mathcal{E}}_\rho; |W(z)|)$ denote the Hilbert space resulting from the completion of $A(\mathcal{E}_\rho)$ with respect to the area integral norm (5), and let $H^2(\mathcal{E}_\rho; |w(z)|)$ denote the Hilbert space resulting from the completion of $A(\mathcal{E}_\rho)$ with respect to the line integral norm (3). The Hilbert space

$L^2(\mathcal{E}_\rho; 1)$ has been used extensively for obtaining estimates for the errors of numerical approximation for functions analytic on $[-1,1]$ (the method suggested by Davis; see, e.g., [4]). As a consequence of our results we show that the area integral inner product (2) is unsuitable for obtaining error estimates for the class of functions $A[-1,1]$; this has already been observed in Chawla [6]. As a side result we obtain relationships between the real integral, with Jacobi weight function, and the (complex) line and area integrals for an $f \in A(\bar{\mathcal{E}}_\rho)$. As an important application of the line integral orthogonality of the Jacobi polynomials on $\mathcal{E}_\rho$, we show that estimates for the errors of rules of numerical approximation for analytic functions can be obtain directly from Cauchy's integral formula.

2. Line Integral Orthogonality of Jacobi Polynomials

The Jacobi polynomials form a line integral orthogonal system on $\mathcal{E}_\rho$ with the Jacobi weight function. This is contained in

THEOREM 1. Let

(6)
$$P_n^*(z) = p_n^*(z)/\|p_n^*\|_{\mathcal{E}_\rho}, \qquad n = 0, 1, 2, \ldots,$$

then, for $\alpha > -1$, $\beta > -1$,

(7)
$$\int_{\mathcal{E}_\rho} |1-z|^\alpha |1+z|^\beta P_n^*(z) \overline{P_m^*(z)}\, ds = \delta_{m,n}, \qquad m, n = 0, 1, 2, \ldots.$$

Proof. We follow Walsh [1, Theorem 12]. For $\alpha > -1$, $\beta > -1$, we have

$$\delta_{m,n} = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta p_n^*(x) p_m^*(x)\, dx$$

$$= \frac{1}{2} \int_{|\xi|=1} \left|1 - \frac{\xi+\xi^{-1}}{2}\right|^\alpha \left|1 - \frac{\xi+\xi^{-1}}{2}\right|^\beta p_n^*\left(\frac{\xi+\xi^{-1}}{2}\right) \overline{p_m^*\left(\frac{\xi+\xi^{-1}}{2}\right)} \left|d\left(\frac{\xi+\xi^{-1}}{2}\right)\right|$$

$$= \frac{1}{2} \frac{1}{\alpha_{n,m}(\rho)} \int_{\mathcal{E}_\rho} |1-z|^\alpha |1+z|^\beta p_n^*(z) \overline{p_m^*(z)}\, |dz|$$

where $\alpha_{n,m}(\rho)$ is a normalizing factor. Since for m=n,

$$2\,\alpha_{n,n}(\rho) = \| p_n^* \|^2_{\mathcal{E}_\rho}$$

the line integral orthogonality (7) of the Jacobi polynomials (6) follows.

Since for $\rho \to 1$, $\alpha_{n,n}(\rho) \to 1$ and $\| p_n^* \|_{\mathcal{E}_\rho} \to (2)^{\frac{1}{2}}$ for all $n \geqslant 0$, there follows

COROLLARY 1. On $-1 \leq z \leq 1$, $P_n^*(z) = (2)^{-\frac{1}{2}}\, p_n^*(z)$.

