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STRONG TURBULENCE AND THE ANOMALOUS
LENGTH OF STORED PARTICLE BEAMS

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ABSTRACT

A theoretical analysis is made of the longitudinal stability of intense beams of charged particles in high energy storage rings. The effect of externally applied radiofrequency fields, i.e., synchrotron oscillations, is included. For electron and positron beams, the damping and quantum excitation due to particle radiation are also included. A criterion for the stability of small longitudinal plasma oscillations is derived and used to obtain the stable size of intense stored beams of electrons and positrons.

Anomalous lengthening of stored electron and positron beams has been experimentally measured. Previous theories have assumed that the effect is due to modification of the potential-well due to self-fields or to the excitation of coherent synchrotron oscillations. One previous calculation assumed that the effect was due to strongly turbulent collective motion, but failed to take into account the effect of the external radio-frequency fields, i.e., bunching. We calculate the bunch lengthening under

the assumption that turbulent oscillations are present and include the effect of bunching and associated synchrotron oscillations. We also include the effect of the modification of the potential-well due to self-fields.

Since strong turbulence is assumed to be partially responsible for bunch lengthening, we first derive, using a random phase approximation, the basic equation of one-dimensional strong turbulence theory. It is then shown, using a criterion due to Chirikov, that particle trajectories are stochastic and that this leads to the mixing of wave phases assumed in the random phase approximation.

The dispersion relation for longitudinal perturbations of turbulent uniform beams is derived and compared with previous results. The integral equation for perturbations of bunched proton beams without turbulence is derived, and the dispersion relation for the short-wavelength case is given. Landau damping is included in an approximate fashion.

The integral equation for perturbations of electron and positron beams is presented. The dispersion relation for short-wavelength perturbations is derived. By assuming that the level of turbulence will increase to a point where small perturbations are stable, a formula for bunch length is derived which, typically, gives values which are greater than the natural bunch length and which depend on current.

The formula for bunch length is compared with experimental data from SPEAR I and from SPEAR II. The data from SPEAR I can be fit very well if sizeable contributions from both turbulence and self-field distortion of the potential well are included. The data from SPEAR II can be fit very well by only including the effect of turbulence, in

probable accord with the absence of ferrite magnets in SPEAR II. The possible relation of a saturation phenomenon, observed on SPEAR II, to nonlinear potential well distortion and to mode-mode coupling is discussed.

I. INTRODUCTION

In the presence of radiofrequency fields, charged particles in storage rings, with energies different from the average energy, undergo longitudinal synchrotron oscillations about a synchronous orbit. Electrons and positrons, in addition, radiate significantly, and this tends to damp out the synchrotron oscillations. However, because the radiation is quantum mechanical, it is "noisy", and this noise causes diffusion. The natural length of electron and positron beams is due to the balance of radiation damping with quantum diffusion and is easily calculated. Charged particles also undergo transverse oscillations, betatron oscillations, about a "design" orbit as a result of magnetic focusing forces. For electrons and positrons, radiation damping and quantum excitation determine part of the natural width for the beams. However, because the radial position of a charged particle depends on energy, the energy spread of the beam contributes to the width. Given the parameters of the particular machine, the natural bunch width can be easily calculated.

It has been observed at Orsay, Frascati,¹ and at Stanford, both with SPEAR I² and with SPEAR II,³ that bunches are anomalously long and that this effect depends on the total current in the bunch. In addition, experiments have shown that the effect is independent of the number of bunches present. Thus, the effect seems to be local.

With SPEAR II, an anomalous, current-dependent widening has also been measured.³ This suggests that an increase in energy spread may contribute to the bunch lengthening phenomenon. A threshold in current has also been observed.

These observations have stimulated considerable theoretical effort,

and three types of theories have been developed. In the first type, the equilibrium theories, the bunch lengthening effect is assumed to be due to the modification of the potential well that individual particles see by the self-fields of the bunch. A general formulation of these theories in linear approximation has been developed by Pellegrini and Sessler.⁴ More recently, the effect on the bunch length of the nonlinear distortion of the potential well due to self-fields has been considered by Keil.⁵ These theories have dependences on current, beam energy, and RF voltage that are roughly in agreement with the data. However, because self-fields are present even at very small currents, these theories cannot account for the observed threshold effect in current. In addition, these theories predict no increase in energy width, and thus cannot account for the observed bunch widening.

The second type of theory, due to Lebedev,⁶ assumes that bunch lengthening is due to the excitation of coherent synchrotron oscillations in the bunch. In response, the energy spread of the bunch increases until the coherent oscillations are stabilized. In this theory, the effect has dependences on current and RF voltage which are in reasonable agreement with the data. In addition, the threshold effect and bunch widening are easily explained by this theory. The threshold effect is simply due to the fact that below a certain current the beam is naturally stable against coherent oscillations. The bunch widening is due to the increase in energy spread. However, this theory has too strong a dependence on beam energy to fit the data.

The third type of theory, related to the second type, was developed by Sessler.⁷ In this theory, if the bunch is unstable against coherent oscillations, a turbulent state develops which increases the diffusion

coefficient, and thus the energy width, until the oscillations are stabilized. The lengthened steady state is thus assumed to be turbulent. The stability criterion is calculated using the strong turbulence theory of Dupree⁸ and others.⁹⁻¹¹ However, the calculation was done for a uniform, i.e., unbunched, beam, and the results were applied in an ad hoc manner to bunched beams. Thus, although the fits to the data were fairly good, the need for a calculation for bunched beams was evident. In fact, we will see that the good fit to the data that this theory gave was, to a certain extent, fortuitous.

In this thesis, we calculate the bunch lengthening effect for bunched beams by using strong turbulence theory. In addition, we include the equilibrium effect, i.e., the potential well distortion due to self-fields. We find that the data from SPEAR I can be fit very well only if we include large contributions from both turbulence and equilibrium effects. The SPEAR II data, on the other hand, can be fit by assuming only turbulence. This is consistent with the observation of a threshold in current at SPEAR II. In addition, the absence of equilibrium effect for SPEAR II is reasonable when the differences between SPEAR I and SPEAR II are taken into account, i.e., the lack of ferrite kicker magnets and the different RF system for SPEAR II.

As can be seen, our calculation encompasses all three previous types of theories. An effect which is not explicitly included in our calculation is the nonlinear distortion of the potential well due to self-fields. We will comment on the possible contribution of this when we examine the data from SPEAR II. Also not included is the effect of mode-mode coupling, but we will comment on its possible contribution.

In Section II, we derive the strong turbulence theory by a Fokker-

Planck technique. Although our derivation is considerably less general than other derivations, the calculation serves to point out the physical assumptions involved and establishes the basic equation which we use. In Section III, the random phase approximation, which was used in deriving the strong turbulence theory, is examined. In particular, it is argued that, under reasonable physical circumstances, trajectories of particles interacting with short wavelength waves are stochastic, i.e., random, and that this leads to a mixing of wave phases.

In Section IV, we convert our integro-differential equation into an integral equation. In Section V, this integral equation is used to derive the dispersion relation for uniform beams. Thus, we reproduce Sessler's previous result.⁷ In Section VI, we convert our integral equation into a simpler integral equation in the case of bunched beams. In Section VII, this integral equation is applied to proton beams. Also, the Landau damping due to nonlinear potential wells is discussed.

In Section VIII, the electron beam case is solved and the dispersion relation is obtained. From this, a formula for the bunch length follows. In Section IX, our formula for bunch length is compared with experimental data from SPEAR I and from SPEAR II. We discuss possible additional effects and suggest experiments to clarify the situation. In Section X, we summarize our results and present our conclusions.

In Appendices A and B, some relevant mathematical calculations are presented. In Appendix C, an alternate derivation for the uniform beam case is presented, and the various results are shown to agree with each other. In Appendix D, a function necessary for the electron beam case is calculated.

II. STRONG TURBULENCE THEORY

In this section we derive our basic equation from strong turbulence theory. Weak turbulence is characterized by the fact that the time for a particle to diffuse through a typical wavelength, τ_{TR} in Dupree's notation,⁸ is long compared to the autocorrelation time, τ_{AC} , of the turbulent electric field; thus $\tau_{TR} \gg \tau_{AC}$. This means that coherent modes "see" a quiescent plasma and can be treated by the usual Vlasov theory (in the collisionless regime that we are concerned with). Only the averaged distribution function is affected by turbulence. In the strong turbulence case, on the other hand, $\tau_{AC} \gg \tau_{TR}$, and coherent modes "see" a turbulent plasma. Thus, in strong turbulence, the dispersion relation for coherent modes is affected by turbulence. The typical effect is to change the usual resonance function, $\delta(\omega - kv)$, into a function $R(\omega, k, v)$, with finite width. We will assume that we are always in the strong turbulence regime, if turbulence is present at all.

To begin our derivation, we write down the equations of motion for a particle. We let θ denote the angle around the machine of a particle measured from the position of the "synchronous" particle. We let ΔE denote the energy deviation of a particle from the average energy. We let f denote the revolution frequency of the bunch. We then define

$$w = \frac{\Delta E}{f} \quad (2.1).$$

The equation for θ then becomes

$$\frac{d\theta}{dt} = 2\pi \frac{df}{dE} \Delta E = k_0 w \quad (2.2),$$

where

$$k_0 = 2\pi f \frac{df}{dE} = \frac{2\pi f^2}{E} \alpha \quad (2.3),$$

where α is the usual momentum compaction factor and depends on the machine parameters. The equation for w is

$$\frac{dw}{dt} = 2\pi R \operatorname{Re} \mathcal{E}(\theta, t) - \frac{w}{\tau_r} + g(t) - \frac{\Omega^2}{k_0} \theta \quad (2.4),$$

where R is the radius of the machine, τ_r is the radiation damping time, and Ω is the synchrotron oscillation frequency. In Eq. (2.4), the first term is the effect of various electric fields on the particle, and the second term is the effect of radiation damping. The third term, $g(t)$, is a random function with zero mean which represents the effect of quantum noise. The last term in (2.4) represents the synchrotron oscillations. In this term we include not only the effect of the RF fields, which is given by a synchrotron frequency Ω_0 , but also the linear effect of the self-fields on the synchrotron motion. Thus, we explicitly include the equilibrium effects in our calculation. As an equation for Ω , we will use the general expression given by Pellegrini and Sessler.⁴

We let $\psi(\theta, w, t)$ denote the distribution function for particles.

Then,

$$\psi(\theta, w, t + \Delta t) = \int \psi(\theta - \Delta\theta, w - \Delta w, t) P(\theta, w, \Delta\theta, \Delta w, \Delta t, t) d\Delta\theta d\Delta w \quad (2.5),$$

where $P(\theta, w, \Delta\theta, \Delta w, \Delta t, t) d\Delta\theta d\Delta w$ is the probability of a change $\Delta\theta, \Delta w$ in time Δt . If we let $\tau_r \rightarrow \infty$, $g(t) \rightarrow 0$, $\Omega \rightarrow 0$, then Eqs. (2.2) and (2.4) imply

$$P(\theta, w, \Delta\theta, \Delta w, \Delta t, t) = \delta(\Delta\theta - k_0 w \Delta t) \delta(\Delta w - 2\pi R \operatorname{Re} \mathcal{E} \Delta t) \quad (2.6).$$

Putting (2.6) into (2.5), expanding to first order in Δt , then letting $\Delta t \rightarrow 0$, we get

$$\frac{\partial \psi}{\partial t} + k_0 w \frac{\partial \psi}{\partial \theta} + 2\pi R \operatorname{Re} \mathcal{E}(\theta, t) \frac{\partial \psi}{\partial w} = 0 \quad (2.7),$$

the usual Vlasov equation for unbunched beams.

In order to take into account turbulence, we will replace Γ by an approximate P for a "test" field $\mathcal{E}_T$ plus a turbulent background $\mathcal{E}_B$. Dupree⁸ has shown that such a separation leads to a convergent perturbation theory. We will then let $\Delta t \rightarrow \tau$, where τ is short compared to times of interest, but long enough for particles to "see" significant turbulence.

Let

$$\theta' = \theta - \theta_0 - k_0 w \Delta t \quad (2.8),$$

and

$$w' = w - w_0 - 2\pi \text{Re } \mathcal{E}_T \Delta t + \frac{w}{\tau} \Delta t + \frac{\Omega^2}{k_0} \theta \Delta t \quad (2.9),$$

where θ_0, w_0 are the starting values of θ, w , and

$$\mathcal{E}(\theta, t) = \mathcal{E}_T(\theta, t) + \mathcal{E}_B(\theta, t) \quad (2.10),$$

as discussed above. Then the equations for θ', w' are

$$\frac{d\theta'}{dt} = k_0 w' \quad (2.11),$$

$$\frac{dw'}{dt} = 2\pi \text{Re } \mathcal{E}_B + g(t) \quad (2.12).$$

With the assumption that $\mathcal{E}_B(\theta, t)$ represents a turbulent background of waves with random phases, we see, from classical diffusion theory,¹² that w' undergoes a diffusion process with diffusion constant

$$d = \frac{D}{f^2} = (2\pi \text{Re})^2 \int_0^\infty \mathcal{E}_B(\tau) \mathcal{E}_B(t+\tau) d\tau + \frac{D_Q}{f^2} \quad (2.13),$$

where D_Q is the usual quantum diffusion constant.¹³ We have assumed no

correlation between quantum noise and the turbulent background. Now, from Benford and Thompson,¹¹ we find that the probability function for θ', w' is

$$P(\theta', w', \Delta t) = \frac{e^{-\frac{w'^2}{4\Delta t}}}{\sqrt{4\pi\Delta t}} \frac{e^{-\frac{(\theta' - k_0/2 w' \Delta t)^2}{\left(\frac{1}{3} d(\Delta t)^3\right)}}}{\sqrt{\frac{1}{3} \pi d(\Delta t)^3}} \quad (2.14).$$

Using Eqs. (2.8) and (2.9), we find

$$\begin{aligned} P(\theta, w, \Delta\theta, \Delta w, \Delta t, t) &= \frac{1}{\sqrt{4\pi\Delta t}} \exp \left\{ - \left[\Delta w - 2\pi \text{Re} \varepsilon_T \Delta t + \frac{w}{\tau_r} \Delta t + \frac{\Omega^2}{k_0} \theta \Delta t \right]^2 / 4\Delta t \right\} \\ &\quad \frac{1}{\sqrt{\frac{1}{3} \pi d(\Delta t)^3}} \times \exp \left\{ - \left[\Delta\theta - k_0 w \Delta t - \frac{\Delta t}{2} (k_0 \Delta w - 2\pi \text{Re} \varepsilon_T \Delta t k_0 \right. \right. \\ &\quad \left. \left. + k_0 \frac{w}{\tau_r} \Delta t + \Omega^2 \theta \Delta t) \right]^2 / \left(\frac{1}{3} d(\Delta t)^3 \right) \right\} \end{aligned} \quad (2.15).$$

Putting (2.15) into (2.5), expanding in Δt , and keeping only first order terms in Δt , we find

$$\frac{\partial \psi}{\partial t} + k_0 w \frac{\partial \psi}{\partial \theta} + 2\pi \text{Re} \varepsilon_T \frac{\partial \psi}{\partial w} - \frac{w}{\tau_r} \frac{\partial \psi}{\partial w} - \frac{\Omega^2 \theta}{k_0} \frac{\partial \psi}{\partial w} - \frac{\psi}{\tau_r} = \frac{D}{f^2} \frac{\partial^2 \psi}{\partial w^2} \quad (2.16).$$

We let

$$D = D_Q + D_T \quad (2.17),$$

where D_Q is the quantum diffusion coefficient, and D_T is the turbulent diffusion coefficient. Equation (2.16) is our basic equation.

Now, we have assumed that $\Delta t \rightarrow \tau > \tau_{AC}$. In order to neglect higher order terms in Δt , several assumptions must be made. First, we have to assume that $\tau_{AC} \ll 1/\Omega$. Since the modes we shall consider oscillate at multiples of Ω , this is basically a high mode number, or short wavelength,

assumption. We have also assumed that D is a constant. For D_Q this is certainly correct, but for D_T this requires a broad band turbulence spectrum. This also follows from the assumption of a short autocorrelation time. Various other terms are negligible in the broad band spectrum case. Finally, we must assume $\tau_{AC} \ll 2d\tau^2/w_{RMS}^2$, where w_{RMS} is the root mean square w . This condition is equivalent to $\tau_T \gg \tau_{AC}$ which will certainly be true in all cases we consider, since $\tau_T \gg 1/\Omega$.

