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LEVEL DEGENERACY OF THE GENERALISED
N-PARTICLE DUAL AMPLITUDE WITH SATELLITES

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July, 1970

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ABSTRACT

We show that the level degeneracy at $\alpha(s) = n$ for the general N-pt. function with satellites is

$$d_n \sim \exp n^\gamma \text{ as } n \rightarrow \infty$$

where $\gamma = 1 - \frac{1}{2k+1}$ $k = 1, 2, \dots$

The form of the single loop constructed from such amplitudes is briefly discussed.

⁺Supported in part by the U. S. Atomic Energy Commission.

Several authors^(1,2,3) have given expressions for dual N-pt. scalar amplitudes with satellites, some of which have been factorised^(4,5).

In this note we examine the level degeneracy of the class of N-pt. functions discussed in Ref. 1, the simplest of which has already been examined in some detail^(5,6).

Our main result is that the N-pt. scalar dual amplitudes with satellites B_N can be classified into sets $\{B_N^k\}$ $k=1,2,\dots$ (as was essentially done in Ref. 1) such that the level degeneracy at $\alpha(s)=n$ for the amplitude B_N^k is

$$d_n^{(k)} \sim \exp b_k n^{\gamma_k} \quad \text{as } n \rightarrow \infty \quad (1)$$

where

$$\gamma_k = 1 - \frac{1}{2k+1}, \quad k \geq 1 \quad (2)$$

We examine the N-pt. function with satellites using the terminology of Ref. 1. The N-pt. scalar amplitude has the form

$$B_N(p_i) = \int \pi dx_i \tilde{\phi}_N(x_i, p_i) = \int \pi dx_i \phi_N(x_i, p_i) F_N(x_i)_{(3)}$$

where $F_N \equiv 1$ corresponds to the usual N-pt. function.

The satellite function $F_N(x_i)$ has the form

$$F_N(x_i) = \sum_k F_N^k(x_i) \quad (4)$$

where $F_N^k(x_i) = \sum f_N^k(u_{l,l+1}, \dots, u_{l-2,l-1}) \quad (5)$

the sum in eqn. (5) being taken over the N sets of u's at the vertices of the dual polygon. Only a finite number of the f_N^k are non-zero and they have the form

$$f_N^k(x_1, \dots, x_{N-3}) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq N-3} x_{i_1} x_{i_2} \dots x_{i_k} (1-x_{i_1})(1-x_{i_2}) \dots (1-x_{i_k}) \cdot \tilde{f}_{k+3}(x_{i_1}, x_{i_2}, \dots, x_{i_k}) \quad (6)$$

Let us factorise the $N=(r+s+4)$ -pt. function given by Fig. 1 in the $s=(\sum p_i)^2 = (\sum q_j)^2$ channel. If z is the variable associated with this channel (Fig. 2) we expand F_N^k as

$$F_N^k = \sum_{n=0}^{\infty} z^n D_n^k(x, y) \quad (7)$$

associating x_i with the incoming momenta p_i , y_j with the outgoing momenta q_j . Suppose D_n^k can be factorised in a symmetric way in the least number of factored terms as

$$D_n^k(x, y) = \sum_{\alpha=1}^{\delta_n^k} L_{n\alpha}^k(x) R_{n\alpha}^k(y) \quad (8)$$

Then from eqn. (4)

$$\begin{aligned} F_N &= \sum_{n=0}^{\infty} z^n \sum_k D_n^k(x, y) \\ &= \sum_{n=0}^{\infty} z^n \sum_{\alpha=1}^{\delta_n} L_{n\alpha}(x) R_{n\alpha}(y) \end{aligned} \quad (9)$$

in an obvious notation, where $\delta_n = \sum_k \delta_n^k < \infty$.

Inserting eqn. (9) in eqn. (3) gives

$$\begin{aligned} B_N &= \int (\prod dx_i \Phi_{r+3}(x_i, p_i)) (\prod dy_j \Phi_{s+3}(y_j, q_j)) \int_0^1 dz z^{-\alpha(s)-1} (1-z)^{\alpha(t)-1} \\ &\quad \cdot \exp \sum_{n=1}^{\infty} z^n \left[\frac{1}{n} P^{(n)} \cdot Q^{(n)} + \sum_{\alpha} L_{n\alpha}(x) R_{n\alpha}(y) \right] \end{aligned} \quad (10)$$

in the usual notation.

