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# COUPLED TRANSMISSION LINES

R. E. Worsham and S. W. Mosko

## Introduction

In a discussion about the design of cyclotron resonators, the suggestion has been made[1] that a wide range variable-frequency system might be constructed with two coupled resonant circuits. It also would present the possibility of tuning without the switching of high-current contacts, a troublesome item now commonly used for cyclotrons. One of the circuits would be the dee and its stem which might, or might not, be tuned. The second circuit could be located out of the magnet gap in a region where more space is usually available. Tuning could be accomplished by variation of the coupling between the two circuits and/or by, say, variable capacitance of the tuned, or tunable, circuit.

## Analysis

In any case, it was necessary to solve the problem of two coupled transmission lines[2] subject to a peculiar set of boundary conditions. Consider two lossless parallel transmission lines surrounded by a common outer conductor. They may be treated from an electrical circuit viewpoint to have distributed inductance and capacitance along their length.

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[1] K. R. MacKenzie, private communication.

[2] A similar problem has been treated extensively for cross-talk problems in telephone lines and for directional couplers. See, for example, B. M. Oliver, Proc. I. R. E. 42, 1686 (Nov. 1954).

The physical arrangement and the definition of the line constants are shown in Fig. 1. Note that the polarity of the voltages is positive from line to sheath, and the polarity of the currents is positive when each is flowing into the coupled region.

From Kirchoff's Laws, the voltage and current increase per unit length along the line are given by:

$$
\begin{aligned}
\frac{\partial v_1}{\partial z} &= -L_{11}\frac{\partial i_1}{\partial t} - L_m\frac{\partial i_2}{\partial t} \\
\frac{\partial i_1}{\partial z} &= -C_1\frac{\partial v_1}{\partial t} - C_m\frac{\partial (v_1 - v_2)}{\partial t} \\
\frac{\partial v_2}{\partial z} &= -L_{22}\frac{\partial i_2}{\partial t} - L_m\frac{\partial i_1}{\partial t} \\
\frac{\partial i_2}{\partial z} &= -C_2\frac{\partial v_2}{\partial t} - C_m\frac{\partial (v_2 - v_1)}{\partial t}
\end{aligned}
\qquad (1)
$$

Since only the steady state solution to these equations is of interest here, the following substitutions are made:

$$v_{1,2} = V_{1,2}\, e^{j\omega t}$$

$$i_{1,2} = I_{1,2}\, e^{j\omega t}$$

Elimination of $I_1$ and $I_2$ then leads to two second-order simultaneous linear differential equations with constant coefficients in terms of $V_1$ and $V_2$:

$$
\begin{aligned}
&\frac{d^2 V_1}{dz^2} + \omega^2 V_1\left[L_{11}C_{11} - L_m C_m\right] + \omega^2 V_2\left[L_m C_{22} - L_{11}C_m\right] = 0 \\
&\frac{d^2 V_2}{dz^2} + \omega^2 V_2\left[L_{22}C_{22} - L_m C_m\right] + \omega^2 V_1\left[L_m C_{11} - L_{22}C_m\right] = 0
\end{aligned}
\qquad (2)
$$

These equations have a solution of the form

$$V_1 = Ae^{p_1 z} + Be^{p_2 z} + Ce^{p_3 z} + De^{p_4 z}$$

where the four values of p are the four roots the determinant of Eq. (2). The four waves, as represented by the four values of p, must all have the same propagation constant. An essential requirement, then, is that $L_{11}C_{11} = L_{22}C_{22}$ and that $k_L = k_C = k$ where

$$k_L = \frac{L_m}{\sqrt{L_{11} L_{22}}} \quad \text{and} \quad k_C = \frac{C_m}{\sqrt{C_{11} C_{22}}} \tag{3}$$

Then, only two values of p remain; they are

$$p_{1,2} = \pm j\omega\sqrt{L_{11}C_{11}(1-k^2)} \tag{4}$$

From the coupling term coefficients in Eq. (2), using Eq. (3)

$$L_m C_{22} - L_{11} C_m = k\sqrt{L_{11} C_{22}}\left(\sqrt{L_{22} C_{22}} - \sqrt{L_{11} C_{11}}\right)$$
$$\equiv 0.$$