Example. Let $\alpha = -\frac{1}{2}$, $\beta = -\frac{1}{2}$, then $p_n^*(x) = (2/\pi)^{\frac{1}{2}}\, T_n(x)$, $n > 0$, $p_0^*(x) = (\pi)^{-\frac{1}{2}}$. Since on $\mathcal{E}_\rho$, $T_n(z) = \frac{1}{2}(\xi^n + \xi^{-n})$,

$$\| p_n^* \|_{\mathcal{E}_\rho} = (\rho^{2n} + \rho^{-2n})^{1/2}, \qquad n \geqslant 0 \tag{8}$$

The corresponding line integral orthonormal system of Chebyshev polynomials is, therefore,

$$\begin{aligned} P_n^*(z) &= \left(\tfrac{2}{\pi}\right)^{1/2} (\rho^{2n} + \rho^{-2n})^{-1/2}\, T_n(z), \qquad n > 0 \\ P_0^*(z) &= (2\pi)^{-1/2} \end{aligned} \tag{9}$$

3. Area Integral Orthogonality of Jacobi Polynomials

We next show that the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ form an orthogonal system on the area of $\mathcal{E}_\rho$ with weight $|1-z|^{\alpha-\frac{1}{2}}\, |1+z|^{\beta-\frac{1}{2}}$.

THEOREM 2. Let

$$\mathcal{P}_n^*(z) = p_n^*(z) / (\beta_n(\rho))^{1/2}, \qquad n = 0, 1, 2, \cdots, \tag{10}$$

where

$$\beta_n(\rho) = \int_1^\rho \| p_n^* \|^2_{\mathcal{E}_R} \frac{dR}{R} \tag{11}$$

Then, for $\alpha > -1$, $\beta > -1$,

$$\iint_{\hat{\mathcal{E}}_\rho} |1-z|^{\alpha-1/2}\, |1+z|^{\beta-1/2}\, \mathcal{P}_n^*(z)\, \overline{\mathcal{P}_m^*(z)}\, dx\,dy = \delta_{m,n}, \quad m, n = 0, 1, 2, \cdots \tag{12}$$

Proof. Let A designate the annulus: $\xi = Re^{i\theta}$, $1 < R < \rho$, $0 \leqslant \theta \leqslant 2\pi$, in the ξ-plane, which is the map of $\hat{\mathcal{E}}_\rho$ in the z-plane. Then, with $z=\frac{1}{2}(\xi+\xi^{-1})$, $\xi=u+iv$,

$$(13)\qquad \iint_{\hat{\mathcal{E}}_\rho} |W(z)|\, p_n^*(z)\overline{p_m^*(z)}\,dx\,dy = \iint_A |W(z)|\, p_n^*(z)\overline{p_m^*(z)}\,\tfrac{1}{4}|1-\xi^{-2}|^2\,du\,dv$$

$$= \int_1^\rho \Big[\int_{|\xi|=R} |W(z)|\, p_n^*(z)\overline{p_m^*(z)}\,\tfrac{1}{4}|1-\xi^{-2}|^2\,|d\xi|\Big]\,dR$$

$$= \int_1^\rho \Big[\int_{\mathcal{E}_R} |W(z)|\, p_n^*(z)\overline{p_m^*(z)}\,|1-z^2|^{1/2}\,|dz|\Big]\,\frac{dR}{R}$$

Now, if $|W(z)| = |1-z|^{\alpha-\frac{1}{2}}\,|1+z|^{\beta-\frac{1}{2}}$, then

$$(14)\qquad \iint_{\hat{\mathcal{E}}_\rho} |1-z|^{\alpha-1/2}|1+z|^{\beta-1/2} p_n^*(z)\overline{p_m^*(z)}\,dx\,dy = \int_1^\rho\Big[\int_{\mathcal{E}_R} |1-z|^\alpha|1+z|^\beta p_n^*(z)\overline{p_m^*(z)}\,|dz|\Big]\frac{dR}{R}$$

From (7), (8) and (14) we obtain

$$(15)\qquad \iint_{\hat{\mathcal{E}}_\rho} |1-z|^{\alpha-1/2}|1+z|^{\beta-1/2} p_n^*(z)\overline{p_m^*(z)}\,dx\,dy = \int_1^\rho\Big[\int_{\mathcal{E}_R} |1-z|^\alpha|1+z|^\beta \mathcal{P}_n^*(z)\overline{\mathcal{P}_m^*(z)}\,|dz|\Big]\times$$

$$\times \|p_n^*\|_{\mathcal{E}_R}\|p_m^*\|_{\mathcal{E}_R}\frac{dR}{R}$$

$$= \delta_{m,n}\cdot\int_1^\rho \|p_n^*\|^2_{\mathcal{E}_R}\frac{dR}{R}$$

and (12) now follows from (15). We note an alternative expression for $\beta_n(\rho)$:

$$(16)\qquad \beta_n^2(\rho) = \iint_{\hat{\mathcal{E}}_\rho} |1-z|^{\alpha-1/2}|1+z|^{\beta-1/2}|p_n^*(z)|^2\,dx\,dy$$