Although our derivation of strong turbulence theory is less general than that of Dupree,⁸ Weinstock,⁹ or Benford and Thompson,¹¹ it has the virtue of simplicity; that is, only assuming that particles see a turbulent background allows one to derive the usual Fokker-Planck equation. In addition, we will see that we reproduce all of the first order results of the aforementioned authors. Thus, for our purposes, Eq. (2.16) should be an adequate representation of strong turbulence theory.

III. THE RANDOM PHASE APPROXIMATION

In deriving Eq. (2.13), it was necessary to assume random phases for the spectrum of coherent oscillations. This assumption is usually justified by saying that mode-mode coupling causes a rapid exchange of energy between modes, and as a result, the phases seem quite random simply because these are very complicated processes. We believe that some such mechanism takes place in the case we are considering. However, in this section we examine this assumption in more detail.

We consider a particle executing transverse betatron oscillations, moving in a nonlinear longitudinal well, and interacting with short wavelength longitudinal waves of definite phases. We will show, using a criterion due to Chirikov,¹⁴ that such a particle has a "stochastic",

i.e., random, trajectory. Since the particle interacts with the wave, this randomness results in a stochastic transfer of energy to and from the wave. Thus, the phase of the wave becomes random. Through mode-mode coupling, which we assume exists, this randomness is fed down into the lower order modes. As a result, it is justified to assume random phases of all waves.

In our model, we assume that the magnitude of the short wavelength modes is proportional to the average particle density at that transverse position; i.e., we assume that it is a Gaussian in the transverse coordinate, x . Thus,

$$E = \sum_{m \neq 0} E_m e^{im\theta - \frac{x^2}{2\sigma^2}} \quad (3.1),$$

where σ is the root mean square transverse displacement. Our equations of motion become

$$\dot{\theta} = k_0 w \quad (3.2),$$

$$\dot{w} = 2\pi \text{Re } \mathcal{E}_0(\theta, t) + 2\pi \text{Re} \sum_{m \neq 0} E_m e^{im\theta - \frac{x^2}{2\sigma^2}} \quad (3.3),$$

$$\ddot{x} = -\omega_\beta^2 x + \frac{2\pi i \text{Re } x}{\sigma^2} \sum_{m \neq 0} \frac{E_m}{m} e^{im\theta - \frac{x^2}{2\sigma^2}} \quad (3.4),$$

where ω_β is the betatron oscillation frequency, and $\mathcal{E}_0$ is the RF field.

In order to show stochasticity of the particle trajectories, we will use the Chirikov criterion.¹⁴ To apply this criterion one first solves the lowest order equations; in our case, the equations with $E_m = 0$. An oscillatory motion must be assumed to be nonlinear; we will assume the synchrotron motion to be nonlinear. One then puts the lowest order solution into the perturbing term and treats it as a time-dependent

perturbation. Clearly this can't be assumed to give the solution for all time, since energy is conserved, but is adequate for deriving the stochasticity threshold. Since the unperturbed motion is nonlinear, its frequency depends on amplitude. Typically, the perturbing terms will have many frequencies present, and at any oscillation amplitude, many of the perturbing terms will be in resonance. Isolating any one resonance, one sees that it gives rise to a resonance region of finite width, since the unperturbed motion is assumed to be nonlinear. We call the width in amplitude of this resonance δ_W . The distance between resonances is called Δ_R . When resonances overlap, i.e. when $\delta_W \gtrsim \Delta_R$, the assumption is that stochasticity sets in. The rough idea is that particles can move from resonance to resonance in a random fashion. Chirikov¹⁴ has investigated this criterion with numerical simulation and found it to be remarkably accurate.

To apply this criterion in our case we write down the lowest order equations. From (3.2), (3.3), and (3.4), these are

$$\ddot{x} = -\omega_\beta^2 x \quad (3.5),$$

$$\ddot{\theta} = 2\pi \operatorname{Re} \mathcal{E}_0(\theta, t) \quad (3.6).$$

The solution to (3.5) is clearly

$$x = x_0 \sin(\omega_\beta t + \gamma) \quad (3.7)$$

where x_0 is the amplitude and γ the initial phase. The solution of (3.6) is, in lowest order,

$$\theta = \theta_0 \cos \Omega_1 t + \frac{k_0 w_0}{\Omega} \sin \Omega_1 t \quad (3.8),$$

where

$$\Omega_1 = \Omega \left[1 - \frac{3\epsilon}{8} \left(\Omega_0^2 + \frac{k_0^2 w_0^2}{\Omega^2} \right) \right] \quad (3.9).$$

Equations (3.8) and (3.9) were obtained by assuming a cubic nonlinearity with coefficient ϵ and using the usual first order solution.¹⁵ Equation (3.9) shows the amplitude dependence of the frequency of oscillation of

9. Let

$$I = \frac{k_0 w_0^2}{2\Omega} + \frac{\Omega \theta^2}{2k_0} \quad (3.10),$$

$$\Omega_1(I) = \Omega \left(1 - \frac{3\epsilon k_0}{4\Omega} I \right) \quad (3.11).$$

The lowest order Hamiltonian is then

$$H_0 \approx \int^I \Omega(I') dI' \quad (3.12).$$

Using the solution (3.7) in the perturbation term of (3.3), the next order Hamiltonian for longitudinal motion becomes

$$H \approx H_0 + 2\pi \operatorname{Re} i \sum_{m \neq 0} \frac{E_m}{m} e^{im\sqrt{2k_0 I/\Omega} \sin\psi} \times \exp \left\{ \frac{-x_0^2}{4\sigma^2} + \frac{x_0^2 \cos[2(\omega_\beta t + \gamma)]}{4\sigma^2} \right\} \quad (3.13),$$

where ψ is the angle variable canonically conjugate to I . We now use

$$e^{im\sqrt{2k_0 I/\Omega} \sin\psi} = \sum_{n=-\infty}^{\infty} (-1)^n J_n(m\sqrt{2k_0 I/\Omega}) e^{-in\psi} \quad (3.14),$$

$$\exp \left\{ \frac{x_0^2}{4\sigma^2} \cos[2(\omega_\beta t + \gamma)] \right\} = \sum_{\ell=-\infty}^{\infty} I_\ell \left(\frac{x_0^2}{4\sigma^2} \right) e^{i\ell[2(\omega_\beta t + \gamma)]} \quad (3.15),$$

where J_n and I_ℓ are the usual Bessel functions. Putting (3.14) and (3.15) into (3.13) we find

$$H \approx H_0 + 2\pi \operatorname{Re} i \exp \left\{ \frac{-x_0^2}{4\sigma^2} \right\} \times \sum_{m \neq 0} \sum_{n=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \frac{E_m}{m} \\ \times (-1)^n J_n(m\sqrt{2k_0 I/\Omega}) I_{\ell} \left(\frac{x_0^2}{4\sigma^2} \right) e^{i\ell[2(\omega_{\beta} t + \gamma)] - in\psi} \quad (3.16).$$

Since we have

$$\psi \sim \Omega_1 t \quad (3.17),$$

we see that resonance occurs when

$$2\ell\omega_{\beta} \approx n\Omega_1 \quad (3.18).$$

Since $\omega_{\beta} \gg \Omega$, we expect that $\ell = \pm 1$ will be the dominant terms. For each n , there is a value of I , call it I_n , for which

$$\pm 2\omega_{\beta} = n\Omega_1(I_n) \quad (3.19).$$

For values of I near I_n , only one term in the sum over n will contribute. Selecting this term and letting

$$\bar{\psi} = n\psi - 2\omega_{\beta} t \quad (3.20),$$

$$\bar{I} = \frac{1}{n} (I - I_n) \quad (3.21),$$

the Hamiltonian is approximately

$$H \approx n^2 \frac{d\Omega_1}{dI} \bigg|_{I_n} \frac{\bar{I}^2}{2} + U_n \cos(\bar{\psi} - 2\gamma) \quad (3.22),$$

where

$$U_n = 4\pi \operatorname{Re} i \exp \left(\frac{-x_0^2}{4\sigma^2} \right) (-1)^n I_1 \left(\frac{x_0^2}{4\sigma^2} \right) \sum_{m \neq 0} \frac{E_m}{m} J_n(m\sqrt{2k_0 I_n/\Omega}) \quad (3.23).$$

Equation (3.22) is the usual Hamiltonian for a pendulum, which has a closed region of bounded trajectories with width in I of

$$\delta_W = 8 \sqrt{\left| \frac{U_n}{3\epsilon k_0 n^2} \right|} \quad (3.24),$$

where we used

$$\frac{d\Omega_1}{dI} = - \frac{3\epsilon k_0}{4} \quad (3.25).$$

Now, from (3.11) and (3.19), we have

$$1 - \frac{3\epsilon k_0}{4\Omega} I_n = \frac{2\omega_\beta}{n\Omega} \quad (3.26).$$

Thus,

$$I_n - I_{n+1} = \frac{8\omega_\beta}{-3\epsilon k_0} \left(\frac{1}{n} - \frac{1}{n+1} \right) \approx \frac{8\omega_\beta}{-3k_0 \epsilon n^2} \quad (3.27).$$

Using (3.27) and (3.21) we find

$$\Delta_R = \left| \frac{8\omega_\beta}{3\epsilon k_0 n^3} \right| \quad (3.28),$$

thus the Chirikov criterion becomes

$$1 \leq \frac{\delta_W}{\Delta_R} = n^2 \frac{\sqrt{|3\epsilon k_0 U_n|}}{\omega_\beta} \quad (3.29).$$

Now,¹⁶

$$2\pi R E_m = -Z_m \lambda_m e\beta c \quad (3.30),$$

where λ_m is the m^{th} Fourier component of charge density, $\beta = v/c \approx 1$ is the velocity in units of c , the speed of light. Z_m is an impedance which depends on the beam surroundings. We want to estimate the size of λ_m/λ_0 for which (3.29) is satisfied. Taking $x_0 = \sigma$, and using (3.30) in (3.23), we get

$$|U_n| \approx .2 \beta c \left\langle \frac{Z_m}{m} \right\rangle \left\langle \frac{\lambda_m}{\lambda_0} \right\rangle e^2 \lambda_0 \sum_{m \neq 0}' J_n(m\sqrt{2k_0 I_n/\Omega}) \quad (3.31),$$

where $\langle Z_m/m \rangle$ is assumed roughly independent of m , in agreement with our fits to the data, and $\langle \lambda_m/\lambda_0 \rangle$ is an average size of the coherent modes. The prime on the sum indicates that phases should be taken into account. Since

$$\sqrt{2k_0 I_n/\Omega} \approx \theta_{\text{RMS}} \quad (3.32),$$

we conclude that $m\sqrt{2k_0 I_n/\Omega} > 1$. Thus,

$$J_n(m\sqrt{2k_0 I_n/\Omega}) \approx \frac{1}{\sqrt{2\pi m \theta_{\text{RMS}}}} \quad (3.33).$$

Since about $\sqrt{n}$ terms should add coherently, we find

$$\sum_{m \neq 0}' J_n(m\sqrt{2k_0 I_n/\Omega}) \sim \frac{1}{\sqrt{\theta_{\text{RMS}}}} \quad (3.34).$$

Thus,

$$|U_n| \approx .2 \left\langle \frac{Z_m}{m} \right\rangle \left\langle \frac{\lambda_m}{\lambda_0} \right\rangle e^2 \frac{\lambda_0}{\sqrt{\theta_{\text{RMS}}}} \beta c \quad (3.35).$$

Let us assume

$$n \approx \frac{2\omega\beta}{\Omega} \quad (3.36),$$

and use machine parameters relevant to SPEAR I, namely

$$\begin{aligned} f &= 1.28 \times 10^6 \text{ sec}^{-1} \\ \lambda_0 &= \frac{N}{2\pi R} = \frac{5 \times 10^{10}}{2\pi(37.3 \text{ m})} = 2.13 \times 10^6 \text{ cm}^{-1} \\ \beta &\approx 1 \end{aligned}$$

$$\theta_{\text{RMS}} \cong 2.68 \times 10^{-3}$$

$$E = 1.5 \text{ GeV}$$

$$|k_0| = \frac{2\pi f^2}{E} \alpha, \quad \alpha = .041$$

$$\omega_B \cong 2.66 \times 10^7 \text{ sec}^{-1}$$

$$\Omega \cong 8 \times 10^3 \text{ sec}^{-1}$$

$$\epsilon \cong 270.$$

We will also assume that

$$\frac{Z_m}{m} = 1 \text{ ohm}.$$

Putting (3.35) into (3.29) and using these values, we get

$$\left\langle \frac{\lambda_m}{\lambda_0} \right\rangle \geq 4 \times 10^{-5} \quad (3.37).$$

Since we have included one term in the sum over l in (3.16), the actual stochasticity boundary is probably somewhat smaller. In any case, we conclude that even very small amplitude waves will give rise to stochastic particle trajectories.

The magnitude of the impedance that we have assumed is reasonable and is about the same as that required to fit the bunch lengthening data. Thus, although some of the approximations we made were very crude, in particular Eqs. (3.31) and (3.34), our result can be taken as an indication that particle trajectories are stochastic, and that the random phase approximation is justified.

IV. CONVERSION TO AN INTEGRAL EQUATION

In this section, we begin the derivation of the dispersion relations from Eq. (2.16). We assume small perturbations about an equilibrium state and linearize in these small terms. The effect of nonlinear terms, namely mode coupling, will be discussed, but not calculated, when we examine the experimental data.

Equation (2.16) has a steady state solution given by

$$\psi_0(\theta, w) = M \exp \left\{ -\frac{f^2}{2D\tau_r} \left(w^2 + \frac{\Omega^2 \theta^2}{k_0^2} \right) \right\} \quad (4.1),$$

where M is a normalization constant. We let

$$\psi = \psi_0 + \psi_1 \quad (4.2),$$

and, keeping only terms linear in ψ_1 , from (2.15) we get

$$\frac{\partial \psi_1}{\partial t} + k_0 w \frac{\partial \psi_1}{\partial \theta} - \left(\frac{w}{\tau_r} + \frac{\Omega^2 \theta}{k_0} \right) \frac{\partial \psi_1}{\partial w} - \frac{\psi_1}{\tau_r} = \frac{D}{f^2} \frac{\partial^2 \psi_1}{\partial w^2} - 2\pi \operatorname{Re} \mathcal{E} \frac{\partial \psi_0}{\partial w} \quad (4.3).$$

In order to solve Eq. (4.3) for ψ_1 , we consider the equation

$$\frac{\partial h}{\partial t} + k_0 w \frac{\partial h}{\partial \theta} - \left(\frac{w}{\tau_r} + \frac{\Omega^2 \theta}{k_0} \right) \frac{\partial h}{\partial w} - \frac{h}{\tau_r} = \frac{D}{f^2} \frac{\partial^2 h}{\partial w^2} \quad (4.4).$$

In terms of the solution to Eq. (4.4), with the initial condition

$$h(\theta, w, t_0; \theta_0, w_0, t_0) = \delta(\theta - \theta_0) \delta(w - w_0) \quad (4.5),$$

the solution of Eq. (4.3) satisfies

$$\begin{aligned} \psi_1(\theta, w, t) = & \int h(\theta, w, t; \theta_0, w_0, 0) \psi_1(\theta_0, w_0, 0) d\theta_0 dw_0 \\ & - 2\pi \operatorname{Re} \int_0^t d\tau \int d\theta_0 dw_0 h(\theta, w, t; \theta_0, w_0, \tau) \left(\mathcal{E}(\psi_1) \frac{\partial \psi_0}{\partial w} \right) \Big|_{\theta_0, w_0, \tau} \end{aligned} \quad (4.6).$$

We have written $\mathcal{E} = \mathcal{E}(\psi_1)$ to indicate its dependence on ψ_1 . We have taken $t_0 = 0$. The method of solution that we are using is a generalization of the method of integration over characteristics of the unperturbed orbits. If there were no diffusion, there would be exact unperturbed orbits. When a diffusion term is present, we are taking a weighted average over an ensemble of unperturbed orbits. In any case, Eq. (4.6) is an exact consequence of Eq. (4.3).

The solution of Eq. (4.4) which satisfies Eq. (4.5) is known.¹² For convenience, we present a derivation of it in Appendix A.