We see from eqn. (9) that we can perform the single factorisation by introducing harmonic oscillator operators $a_\mu^{(s)}$ ($\mu=0,1,2,3$), $A_\alpha^{(r)}$ ($\alpha=1,2,\dots, \delta_r$) satisfying

$$[a_\mu^{(r)}, a_\nu^{(s)}] = -\delta_{rs} g_{\mu\nu}, \quad [A_\alpha^{(r)}, A_\beta^{(s)}] = \delta_{rs} \eta_{\alpha\beta} \quad (11)$$

with all other commutators zero. The $a^{(r)}$ are the usual 4-dimensional operators⁽⁸⁾ that couple to the external momenta, and the $A^{(r)}$ couple to the additional internal degrees of freedom. The metric $\eta_{\alpha\beta}$ is determined by the details of F_N .

Assuming that there is no difficulty in performing multiple factorisation eqn. (11) implies an internal Hamiltonian

$$H = \sum_{r=1}^{\infty} -r (a_\mu^{(r)} g_{\mu\nu} a_\nu^{(r)} - A_\alpha^{(r)} \eta_{\alpha\beta} A_\beta^{(r)}) \quad (12)$$

If d_n is the degeneracy of H with eigenvalue n we have

$$P(x) = \sum_{n=0}^{\infty} d_n x^n = \text{Tr } x^H = \prod_{r=1}^{\infty} (1-x^r)^{-4-\delta_r} \quad (13)$$

We now compute δ_n^k (eqn. 8) for large n .

a) $k=1$ ⁽⁵⁾. Expand $\tilde{f}_4(x)$ as

$$x(1-x)\tilde{f}_4(x) = \sum_{k \geq 1} e_k (1-x)^k \quad (14)$$

where $\sum e_k = 0$. The main contribution to δ_n^1 comes from the sum

$$F_N^{1/4} = \sum_{\substack{1 \leq p \leq r+1 \\ 1 \leq q \leq s+1}} d_{pq} (1-d_{pq}) \tilde{f}_4(d_{pq}) \quad (15)$$

where d_{pq} are the lines intersecting z . The d_{pq} satisfy

$$(1 - d_{pq})^k = \sum_{st \leq r=k}^{\infty} C_r^{st}(k) x_p^s (\rho_{p-1} z \sigma_{q-1})^r \gamma_q^t \quad (16)$$

where $\rho_p = x_1 x_2 \dots x_p$, $\sigma_q = \gamma_1 \gamma_2 \dots \gamma_q$ and

$$C_r^{st}(k) = \sum_{p+q=r-k} (-1)^{s+t} \binom{-k}{p} \binom{-k}{q} \binom{k}{s-p} \binom{k}{t-q} \quad (17)$$

Inserting eqn. (14) into eqn. (15) we have

$$F_N'^1 = \sum_{r=1}^{\infty} z^r \sum_{st \leq r=1}^{\infty} L_{rs}(x) C_r^{st} R_{rt}(y) \quad (18)$$

where $C_r^{st} = \sum_{k \geq 1} C_r^{st}(k) e_k \quad (19)$

and $L_{rs}(x) = \sum_{p=1}^{r+1} x_p^s \rho_{p-1}^r$, $R_{rt}(y) = \sum_{q=1}^{s+1} \gamma_q^t \sigma_{q-1}^r \quad (20)$

Diagonalising C_r we write

$$F_N'^1 = \sum_{r=1}^{\infty} z^r \sum_{\alpha=0}^r L'_{r\alpha}(x) \lambda_{\alpha\beta}^r R'_{r\beta}(y) \quad (21)$$

where $\lambda_{\alpha\beta}^r$ are the signs of the eigenvalues of C_r .