The coupling term vanishes, thus

$$\begin{aligned} V_1 &= Ae^{pz} + Be^{-pz} \\ V_2 &= Ce^{pz} + De^{-pz} \end{aligned} \tag{5}$$

where the value of p and the propagation constant n are given by

$$p = jn = j\omega\sqrt{L_{11}C_{11}(1-k^2)} = j\omega\sqrt{L_{22}C_{22}(1-k^2)} \qquad (6)$$

Boundary Conditions

For the particular application in a cyclotron resonator, the coupled transmission lines will have uniform coupling over a length, $\ell$, and will be grounded at one end. Thus:

$$\begin{aligned} &\text{at } Z = 0: \quad V_1 = V_{01},\ V_2 = V_{02} \\ &\text{at } Z = \ell: \quad V_1 = 0,\ V_2 = 0. \end{aligned} \qquad (7)$$

Evaluation of the constants in Eq. (5) leads to:

$$\begin{aligned} V_1 &= V_{01}\,\frac{\sin n(\ell - Z)}{\sin n\ell} \\ V_2 &= V_{02}\,\frac{\sin n(\ell - Z)}{\sin n\ell} \quad . \end{aligned} \qquad (8)$$

Therefore, the ratio of voltages at any point, Z, on the lines is the same.

Now to find the currents in the two lines as a function of their length, from Eq (1):

$$\frac{dI_1}{dz} = -j\omega C_{11}V_1 + j\omega C_m V_2$$

and similarly for $I_2$. Then

$$I_1 = -j\omega\left(C_1 V_{01} - C_m V_{02}\right)\frac{\cos n(l-z)}{n \sin nl}$$

or

$$I_1 = -j\frac{1}{\sqrt{\frac{L_{11}}{C_{11}}(1-k^2)}}\left[V_{01} - k V_{02}\sqrt{\frac{L_{11}}{L_{22}}}\right]\frac{\cos n(l-z)}{\sin nl}$$

Also

$$I_2 = -j\frac{1}{\sqrt{\frac{L_{22}}{C_{22}}(1-k^2)}}\left[V_{02} - k V_{01}\sqrt{\frac{L_{22}}{L_{11}}}\right]\frac{\cos n(l-z)}{\sin nl} \tag{9}$$

The secondary quantities $V_{02}$ and $I_{02}$ may be stated in terms of the primary quantities, $V_{01}$ and $I_{01}$:

$$V_{02} = \frac{V_{01}\sqrt{\frac{C_{11}}{C_{22}}} - j I_{01} \tan nl \sqrt{\frac{L_{11}}{C_{22}}(1-k^2)}}{k} \tag{10}$$

and

$$I_{02} = \frac{-I_{01}\sqrt{\frac{C_{22}}{C_{11}}} - j\frac{V_{01}}{\tan nl}\sqrt{\frac{C_{11}}{L_{22}}(1-k^2)}}{k} \tag{11}$$

<u>Solution for Currents and Voltages with Coupling Near Unity</u>

The capacitive coefficient of coupling, Eq. (3) may be re-written

$$k_C = \frac{C_m}{\sqrt{(C_m + C_1)(C_m + C_2)}} \tag{12}$$

The value of $k_C$ is made to approach unity by increasing $C_m$ so that as $k_C \to 1$, $\frac{C_1}{C_m}$ and $\frac{C_2}{C_m} \to 0$. Also, the inductive coefficient of coupling has to approach unity, as was shown previously. As it does so, the number of magnetic lines or tubes of flux that surround just one of the transmission lines approaches zero so that the magnetic flux must surround both lines, making $L_{11} \to L_m$ and $L_{22} \to L_m$.

The propagation constant may be written out as

$$n = \omega\sqrt{L_{11}\,\frac{C_1C_2 + C_m(C_1+C_2)}{C_m+C_2}} \tag{13}$$

as $k \to 1$, $L_{11} \to L_m$, $\frac{C_1}{C_m} \to 0$, $\frac{C_2}{C_m} \to 0$, so

$$n \to \omega\sqrt{L_m(C_1+C_2)} \quad (\text{as } k \to 1) \tag{14}$$

which is the propagation constant for the two lines electrically connected at every point.