Since Eq. (4.4) is homogeneous in time, the solution, h , satisfies

$$h(\theta, w, t; \theta_0, w_0, t_0) = h(\theta, w; \theta_0, w_0; t - t_0) \quad (4.7).$$

Thus, we can Laplace transform Eq. (4.6) in time and use the convolution theorem for Laplace transforms to obtain

$$\begin{aligned} \tilde{\psi}_1(\theta, w, s) &= \int \tilde{h}(\theta, w; \theta_0, w_0, s) \psi_1(\theta_0, w_0, 0) d\theta_0 dw_0 \\ &\quad - 2\pi \operatorname{Re} \int d\theta_0 dw_0 \tilde{h}(\theta, w; \theta_0, w_0, s) \left(\mathcal{E}(\tilde{\psi}_1) \frac{\partial \psi_0}{\partial w} \right) \Big|_{\theta_0, w_0, s} \end{aligned} \quad (4.8),$$

where we have assumed that $\mathcal{E}$ is a linear operator on ψ_1 . Let

$$F(\theta, w, s) = \int \tilde{h}(\theta, w; \theta_0, w_0, s) \psi_1(\theta_0, w_0, 0) d\theta_0 dw_0 \quad (4.9),$$

which is the initial value term. From Eq. (4.1), we have

$$\psi_0(\theta_0, w_0) = M P(\theta_0) W(w_0) \quad (4.10).$$

Let

$$L(\theta, w, \theta_0, s) = \int dw_0 \tilde{h}(\theta, w; \theta_0, w_0, s) \frac{dW(w_0)}{dw} \quad (4.11).$$

Equation (4.8) is now

$$\tilde{\psi}_1(\theta, w, s) = F(\theta, w, s) - 2\pi \operatorname{Re} M \int d\theta_0 L(\theta, w, \theta_0, s) P(\theta_0) \mathcal{E}(\tilde{\psi}_1) \Big|_{\theta_0, s} \quad (4.12).$$

We have used the fact that $\mathcal{E}$ is a function only of θ_0 and not of w_0 .

Now we use

$$\mathcal{E}(\theta_0, t) = \frac{fR}{\beta c} \int d\theta' \lambda(\theta', t) G(R(\theta' - \theta_0)) \quad (4.13),$$

where $\lambda(\theta', t)$, the charge density per unit length, is given by

$$\lambda(\theta', t) = \frac{e}{R} \int \psi(\theta', w, t) dw \quad (4.14).$$

Equation (4.13) is a consequence of the linearity of Maxwell's equations.

In particular, it is exactly the same form used by Pellegrini and Sessler.⁴

G is a Green's function which includes the effect of the beam surroundings. In principle G is calculable once the details of the particular machine are known. In fact G has been calculated for various types of structures.⁴ In practice, however, G is very poorly known. Fortunately our results are relatively insensitive to the details of this function.

Putting Eqs. (4.13) and (4.14) into Eq. (4.12), we obtain

$$\begin{aligned} \tilde{\psi}_1(\theta, w, s) = & F(\theta, w, s) - 2\pi \operatorname{Re} M \int d\theta_0 L(\theta, w, \theta_0, s) P(\theta_0) \left(\frac{fe}{\beta c} \right) \\ & \times \int d\theta' G(R(\theta' - \theta_0)) \int \tilde{\psi}_1(\theta', w, s) dw \end{aligned} \quad (4.15).$$

Equation (4.15) is a linear integral equation for $\tilde{\psi}_1$ from which we will obtain a dispersion relation.

V. UNIFORM BEAMS

In this section we obtain the dispersion relation for a uniform, i.e., unbunched beam. Although the complete solution has no obvious applications, the special case in which radiation damping is absent describes turbulent, uniform, proton beams. Our derivation also reproduces the previous result of Sessler.⁷

The uniform beam case results when we take the limit $\Omega \rightarrow 0$. In this limit, Eq. (4.4) is translationally invariant in θ , and thus

$$h(\theta, w; \theta_0, w_0, t - t_0) = h(\theta - \theta_0, w, w_0, t - t_0) \quad (5.1).$$

Thus, L in Eq. (4.11) depends only on $\theta - \theta_0$. Also, in (4.10),

$$P(\theta_0) = 1 \quad (5.2).$$

Thus we can Fourier transform Eq. (4.15) in θ , and use the convolution theorem for Fourier transforms. We get

$$\tilde{\psi}_1(p, w, s) = \tilde{F}(p, w, s) - (2\pi)^2 \operatorname{Re} M\left(\frac{f\epsilon}{\beta C}\right) \tilde{L}(p, w, s) \left(\int d\theta' G \int \tilde{\psi}_1 dw \right) \quad (5.3),$$

In Eq. (5.3) we again use the convolution theorem for the last term along with the fact that

$$\tilde{G}(p) = \frac{\beta C}{2\pi R} Z\left(-\frac{p\beta C}{R}\right) \quad (5.4).$$

In Eq. (5.4), Z is the impedance function. The relation between this impedance function and that defined by Eq. (3.30) is

$$Z(-p\omega_0) = -Z_p \quad (5.5),$$

where

$$\omega_0 = \frac{\beta C}{R} = 2\pi f \quad (5.6),$$

Equation (5.3) becomes

$$\tilde{\psi}_1(p, w, s) = \tilde{F}(p, w, s) - \frac{(2\pi)^3 \text{Re}^2 M f^2}{\beta c} \tilde{L}(p, w, s) Z(-p\omega_0) \int \tilde{\psi}_1 dw \quad (5.7).$$

If we define

$$\phi(p, s) = \int dw \tilde{\psi}_1(p, w, s) \quad (5.8),$$

$$\bar{F}(p, s) = \int dw \tilde{F}(p, w, s) \quad (5.9),$$

$$\bar{L}(p, s) = \int dw \tilde{L}(p, w, s) \quad (5.10),$$

then from Eq. (5.7) we get

$$\phi(p, s) = \bar{F}(p, s) - \frac{(2\pi)^3 \text{Re}^2 M f^2}{\beta c} \bar{L}(p, s) Z(-p\omega_0) \phi(p, s) \quad (5.11).$$

We recall that $\bar{F}(p, s)$ is an initial value term.

In this case the normalization constant is

$$M = \frac{fN}{\sqrt{2\pi D \tau_r}} \quad (5.12),$$

where N is the total number of particles in the beam. Thus from Eq.

(5.11) we find

$$\phi(p, s) = \frac{\bar{F}(p, s)}{1 + \frac{(2\pi)^3 \text{Re}^2 f^3 N Z(-p\omega_0)}{\beta c \sqrt{2\pi D \tau_r}} \bar{L}(p, s)} \quad (5.13).$$

The zeros of the denominator of this expression give, as usual, the dispersion relation. In Appendix B the function $\bar{L}(p, s)$ is calculated.

The result is

$$\bar{L}(p,s) = i p k_0 \sqrt{\frac{D\tau_r^5}{2\pi f^2}} \int_0^\infty dx x \exp \left\{ -vx + (s\tau_r - 1 + v) \ln(x+1) \right\} \quad (5.14)$$

In Eq. (5.14) we have defined

$$v = D \left(\frac{pk_0}{f} \right)^2 \tau_r^3 \quad (5.15),$$

thus the dispersion relation is

$$1 + Ne^2 f Z(-p\omega_0) i p k_0 \tau_r^2 \int_0^\infty dx x \exp \left\{ -vx + (s - 1 + v) \ln(1+x) \right\} = 0 \quad (5.16)$$

In Appendix C, we present another derivation of Eq. (5.16). We show that this result is identical with that obtained by Sessler,⁷ even though Sessler manipulated divergent integrals to obtain his result.

VI. INTEGRAL EQUATION FOR BUNCHED BEAMS

In this section we will derive an integral equation for bunched beams. Since we are concerned with bunched beams whose length is much smaller than the machine circumference, we extend the integrations in angle in Eq. (4.15) to the interval $[-\infty, \infty]$. Though not exact, the rapid fall-off in angle of $\psi_0(\theta, w)$ makes this an excellent approximation.

The usual method^{17,18} of solving Eq. (2.16), when damping and diffusion are absent, is to transform to action-angle variables

$$J = w^2 + \frac{\Omega^2 \theta^2}{k_0^2}$$

$$\phi = \cos^{-1} \frac{w}{\sqrt{J}} .$$

The equation is then Fourier transformed in ϕ . In this approach the impedance function enters in a very complicated fashion, and only special types of impedance functions can be treated. The presence of the diffusion term makes the transformation to action-angle variables ineffective.

Instead, we must work with the Fourier transform in the physical angle θ . As a result the impedance functions enters in a relatively simple manner.

Since we are only interested in the response of the beam as a function of angle, let us define

$$\xi(\theta, s) = \int d\omega \tilde{\psi}_1(\theta, \omega, s) \quad (6.1),$$

$$\sigma(\theta, s) = \int d\omega F(\theta, \omega, s) \quad (6.2),$$

$$K(\theta, \theta_0, s) = \int d\omega L(\theta, \omega, \theta_0, s) \quad (6.3).$$

Using these definitions, Eq. (4.15) implies

$$\xi(\theta, s) = \sigma(\theta, s) - \frac{2\pi R e^2 M f}{8c} \int d\theta_0 K(\theta, \theta_0, s) P(\theta_0) \int d\theta' G(R(\theta' - \theta_0)) \xi(\theta', s) \quad (6.4).$$

Let us Fourier transform in θ .

$$\tilde{\xi}(p, s) = \tilde{\sigma}(p, s) - \frac{2\pi R e^2 M f}{8c} \int d\theta_0 \tilde{K}(p, \theta_0, s) P(\theta_0) \int d\theta' G(R(\theta' - \theta_0)) \xi(\theta', s) \quad (6.5).$$

Recall that

$$\int d\theta_0 f(\theta_0) = 2\pi \tilde{f}(0) \quad (6.6).$$

The last term in Eq. (6.5) is precisely of this form. Since the integrand of the last term in Eq. (6.5) is a product, its Fourier transform is a convolution, i.e.

$$\begin{aligned} \widetilde{K}(p, \theta_0, s) P(\theta_0) \int d\theta' G(R(\theta' - \theta_0)) \xi(\theta', s) \Big|_{p, r, s} = \\ \int dr_1 \widetilde{K}(p, \theta_0, s) P(\theta_0) \Big|_{p, r - r_1, s} \int d\theta' G(R(\theta' - \theta_0)) \xi(\theta', s) \Big|_{p, r_1, s} \end{aligned} \quad (6.7).$$

Using Eq. (5.4) and the convolution theorem, we see that

$$\int d\theta' G(R(\theta' - \theta_0)) \xi(\theta', s) \Big|_{p, r_1, s} = \frac{\beta c}{R} Z(-r_1 \omega_0) \widetilde{\xi}(r_1, s) \quad (6.8).$$

Let us define

$$I(p, r_1, s) = \widetilde{K}(p, \theta_0, s) P(\theta_0) \Big|_{p, -r_1, s} \quad (6.9).$$

Equations (6.6), (6.7), (6.8), and (6.9) imply that Eq. (6.5) is equivalent to

$$\widetilde{\xi}(p, s) = \widetilde{\sigma}(p, s) - (2\pi)^2 e^2 M f \int dr_1 I(p, r_1, s) Z(-r_1 \omega_0) \widetilde{\xi}(r_1, s) \quad (6.10).$$

Equation (6.10) is the basic integral equation for bunched beams. $\widetilde{\sigma}(p, s)$ is an initial value term, and we want to solve for $\widetilde{\xi}(p, s)$. For bunched beams the normalization constant is given by

$$M = \frac{f^2 \Omega N}{2\pi D \tau_r |k_0|} \quad (6.11),$$

where N is the number of particles in the beam. From Eqs. (4.11), (6.3), and (6.9), we get

$$\begin{aligned} I(p, r_1, s) = \frac{1}{(2\pi)^2} \int_0^\infty dt e^{-st} \int d\theta_0 d\theta dw dw_0 e^{ir_1 \theta_0 - ip\theta} \times \\ \times P(\theta_0) \frac{dW(w_0)}{dw} h(\theta, w, \theta_0, w_0, t) \end{aligned} \quad (6.12).$$

In Eq. (6.10), we see that the contribution of $\tilde{\xi}(r_1, s)$ is weighted by the impedance at the appropriate frequency, $Z(-r_1 \omega_0)$; a fact which agrees with our physical intuition. The contribution of $\tilde{\xi}(r_1, s)$ is also weighted by the function $I(p, r_1, s)$, which depends on the unperturbed distribution function and on the solution to Eq. (4.4). Fortunately, $I(p, r_1, s)$ can be calculated in the cases of interest. In fact, when damping and diffusion are absent, the calculation of $I(p, r_1, s)$ is rather simple, as we shall show in the next section.

VII. BUNCHED PROTON BEAMS WITHOUT TURBULENCE

In this section we obtain the dispersion relation for proton beams from Eq. (6.10). For proton beams, radiation damping and quantum diffusion are absent. If turbulent diffusion is present, the situation is rather complicated. Because radiation damping is absent, particles will diffuse to such large synchrotron amplitudes that they are lost, or to such large amplitudes that Landau damping due to nonlinearities will damp the coherent modes. The latter case requires an extensive study of Landau damping. Though we will give an approximate analysis of Landau damping, it will not be exact enough to consider this case. In any event, it seems desirable to avoid, if possible, the onset of turbulence in proton beams. Thus, one should require, if possible, the stability of coherent oscillations. We will assume that turbulent diffusion is absent.

The dispersion relation for proton beams has been considered previously.^{17,18} Sacherer,¹⁷ in particular, has given a description of the modes in special cases. However, these treatments have been hindered by two problems. First, it was difficult to include Landau damping in any simple way; only in special cases could Landau damping be included. Secondly, as commented on previously, the impedance function entered into the calculations in a very complicated way, and as a result, only special impedance functions could be used. In our derivation, the impedance enters in a simple way, and general impedance functions can be used. In addition, although we cannot give an exact treatment of Landau damping, we do give an approximate analysis for all cases. The exact treatment of Landau damping, in our approach, reduces to the analysis of a complicated integral. Furthermore, the integral is the same for all cases if the same distribution

function is assumed. Thus, at worst, a detailed treatment of Landau damping requires only a numerical investigation of this one integral.

Since we have assumed no radiation damping and no diffusion, the solution of Eq. (4.4) which satisfies (4.5) is

$$h(\theta, w, t; \theta_0, w_0, 0) = \delta(\theta - A(\theta_0, w_0, t)) \delta(w - B(\theta_0, w_0, t)) \quad (7.1)$$

where

$$A(\theta_0, w_0, t) = \theta_0 \cos \Omega t + \frac{k_0 w_0}{\Omega} \sin \Omega t \quad (7.2),$$

$$B(\theta_0, w_0, t) = w_0 \cos \Omega t + \frac{\Omega \theta_0}{k_0} \sin \Omega t \quad (7.3).$$

Using Eqs. (7.1) and (7.2) in Eq. (6.12), we easily find

$$\begin{aligned} I(p, r_1, s) &= \frac{1}{(2\pi)^2} \int_0^\infty dt e^{-st} \int d\theta_0 dw_0 P(\theta_0) \frac{dW(w_0)}{dw} \\ &\times \exp \left\{ i r_1 \theta_0 - i p \left(\theta_0 \cos \Omega t + \frac{k_0 w_0}{\Omega} \sin \Omega t \right) \right\} \end{aligned} \quad (7.4).$$

Since $P(\theta_0)$ and $W(w_0)$ are Gaussians, given by Eqs. (4.1) and (4.10), the integrals over θ_0 and w_0 in (7.4) are easily done. The result is

$$\begin{aligned} I(p, r_1, s) &= \frac{D\tau_r |k_0| i p k_0}{2f^2 \Omega^2} \exp \left\{ - \frac{D\tau_r k_0^2 (p^2 + r_1^2)}{2f^2 \Omega^2} \right\} \\ &\times \int_0^\infty dt e^{-st} \sin \Omega t \exp \left\{ \frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \cos \Omega t \right\} \end{aligned} \quad (7.5).$$

Note that although we have taken the limit $D \rightarrow 0$, $\tau_r \rightarrow \infty$, we have kept $D\tau_r$, the energy spread of the beam, fixed. Thus only the combination, $D\tau_r$, appears in Eq. (7.5).