From (11) we observe that as $\rho\to 1$, $\beta_n(\rho)\to 0$, and since $p_n^*(z)$ are uniformly bounded on $-1\le z\le 1$, there follows

<u>COROLLARY 2</u>. The polynomials $\mathcal{P}_n^*(z)$ are unbounded on $-1\le z\le 1$, and hence unbounded on every $\overline{\mathcal{E}}_\rho$, $\rho>1$.

See also Sewell [7] , Theorem 5.3.1 and remarks on p. 154.

<u>Examples</u>. 1. Let $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$, then $p_n^*(x) = (2/\pi)^{\frac{1}{2}}\,U_n(x)$, $n\geq 0$. Since on $\hat{\mathcal{E}}_\rho$, $U_n(z) = (\xi^{n+1} - \xi^{-n-1})/(\xi-\xi^{-1})$,

$$\| p_n^* \|_{\mathcal{E}_\rho} = (\rho^{2n+2} + \rho^{-2n-2})^{1/2} \tag{17}$$

so that

$$P_n^*(z) = \left(\frac{2}{\pi}\right)^{1/2} (\rho^{2n+2} + \rho^{-2n-2})^{-1/2} U_n(z) \tag{18}$$

are line integral orthonormal over $\mathcal{E}_\rho$. Now,

$$\beta_n(\rho) = \int_1^\rho (R^{2n+2} + R^{-2n-2}) \frac{dR}{R} \tag{19}$$

$$= (\rho^{2n+2} - \rho^{-2n-2})/(2n+2)$$

therefore the polynomials

$$\mathcal{P}_n^*(z) = 2 \left(\frac{n+1}{\pi}\right)^{1/2} (\rho^{2n+2} - \rho^{-2n-2})^{-1/2} U_n(z) \tag{20}$$

are area integral orthonormal over $\hat{\mathcal{E}}_\rho$ with weight $|W(z)| = 1$.

2. Let $\alpha = -\frac{1}{2}$, $\beta = -\frac{1}{2}$; then from (8) and (11)

$$\beta_n(\rho) = \int_1^\rho (R^{2n} + R^{-2n}) \frac{dR}{R}$$

$$= (\rho^{2n} - \rho^{-2n})/(2n), \quad n > 0$$

while

$$\beta_0(\rho) = 2 \int_1^\rho \frac{dR}{R}$$

$$= 2 \log \rho, \quad (\rho > 1).$$

Therefore, the Chebyshev polynomials of the first kind,

$$\begin{aligned} \mathcal{P}_n^*(z) &= 2 \left(\frac{n}{\pi}\right)^{1/2} (\rho^{2n} - \rho^{-2n})^{-1/2} T_n(z), \quad n > 0 \\ \mathcal{P}_0^*(z) &= (2\pi \log \rho)^{-1/2}, \end{aligned} \tag{21}$$

are area integral orthonormal over $\hat{\mathcal{E}}_\rho$ with weight $|1-z^2|^{-1}$.

4. Relations between Real, Line and Area Integrals of an Analytic Function

Assume that $f \in A(\bar{\mathcal{E}}_\rho)$, $\rho > 1$. Then f possesses the uniformly convergent Fourier expansions: On $-1 \le x \le 1$,

$$f(x) = \sum_{n=0}^{\infty} a_n^* p_n^*(x) = \sum_{n=0}^{\infty} \left(\frac{a_n^*}{\| P_n^{(\alpha,\beta)} \|} \right) P_n^{(\alpha,\beta)}(x) \tag{22}$$

where

$$a_n^* = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x) p_n^*(x) dx \tag{23}$$