The z -dependent terms in F_N^1 omitted in $F_N'^1$ add only a constant number of factorised terms to each mode. Thus

$$\delta_r^1 \sim r \quad \text{as } r \rightarrow \infty \quad (22)$$

b) $k \geq 1$: One of the terms giving the asymptotic forms of

δ_n^k is $F_N'^k = F_N^{(1)k} + F_N^{(2)k}$, where

$$F_N^{(1)k} = \sum_{1 \leq p \leq r+1} \sum_{1 \leq q_1 < \dots < q_{k-1} \leq s+1} l_{1p} (1-l_{1p}) d_{p1} \dots d_{pq_{k-1}} (1-d_{pq_1}) \cdot (1-d_{pq_2}) \dots (1-d_{pq_{k-1}}) \tilde{f}_{k+3}(l_{1p}, d_{pq_1}, \dots, d_{pq_{k-1}}) \quad (23)$$

$$F_N^{(2)k} = \sum_{1 \leq q \leq s+1} \sum_{1 \leq p_1 < \dots < p_{k-1} \leq r+1} r_{1q} (1-r_{1q}) d_{1q} \dots d_{p_{k-1}q} (1-d_{p_1q}) \cdot (1-d_{p_2q}) \dots (1-d_{p_{k-1}q}) \tilde{f}_{k+3}(r_{1q}, d_{p_1q}, \dots, d_{p_{k-1}q}) \quad (24)$$

The l_{1p} , r_{1q} (Fig. 2) satisfy

$$(1-l_{1p})^k = \sum_{st \leq r=k}^{\infty} C_r^{st}(k) x_p^s \rho_{p-1}^r z^t \quad (25)$$

with a similar expression for r_{1q} .

Expand $\tilde{f}_{k+3}$ as

$$x_1(1-x_2) \dots (1-x_k) x_k \tilde{f}_{k+3}(x_1, x_2, \dots, x_k) = \sum_{\substack{l_i \geq 1 \\ i=1,2,\dots,k}} e_{l_1, l_2, \dots, l_k} (1-x_1)^{l_1} \dots (1-x_k)^{l_k} \quad (26)$$

where

$$\sum_{l_1 \geq 1} e_{l_1, \dots, l_k} = \sum_{l_k \geq 1} e_{l_1, \dots, l_k} = 0$$

Then, from eqns. (16), (25) and

$$d_{p1} \dots d_{pq_{k-1}} = 1 - \sum_{st \leq r=1}^{\infty} C_r^{st}(1) x_p^s \rho_{p-1}^r z^t \sigma_{q_{k-1}}^{-r} \quad (27)$$

$$F_N^{(1)k} = \sum_{1 \leq p \leq s+1} \sum_{1 \leq q_1 < \dots < q_{k-1} \leq s+1} \left\{ \sum_{\substack{a_i, b_i \leq c_i=1 \\ i=0,1,\dots,k-1}} E_{\substack{a_0, \dots, a_{k-1}; b_0, \dots, b_{k-1} \\ c_0, \dots, c_{k-1}}} x_p^{a+a_0} \rho_{p-1}^{c+c_0} \cdot z^{c+b_0} \prod_{i=1}^{k-1} \sigma_{q_i-1}^{c_i} \gamma_{q_i}^{b_i} \left[1 - \sum_{a_k b_k \leq c_k=1}^{\infty} C_{c_k}^{a_k b_k}(1) x_p^{a_k} \rho_{p-1}^{c_k} z^{c_k} \sigma_{q_{k-1}}^{-b_k} \right] \right\} \quad (28)$$

where

$$E_{c_0, \dots, c_{k-1}}^{a_0, \dots, a_{k-1}; b_0, \dots, b_{k-1}} = \sum_{\substack{l_i \geq 0 \\ i=0, \dots, k-1}} e_{l_0, \dots, l_{k-1}} C_{c_0}^{a_0 b_0}(l_0) C_{c_1}^{a_1 b_1}(l_1) \dots C_{c_{k-1}}^{a_{k-1} b_{k-1}}(l_{k-1}) \quad (29)$$

where $a = \sum_{i=1}^{k-1} a_i$, $c = \sum_{i=1}^{k-1} c_i$

In an obvious notation we can write eqn. (28) as

$$F_N^{(1)k} = \sum_{b_0} \sum_{\substack{b_i \leq c_i=1 \\ i=1, \dots, k-1}}^{\infty} L_0^{c_1, \dots, c_{k-1}; b_0, \dots, b_{k-1}}(x) z^{c+b_0} R_2^{c_1, \dots, c_{k-1}; b_1, \dots, b_{k-1}}(y) \\ - \sum_{b_0} \sum_{\substack{b_i \leq c_i=1 \\ i=1, \dots, k}} L_1^{c_1, \dots, c_k; b_0, \dots, b_k}(x) z^{c+c_k+b_0} R_2^{c_1, \dots, c_{k-1}+b_k; b_1, \dots, b_{k-1}}(y) \quad (30)$$