Now let us use the fact that¹⁹

$$\exp \left\{ \frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \cos \Omega t \right\} = I_0 \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \right) + 2 \sum_{k=1}^{\infty} I_k \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \right) \cos k \Omega t \quad (7.6),$$

where I_k denote the usual Bessel functions. Putting Eq. (7.6) into Eq. (7.5) and performing the integration in t , we get

$$\begin{aligned} I(p, r_1, s) &= \frac{D\tau_r |k_0| i p k_0}{2f^2 \Omega^2} \exp \left\{ \frac{-D\tau_r k_0^2 (p^2 + r_1^2)}{2f^2 \Omega^2} \right\} \\ &\times \left\{ I_0 \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \right) \frac{\Omega}{s^2 + \Omega^2} + \sum_{k=1}^{\infty} i k \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \right) \right. \\ &\times \left. \left[\frac{\Omega(k+1)}{s^2 + \Omega^2 (k+1)^2} - \frac{\Omega(k-1)}{s^2 + \Omega^2 (k-1)^2} \right] \right\} \quad (7.7). \end{aligned}$$

We note that $I(p, r_1, s)$ is large when $s \approx i n \Omega$, with n a positive or negative integer. Thus, coherent oscillations occur at multiples of the synchrotron frequency, as was to be expected. When $s \approx i n \Omega$, we shall keep only the largest terms, and hence

$$\begin{aligned} I(p, r_1, s) &\approx \frac{D\tau_r |k_0| i p k_0}{2f^2 \Omega^2} \exp \left\{ \frac{-D\tau_r k_0^2 (p^2 + r_1^2)}{2f^2 \Omega^2} \right\} \\ &\times \frac{\Omega n}{s^2 + n^2 \Omega^2} \left[I_{n-1} \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \right) - I_{n+1} \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \right) \right] \quad (7.8). \end{aligned}$$

We now use the identity,¹⁹

$$I_{n-1}(Z) - I_{n+1}(Z) = \frac{2n}{Z} I_n(Z) \quad (7.9),$$

to find

$$I(p, r_1, s) = \frac{i \frac{1}{2} \frac{\partial}{\partial p}}{r_1 |k_0|} \frac{(n^2)}{s^2 + n^2 \Omega^2} \exp \left\{ \frac{-D \tau_r k_0^2 (n^2 + r_1^2)}{2 f^2 \Omega^2} \right\} I_n \left(\frac{D \tau_r k_0^2 r_1 p}{f^2 \Omega^2} \right) \quad (7.10).$$

We have not, up to this point, included Landau damping in the calculation since we have assumed linear binding forces. Because particles with different synchrotron amplitudes rotate at the same rate in phase space, mixing of wave phases does not occur. Thus in order to include Landau damping, nonlinear binding forces must be assumed.

It is clear that in order to include Landau damping all we have to do is to replace $A(\theta_0, w_0, t)$ in Eq. (7.2) with $\bar{A}(\theta_0, w_0, t)$, the exact $\theta(t)$ for the nonlinear force under consideration. For example, if the potential well is sinusoidal instead of harmonic, $\bar{A}(\theta_0, w_0, t)$ is an elliptic function.¹⁵ In order to deal with different equilibrium distribution functions, $\psi_0(\theta_0, w_0)$, one merely makes the replacement

$$P(\theta_0) \frac{dW(w_0)}{dw} \longrightarrow \frac{1}{M} \frac{\partial \psi_0(\theta_0, w_0)}{\partial w} \quad (7.11),$$

where M is the normalization constant. Thus in the general case, Eq. (7.4) becomes

$$I(p, r_1, s) = \frac{1}{(2\pi)^2 M} \int_0^\infty dt e^{-st} \int d\theta_0 dw_0 \frac{\partial \psi_0(\theta_0, w_0)}{\partial w} \exp \{ i r_1 \theta_0 - i p \bar{A}(\theta_0, w_0, t) \} \quad (7.12).$$

It is clear that the integrals in Eq. (7.12) are very difficult to do. Therefore, we will present an approximate treatment for the case of cubic nonlinearity. The approximate solution in this case is¹⁵

$$\bar{A}(\theta_0, w_0, t) = \theta_0 \cos \Omega_1 t + \frac{k_0 w_0}{\Omega} \sin \Omega_1 t \quad (7.13),$$

where

$$\Omega_1(\theta_0, w_0) = \Omega \left[1 - \frac{3\epsilon}{8} \left(\theta_0^2 + \frac{k_0^2 w_0^2}{\Omega^2} \right) \right] \quad (7.14),$$

and ϵ is a nonlinearity parameter. When $\epsilon \rightarrow 0$, this reduces to the previous solution. Actually, to the same order there are additional terms in Eq. (7.13) with frequency $3\Omega_1$. These terms contribute additional resonant terms which are at the same frequencies and are smaller than the resonances already found. We will ignore these additional terms.

Using a Gaussian distribution function, Eq. (7.12) is

$$I(p, r_1, s) = \frac{1}{(2\pi)^2} \int_0^\infty dt e^{-st} Q_\epsilon(p, r_1, s, t) \quad (7.15),$$

where

$$Q_\epsilon(p, r_1, s, t) = \int d\theta_0 dw_0 P(\theta_0) \frac{dW(w_0)}{dw} \exp \left\{ i r_1 \theta_0 - i p \left(\theta_0 \cos \Omega_1 t + \frac{k_0 w_0}{\Omega} \sin \Omega_1 t \right) \right\} \quad (7.16).$$

Because $\Omega_1 = \Omega_1(\theta_0, w_0)$, we cannot do this integral. Observe that $Q_0(p, r_1, s, t)$ is a periodic function of Ωt . Now, however, the amplitude dependence of Ω_1 makes Q_ϵ aperiodic. Q_ϵ is a superposition of functions with different frequencies, i.e., a "wave packet" of finite width. As a result, we expect Q_ϵ to damp out as a function of time. In fact, a rough asymptotic argument shows that the damping coefficient should be

$$\delta_L \sim \frac{p \theta_{\text{RMS}} \Delta \Omega_{\text{RMS}}}{2\pi} \quad (7.17),$$

where θ_{RMS} is the root mean square angular spread in the beam, and $\Delta \Omega_{\text{RMS}}$ is the root mean square synchrotron-frequency spread in the beam. Of course, $\Delta \Omega_{\text{RMS}}$ is a function of ϵ and θ_{RMS} , which is given by

$$\Delta\Omega_{\text{RMS}} \approx \frac{3\epsilon \Omega \theta_{\text{RMS}}^2}{8} \quad (7.18).$$

δ_L is the coefficient in the exponential damping term. To understand Eq. (7.17), consider a wave with wave number p . The time, T , for particles to spread one wavelength is given by

$$\Delta\Omega_{\text{RMS}} T \sim \frac{2\pi}{p} \quad (7.19).$$

Since a coherent mode must be confined to the bunch, a packet of sinusoidal waves must exist. There are about $1/\theta_{\text{RMS}}$ sinusoidal waves in such a packet. Because the energy of a sinusoidal wave will be transferred to a wave with mode number $p \pm 1$, the time for the packet to damp is

$$T_L \sim \frac{T}{\theta_{\text{RMS}}} \quad (7.20).$$

Combining Eqs. (7.19) and (7.20), we see that

$$T_L = \frac{1}{\delta_L} \sim \frac{2\pi}{p \theta_{\text{RMS}} \Delta\Omega_{\text{RMS}}} \quad (7.21).$$

Equation (7.21) of course only gives the order of magnitude of δ_L .

The preceding argument is very crude, and, as a result, we found it necessary to investigate $Q_\epsilon(p, r_1, s, t)$ numerically. In order to put $Q_\epsilon(p, r_1, s, t)$ into a form convenient for numerical work, let us define

$$u^2 = \theta_0^2 + \frac{k_0^2 w_0^2}{\Omega^2} \quad (7.22),$$

$$\phi = \sin^{-1} \frac{\theta_0}{u} \quad (7.23),$$

then

$$Q_{\epsilon}(p, r_1, s, t) = \frac{f^2 \Omega^2}{D \tau_r k_0^2} \int_0^{\infty} du u^2 e^{-\frac{\Omega^2 f^2 u^2}{2 D \tau_r k_0^2}} \int_0^{2\pi} d\phi \cos \phi \\ \times \exp \left\{ i u \sin \phi (r_1 - p \cos \Omega_1 t) - i u p \cos \phi \sin \Omega_1 t \right\} \quad (7.24).$$

The ϕ integration can be done,¹⁹ and we find

$$Q_{\epsilon}(p, r_1, s, t) = -\frac{2\pi p f^2 \Omega^2 i}{D \tau_r k_0^2} \int_0^{\infty} du u^2 e^{-\frac{\Omega^2 f^2 u^2}{2 D \tau_r k_0^2}} \\ \times \frac{\sin \Omega_1(u) t}{\sqrt{p^2 + r_1^2 - 2pr_1 \cos \Omega_1(u) t}} J_1 \left(u \sqrt{p^2 + r_1^2 - 2pr_1 \cos \Omega_1(u) t} \right) \quad (7.25),$$

where J_1 is the Bessel function of the first kind. Recall that

$$\Omega_1(u) = \Omega \left(1 - \frac{3\epsilon u^2}{8} \right) \quad (7.26).$$

The integral in Eq. (7.25) is very complicated, but it is in a form which is convenient for numerical computation. With L. J. Laslett's help we have programmed Eq. (7.25) and investigated several cases numerically. Our results can be roughly described as follows: for a couple of damping times, the function is similar to Q_0 but with an exponential damping. After that, the amplitude of the oscillations remains approximately constant, but the phases of the oscillations shift rather strongly. Since both of these behaviors result in damping when inserted in Eq. (7.15), we assume that Q_{ϵ} is adequately represented by

$$Q_e(p, r_1, s, t) \approx e^{-\delta_L t} Q_0(p, r_1, s, t) \quad (7.27).$$

The magnitudes of the maxima, for a typical case, as a function of time are shown in Fig. 1 plotted on log paper. As can be seen, they lie approximately on a straight line. The slope of this line we take to be the damping coefficient. Figure 2 shows the damping coefficient as a function of θ_{RMS} . Figure 3 shows the damping coefficient as a function of $\Delta\Omega_{\text{RMS}}$. Figure 4 shows the damping coefficient as a function of p . In each figure, the straight line is a fit to the data. All of our results can be summarized by the formula

$$\delta_L = \frac{p \theta_{\text{RMS}} \Delta\Omega_{\text{RMS}}}{2} \quad (7.28).$$

Comparing this with Eq. (7.17), we see that the rough physical argument gave a remarkably accurate result. Examining Eqs. (7.27) and (7.15), we see that Landau damping can be approximately included if we make the substitution

$$s \rightarrow s + \delta_L \quad (7.29).$$

Using (7.29) and (7.10), our integral equation, (6.9), becomes

$$\begin{aligned} \tilde{\xi}(p, s) = & \tilde{\sigma}(p, s) - \frac{2\pi e^2 f^3 N i \Omega^2 n^2}{D\tau_r k_0 ((s + \delta_L)^2 + n^2 \Omega^2)} \int dr_1 \frac{1}{r_1} \\ & \times \exp \left\{ -\frac{D\tau_r k_0^2 (p^2 + r_1^2)}{2f^2 \Omega^2} \right\} I_n \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \right) Z(-r_1 \omega_0) \tilde{\xi}(r_1, s) \end{aligned} \quad (7.30).$$

The singularity at $r_1 = 0$ is only apparent, since $Z(0) = 0$.

If we were to expand the kernel of Eq. (7.30) in a power series in r_1 and keep only m terms, we would obtain, in the usual way,²⁰ m possible solutions. Therefore, in general, there are an infinite number of solutions of Eq. (7.30). Taking into account the infinite number of harmonics of Ω , we have a doubly infinite set of coherent oscillations. From Eq. (4.1), we see that

$$\frac{D\tau_r k_0^2}{f^2 \Omega^2} = \theta_{RMS}^2 \quad (7.31).$$

For wavelengths shorter than, or equal to the bunch length we have

$$\frac{D\tau_r k_0^2}{f^2 \Omega^2} pr_1 \geq (2\pi)^2 \gg 1 \quad (7.32).$$

Since the arguments of functions in Eq. (7.30) are large, a power series solution converges slowly. We can, however, use the large argument expansion of the Bessel function,¹⁹

$$I_n(z) \cong \frac{e^z}{\sqrt{2\pi z}} \quad (7.33).$$

Using Eq. (7.33) in (7.30), we get

$$\begin{aligned} \tilde{\xi}(p,s) = & \tilde{\sigma}(p,s) - \frac{\sqrt{2\pi} e^2 f^4 N i \Omega^3 n^2}{(D\tau_r)^{3/2} k_0 |k_0| [(s + \delta_L)^2 + n^2 \Omega^2]} \int dr_1 \frac{1}{r_1} \\ & \times \frac{\exp \left\{ -\frac{D\tau_r k_0^2 (p - r_1)^2}{2f^2 \Omega^2} \right\}}{\sqrt{pr_1}} Z(-r_1 \omega_0) \tilde{\xi}(r_1, s) \end{aligned} \quad (7.34)$$

All of the approximations which have been used in deriving Eq. (7.34) are usually quite good. For special assumptions about $Z(-r_1 \omega_0)$, Eq. (7.34) should be easy to solve.

We can simplify Eq. (7.34) considerably if we make one additional approximation. In the short wavelength case, i.e., $p\theta_{\text{RMS}} \gg 2\pi$, a good approximation is

$$\exp \left\{ - \frac{D_{\text{T}_\text{r}} k_0^2 (p - r_1)^2}{2f^2 \Omega^2} \right\} \approx \sqrt{\frac{2\pi f^2 \Omega^2}{D_{\text{T}_\text{r}} k_0^2}} \delta(p - r_1) \quad (7.35).$$

Using this, Eq. (7.34) becomes

$$\tilde{\xi}(p, s) = \tilde{\sigma}(p, s) - \frac{2\pi e^2 f^5 N i \Omega^4 n^2 Z(-p\omega_0)}{(D_{\text{T}_\text{r}})^2 k_0^3 [(s + \delta_L)^2 + n^2 \Omega^2] p^2} \tilde{\xi}(p, s) \quad (7.36).$$

Thus, the dispersion relation is

$$(s + \delta_L)^2 = -n^2 \Omega^2 - \frac{2\pi e^2 f^5 N i \Omega^4 n^2 Z(-p\omega_0)}{(D_{\text{T}_\text{r}})^2 p^2 k_0^3} \quad (7.37).$$

Instability occurs when $\text{Re}(s) > 0$. Equation (7.37) is a particularly simple result but should be applied cautiously to real proton beams. Both the treatment of Landau damping and the approximation (7.35) may be inadequate in cases of physical interest. For the longer wavelengths, it is necessary to solve Eq. (7.34).

Let us note that only resistance, i.e., $\text{Re}(Z) \neq 0$, can give rise to instability. This agrees with earlier work on this problem.^{17,18} In particular, if we take $\delta_L = 0$, then any resistance at all causes instab-

ility. Since Landau damping provides the only stabilization for proton beams, an accurate treatment is necessary, and the approximation we have presented may not be adequate for detailed applications.

VIII. ELECTRON BEAMS

In this Section, we derive the dispersion relation for electron beams. We then assume that the waves are stable; i.e. we assume that diffusion has lengthened the bunch so that the beam is stable. We can then find a formula for bunch length. If the beam, as lengthened by equilibrium effects, is stable, then no turbulent bunch lengthening occurs. Thus, a threshold effect is predicted.

In order to derive the dispersion relation for electron beams, we need to know the function $I(p, r_1, s)$ in Eq. (6.10). Remarkably enough, the function, I , can be calculated exactly. We present the derivation in Appendix D. Let us define

$$\omega = \sqrt{\Omega^2 - \frac{1}{4\tau_r^2}} = \Omega \sqrt{1 - \frac{1}{4\tau_r^2 \Omega^2}} \quad (8.1),$$

$$x = \sin^{-1} \frac{1}{2\Omega\tau_r} \quad (8.2),$$

then

$$\begin{aligned} I(p, r_1, s) = & \left(\frac{D\tau_r |k_0|}{f^2 \Omega \omega} \right) \left(\frac{i p k_0}{2} \right) \exp \left\{ \frac{-D\tau_r k_0^2 (p^2 + r_1^2)}{2f^2 \Omega^2} \right\} \\ & \times \int_0^\infty dt \, e^{-st - \frac{t}{2\tau_r}} \sin \omega t \exp \left\{ \frac{D\tau_r r_1 p k_0^2}{f^2 \Omega \omega} e^{-\frac{t}{2\tau_r}} \cos(\omega t - x) \right\} \end{aligned} \quad (8.3).$$

Let us note that in the limit, $D \rightarrow 0$, $\tau_r \rightarrow \infty$, $D\tau_r$ fixed, Eq. (8.3) goes over into Eq. (7.5). (The fact that the very long calculation of Appendix D reduces to the correct result in this special case is a comfort.)

