Longitudinal phase space parameters

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1 Introduction

Occasionally there is minor confusion about how longitudinal phase space parameters, such as bunch area $S$, emittance $\epsilon_s$, beta function $\beta_s$, and tune $Q_s$, are related. Although it rarely takes much time to clear up each case of confusion, the exercise can be frustrating, and the integrated amount of time that is wasted continues to accumulate. This, then, is our attempt to unambiguously state those relations, in a single convenient reference location.

2 Parametric relationships

2.1 Bunch area

The canonical coordinates often used to study longitudinal motion in an RF bucket are $(\phi_{RF}, W)$, where $\phi_{RF}$ is the RF phase of a test particle or ion, and

$$W = \frac{E - E_s}{\omega_{RF}}$$

(1)

is the total energy offset of the test particle relative to the synchronous particle, scaled by $\omega_{RF}$, the angular frequency of the RF system [1, 2]. The bunch area is defined in this coordinate frame. If the bunch motion is linear - nowhere near the bucket separatrix - then the RMS bunch area is conveniently defined as

$$S_{RMS} = \frac{\pi}{A} \sigma_W \sigma_\phi \quad [\text{eV s/u}]$$

(2)

where $\sigma_W$ and $\sigma_\phi$ are the standard deviations of the distribution, and the mass number, $A$, is an integer. For protons, $A = 1$ and the units are [eV s], not [eV s/u]. If the distribution is bi-gaussian, the 95% bunch area is given by

$$S_{95} = \frac{6\pi}{A} \sigma_W \sigma_\phi \quad [\text{eV s/u}]$$

(3)

Note that the factor of $\pi$ is usually explicitly included in numerical values of the bunch area. For example, it is common to see $S_{RMS} = 0.1 \text{ [eV s]}$, and rare to see $S_{RMS} = 0.032 \pi \text{ [eV s]}$, expressions which are (approximately) identical.
2.2 Emittance

The RMS bunch length $\sigma_s$ and the RMS relative momentum spread $\sigma_p$ are very often the practical parameters of choice, rather than $\sigma_W$ and $\sigma_\phi$. In direct analogy to transverse phase space, the normalized longitudinal emittance $\epsilon_s$ and the longitudinal beta function $\beta_s$ are defined such that

$$
\sigma_s = \sqrt{\frac{\beta_s \epsilon_s}{\beta \gamma}}
$$

(4)

$$
\sigma_p = \sqrt{\frac{\epsilon_s}{\beta_s (\beta \gamma)}}
$$

(5)

where $\beta \gamma$ is the usual Lorentz factor. These equations may also be written

$$
\epsilon_s = \sigma_s \sigma_p (\beta \gamma)
$$

(6)

$$
\beta_s = \frac{\sigma_s}{\sigma_p}
$$

(7)

It is readily shown that

$$
\sigma_W \sigma_\phi = \frac{m_0 c^2}{c} (\beta \gamma) \sigma_s \sigma_p
$$

(8)

where $c$ is the speed of light, and $m_0$ is the rest mass per nucleon - the total rest mass divided by $A$. Table 1 in the last section of this paper records $m_0$ values for prominent ion species.

The RMS emittance is therefore related to the 95\% area by

$$
\epsilon_s = \frac{1}{6\pi} \frac{c}{m_0 c^2} S_{95}
$$

(9)

If the test particle is a proton, then

$$
\epsilon_s [\text{m}] = 0.01695 \times S_{95} \ [\text{eV s}]
$$

(10)

or, equivalently, the RMS emittance is related to the RMS area by

$$
\epsilon_s [\text{m}] = 0.1017 \times S_{\text{RMS}} \ [\text{eV s}]
$$

(11)

Strictly speaking, the coefficients in the last two equations above should be modified for ion species other than protons. In practice, the rest mass values listed in Table 1 show that, if the same coefficients are used verbatim for all ion species, then the maximum error incurred is only about 2\%. For most practical purposes this is negligible.
2.3 Beta function

The only parameter left hanging at this point is the longitudinal beta function, which can be shown \([3]\) to be given by the remarkably simple expression

\[
\beta_s = \frac{C}{2\pi Q_s} |\eta|
\]  
(12)

where \(C\) is the circumference of the machine, and \(Q_s\) is the synchrotron tune. The slip factor (or momentum compaction factor) \(\eta\) is given by

\[
\eta = \frac{1}{\gamma_T} - \frac{1}{\gamma^2}
\]  
(13)

where \(\gamma_T\) is the transition gamma of the machine. For RHIC, with \(C = 3833.845\) [m] and \(\gamma_T = 22.89\), if it is assumed that \(\gamma \gg \gamma_T\), then

\[
\beta_{s,RHIC} \approx \frac{1.165}{Q_s} \text{ [m]}
\]  
(14)

Note that the synchrotron tune is of order \(Q_s \sim 10^{-3}\). If the slip factor is considered to be a fixed property of the transverse optics, then the sole independent variable is \(Q_s\). The synchrotron tune is (arguably) a much more natural independent parameter than the RF voltage \(V_{RF}\), or the voltage slope \(V'_{RF}\).

2.4 RF voltage

With \(t\) the turn number, \(\delta = \Delta p/p\) the off momentum parameter, and \(s\) the longitudinal displacement, small amplitude motion in a stationary bucket is described by

\[
\frac{ds}{dt} = -\eta C \delta
\]  
(15)

\[
\frac{d\delta}{dt} = \frac{1}{\beta^2 \frac{z}{A}} \frac{Z e V'_{RF}}{E_n} s
\]  
(16)

where \(Z\) is the atomic number and \(E_n\) is the nominal total energy per nucleon. After solving for the synchrotron tune in the relativistic limit when \(\beta \approx 1\), the required value of the voltage slope is given by

\[
V'_{RF} = \frac{(2\pi Q_s)^2 A E_n}{\eta C \frac{Z}{e}}
\]  
(17)

The required RF voltage \(V_{RF}\) is related to the voltage slope through

\[
|V'_{RF}| = \frac{2\pi}{\lambda_{RF}} V_{RF}
\]  
(18)

where \(\lambda_{RF}\) is the RF wavelength. Putting all this together gives, finally

\[
V_{RF} = \lambda_{RF} \frac{2\pi Q_s^2 A E_n}{|\eta| C \frac{Z}{e}}
\]  
(19)
3 Rest mass per nucleon

Except for Carbon ions, the rest mass per nucleon $m_0$ is NOT the same as $m_u$, the atomic mass unit. One atomic mass unit is defined to be 1/12 of the mass of the most abundant isotope of Carbon. The value for $m_u$ recorded in Table 1 was published by NIST in 1986. Note that the mass number, $A$, is the number of nucleons in an ion - an integer.

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Atomic number</th>
<th>Mass number</th>
<th>Rest mass per nucleon $m_0$ [GeV/c$^2$]</th>
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<tr>
<td>Proton</td>
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</tr>
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<td>79</td>
<td>197</td>
<td>.93113</td>
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</tbody>
</table>

Table 1: The rest mass per nucleon, for various ion species.

References

[1] See, for example, Conte & Mackay, “An Introduction to the Physics of Particle Accelerators”, p. 116 et seq.


[3] See, for example, Conte & Mackay (ibid), eqns 7.32, 7.103, 7.106 and 7.107, on pages 113, 125, and 126.