On $\bar{\mathcal{E}}_\rho$:

$$f(z) = \sum_{n=0}^{\infty} b_n^* P_n^*(z) = \sum_{n=0}^{\infty} \left(\frac{b_n^*}{\| p_n^* \|_{\mathcal{E}_\rho} \| P_n^{(\alpha,\beta)} \|} \right) P_n^{(\alpha,\beta)}(z) \tag{24}$$

where

$$b_n^* = \int_{\mathcal{E}_\rho} |1-z|^\alpha |1+z|^\beta f(z) \overline{P_n^*(z)} |dz| \tag{25}$$

and also the Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} c_n^* \mathcal{P}_n^*(z) = \sum_{n=0}^{\infty} \left(\frac{c_n^*}{(\beta_n(\rho))^{1/2} \| P_n^{(\alpha,\beta)} \|} \right) P_n^{(\alpha,\beta)}(z) \tag{26}$$

where

$$c_n^* = \iint_{\hat{\mathcal{E}}_\rho} |1-z|^{\alpha-1/2} |1+z|^{\beta-1/2} f(z) \overline{\mathcal{P}_n^*(z)} \, dx\, dy \tag{27}$$

Since all these Fourier expansions (22), (24) and (26) of the same $f \in A(\bar{\mathcal{E}}_\rho)$ are identical on $-1 \le z \le 1$, the coefficients of $P_n^{(\alpha,\beta)}(z)$ in these expansions must be the same. Equating the coefficients in (22) and (24),

$$\| p_n^* \|_{\mathcal{E}_\rho} a_n^* = b_n^* , \qquad n \geq 0 \tag{28}$$

Substituting for a_n^* and b_n^* from (23) and (25), we have

$$(29)\qquad \int_{\mathcal{E}_\rho} |1-z|^{\alpha}|1+z|^{\beta} f(z)\,\overline{p_n^*(z)}\,|dz| = \|p_n^*\|^2_{\mathcal{E}_\rho} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\,p_n^*(x)\,dx$$

For n=0, (29) gives

$$(30)\qquad \int_{\mathcal{E}_\rho} |1-z|^{\alpha}|1+z|^{\beta} f(z)\,|dz| = \|p_0^*\|^2_{\mathcal{E}_\rho} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\,dx$$

since p_0^* is a constant.

Next, equating the coefficients in the expansions (24) and (26), we obtain

$$(31)\qquad (\beta_n(\rho))^{1/2}\, b_n^* = \|p_n^*\|_{\mathcal{E}_\rho}\, c_n^*, \qquad n \geq 0$$

Substituting for b_n^* and c_n^* from (25) and (27), we obtain

$$(32)\qquad \beta_n(\rho)\int_{\mathcal{E}_\rho} |1-z|^{\alpha}|1+z|^{\beta}\,\overline{p_n^*(z)}\, f(z)\,|dz| = \|p_n^*\|^2_{\mathcal{E}_\rho} \iint_{\hat{\mathcal{E}}_\rho} |1-z|^{\alpha-1/2}|1+z|^{\beta-1/2}\,\overline{p_n^*(z)}\, f(z)\,dx\,dy$$

For n=0, (32) gives

$$(33)\qquad \beta_0(\rho)\int_{\mathcal{E}_\rho} |1-z|^{\alpha}|1+z|^{\beta} f(z)\,|dz| = \|p_0^*\|^2_{\mathcal{E}_\rho} \iint_{\hat{\mathcal{E}}_\rho} |1-z|^{\alpha-1/2}|1+z|^{\beta-1/2} f(z)\,dx\,dy$$

Finally, comparing (30) and (33) we obtain

$$(34)\qquad \beta_0(\rho)\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\,dx = \iint_{\hat{\mathcal{E}}_\rho} |1-z|^{\alpha-1/2}|1+z|^{\beta-1/2} f(z)\,dx\,dy$$

We note the following special cases of (34).

(i) $\alpha=\frac{1}{2},\ \beta=\frac{1}{2}$:

$$(35)\qquad \tfrac{1}{2}(\rho^2-\rho^{-2})\int_{-1}^{1} (1-x^2)^{1/2} f(x)\,dx = \iint_{\hat{\mathcal{E}}_\rho} f(z)\,dx\,dy$$

This particular relation was derived by Davis[5] in connection with simple quadratures.