We now use the identity¹⁹

$$\exp \left\{ \frac{D\tau_r r_1 p k_0^2}{f^2 \Omega \omega} e^{-\frac{t}{2\tau_r}} \cos(\omega t - x) \right\} = I_0 \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega \omega} e^{-\frac{t}{2\tau_r}} \right) + 2 \sum_{\ell=1}^{\infty} I_{\ell} \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega \omega} e^{-\frac{t}{2\tau_r}} \right) \cos \ell(\omega t - x) \quad (8.4),$$

where I_{ℓ} are Bessel functions. Since, in all cases that we are interested in, $\tau_r \Omega \gg 1$, we will expand to first order in $1/2\tau_r \Omega$. In this case

$$\omega \approx \Omega$$

$$x \approx \frac{1}{2\tau_r \Omega} \quad (8.5).$$

We then have

$$I(p, r_1, s) = \left(\frac{D\tau_r |k_0|}{f^2 \Omega^2} \right) \left(\frac{i p k_0}{2} \right) \exp \left\{ \frac{-D\tau_r k_0^2 (p^2 + r_1^2)}{2f^2 \Omega^2} \right\} \times \int_0^{\infty} dt e^{-st - \frac{t}{2\tau_r}} \sin \Omega t \left\{ I_0 \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} e^{-\frac{t}{2\tau_r}} \right) + 2 \sum_{\ell=1}^{\infty} I_{\ell} \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} e^{-\frac{t}{2\tau_r}} \right) [\cos \ell \Omega t + \ell x \sin \ell \Omega t] \right\} \quad (8.6).$$

We have expanded the cosine. Clearly this requires $\Delta x \ll 1$. Because τ_r is usually large compared to the synchrotron period, we expect a qualitative behavior similar to the proton beam case. Thus, we expect resonances at multiples of the synchrotron frequency. If we assume $s \approx i n \Omega$ and keep only the dominant terms, we get

$$\begin{aligned}
 I(p, r_1, s) &\approx \left(\frac{D\tau_r |k_0|}{f^2 \Omega^2} \right) \left(\frac{i p k_0}{2} \right) \exp \left\{ \frac{-D\tau_r k_0^2 (p^2 + r_1^2)}{2 f^2 \Omega^2} \right\} \\
 &\times \int_0^\infty dt \, e^{-st - \frac{t}{2\tau_r}} + i n \Omega t \left\{ \frac{1}{2i} \left[I_{n-1} \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} e^{-\frac{t}{2\tau_r}} \right) \right. \right. \\
 &\left. \left. - I_{n+1} \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} e^{-\frac{t}{2\tau_r}} \right) \right] + \frac{x}{2} \right. \\
 &\left. \times \left[(n+1) I_{n+1} \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} e^{-\frac{t}{2\tau_r}} \right) - (n-1) I_{n-1} \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} e^{-\frac{t}{2\tau_r}} \right) \right] \right\} \\
 &\quad (8.7).
 \end{aligned}$$

As discussed previously,

$$\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \gg 1.$$

As a result, for times $t < 6\tau_r$, the large argument expansion can be used for the Bessel functions, i.e. Eq. (7.31). Thus we have

$$I_n \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} e^{-\frac{t}{2\tau_r}} \right) \approx \frac{f \Omega e^{\frac{t}{4\tau_r}}}{\sqrt{2\pi D\tau_r k_0^2 r_1 p}} \exp \left\{ \frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} e^{-\frac{t}{2\tau_r}} \right\} \quad (8.8).$$

For times $t < 6\tau_r$, this is a good approximation. However, as $t \rightarrow \infty$,

the left hand side of Eq. (8.8) goes to zero while the right hand side goes to infinity. A better approximation results if we expand the exponential inside the exponential and keep the first two terms. Then

$$I_n \left(\frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} e^{-\frac{t}{2\tau_r}} \right) \approx \frac{f \Omega e^{\frac{t}{4\tau_r}}}{\sqrt{2\pi D\tau_r k_0^2 r_1 p}} \exp \left\{ \frac{D\tau_r k_0^2 r_1 p}{f^2 \Omega^2} \left(1 - \frac{t}{2\tau_r} \right) \right\} \quad (8.9).$$

For times $t < 6\tau_r$, Eq. (8.9) is an excellent approximation. Also, as $t \rightarrow \infty$, both sides of Eq. (8.9) go to zero exponentially. Since there are damping terms much larger than t/τ_r , we drop t/τ_r terms from now on. Using Eqs. (7.9) and (8.9), Eq. (8.7) becomes

$$\begin{aligned} I(p, r_1, s) &\approx \left(\frac{D\tau_r |k_0|}{f^2 \Omega^2} \right) \left(\frac{i p k_0}{2} \right) \exp \left\{ \frac{-D\tau_r k_0^2 (p - r_1)^2}{2f^2 \Omega^2} \right\} \\ &\times \int_0^\infty dt \exp \left\{ -st + i\Omega t - \frac{Dk_0^2 r_1 p}{2f^2 \Omega^2} t \right\} \\ &\times \left\{ \frac{-i\Omega^3}{\sqrt{2\pi} (D\tau_r k_0^2 r_1 p)^{3/2}} + \frac{x f \Omega}{\sqrt{2\pi D\tau_r k_0^2 r_1 p}} \right\} \quad (8.10) \end{aligned}$$

Let us note that the diffusion damping term is the same as that derived by Dupree²¹ for a magnetized plasma. Thus, our theory does reproduce the first order results of previous strong turbulence theories.

The integral in Eq. (8.10) is easily performed. If the result is substituted into Eq. (6.9), we obtain

$$\begin{aligned} \tilde{\xi}(p,s) = \tilde{\alpha}(p,s) - \frac{\pi e^2 N f^2 i p k_0}{\sqrt{2\pi D \tau_I k_0^2}} \int dr_1 \frac{\exp \left\{ \frac{-D \tau_I k_0^2 (p - r_1)^2}{2 f^2 \Omega^2} \right\}}{\sqrt{p r_1}} \\ \times \frac{Z(-r_1 \omega_0) \tilde{\xi}(r_1, s)}{s - i n \Omega + \frac{D k_0^2 r_1 p}{2 f^2 \Omega^2}} \left\{ \frac{1}{2 \tau_I \Omega} - \frac{i n f^2 \Omega^2}{D \tau_I k_0^2 r_1 p} \right\}. \end{aligned} \quad (8.11).$$

Now, if we use the short wavelength approximation, i.e. Eq. (7.55), we get

$$\tilde{\xi}(p,s) = \tilde{\alpha}(p,s) - \frac{\pi e^2 N f^3 i \Omega Z(-p \omega_0) \tilde{\xi}(p,s)}{D \tau_I k_0 \left[s - i n \Omega + \frac{D k_0^2 p^2}{2 f^2 \Omega^2} \right]} \left[\frac{1}{2 \tau_I \Omega} - \frac{i n f^2 \Omega^2}{D \tau_I k_0^2 p^2} \right] \quad (8.12).$$

Thus, the dispersion relation is

$$s = i n \Omega - \frac{D k_0^2 p^2}{2 f^2 \Omega^2} - \frac{\pi e^2 N f^3 i \Omega Z(-p \omega_0)}{D \tau_I k_0} \left[\frac{1}{2 \tau_I \Omega} - \frac{i n f^2 \Omega^2}{D \tau_I k_0^2 p^2} \right] \quad (8.13).$$

The stability criterion is $\text{Re}(s) \leq 0$. Let us note several things about Eq. (8.13). First, both resistance and inductance can give rise to instability. Previously, only the resistive instability has been found, because only proton beams have been treated. For electron beams, an additional dissipative process, namely radiation damping, is present. Thus, inductance, i.e. $\text{Im}(z) \neq 0$, can also cause instability. In fact, the inductive term decreases less rapidly with energy than the resistive term because of the increase of radiation damping. Thus, while the resistive term, considered by Lebedev,⁶ has a strong dependence on energy, the inductive term has a dependence on energy which is much

weaker. Let us observe that not only the imaginary part of Z , but also the real part of Z contributes to a shift in frequency. At high energy, the resistive frequency shift dominates. This suggests that calculations of the equilibrium lengthening effect should take into account a contribution from resistance. Previous theories⁴ have only included the inductive bunch lengthening.

It is clear that it would be difficult to include the effects of Landau damping in the calculation. However, we will assume that Landau damping can be included simply by the substitution

$$s \rightarrow s + \delta_L \quad (8.14),$$

with δ_L given by Eq. (7.28). That this must be approximately correct follows from the fact that the rough physical argument that we gave previously remains unchanged. This treatment of Landau damping is subject to the same criticism as in the proton beam case. However, in this case there is another damping mechanism, namely diffusion. In fact, as we will see the diffusion damping usually dominates the Landau damping. As a result, only an estimate of the magnitude of Landau damping is needed, and, for this, Eq. (7.28) should be adequate.

The stability criterion is now

$$\delta_L + \frac{Dk_0^2 p^2}{2f^2 \Omega^2} \geq \frac{\pi e^2 N f^3 \Omega}{D\tau_r |k_0|} \left[\frac{|Im(Z)|}{2\tau_r \Omega} + \frac{|Re(Z)| n f^2 \Omega^2}{D\tau_r k_0^2 p^2} \right] \quad (8.15).$$

Using Eqs. (7.18), (7.28), and (7.31), this becomes

$$\frac{3\epsilon_0 p \theta_{RMS}^5}{16} + \frac{p^2 \theta_{RMS}^4}{2\tau_r} \geq \frac{\pi e^2 N f |k_0|}{\Omega} \left[\frac{|Im(Z)|}{2\tau_r \Omega} + \frac{n |Re(Z)|}{p^2 \theta_{RMS}^2} \right] \quad (8.16).$$

Equation (8.16), with equality, gives us a formula for the bunch length. Recall that Eq. (8.16) should be applied only when the bunch length which it predicts is longer than the bunch length as given by the equilibrium theory. If the bunch length that Eq. (8.16) predicts is shorter than that given by equilibrium theory, turbulence is absent.

The equilibrium effect is included in Eq. (8.16), since Ω , and not Ω_0 , is present. Note that the equilibrium effect does not enter in a simple additive fashion. Recall, by the way, that equilibrium bunch lengthening occurs when $\Omega < \Omega_0$, since this corresponds to a weakening of the binding force.

Finally, let us note that nonlinearities enter into Eq. (8.16) through ϵ . In our simple model, ϵ is constant. However, if nonlinear distortion of the potential well due to self-fields is significant, then ϵ may depend on current.⁵

IX. COMPARISON WITH EXPERIMENTS

In this section, we apply Eq. (8.16) to the experimental data. We first examine the data from SPEAR I, and then that from SPEAR II. Finally, we make predictions about turbulent bunch lengthening for PEP.

First, let us note that the ratio of turbulent damping to Landau damping is

$$\frac{8p}{3\epsilon \Omega \theta_{\text{RMS}} \tau_r} \quad (9.1).$$

For the cases of interest, this ratio is large, so we will drop the Landau damping term. As a result, our approximate treatment of Landau

damping should be adequate for this application.

Now we need to specify the impedance function, $Z(-p\omega_0)$. Ideally, $Z(-p\omega_0)$ should be known once the details of the machine are known. In practice, $Z(-p\omega_0)$ is not well known. We will assume that

$$\frac{Z(-p\omega_0)}{p} = \text{constant} \quad (9.2).$$

This is physically reasonable for a high frequency resonant structure or for the radiation dominated regime.¹⁶

We also need to specify the wave number p . We will assume that

$$p_{\text{RMS}}^0 = \text{constant} \quad (9.3).$$

If we consider a single mode, as the length of the bunch changes the "size" of the mode changes in proportion. Thus, the wavelengths which make up the mode should be proportional to the bunch length. As a result, we get Eq. (9.3).

We also need an equation for Ω . We will use the expression of Pellegrini and Sessler⁴

$$\frac{\Omega_0^2}{\Omega^2} = 1 + \frac{kI}{E^3 \Delta} \quad (9.4),$$

where I is the current in the bunch, k is a constant, E is the beam energy, and Δ is the bunch length. We will fit the constant, k , from the data. As noted earlier, Eq. (9.4) does not include resistance-induced equilibrium bunch lengthening, however, in the absence of a theory which does, we must use (9.4). For SPEAR II, as we will see, this makes no difference. Also, for SPEAR I, it might make little difference

if the resistance which contributes to the equilibrium effect is small. Let us also remark that the impedances in Eq. (9.2), and those contained in the constant, k , may be quite different, since they correspond to different frequencies.

Our formula for the bunch length finally becomes

$$R^3 = \left(\frac{\theta_{\text{RMS}}}{\theta_{\text{RMSO}}} \right)^3 = \frac{2\pi e^2 N f |k_0| \tau_r}{c \Omega \theta_{\text{RMSO}}^3} \left(\frac{A}{2\tau_r \Omega} + B \right) \quad (9.5),$$

where A and B are dimensionless constants that we will fit to the data, and c is the speed of light. We have written Eq. (9.5) as a ratio cubed, where θ_{RMSO} is the natural bunch length, i.e.

$$\theta_{\text{RMSO}}^2 = \frac{D_Q \tau_r k_0^2}{f^2 \Omega_0^2} \quad (9.6).$$

In Eq. (9.5), there are three constants to fit - A , B , and the constant k which enters through Eq. (9.4).

Let us consider the data from SPEAR I.² There are data with three energies, 1.5, 2.2, and 2.5 GeV, and with a variety of rf voltages. It turns out that the bunch lengthening at the higher energies is due entirely to equilibrium effects. Thus, we can determine the constant k from one point at 2.5 GeV. The resulting curve is shown in Fig. 5. The ratio R is plotted as a function of beam current. The solid line is the theoretical curve, the squares are the experimental points. The point with the circle was used to determine $k = .35 \text{ GeV}^3 \text{ ns./mA}$. As can be seen, the fit is excellent. We remark that turbulence should begin to contribute at a current of about 285 milliamps (mA). At 2.2

GeV there is also no turbulence. The fit is shown in Fig. 6 and is excellent. Turbulence should begin at a current of 220 mA.

At 1.5 GeV, turbulence contributes significantly. Two circled points, one in Fig. 7 and one in Fig. 9, were fit to obtain the constants A and B, which are given by

$$\begin{aligned} A &= 2.39 \times 10^{-4} \\ B &= 8.6 \times 10^{-8} \end{aligned} \quad (9.7).$$

The fits for various voltages are shown in Figs. 7 through 10. The dashed lines represent the contribution of the equilibrium effect. As can be seen, the equilibrium contribution is substantial at all voltages. The contribution from turbulence is large at high voltages, but is relatively small at lower voltages. The experimentalists reported² that when half of the ferrite kicker magnets were removed, the bunch lengthening effect was reduced by about 50% at lower voltages, but only by about 20% at higher voltages. If we assume that the ferrite kicker magnets contributed to the equilibrium lengthening, but not to the turbulent lengthening, then these observations are perfectly consistent with Figs. 7 through 10. We remark, by the way, that all of the ferrite magnets were removed in making the transition to SPEAR II.

On SPEAR I, energy widths were calculated by measuring the quantum lifetime of the beam. The results seemed to show only small increases in energy width. Unfortunately, quantum lifetime measurements only reveal the diffusion coefficient at large synchrotron amplitudes, not the energy width of the beam. The energy width is determined by

the diffusion coefficient near small synchrotron amplitudes, where most of the particles are. It is obvious that the turbulent diffusion coefficient is small at large synchrotron amplitudes, where there are few particles. Thus the quantum lifetime should be near the natural lifetime even if the energy width is increased considerably. Although we have assumed a constant diffusion coefficient in our calculations, it is clear that it cannot be strictly constant. However, for most of the particles, near the center of the beam, the assumption of a constant diffusion coefficient is probably pretty good. The experimental evidence for a nonconstant diffusion coefficient would be a deviation of the particle distribution from a Gaussian at large amplitudes. Since nonlinearity of the potential well also contributes to a non-Gaussian distribution function, it is probably not possible to extract the variation of the diffusion coefficient.