In the limit Eq. (10) gives

$$V_{02} \to V_{01} \quad (\text{as } k \to 1). \tag{15}$$

For Eq. (11), however, we must first evaluate

$$Z = k\sqrt{\frac{L_{22}}{C_{11}(1-k^2)}} \tag{16}$$

$$Z = \sqrt{\frac{L_{22}}{\left(1+\frac{C_1}{C_m}\right)\left(\frac{C_1 C_2}{C_m}+C_1+C_2\right)}} \tag{17}$$

*in the limit as* $k \rightarrow 1$

$$Z \rightarrow Z_{0\ell} = \sqrt{\frac{L_m}{C_1+C_2}} \tag{18}$$

where $Z_{0\ell}$ is the characteristic impedance the coupled line would have if the two lines were electrically connected at every point.

Finally, then, as $k \rightarrow 1$

$$I_{02} \rightarrow -I_{01} + \frac{V_{01}}{jZ_{0\ell}\tan nl} \tag{19}$$

## Experimental Work

An r-f resonator with coupled transmission lines was built as shown in Fig. 2. The characteristic impedance of line A was 15Ω. Inside the box, the characteristic impedance of the lines was 50Ω, and both lines were shorted at C. Various coefficients of coupling could be obtained between points "e" and "C" by changing the spacing between lines, and by varying the geometry of the lines in that region. The system was designed to be in resonance at approximately 90 Mc/s if the coupling coefficient were unity. The system has other resonant frequencies which are not necessarily of interest.

The resonator described above is to be used for verifying the theory of coupled transmission lines. From the discussion above, as $k \longrightarrow 1$, the voltages and currents should satisfy Eqs. (15) and (19). For k other than unity, we may obtain from Eq. (9):

$$k = \frac{\dfrac{V_{01}}{V_{02}}\sqrt{\dfrac{C_{11}}{C_{22}}} - \dfrac{I_{01}}{I_{02}}\sqrt{\dfrac{L_{11}}{L_{22}}}}{1 - \dfrac{I_{01}}{I_{02}} \cdot \dfrac{V_{01}}{V_{02}}} \qquad (20)$$

From Eq. (3), k may be predicted for a given system. If we measured the voltage standing waves in the system, and computed the characteristic impedance of the lengths of line outside the coupled region from their geometry, we would be able to find values for $V_{01}$, $V_{02}$, $I_{01}$, and $I_{02}$[3]. These values may be used in Eq. (20) to give an experimental value of k.

If we use semi-cylindrical geometry for the coupled section, as shown in Fig. 3(a), we can obtain better coupling than for cylinders as shown in Fig. 3(b) when the separation of the lines is small. For example, in the semi-cylindrical case, if each line was 1/2 of a 3/8-in. round rod, and with a line spacing of 0.010 in. we get a value of $C_m$ = 332 μμf/meter and k = 0.907. Similarly, for 3/8-in. cylinders with the same 0.010-in. spacing, we get $C_m$ = 9.4 μμf/meter and k = 0.123.

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[3]Bronwell, A. B., and Beam, R. E., <u>Theory and Application of Microwaves</u>, McGraw-Hill, New York, Chapter 8.

Measurements

A series of four sets of measurements was made. Experimental data was obtained for the resonator as follows:

1. The line geometry was that of Fig. 3(a), and the clearance between the lines was set to 0.045 in; thus, $C_m = 73.7$ μμf/m. The voltage standing wave was measured at various points along the line. When the observed VSW values were substituted in the transmission line equations, the voltages and currents at the beginning of the coupled line region were found to be:

| | |
|---|---|
| $V_{01} = 2.8$ volts | $I_{01} = 0.754$ amperes |
| $V_{02} = 22$ volts | $I_{02} = 1.264$ amperes |

The values of $I_{02}$ and $V_{02}$ from Eqs. (10), (11) were:

$I_{02} = 1.20$ amperes

$V_{02} = 23.3$ volts.

Also, the values of k were:

k from geometry of coupled section = 0.685

k computed from measurements = 0.67.