where

$$R_2^{c_1, \dots, c_{k-1}; b_1, \dots, b_{k-1}}(y) = \sum_{1 \leq q_1 < \dots < q_{k-1} \leq N-3} \prod_{i=1}^{k-1} \sigma_{q_i-1}^{c_i} \gamma_{q_i}^{b_i} \\ L_0^{c_1, \dots, c_{k-1}; b_0, \dots, b_{k-1}}(x) = \sum_{1 \leq p \leq s+1} \sum_{c_0} \sum_{\substack{a_i \leq c_i \\ i=0, \dots, k-1}} E_{c_0, \dots, c_{k-1}}^{a_0, \dots, a_{k-1}; b_0, \dots, b_{k-1}} x_p^{a+a_0} p_{p-1}^{c+c_0} \\ L_1^{c_1, \dots, c_k; b_0, \dots, b_k}(x) = \sum_{1 \leq p \leq s+1} \sum_{c_0} \sum_{\substack{a_i \leq c_i \\ i=0, \dots, k-1}} C_{c_k}^{a_k b_k}(1) E_{c_0, \dots, c_{k-1}}^{a_0, \dots, a_{k-1}; b_0, \dots, b_{k-1}} x_p^{a+a_0+a_k} \cdot p_{p-1}^{c+c_0+c_k} \quad (31)$$

Eqn. (30) can be written formally as

$$F_N^{(1)k} = \sum_{n=k-1}^{\infty} z^n \sum_{\alpha=1}^{S_n^{k(0)}} L_{0\alpha}^{(n)}(x) R_{2\alpha}^{(n)}(y) - \sum_{n=k}^{\infty} z^n \sum_{\alpha=1}^{S_n^{k(1)}} L_{1\alpha}^{(n)}(x) R_{2\alpha}^{(n)}(y) \quad (32)$$

$F_N'^k$ (eqn. 23)) then has the factorised form

$$F_N'^k = \sum_{n=k-1}^{\infty} z^n \sum_{\alpha=1}^{\delta_n^{k(0)}} (L_{0\alpha}^{(n)}(x) R_{2\alpha}^{(n)}(y) + L_{2\alpha}^{(n)}(x) R_{0\alpha}^{(n)}(y)) -$$

$$- \sum_{n=k}^{\infty} z^n \sum_{\alpha=1}^{\delta_n^{k(1)}} (L_{1\alpha}^{(n)}(x) R_{2\alpha}^{(n)}(y) + L_{2\alpha}^{(n)}(x) R_{1\alpha}^{(n)}(y))$$

(33)

where a similar expansion has been performed for $F_N^{(2)k}$.

To symmetrically factorise $F_N'^k$ we have to make the substitution

$$-(L_1 R_2 + L_2 R_1) = L_1 R_1 + L_2 R_2 - (L_1 + L_2)(R_1 + R_2)$$

(34)

This gives rise to a large number of components with negative metric which will give ghosts in the level degeneracy.

We observe immediately that $\delta_n^{k(1)} > \delta_n^{k(0)}$ so we restrict ourselves to examining the second term on the right hand side of eqn. (30). In order that $\delta_n^{k(1)}$ be the least number of factored terms there must be no linear dependences between the $R_{\alpha}^{(n)}$ or between the $L_{\alpha}^{(n)}$. That this is the case for the $R_{\alpha}^{(n)}(y)$ follows directly from eqn. (31). It is also apparent that for general values of the $e_{l_0 \dots l_{k-1}}$ the $L_{1\alpha}^{(n)}(x)$ will be linearly independent. (9)

Fixing the power of z at $c + c_k + b_k = n$ in the second term of eqn. (30) we thus have

$$\delta_n^{k(1)} \sim b_k^{(1)} n^{2k-1}$$

(35)

Since the other z -dependent terms in F_N^k give similar power behaviour [if they contain $l_{1p}, d_{pq_1}, i=1,2$] or less we have

$$\delta_n^k \sim b_k n^{2k-1} \text{ as } n \rightarrow \infty, k > 1 \quad (36)$$

Let $\{B_N^k\}$ be the set of N -pt. functions for which $f_N^k \neq 0, f_N^l \equiv 0, l > k$ (i.e. the simplest set of N -pt. functions ($N > k+3$) compatible with arbitrary $(k+3)$ and lesser point functions). Then, for a particular B_N^k

$$\delta_n^{(k)} = \sum_l \delta_n^l \sim b_k n^{2k-1}, k \geq 1 \quad (37)$$

where the sum is over the non-zero $\delta_n^l, l \leq k$.