The curves in Figs. 5 through 10 are not best fits to the data. They were obtained simply by fitting the three circled points. The closeness of the fits is thus an indication that the theory is basically right. Also, we note that it is remarkably difficult to obtain good fits with assumptions other than (9.2) and (9.3).

The magnitude of the impedances that (9.7) implies depend on an assumption about wave number, p . With a reasonable assumption, however, we obtain

$$\frac{|\operatorname{Re}(Z(-p\omega_0))|}{p} \sim \frac{|\operatorname{Im}(Z(-p\omega_0))|}{p} \sim 1 \text{ ohm} \quad (9.8).$$

This is a reasonable set of values.²²

SPEAR II has an rf system which is considerably different from

that of SPEAR I. Also, all of the ferrite kicker magnets have been removed. The energy width can now be directly measured by observing the beam width. Thus we are confronted with a different machine and with additional measurements.

The data from SPEAR II are plotted in Figs. 11 through 13.³ In Figs. 11 and 12, the bunch length is plotted as a function of current. In Fig. 13, the ratio of bunch length to natural bunch length, dots, and the ratio of energy spread to natural energy spread, crosses, is plotted versus current.

At 3 GeV, in Fig. 12, we notice that there seems to be a threshold in current. The equilibrium theories cannot account for such an effect. At 1.5 GeV and 1.7 mV, in Fig. 13, we notice that the energy spread ratio and the bunch length ratio seem to be identical. Thus we can conclude that equilibrium effects are negligible for SPEAR II; i.e. we take $k=0$.

Signals from coherent modes can also be measured. With small bunch lengthening, only a few modes are present. As bunch lengthening increases, the number of modes increases. Also, the spectrum looks quite turbulent; i.e., the amplitudes of the modes fluctuate in a seemingly random fashion. Thus, all of the data seems consistent with the interpretation that the bunch lengthening is due to turbulence. The absence of equilibrium bunch lengthening seems natural when it is remembered that all of the ferrite magnets were removed.

Since $k=0$, we have only two constants to fit: A and B. Figures 14 through 18 show the fit to the data. The ratio, R, is plotted as a function of the current. The solid lines are the theoretical curves, the squares are the experimental data. The circled points were used to determine A and B. The values of these constants are

$$A = 4.2 \times 10^{-4}$$

$$B = 4. \times 10^{-7} \quad (9.9).$$

From Figs. 14 and 15, we see that the fits at 1.5 GeV with 1.7 MV and 1 MV are quite good. From Fig. 16, we see that the fit at .6 MV is fairly good, but there seems to be a saturation phenomenon. We will discuss this phenomenon in more detail presently. In Fig. 17, we see that the fit at 3 GeV is pretty good. The points at higher current seem high, but this is probably due to an inadequate control over the rf system, since measurements of the synchrotron frequency varied by about 20% in this case, but not in the others. In Fig. 18, we see that the threshold for bunch lengthening occurs at about the current at which the measurements stopped. Higher currents have not been achieved yet because of synchrotron-betatron coupling resonances. In general, the fit to the data is quite good.

The saturation phenomenon at 1.5 GeV, .6 MV, seen in Fig. 16 may only be due to inaccurate measurements. However, if it is an actual phenomenon, there are two possible explanations for it. The first explanation is that it is due to Landau damping. The simple form that we have used, Eq. (8.16), cannot fit the data. This may be due to the fact that our treatment of Landau damping in this model is inadequate. It may also be due to the fact that our model, which does not include nonlinear distortion of the potential well due to self fields,⁵ may be inadequate. In the latter case, the saturation is due to the fact that ϵ , the nonlinearity parameter, is increasing rapidly as a function of current because of the rapidly increasing nonlinearity due to self-fields. In this case, the experimental evidence would be a deviation of the beam

from a Gaussian shape. However, since a nonconstant diffusion coefficient would also imply a non-Gaussian beam, it is probably difficult to extract the amount of deviation due to nonlinearity.

The second explanation of the saturation phenomenon is that it is due to mode-mode coupling. In other words, the beam stabilizes itself not by increasing its length, but by feeding energy into more and more coherent oscillations. Only a higher order perturbation calculation than ours could predict the contribution of mode-mode coupling. The experimental evidence for mode-mode coupling would consist of the rapid appearance of many higher order coherent oscillations when saturation sets in.

It is possible that both Landau damping and mode-mode coupling contribute to the saturation phenomenon. In any event, it is clear that this is an interesting effect which should be studied in more detail.

Note that the constants given by Eqs. (9.7) and (9.9) are fairly close. Thus while the impedances causing the equilibrium effect were virtually eliminated in SPEAR II, the impedances causing turbulence were increased by a small amount.

In order to apply our theory to PEP, we must estimate the impedances. Let us simple-mindedly just multiply the values of (9.9) by a factor of 10, since PEP will have 10 times greater circumference than SPEAR. Let us also assume that $k \approx 0$. Using PEP parameters, namely

$$\begin{aligned} R &= 345 \text{ m} \\ \Omega_y &= 6.3 \times 10^4 \text{ sec}^{-1} \\ E &= 15 \text{ GeV} \\ f &= 1.38 \times 10^5 \text{ sec}^{-1} \end{aligned}$$

$$\begin{aligned}\alpha &= .00455 \\ \tau_r &= .00823 \text{ sec} \\ \theta_{\text{RMSO}} &= 8.65 \times 10^{-5}\end{aligned}\tag{9.10},$$

we find

$$R^3 = 1.1 \times 10^{-2} I' \tag{9.11},$$

where I' is the current in units of milliamps. Thus, $R \geq 1$ implies

$$I \geq 100 \text{ mA} \tag{9.12}.$$

Since PEP plans to operate at currents-per-bunch less than this, turbulent bunch lengthening should be no problem, if impedances per unit length similar to SPEAR II are attained.

X. SUMMARY AND CONCLUSIONS

We have derived the strong turbulence theory by a Fokker-Planck technique and have investigated the random phase approximation in some detail. Our calculations have shown that we reproduce the first-order results of previous strong turbulence theories.

We have applied our theory to the uniform beam case and have reproduced the earlier result of Sessler. We have also obtained the integral equation for bunched proton beams with general distribution function and potential wells. In the short wavelength case, with a Gaussian distribution function, we were able to obtain a fairly simple dispersion relation for proton beams. The only inadequacy in our treatment of proton beams was the approximate analysis of Landau damping.

We were able to obtain the exact integral equation for electrons in a harmonic potential. The dispersion relation in the short wavelength limit was used to obtain a formula for the bunch length. This bunch length formula was applied to the experimental data from SPEAR I and from SPEAR II and good fits to the data resulted from physically reasonable assumptions.

We can conclude that the theory presented in this thesis is adequate to fit almost all of the data. A complete analysis would require solving the integral equation, (8.11), more accurately.

It is clear that more experimental measurements of the saturation phenomenon are needed. More measurements of the energy spread and its correlation with bunch length would allow a more rigorous assessment of the role of turbulence. In particular, energy spread measurements in the saturation regime would help clarify the situation. A detailed study of the coherent modes would also be of use. Observations of the

bunch shape are needed, although the interpretation of the data would be difficult, because both a nonconstant diffusion coefficient and a nonlinear potential well can affect the bunch shape. Finally, the effect on the bunch length of various parts of the beam environment should be measured, if possible. This would shed light both on the types of impedances which contribute to bunch lengthening and on the possible effects for PEP.

The equilibrium theory should be generalized in two ways. First, the effect of resistive impedances should be included, since it is possible that the resistive effect dominates at high energy. Second, the effect of nonlinearities of the potential well due to self-fields should be studied. Since it seems likely that turbulent bunch lengthening will be negligible at higher energies for PEP, the possible role of equilibrium effects is quite important. Thus, the theory should be developed in more detail.

The contribution of Landau damping should be analyzed much more extensively. This is particularly true for applications to proton beams where Landau damping provides the only stabilization, in linear approximation, of the coherent modes. Also, for electron beams, we have seen that Landau damping may be important in explaining the saturation phenomenon. In particular, the effect on Landau damping of the nonlinear distortion of the potential well due to self-fields should be studied carefully.

The theory of mode-mode coupling should be derived, since it may play an important role in explaining the saturation phenomenon. The theory of mode-mode coupling in uniform plasmas is well known,²⁵ but a theory adequate for these applications has not been developed. In fact,

except for our ignorance of the impedance function, the study of stored particle beams would be an excellent context in which to observe and analyze many nonlinear plasma phenomena.

APPENDIX A. SOLUTION TO HOMOGENEOUS EQUATION

In this Appendix, we want to derive the solution to Eq. (4.4) that satisfies (4.5). The solution has been obtained previously by Chandrasekhar,¹² and we reproduce the calculation here merely for the convenience of the reader.

Let us first define

$$\phi = \frac{\theta}{k_0}, \quad \beta_1 = \frac{1}{\tau_r}, \quad q = \frac{D}{f^2} \quad (A1).$$

Using (A1), Eq. (4.4) becomes

$$\frac{\partial h}{\partial t} + w \frac{\partial h}{\partial \phi} - (\beta_1 w + \Omega^2 \phi) \frac{\partial h}{\partial w} - \beta_1 h = q \frac{\partial^2 h}{\partial w^2} \quad (A2),$$

In order to proceed, we need a *lemma*. Let $g(t)$ and $f(t)$ be two functions of time. Then the solution of

$$\frac{\partial F}{\partial t} = f^2(t) \frac{\partial^2 F}{\partial x^2} + 2f(t)g(t) \frac{\partial^2 F}{\partial x \partial y} + g^2(t) \frac{\partial^2 F}{\partial y^2} \quad (A3),$$

which has a source at $x=y=0$ when $t=0$, is

$$F = \frac{1}{2\pi\Delta^2} \exp \left\{ \frac{-(ax^2 + 2lxy + by^2)}{2\Delta} \right\} \quad (A4),$$

where

$$a = 2 \int_0^t g^2(t') dt' \quad (A5),$$

$$b = 2 \int_0^t f^2(t') dt' \quad (A6),$$

$$\ell = -2 \int_0^t g(t') f(t') dt' \quad (A7),$$

$$\Delta = ab - \ell^2 \quad (A8).$$

To prove this lemma, one merely substitutes (A4) into (A3) and uses (A5) through (A8). The calculation is tedious but straightforward.

Now let

$$\mu_1 = -\frac{1}{2} \beta_1 + \sqrt{\frac{1}{4} \beta_1^2 - \Omega^2} \quad (A9),$$

$$\mu_2 = -\frac{1}{2} \beta_1 - \sqrt{\frac{1}{4} \beta_1^2 - \Omega^2} \quad (A10).$$

If we define new variables

$$\xi = (\phi \mu_1 - w) \exp(-\mu_2 t) \quad (A11),$$

$$\eta = (\phi \mu_2 - w) \exp(-\mu_1 t) \quad (A12),$$

then Eq. (A2) becomes

$$\frac{\partial h}{\partial t} = \beta_1 h + q \left\{ e^{-2\mu_2 t} \frac{\partial^2 h}{\partial \xi^2} + 2 e^{-(\mu_1 + \mu_2)t} \frac{\partial^2 h}{\partial \xi \partial \eta} + e^{-2\mu_1 t} \frac{\partial^2 h}{\partial \eta^2} \right\} \quad (A13).$$

Now, let

$$G = h e^{\beta_1 t} \quad (A14).$$

Equation (A13) becomes

$$\frac{\partial G}{\partial t} = q \left(e^{-2\mu_2 t} \frac{\partial^2 G}{\partial \xi^2} + e^{-(\mu_1 + \mu_2)t} \frac{\partial^2 G}{\partial \xi \partial \eta} + e^{-2\mu_1 t} \frac{\partial^2 G}{\partial \eta^2} \right) \quad (A15).$$

Equation (A15) is of the same form as Eq. (A3). Thus, the solution is

$$G = \frac{1}{2\pi\Delta^{\frac{1}{2}}} \exp \left\{ - [a(\xi - \xi_0)^2 + 2\ell(\xi - \xi_0)(\eta - \eta_0) + b(\eta - \eta_0)^2] / 2\Delta \right\} \quad (\text{A16}),$$

where

$$a = \frac{q}{\mu_1} \left(1 - e^{-2\mu_1 t} \right) \quad (\text{A17}),$$

$$b = \frac{q}{\mu_2} \left(1 - e^{-2\mu_2 t} \right) \quad (\text{A18}),$$

$$\ell = \frac{-2q}{\mu_1 + \mu_2} \left(1 - e^{-(\mu_1 + \mu_2)t} \right) \quad (\text{A19}).$$

Requiring condition (4.5), we get

$$h(\theta, w, t; \theta_0, w_0, 0) = \frac{\left(\sqrt{1/\tau_r^2 - 4\Omega^2} \right) e^{t/\tau_r}}{2\pi |k_0| \sqrt{\Delta(t)}} \exp \left\{ - [a(t)(\xi - \xi_0)^2 + 2\ell(t)(\xi - \xi_0)(\eta - \eta_0) + b(t)(\eta - \eta_0)^2] / 2\Delta(t) \right\} \quad (\text{A20}).$$

APPENDIX B: UNIFORM BEAM KERNEL

In this Appendix, we calculate the function $\bar{L}(p,s)$ defined by Eq. (5.10). Combining Eqs. (5.10) and (4.11), we find

$$\bar{L}(p,s) = \frac{1}{2\pi} \int_0^\infty dt e^{-st} \int d\theta dw dw_0 \frac{dw(w_0)}{dw} h(\theta, w, w_0, t) e^{-ip\theta} \quad (B1).$$

We now need the function h in the limit $\Omega \rightarrow 0$. Using the notation of Appendix A, we find

$$\begin{aligned} \mu_1 &= 0 \\ \mu_2 &= -\beta_1 \\ a(t) &= 2qt \\ b(t) &= -\frac{q}{\beta_1} \left(1 - e^{2\beta_1 t} \right) \\ \ell(t) &= \frac{2q}{\beta_1} \left(1 - e^{\beta_1 t} \right) \end{aligned} \quad (B2).$$

We thus find that

$$h = \frac{\beta_1 e^{\beta_1 t}}{2\pi \sqrt{\Delta(t)} |k_0|} \exp \left\{ -A_1 \theta^2 - A_2 w^2 - A_3 w_0^2 - A_4 \theta w - A_5 \theta w_0 - A_6 w w_0 \right\} \quad (B3),$$

where

$$A_1(t) = \frac{q\beta_1}{2\Delta k_0^2} \left(e^{2\beta_1 t} - 1 \right) \quad (B4)$$

$$A_2(t) = \frac{q}{2\beta_1 \Delta} \left\{ (2\beta_1 t - 3) e^{2\beta_1 t} + 4e^{\beta_1 t} - 1 \right\} \quad (B5)$$

$$A_3(t) = \frac{q}{2\beta_1 \Delta} \left\{ e^{2\beta_1 t} - 4e^{\beta_1 t} + 2\beta_1 t + 3 \right\} \quad (B6)$$

$$A_u(t) = \frac{-q}{\Delta k_0} (e^{\beta_1 t} - 1)^2 = A_s(t) \quad (B7),$$

$$A_e(t) = \frac{q}{\beta_1 \Delta} \left[e^{2\beta_1 t} - 2\beta_1 t e^{\beta_1 t} - 1 \right] \quad (B8).$$

Recall that

$$W(w_0) = \exp \left\{ -\frac{w_0^2 \beta_1}{2q} \right\} \quad (B9).$$

With (B3), we note that the θ, w, w_0 integrals in (B1) are Gaussian and are easily done. Doing the θ and w integrals, we find

$$\begin{aligned} \bar{L}(p, s) = & \frac{-\beta_1^2}{4\pi q |k_0|} \int_0^\infty dt \frac{e^{-st + \beta_1 t}}{\sqrt{\Delta(t) A_2(t) (A_1(t) - A_4^2/4A_2)}} \\ & \times \int dw_0 w_0 \exp \left\{ -\frac{\beta_1 w_0^2}{2q} - A_3 w_0^2 + \frac{A_6^2}{4A_2} w_0^2 - \frac{p^2}{4(A_1 - A_4^2/4A_2)} \right. \\ & \left. - \frac{2ipw_0(-A_5 + A_4 A_6/2A_2)}{4(A_1 - A_4^2/4A_2)} + \frac{w_0^2(-A_5 + A_4 A_6/2A_2)^2}{4(A_1 - A_4^2/4A_2)} \right\} \quad (B10). \end{aligned}$$