The resonant frequency of the system for the above case was 73 Mc/s. The circuit has two modes of oscillation of which this was the upper mode; the lower mode was not checked.

2. With the same geometry as above, a polystyrene dielectric was inserted, and the gap was reduced to 0.001 in. The dielectric constant was 2.56. Then,

$C_m = 8300$ μμf

$k_C = 0.996$.

Actually, there was some air between the lines due to warping. If this air gap were another 0.001 in., the mutual capacity, $C_m$, would be about 2330 $\mu\mu f$, and $k_C = 0.986$. From the VSW data:

$V_{01} = 3.6$ volts $\qquad I_{01} = .693$ amperes

$V_{02} = 11.6$ volts $\qquad I_{02} = .775$ amperes

From these measurements, we compute, $k = 0.955$. The resonant frequency for this test was 80 Mc/s. The computed and measured values of k agree within about 3%, which is better than the accuracy of voltage measurements. The inaccuracy in the gap has only a slight effect on k, since $C_m >> C_1$ or $C_2$.

3. An effort was made to eliminate the air gap leaving only the polystyrene dielectric between the coupled lines. As a result, the resonant frequency was raised to 84 Mc/s.

From VSW data:

$V_{01} = 5.0$ volts $\qquad I_{01} = 0.752$ amperes

$V_{02} = 8.3$ volts $\qquad I_{02} = 0.700$ amperes

From these measurements, $k_C = 1.00$; the predicted value of k from geometry was 0.996.

4. Another effort was made to reduce the air space around the polystyrene dielectric. As a result, the uniformity of the spacing improved as the following measurements show:

$V_{01} = 4.93$ volts $\qquad I_{01} = 0.814$ amperes

$V_{02} = 6.00$ volts $\qquad I_{02} = 1.039$ amperes

At a point 2 in. from where coupling began:

$V_{11} = 4.95$ volts $\qquad\qquad V_{22} = 5.05$ volts, and

$I_{02}$ predicted by Eq. (19) is: 0.970 amperes.

## Conclusion

The tests on the coupled transmission lines resonator show that we are able to predict a fairly accurate value of the coupling coefficient by considering line geometry. The measured values of $V_{02}$ and $I_{02}$ appear to agree with the values predicted by Eqs. (10), (11), and (19).

Development of a resonator for ORIC with two coupled transmission lines is now under way. Details on the performance of the resonator will be given in a report in the near future.

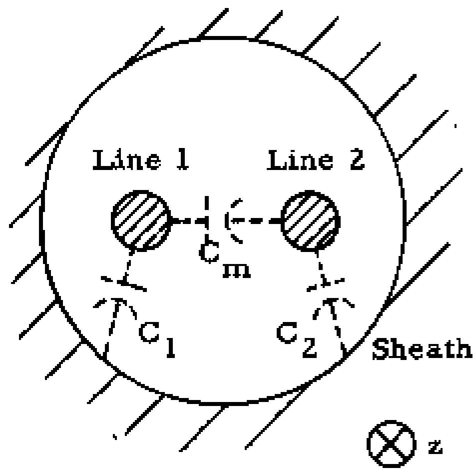


Fig. 1 (a)