(ii) $\alpha = -\frac{1}{2}$, $\beta = -\frac{1}{2}$:

(36) $$2 \log \rho \int_{-1}^{1} (1-x^2)^{-1/2} f(x)\,dx = \iint_{\hat{\mathcal{E}}_\rho} |1-z^2|^{-1} f(z)\,dx\,dy$$

(iii) $\alpha = 0$, $\beta = 0$:

(37) $$\beta_0(\rho) \int_{-1}^{1} f(x)\,dx = \iint_{\hat{\mathcal{E}}_\rho} |1-z^2|^{-1/2} f(z)\,dx\,dy$$

where

$$\beta_0(\rho) = \frac{1}{2} \int_1^{\rho} \frac{L(\mathcal{E}_R)}{R}\,dR$$

and $L(\mathcal{E}_R)$ = length of $\mathcal{E}_R$.

5. Errors of Numerical Approximation for Analytic Functions through Cauchy's Integral Formula

For $f \in A(\overline{\mathcal{E}}_\rho)$, $\rho > 1$, we have from Cauchy's integral formula

(38) $$f(x) = \frac{1}{2\pi i} \int_{\mathcal{E}_\rho} \frac{f(z)\,dz}{z-x}$$

$x \in [-1,1]$; and, on $\mathcal{E}_\rho$, $f(z)$ possesses the uniformly convergent Fourier expansion (24) in terms of the line integral orthonormal Jacobi polynomials $P_n^*(z)$. Substituting (24) in (38), we have

(39) $$f(x) = \sum_{k=0}^{\infty} b_k^* \frac{p_k^*(x)}{\| p_k^* \|_{\mathcal{E}_\rho}}$$

since $P_k^*(z) = p_k^*(z) / \| p_k^* \|_{\mathcal{E}_\rho}$. Also, from the equation of closure,

(40) $$\| f \|^2_{\mathcal{E}_\rho} = \int_{\mathcal{E}_\rho} |1-z|^{\alpha} |1+z|^{\beta} |f(z)|^2\,ds = \sum_{k=0}^{\infty} | b_k^* |^2$$

If E denotes the error of a linear process of numerical approximation for an $f \in (\overline{\mathcal{E}}_\rho)$, $\rho > 1$, then from (39)

(41) $$E(f) = \sum_{k=0}^{\infty} b_k^* \frac{E(p_k^*(x))}{\| p_k^* \|_{\mathcal{E}_\rho}}$$

From Schwarz inequality the error E(f) can now be bounded by

$$(42) \qquad |E(f)| \leq \|f\|_{\mathcal{E}_\rho} \left(\sum_{k=0}^{\infty} \frac{|E(p_k^*(x))|^2}{\|p_k^*\|^2_{\mathcal{E}_\rho}} \right)^{1/2}$$

If $\|f\|_{\mathcal{E}_\rho}$ is estimated from

$$(43) \qquad \|f\|_{\mathcal{E}_\rho} \leq M(\mathcal{E}_\rho) \left(\int_{\mathcal{E}_\rho} |1-z|^\alpha |1+z|^\beta \, ds \right)^{1/2}$$

where $M(\mathcal{E}_\rho) = \max |f(z)|$ on $\mathcal{E}_\rho$, then we can estimate the error from:

THEOREM 3. Let $f \in A(\bar{\mathcal{E}}_\rho)$, $\rho > 1$. Then,

$$(44) \qquad |E(f)| \leq \left(\int_{\mathcal{E}_\rho} |1-z|^\alpha |1+z|^\beta \, ds \right)^{1/2} \left(\sum_{k=0}^{\infty} \frac{|E(p_k^*(x))|^2}{\|p_k^*\|^2_{\mathcal{E}_\rho}} \right)^{1/2} M(\mathcal{E}_\rho)$$

For various error functionals of numerical analysis one can use a fixed Jacobi weight function and the corresponding orthonormal system of polynomials $P_n^*(z)$ in the estimate (44); however, for certain particular error functionals it might be more appropriate to use the related Jacobi weight function and the orthonormal system of polynomials. A striking example is the following.