Using (B4)-(B8), straightforward calculations give

$$A_3 - \frac{A_6^2}{4A_2} - \frac{(-A_5 + A_4 A_6/2A_2)^2}{4(A_1 - A_4^2/4A_2)} = 0 \quad (B11),$$

$$\frac{-A_5 + A_4 A_6/2A_2}{A_1 - A_4^2/4A_2} = \frac{2k_0}{\beta_1} \left(1 - e^{-\beta_1 t} \right) \quad (B12),$$

$$\Delta(t)A_2(t) \left(A_1 - \frac{A_4^2}{4A_2} \right) = \frac{\beta_1^2}{4k_0^2} e^{2\beta_1 t} \quad (B13),$$

$$\Delta(t) = \frac{q^2}{\beta_1^2} (e^{\beta_1 t} - 1) \left[e^{\beta_1 t} (2\beta_1 t - 4) + (2\beta_1 t + 4) \right] \quad (B14),$$

$$\frac{1}{4 \left(A_1 - \frac{A_4^2}{4A_2} \right)} = k_0^2 \frac{q}{2\beta_1^3} \left[(2\beta_1 t - 3) + 4e^{-\beta_1 t} - e^{-2\beta_1 t} \right] \quad (B15),$$

Thus,

$$\begin{aligned} \bar{L}(p,s) &= \frac{-\beta_1}{2\pi q} \int_0^\infty dt e^{-st} \exp \left\{ \frac{-p^2 q k_0^2}{2\beta_1^3} \left[(2\beta_1 t - 3) + 4e^{-\beta_1 t} - e^{-2\beta_1 t} \right] \right\} \\ &\times \int dw_0 w_0 \exp \left\{ \frac{-\beta_1 w_0^2}{2q} - \frac{i p w_0 k_0}{\beta_1} (1 - e^{-\beta_1 t}) \right\} \quad (B16) \end{aligned}$$

Performing the integration over w_0 , we get

$$\bar{L}(p,s) = i p k_0 \sqrt{\frac{q}{2\pi\beta_1^3}} \int_0^\infty dx e^{-sx/\beta_1} (1 - e^{-x}) \exp \{ -v[x - 1 + e^{-x}] \} \quad (B17)$$

where we defined

$$\begin{aligned} x &= \beta_1 t \\ v &= q p^2 k_0^2 \tau_r^3 \end{aligned} \quad (B18)$$

Let $y = -x$, and swing the integration contour around in the complex plane from $-\infty$ to $+\infty$. We get

$$\bar{L}(p,s) = i p k_0 \sqrt{\frac{q}{2\pi\beta_1^5}} \int_0^\infty dy e^{s y/\beta_1} (e^y - 1) \exp \left\{ v(y+1 - e^y) \right\} \quad (B19).$$

Let $x_1 = e^y - 1$. Then (B19) becomes

$$\bar{L}(p,s) = i p k_0 \sqrt{\frac{q}{2\pi\beta_1^5}} \int_0^\infty dx_1 x_1 \exp \left\{ -v x_1 + \left(\frac{s}{\beta_1} - 1 + v \right) \ln(x_1 + 1) \right\} \quad (B20).$$

Using the definition of β_1 and q , this is seen to be identical with

Eq. (5.14).

APPENDIX C: ALTERNATIVE DERIVATION OF UNIFORM BEAM CASE

In this Appendix, we present another derivation of Eq. (5.16). Sessler⁷ has previously derived this equation, but, in the course of the calculation, divergent integrals were manipulated. We present a derivation which is similar to Sessler's but which avoids divergent quantities.

For uniform beams $\Omega = 0$. Let us assume

$$\psi(\theta, w, t) = \sum_n \psi_n(w) e^{in\theta - i\omega_n t} \quad (C1),$$

Then Eq. (2.16) implies

$$\frac{D}{f^2} \frac{d^2 \psi_n}{dw^2} + \frac{w}{\tau_r} \frac{d\psi_n}{dw} + (i\omega_n - ik_0 w n + \frac{1}{\tau_r}) \psi_n = -\beta c e^2 Z_n \frac{d\psi_0}{dw} \int \psi_n(w) dw \quad (C2),$$

where Z_n is defined by Eq. (3.30). Let

$$K_n = -\frac{\beta c e^2 Z_n f^2}{D} \int \psi_n(w) dw \quad (C3).$$

Then Eq. (C2) becomes

$$\frac{d^2 \psi_n}{dw^2} + \left(\frac{f^2}{D\tau_r} \right) w \frac{d\psi_n}{dw} + \frac{f^2}{D\tau_r} \psi_n + \frac{if^2 \omega_n}{D} \psi_n - \frac{ink_0 f^2}{D} w \psi_n = K_n \frac{d\psi_0}{dw} \quad (C4).$$

Let

$$\psi_n(w) = \int e^{iwx} \tilde{\psi}_n(x) dx \quad (C5),$$

then,

$$\frac{d\tilde{\psi}_n}{dx} + \tilde{\psi}_n \left[\frac{-x^2 + if^2 \omega_n / D}{\frac{nk_0 f^2}{D} - \frac{xf^2}{D\tau_r}} \right] = \frac{i\lambda_0 K_n}{2\pi} \frac{x e^{-D\tau_r x^2 / 2f^2}}{\left(\frac{nk_0 f^2}{D} - \frac{xf^2}{D\tau_r} \right)} \quad (C6).$$

Equation (C6) is a first order differential equation that we can easily solve. Let

$$r(x) = \frac{-x^2 + \frac{if^2\omega_n}{D}}{\frac{nk_0 f^2}{D} - \frac{xf^2}{D\tau_r}} \quad (C7),$$

$$g(x) = \frac{i\lambda_0 K_n}{2\pi} \frac{x e^{-D\tau_r x^2/2f^2}}{\left(\frac{nk_0 f^2}{D} - \frac{xf^2}{D\tau_r}\right)} \quad (C8),$$

then

$$\tilde{\psi}_n(x) = \exp\left\{-\int_a^x r(x') dx'\right\} \int_a^x \exp\left\{\int_a^{x'} r(x'') dx''\right\} g(x') dx' + C \exp\left\{-\int_a^x r(x') dx'\right\} \quad (C9),$$

where a and C are constants. Recall that

$$\int dw \psi_n(w) = 2\pi \tilde{\psi}_n(0) \quad (C10).$$

Thus,

$$K_n = -\frac{\beta c e^2 Z_n 2\pi f^2}{D} \tilde{\psi}_n(0) \quad (C11).$$

Now,

$$\begin{aligned} \int_a^x r(x') dx' &= \frac{D\tau_r}{f^2} \left\{ -\frac{if^2\omega_n}{D} \ln\left(\frac{x - nk_0\tau_r}{a - nk_0\tau_r}\right) + \frac{x^2}{2} - \frac{a^2}{2} \right. \\ &\quad \left. + nk_0\tau_r(x - a) + n^2k_0^2\tau_r \ln\left(\frac{x - nk_0\tau_r}{a - nk_0\tau_r}\right) \right\} \quad (C12). \end{aligned}$$

Thus,

$$\begin{aligned}
 \tilde{\psi}_n(0) = & \left(\frac{D\tau_r}{f^2} \right) \left(\frac{i\lambda_0 k_n}{2\pi} \right) \left(\frac{1}{nk_0 \tau_r} \right) \int_a^0 dx \times \exp \left\{ \frac{nk_0 D\tau_r^2}{f^2} x \right. \\
 & + \left. \left(\frac{Dn^2 k_0^2 \tau_r^3}{f^2} - i\omega \tau_r - 1 \right) \ln \left(1 - \frac{x}{nk_0 \tau_r} \right) \right\} \\
 & + C \exp \left\{ \frac{-D\tau_r}{f^2} \left[-\frac{a^2}{2} - nk_0 \tau_r a + \left(n^2 k_0^2 \tau_r^2 - \frac{if^2 \omega_n}{D} \right) \right. \right. \\
 & \times \left. \left. \ln \left(\frac{-nk_0 \tau_r}{a - nk_0 \tau_r} \right) \right] \right\}
 \end{aligned} \tag{C13}$$

In Eq. (C13), we must choose the constants a and C so that the answer is finite. Thus, we choose

$$\begin{aligned} a &\rightarrow -\infty, & C &= 0 & \text{if} & & nk_0 > 0 \\ a &\rightarrow +\infty, & C &= 0 & \text{if} & & nk_0 < 0. \end{aligned}$$

With these choices we get for both cases

$$\begin{aligned} \tilde{\psi}_n(0) &= \left(\frac{D\tau_r}{f^2} \right) \left(\frac{i\lambda_0 K_n}{2\pi} \right) \left(\frac{-1}{nk_0 \tau_r} \right) \int_0^\infty dx \, x \\ &\times \exp \left\{ - \frac{|nk_0| D\tau_r^2 x}{f^2} + \left(\frac{Dn^2 k_0^2 \tau_r^3}{f^2} - i\omega\tau_r - 1 \right) \ln \left(1 + \frac{x}{|nk_0| \tau_r} \right) \right\} \end{aligned} \quad (C14).$$

Using Eqs. (C11) and (C14) and defining

$$y = \frac{x}{|nk_0| \tau_r} \quad (C15),$$

$$v = \frac{n^2 k_0^2 D\tau_r^3}{f^2} \quad (C16),$$

$$\mu = i\omega\tau_r + 1 - v \quad (C17),$$

the dispersion relation becomes

$$1 = i\lambda_0 \beta c e^2 Z_n nk_0^2 \tau_r^2 \int_0^\infty dy \, y \exp \{ -vy - \mu \ln(1+y) \} \quad (C18).$$

Recalling Eq. (5.5), we see that Eqs. (C18) and (5.16) are identical.

Sessler derived a dispersion equation which was

$$1 = \frac{if^2 \lambda_0 \beta c e^2 Z_n}{nk_0 D\tau_r} [1 - i\omega\tau_r K(\mu, v)] \quad (C19),$$

where

$$K(\mu, \nu) = \int_0^{\infty} dy \exp [-\nu y - \mu \ln(1+y)] \quad (C20),$$

Equations (C18) and (C19) are the same if

$$\nu \int_0^{\infty} dy y e^{-\nu y - \mu \ln(1+y)} + (\mu + \nu - 1) \int_0^{\infty} dy e^{-\nu y - \mu \ln(1+y)} \stackrel{?}{=} 1 \quad (C21),$$

i.e., if

$$-\nu \frac{d}{d\nu} \left(\int_0^{\infty} dy e^{-\nu y - \mu \ln(1+y)} \right) + (\mu + \nu - 1) \int_0^{\infty} dy e^{-\nu y - \mu \ln(1+y)} \stackrel{?}{=} 1 \quad (C22).$$

Let $\bar{K}$ satisfy

$$-\nu \frac{d \bar{K}(\mu, \nu)}{d\nu} + (\mu + \nu - 1) \bar{K}(\mu, \nu) = 1 \quad (C23).$$

Let us solve Eq. (C23), which is a first order differential equation.

The solution is

$$\begin{aligned} \bar{K}(\mu, \nu) &= \int_a^{\nu} \exp \left\{ -(v' - \nu) - (\mu - 1) \ln \frac{v'}{\nu} \right\} \frac{-1}{v'} dv' \\ &+ f(\mu) \exp \left\{ \nu - a + (\mu - 1) \ln \frac{\nu}{a} \right\} \end{aligned} \quad (C24),$$

where a is a constant and $f(\mu)$ is a function of μ . Let

$$r = \frac{v'}{\nu} - 1 \quad (C25),$$

then (C24) becomes

$$\bar{K}(\mu, \nu) = - \int_{a-\nu}^0 dr \exp \left\{ -\nu r - \mu \ln(1+r) \right\} + f(\mu) \exp \left\{ \nu - a + (\mu - 1) \ln \frac{\nu}{a} \right\} \quad (C26).$$

Choosing $a = \infty$, $f(\mu) = 0$, we see that

$$\bar{K}(\mu, \nu) = K(\mu, \nu) \quad .$$

Thus, Eqs. (C18) and (C19) are identical.

Let us finally note that¹⁹

$$K(\mu, \nu) = \nu^{-\mu-1} e^{\nu} \Gamma(\mu+1, \nu) \quad (C27),$$

where $\Gamma(\mu+1, \nu)$ is an incomplete gamma function.

APPENDIX D: THE FUNCTION $I(p, r_1, s)$

In this Appendix, we derive Eq. (8.3). Using Eqs. (6.12) and (A20), we find, using the notation of Appendix A,

$$\begin{aligned}
 I(p, r_1, s) = & - \frac{f^2}{(2\pi)^2 D_{T_1}} \int_0^\infty dt e^{-st} \int d\theta_0 d\theta dw_0 dw e^{ir_1 \theta_0 - ip\theta_{w_0}} \\
 & \times \exp \left\{ - \frac{f^2 w_0^2}{2D_{T_1}} - \frac{\theta_0^2 f^2 \Omega^2}{2D_{T_1} k_0^2} - B_1 \theta^2 - B_2 \theta w - B_3 w^2 \right. \\
 & - B_4 \theta_0^2 - B_5 \theta_0 w_0 - B_6 w_0^2 - B_7 \theta \theta_0 - B_8 \theta w_0 - B_9 \theta_0 w \\
 & \left. - B_{10} w w_0 \right\} \times \frac{\sqrt{1/\tau_r^2 - 4\Omega^2}}{2\pi |k_0| \sqrt{\Delta(t)}} e^{\beta_1 t} \quad (D1)
 \end{aligned}$$

where

$$B_1(t) = \frac{1}{2\Delta k_0^2} \left[a(t) \mu_1^2 e^{-2\mu_2 t} + 2\ell(t) \mu_1 \mu_2 e^{-(\mu_1 + \mu_2)t} + b(t) \mu_2^2 e^{-2\mu_1 t} \right] \quad (D2)$$

$$\begin{aligned}
 B_2(t) = & \frac{1}{2\Delta k_0} \left[-2a(t) \mu_1 e^{-2\mu_2 t} - 2\ell(t) (\mu_1 + \mu_2) e^{-(\mu_1 + \mu_2)t} \right. \\
 & \left. - 2b(t) \mu_2 e^{-2\mu_1 t} \right] \quad (D3)
 \end{aligned}$$

$$B_3(t) = \frac{1}{2\Delta} \left[a(t) e^{-2\mu_2 t} + 2\ell(t) e^{-(\mu_1 + \mu_2)t} + b(t) e^{-2\mu_1 t} \right] \quad (D4)$$

$$B_4(t) = \frac{1}{2\Delta k_0^2} \left[a(t) \mu_1^2 + 2\ell(t) \mu_1 \mu_2 + b(t) \mu_2^2 \right] \quad (D5)$$

$$B_5(t) = \frac{1}{2\Delta k_0} [-2\mu_1 a(t) - 2\ell(t)(\mu_1 + \mu_2) - 2b(t)\mu_2] \quad (D6)$$

$$B_6(t) = \frac{1}{2\Delta} [a(t) + 2\ell(t) + b(t)] \quad (D7)$$

$$B_7(t) = \frac{1}{2\Delta k_0^2} \left[-2\mu_1^2 a(t) e^{-\mu_2 t} - 2\ell(t)\mu_1\mu_2 (e^{-\mu_1 t} + e^{-\mu_2 t}) - 2\mu_2^2 b(t) e^{-\mu_1 t} \right]$$

$$B_8(t) = \frac{1}{2\Delta k_0} \left[2\mu_1 a(t) e^{-\mu_2 t} + 2\ell(t)\mu_2 e^{-\mu_1 t} + 2\ell(t)\mu_1 e^{-\mu_2 t} + 2b(t)\mu_2 e^{-\mu_1 t} \right] \quad (D9)$$

$$B_9(t) = \frac{1}{2\Delta k_0} \left[2a(t)\mu_1 e^{-\mu_2 t} + 2\ell(t)\mu_1 e^{-\mu_1 t} + 2\ell(t)\mu_2 e^{-\mu_2 t} + 2b(t)\mu_1 e^{-\mu_1 t} \right] \quad (D10)$$

$$B_{10}(t) = \frac{1}{2\Delta} \left[-2a(t) e^{-\mu_2 t} - 2\ell(t) (e^{-\mu_1 t} + e^{-\mu_2 t}) - 2b(t) e^{-\mu_1 t} \right] \quad (D11)$$

$$\Delta(t) = \frac{q^2}{\mu_1 \mu_2} \left(1 - e^{-2\mu_1 t} - e^{-2\mu_2 t} + e^{2\beta_1 t} \right) - \frac{4q^2}{\beta_1^2} \left(1 - 2e^{\beta_1 t} + e^{2\beta_1 t} \right) \quad (D12)$$