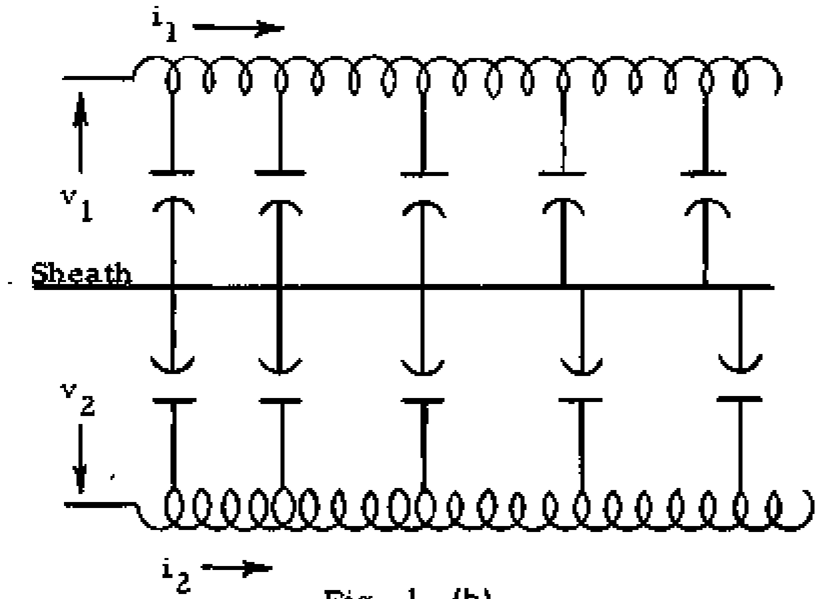


Fig. 1 (b)

$C_m$ = mutual capacitance/unit length

$C_{1,2}$ = line-to-sheath capacity/unit length of line 1, 2

$C_{11} = C_m + C_1$

$C_{22} = C_m + C_2$

$L_m$ = mutual inductance/unit length

$L_{1,2}$ = self-inductance of line 1, 2/unit length

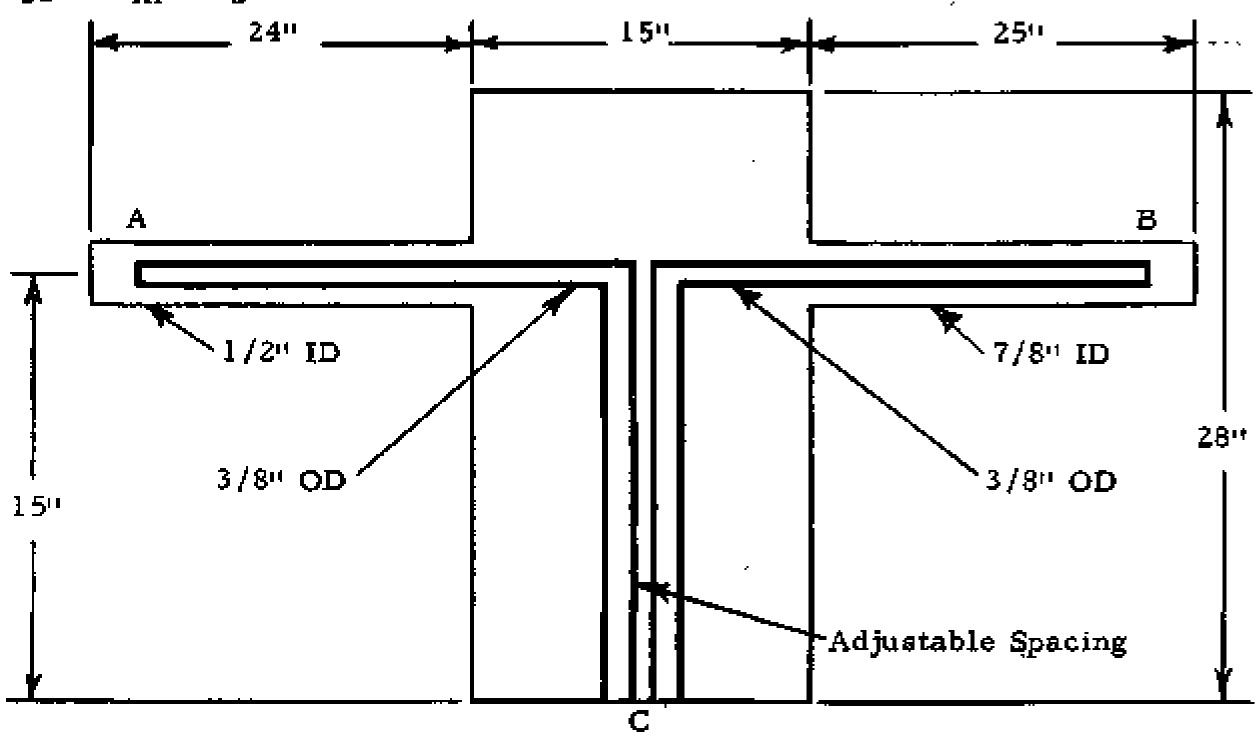


Fig. 2. Coupled-Line Resonator

(a) (b)

Fig. 3. Cross Section for Coupled Lines