The level degeneracy of B_N^k at $\alpha(s)=n$ is d_n^k where⁽¹⁰⁾

$$P^{(k)}(x) = \sum_{n=0}^{\infty} d_n^{(k)} x^n = \prod_{r=1}^{\infty} (1-x^r)^{-\delta_r^{(k)}-4} \quad (38)$$

In Ref. 4 it has been shown that

$$\ln P^{(1)}(x) \sim b_1 \zeta(3) (1-x)^{-2} \text{ as } x \rightarrow 1-. \quad (39)$$

By a straightforward generalisation of this method it can be shown that

$$\ln P^{(k)}(x) \sim b_k \Gamma(2k) \zeta(2k+1) (1-x)^{-2k} \text{ as } x \rightarrow 1-. \quad (40)$$

We can evaluate the asymptotic form $d_n^{(k)}$ by applying the saddle-point method to

$$d_n^{(k)} = \frac{1}{2\pi i} \oint_C \frac{P^{(k)}(z)}{z^{n+1}} \quad (41)$$

where the contour C enclosing the origin lies within $|z| < 1$.

This gives

$$d_n^{(k)} \sim \exp \left[\left(\frac{2k+1}{2k} \right) (b_k \Gamma(2k) \zeta(2k+1))^{\frac{1}{2k+1}} n^{\frac{2k}{2k+1}} \right] \quad (42)$$

This is the result quoted in eqn. (1).

We conclude with a brief discussion of the renormalization of the single loop constructed from the N -pt. function with satellites.

Let us suppose that the vertex and propagator for a given amplitude B_N^k have been constructed as operators in the occupation number space of the relevant harmonic oscillators. For $k \geq 1$ there will probably be no spurious states in general⁽¹¹⁾.

Because the trace factors into the usual trace without satellites and a momentum independent term that depends on the satellite coefficients the planar loop (Fig. 3) has the form

$$L_N(p_i) = \int \pi dx_i L_N(x_i, p_i) \Gamma_N(x_i) \quad (43)$$

where $\Gamma_N(x_i) = 1$ corresponds to the absence of satellites.

Near $W = x_1 x_2 \dots x_N = 1$ ^(12,13)

$$L_N(x_i, p_i) \sim P^{(0)}(w) = \prod_{r=1}^{\infty} (1 - w^r)^{-4}$$

$$\text{and} \quad \Gamma_N(x_i) \sim P^{(k)}(w) \quad (44)$$

To examine this essential singularity it is convenient to make the change of variables⁽¹³⁾

$$q = \exp \left(\frac{2\pi^2}{\ln w} \right) \quad (45)$$

Near $q = 0$,

$$P^{(0)} \sim q^{-1/3} \quad (13)$$

$$P^{(k)} \sim \exp[c_k \ln^2 q] \quad , \quad k \geq 1 \quad (46)$$

In the absence of satellites it is possible to construct a dual counterterm ⁽¹³⁾ because of the power behaviour of $P^{(0)}$. For $k \geq 1$ the singularity of $P^{(k)}$ dominates that of $P^{(0)}$. Because of the non-power behaviour of this singularity we are unable to construct a counterterm.

Another interesting question is that of the singularities of the orientable non-planar loop of Fig. 4 constructed from an N-pt. function with satellites. (Assuming that a twisting operator exists in this case). In the absence of satellites the loop converges for $u < -\frac{4}{3}$ because of the power behaviour of $P^{(0)}$. With satellites the singularity of $P^{(k)}$ (eqn. 46)) would cause the loop to diverge for all values of u . As a corollary, the $s^{1/3}$ behaviour^(13,14) associated with the non-physical branchpoints will be destroyed.

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FIGURE CAPTIONS

Fig. 1. The $N=r+s+4$ pt. function in multiperipheral configuration.

Fig. 2. The momentum space dual diagram of Fig. 1.

Fig. 3. The N -particle planar loop.

Fig. 4. An orientable non-planar 4-particle loop.

Fig. 3

Fig. 4