Let $E_n(f)$ denote the error of an n-point Gauss-Jacobi quadrature formula with weight $(1-x)^\alpha (1+x)^\beta$:

$$(45) \qquad E_n(f) = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)\,dx - \sum_{k=1}^{n} \lambda_{n,k} f(x_{n,k})$$

Here $x_{n,k}$, k=1, ... , n, are the n real and distinct abscissas which are the zeros of the corresponding Jacobi polynomial $p_n^*(x)$ and $\lambda_{n,k} = (1/p_n^{*\prime}(x_{n,k}))^{-1} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta (p_n^*(x)/(x - x_{n,k}))dx$, k=1, ... , n, are the corresponding positive Christoffel numbers. Since $E_n(f) = 0$ whenever f is a polynomial of degree $\leq$ 2n-1, we have from (44) the following estimate for the error of the Gauss-Jacobi quadrature formula.

THEOREM 3a.

$$|E_n(f)| \le \left(\int_{\mathcal{E}_\rho} |1-z|^\alpha |1+z|^\beta \, ds \right)^{1/2} \left(\sum_{k=2n}^{\infty} \frac{|E_n(p_k^*(x))|^2}{\| p_n^* \|^2_{\mathcal{E}_\rho}} \right)^{1/2} M(\mathcal{E}_\rho) \tag{46}$$

6. Asymptotic Estimates

The second factor on the right side of (46) can be recognized as the line integral norm of E_n over $H^2(\mathcal{E}_\rho;\ |1-z|^\alpha\ |1+z|^\beta)$:

$$\| E_n \|^2_{\mathcal{E}_\rho} = \sum_{k=2n}^{\infty} |E_n(P_k^*)|^2 \tag{47}$$

Similarly, the area integral norm of E_n over $L^2(\mathcal{E}_\rho;\ |1-z|^{\alpha-\frac{1}{2}}\ |1+z|^{\beta-\frac{1}{2}})$ can be obtained from the Riesz representation theorem (Davis [3], (9.3.13)) in terms of the area orthonormal polynomials $\mathcal{P}_n^*(z)$:

$$\| E_n \|^2_{D} = \sum_{k=2n}^{\infty} |E_n(\mathcal{P}_k^*)|^2 \tag{48}$$

Since for $z \in \mathcal{E}_\rho$, $\rho > 1$, and n sufficiently large (Szego [8], (8.21.9)),

$$P_n^{(\alpha,\beta)}(z) \simeq \frac{2^{\alpha+\beta}}{(n\pi)^{1/2}} (\xi-1)^{-\alpha-1/2} (\xi+1)^{-\beta-1/2} \xi^{n+\alpha+\beta+1} \tag{49}$$

it follows that

$$\| P_n^{(\alpha,\beta)} \|^2_{\mathcal{E}_\rho} \simeq \frac{2^{\alpha+\beta}}{n} \rho^{2n+\alpha+\beta+1} \tag{50}$$

and since for large n, $\| P_n^{(\alpha,\beta)} \|^2 \simeq 2^{\alpha+\beta}/n$, therefore

$$\| p_n^* \|^2_{\mathcal{E}_\rho} \simeq \rho^{2n+\alpha+\beta+1} \tag{51}$$

From (11) and (51),

$$\beta_n(\rho) \simeq (2n)^{-1} \rho^{2n+\alpha+\beta+1} . \tag{52}$$

therefore from (10) we have for large n and $\rho > 1$,

$$\mathcal{P}_n^*(z) \simeq (2n)^{1/2} P_n^*(z) \tag{53}$$

Now, from (48) and (53), for large n,

(54) $$\|E_n\|_D^2 \simeq 2 \sum_{k=2n}^{\infty} k \, |E_n(P_k^*)|^2$$

and since

(55) $$\|E_n\|_{\mathcal{E}_\rho}^2 \leq \frac{1}{2n} \sum_{k=2n}^{\infty} k \, |E_n(P_k^*)|^2$$

Combining (54) and (55) it follows that for large n,

(56) $$\|E_n\|_{\mathcal{E}_\rho} \lesssim \frac{1}{2}(n)^{-1/2} \|E_n\|_D$$

Thus, the area integral norm will overestimate the errors in Davis' method([4]). See also Chawla [6].

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