Looking at Eq. (D1), we see that the θ , θ_0 , w , and w_0 integrals are Gaussian integrals, and can thus be done. Doing first the w_0 integral, we find

$$\begin{aligned}
 I(p, r_1, s) = & - \frac{f^2 \sqrt{\beta_1^2 - 4\Omega^2}}{(2\pi)^3 |k_0| D\tau_r} \int_0^\infty dt \frac{e^{-st} e^{\beta_1 t}}{\sqrt{\Delta(t)}} \sqrt{\frac{\pi}{B_3(t)}} \int dw_0 d\theta_0 d\theta w_0 \\
 & \times \exp \left\{ - \frac{f^2 w_0^2}{2D\tau_r} + i r_1 \theta_0 - i p \theta - \frac{\theta_0^2 f^2 \Omega^2}{2D\tau_r k_0^2} - B_1 \theta^2 - B_4 \theta_0^2 - B_5 \theta_0 w_0 \right. \\
 & - B_6 w_0^2 - B_7 \theta \theta_0 - B_8 \theta w_0 + \frac{B_2^2 \theta^2 + B_2^2 \theta_0^2 + B_2^2 w_0^2}{4B_3} \\
 & \left. + \frac{B_2 B_9 \theta \theta_0 + B_2 B_{10} \theta w_0 + B_9 B_{10} \theta_0 w_0}{2B_3} \right\} \quad (D13)
 \end{aligned}$$

Doing next the θ integral, we find

$$\begin{aligned}
 I(p, r_1, s) = & - \frac{f^2 \sqrt{\beta_1^2 - 4\Omega^2}}{(2\pi)^3 |k_0| D\tau_r} \int_0^\infty dt \frac{e^{-st + \beta_1 t}}{\sqrt{\Delta(t)}} \sqrt{\frac{\pi}{B_3(t)}} \sqrt{\frac{\pi}{B_1 - B_2^2/4B_3}} \\
 & \times \exp \left\{ \frac{-p^2}{4(B_1 - B_2^2/4B_3)} \right\} \int dw_0 d\theta_0 \exp \left\{ - \frac{f^2 w_0^2}{2D\tau_r} + i r_1 \theta_0 \right. \\
 & - \frac{\theta_0^2 f^2 \Omega^2}{2D\tau_r k_0^2} - B_4 \theta_0^2 - B_5 \theta_0 w_0 - B_6 w_0^2 + \frac{B_2^2 \theta_0^2}{4B_3} + \frac{B_2^2 w_0^2}{4B_3} + \frac{B_9 B_{10} \theta_0 w_0}{2B_3} \\
 & + \left[\theta_0^2 \left(B_7 - \frac{B_2 B_9}{2B_3} \right)^2 + w_0^2 \left(B_8 - \frac{B_2 B_{10}}{2B_3} \right)^2 - 2ip\theta_0 \left(-B_7 + \frac{B_2 B_9}{2B_3} \right) - 2ipw_0 \left(-B_8 + \frac{B_2 B_{10}}{2B_3} \right) \right. \\
 & \left. \left. + 2\theta_0 w_0 \left(-B_7 + \frac{B_2 B_9}{2B_3} \right) \left(-B_8 + \frac{B_2 B_{10}}{2B_3} \right) \right] / 4 \left(B_1 - \frac{B_2^2}{4B_3} \right) \right\} \quad (D14)
 \end{aligned}$$

If we select out the θ_0 part of the integral, we find

$$\begin{aligned}
 & \int d\theta_0 \exp \left\{ i r_1 \theta_0 - \frac{\theta_0^2 f^2 \Omega^2}{2 D \tau_I k_0^2} - B_4 \theta_0^2 - B_5 \theta_0 w_0 + \frac{B_9^2 \theta_0^2}{4 B_3} + \frac{B_9 B_{10}}{2 B_3} \theta_0 w_0 \right. \\
 & + \frac{\theta_0^2 (B_7 - B_2 B_9 / 2 B_3)^2}{4 (B_1 - B_2^2 / 4 B_3)} - \frac{i p \theta_0 (-B_7 + B_2 B_9 / 2 B_3)}{2 (B_1 - B_2^2 / 4 B_3)} \\
 & \left. + \frac{\theta_0 w_0 (-B_7 + B_2 B_9 / 2 B_3) (-B_8 + B_2 B_{10} / 2 B_3)}{2 (B_1 - B_2^2 / 4 B_3)} \right\} \\
 & = \sqrt{\frac{\pi}{\frac{f^2 \Omega^2}{2 D \tau_I k_0^2} + B_4 - \frac{B_9^2}{4 B_3} - \frac{(B_7 - B_2 B_9 / 2 B_3)^2}{4 (B_1 - B_2^2 / 4 B_3)}}} \exp \left\{ \left[-r_1^2 - \frac{p^2 (-B_7 + B_2 B_9 / 2 B_3)^2}{4 (B_1 - B_2^2 / 4 B_3)^2} \right. \right. \\
 & + w_0^2 \left(-B_5 + \frac{B_9 B_{10}}{2 B_3} + \frac{(-B_7 + B_2 B_9 / 2 B_3) (-B_8 + B_2 B_{10} / 2 B_3)}{2 (B_1 - B_2^2 / 4 B_3)} \right)^2 \\
 & + 2 i r_1 w_0 \left(-B_5 + \frac{B_9 B_{10}}{2 B_3} + \frac{(-B_7 + B_2 B_9 / 2 B_3) (-B_8 + B_2 B_{10} / 2 B_3)}{2 (B_1 - B_2^2 / 4 B_3)} \right) \\
 & + r_1 p \frac{(-B_7 + B_2 B_9 / 2 B_3)}{(B_1 - B_2^2 / 4 B_3)} - i p w_0 \frac{(-B_7 + B_2 B_9 / 2 B_3)}{(B_1 - B_2^2 / 4 B_3)} \left(-B_5 + \frac{B_9 B_{10}}{2 B_3} \right. \\
 & + \left. \left. \frac{(-B_7 + B_2 B_9 / 2 B_3) (-B_8 + B_2 B_{10} / 2 B_3)}{2 (B_1 - B_2^2 / 4 B_3)} \right) \right] \times \frac{1}{4} \left(\frac{f^2 \Omega^2}{2 D \tau_I k_0^2} + B_4 \right. \\
 & \left. \left. - \frac{B_9^2}{4 B_3} - \frac{(B_7 - B_2 B_9 / 2 B_3)^2}{4 (B_1 - B_2^2 / 4 B_3)} \right) \right\} \quad (D15)
 \end{aligned}$$

Using Eqs. (A-17) through (A-19), and Eqs. (D2) through (D12), very long but straightforward calculations, give

$$B_4 - \frac{B_9^2}{4B_3} - \frac{(B_7 - B_2 B_9 / 2B_3)^2}{4(B_1 - B_2^2 / 4B_3)} = 0 \quad (D16),$$

$$-B_5 + \frac{B_9 B_{10}}{2B_3} + \frac{(-B_7 + B_2 B_9 / 2B_3)(-B_8 + B_2 B_{10} / 2B_3)}{2(B_1 - B_2^2 / 4B_3)} = 0 \quad (D17),$$

$$-B_6 + \frac{B_{10}^2}{4B_3} + \frac{(B_8 - B_2 B_{10} / 2B_3)^2}{4(B_1 - B_2^2 / 4B_3)} = 0 \quad (D18),$$

thus,

$$\begin{aligned} I(p, r_1, s) &= \frac{-f \sqrt{B_1^2 - 4\Omega^2}}{(2\pi)^2 \Omega \sqrt{2\pi D\tau_r}} \int_0^\infty dt \frac{e^{-st + \beta_1 t}}{\sqrt{\Delta(t)}} \cdot \frac{\pi}{B_3(t)} \sqrt{\frac{\pi}{B_1 - B_2^2 / 4B_3}} \\ &\times \exp \left\{ \frac{-p^2}{4(B_1 - B_2^2 / 4B_3)} - \frac{D\tau_r k_0^2 r_1^2}{2f^2 \Omega} - \frac{p^2 (-B_7 + B_2 B_9 / 2B_3)^2 D\tau_r k_0^2}{8f^2 \Omega^2 (B_1 - B_2^2 / 4B_3)^2} \right. \\ &\left. + \frac{D\tau_r k_0^2 r_1 p (-B_7 + B_2 B_9 / 2B_3)}{2f^2 \Omega^2 (B_1 - B_2^2 / 4B_3)} \right\} \int dw_0 w_0 \\ &\times \exp \left\{ -\frac{f^2 w_0^2}{2D\tau_r} - \frac{ipw_0 (-B_8 + B_2 B_{10} / 2B_3)}{2(B_1 - B_2^2 / 4B_3)} \right\} \end{aligned} \quad (D19).$$

Doing the integral over w_0 , we find

$$\begin{aligned}
 I(p, r_1, s) = & \frac{i p D \tau_{\mathbf{r}} \sqrt{\beta_1^2 - 4\Omega^2}}{2(2\pi)^2 f^2 \Omega} \int_0^\infty dt e^{\frac{-st + \beta_1 t}{\sqrt{\Delta(t)}}} \sqrt{\frac{\pi}{B_3(t)}} \sqrt{\frac{\pi}{B_1 - B_2^2/4B_3}} \\
 & \times \frac{(-B_8 + B_2 B_{10}/2B_3)}{(B_1 - B_2^2/4B_3)} \exp \left\{ \frac{-p^2}{4(B_1 - B_2^2/4B_3)} - \frac{D \tau_{\mathbf{r}} k_0^2 r_1^2}{2f^2 \Omega^2} \right. \\
 & - \frac{p^2 (-B_7 + B_2 B_9/2B_3)^2 D \tau_{\mathbf{r}} k_0^2}{8f^2 \Omega^2 (B_1 - B_2^2/4B_3)^2} + \frac{D \tau_{\mathbf{r}} k_0^2 r_1 p (-B_7 + B_2 B_9/2B_3)}{2f^2 \Omega^2 (B_1 - B_2^2/4B_3)} \\
 & \left. - \frac{D \tau_{\mathbf{r}} p^2 (-B_8 + B_2 B_{10}/2B_3)^2}{8f^2 (B_1 - B_2^2/4B_3)^2} \right\} \quad (D20).
 \end{aligned}$$

Using Eqs. (A17) through (A19), and (D2) through (D12), we see that

$$\frac{(-B_8 + B_2 B_{10}/2B_3)}{(B_1 - B_2^2/4B_3)} = \frac{-2}{\mu_1 - \mu_2} (e^{\mu_2 t} - e^{\mu_1 t}) \quad (D21),$$

$$\Delta(t) B_3(t) \left(B_1 - \frac{B_2^2}{4B_3} \right) = \frac{1}{4} (\mu_1 - \mu_2)^2 e^{2\beta_1 t} \quad (D22),$$

$$\frac{(-B_7 + B_2 B_9/2B_3)}{(B_1 - B_2^2/4B_3)} = \frac{2}{\mu_1 - \mu_2} (\mu_1 e^{\mu_2 t} - \mu_2 e^{\mu_1 t}) \quad (D23),$$

$$\frac{1}{4(B_1 - B_2^2/4B_3)} + \frac{D \tau_{\mathbf{r}} k_0^2 (-B_7 + B_2 B_9/2B_3)^2}{8f^2 \Omega^2 (B_1 - B_2^2/4B_3)^2} + \frac{D \tau_{\mathbf{r}} (-B_8 + B_2 B_{10}/2B_3)^2}{8f^2 (B_1 - B_2^2/4B_3)^2} = \frac{D \tau_{\mathbf{r}} k_0^2}{2f^2 \Omega^2} \quad (D24).$$

Using Eqs. (D21) through (D24) in Eq. (D20), we see that

$$\begin{aligned}
 I(p, r_1, s) &= \frac{D\tau_R |k_0|}{f^2 \Omega \sqrt{\Omega^2 - E_1^2/4}} \frac{pk_0}{4} \exp \left\{ - \frac{D\tau_R k_0^2 (p^2 + r_1^2)}{2f^2 \Omega^2} \right\} \\
 &\times \int_0^\infty dt e^{-st} (e^{\mu_2 t} - e^{\mu_1 t}) \exp \left\{ \frac{D\tau_R k_0^2 r_1 p}{f^2 \Omega^2} \frac{(\mu_1 e^{\mu_2 t} - \mu_2 e^{\mu_1 t})}{\mu_1 - \mu_2} \right\} \\
 &\hspace{15em} (D25)
 \end{aligned}$$

Using the various definitions, we see that this is identical with Eq. (8.3).

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FIGURE CAPTIONS

- Fig. 1. Height of maxima of $Q_c(p, r_1, s, t)$ versus t on a logarithmic scale; $p = 100.$, $\Delta\Omega_{RMS} = .005$, $\Omega = 1.$, $\theta_{RMS}/2\pi = .02$.
- Fig. 2. δ_L versus $\theta_{RMS}/2\pi$; $\Omega = 1$, $\Delta\Omega_{RMS} = .005$, $p = 100$.
- Fig. 3. δ_L versus $\Delta\Omega_{RMS}$; $\Omega = 1$, $p = 100$, $\theta_{RMS}/2\pi = .04$.
- Fig. 4. δ_L versus p ; $\Omega = 1$, $\Delta\Omega_{RMS} = .005$, $\theta_{RMS}/2\pi = .04$.
- Fig. 5. R versus I (current); $E = 2.5$ GeV, $V_{RF} = 475$ kV; in Figs. 5-10, experimental points are squares, theory is the solid line, circled points were used to fit the theory.
- Fig. 6. R versus I ; $E = 2.2$ GeV, $V_{RF} = 300$ kV
- Fig. 7. R versus I ; $E = 1.5$ GeV, $V_{RF} = 525$ kV; in Figs. 7-10, the dotted lines show the contribution of the equilibrium theory.
- Fig. 8. R versus I ; $E = 1.5$ GeV, $V_{RF} = 275$ kV.
- Fig. 9. R versus I ; $E = 1.5$ GeV, $V_{RF} = 150$ kV.
- Fig. 10. R versus I ; $E = 1.5$ GeV, $V_{RF} = 90$ kV; the reason for the high values of R at intermediate currents is unknown; it may only be due to lack of experimental control over the RF system.
- Fig. 11. Bunch length, σ_z , versus I ; $E = 1.5$ GeV, $V_{RF} = .6, 1.0, 1.7$ mV; in Figs. 11-12, circles are experimental points, lines are interpolations by hand.
- Fig. 12. σ_z versus I ; $E = 3.0$ GeV, $V_{RF} = 3.0$ MV; $E = 3.8$ GeV, $V_{RF} = 5.2$ MV.
- Fig. 13. Bunch lengthening, σ_z/σ_{z0} , and energy spread increase, σ_E/σ_{E0} , versus I ; $E = 1.5$ GeV, $V_{RF} = 1.7$ MV; the solid line is an interpolation by hand.

Fig. 14. R versus I ; $E = 1.5$ GeV, $V_{RF} = 1.7$ MV; in Figs. 14-18, squares are experimental points, solid lines are the theory, circled points were used to fit the constants of the theory.

Fig. 15. R versus I ; $E = 1.5$ GeV, $V_{RF} = 1.0$ MV.

Fig. 16. R versus I ; $E = 1.5$ GeV, $V_{RF} = 0.6$ MV; the saturation phenomenon may be due to Landau damping or to mode-mode coupling; it is discussed in the text.

Fig. 17. R versus I ; $E = 3.0$ GeV, $V_{RF} = 3.0$ MV.

Fig. 18. R versus I ; $E = 3.8$ GeV, $V_{RF} = 5.2$ MV.

Figure 1

Figure 5

Figure 12