Rain Scavenging Studies
Progress Report No. 11

A. NELSON DINGLE

May 1975

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Department of Atmospheric and Oceanic Science

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College of Engineering
Department of Atmospheric and Oceanic Science

Progress Report No. II
RAIN SCAVENGING STUDIES

A. Nelson Dingle

U.S. Energy Research & Development Administration
Contract No. AT(11-1)-1407
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May 1975

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List of Symbols, CHAPTER II

\[ a = \text{water activity} \]
\[ c = \text{coefficient of particle distribution} \]
\[ \text{cpd} = \text{specific heat of dry air, erg/(gm}^0\text{K)} \]
\[ \text{cpm} = \text{specific heat of moist air, erg/(gm}^0\text{K)} \]
\[ \text{cpv} = \text{specific heat of water vapor, erg/(gm}^0\text{K)} \]
\[ \text{cw} = \text{specific heat of liquid water, erg/(gm}^0\text{K)} \]
\[ D = \text{coefficient of diffusion of water vapor in air, cm}^3/\text{sec} \]
\[ e_a = \text{ambient vapor pressure, dyne/cm}^2 \]
\[ e_r = \text{vapor pressure at the droplet surface, dyne/cm}^2 \]
\[ e_s = \text{saturation vapor pressure over a flat surface of pure water, dyne/cm}^2 \]
\[ e_s' = \text{equilibrium vapor pressure over a flat surface of solution, dyne/cm}^2 \]
\[ F_h = \text{Fuchs' correction for heat diffusion} \]
\[ F_v = \text{Fuchs' correction for vapor diffusion} \]
\[ g = \text{acceleration of gravity, cm/sec}^2 \]
\[ i = \text{van't Hoff factor} \]
\[ K = \text{thermal conductivity of air, ergs/(cm}^0\text{K sec)} \]
\[ K_a = \text{Kelvin curvature term using air temperature} \]
\[ K_r = \text{Kelvin curvature term using droplet temperature} \]
\[ L = \text{latent heat of condensation, ergs/gm} \]
\[ m = \text{molality of droplet, mole/kg} \]
\[ M = \text{total particulate mass, \mu g/m}^3 \]
\[ m_n = \text{mass of nucleus, gm} \]
\[ M_n = \text{molecular mass of nucleus, gm/mole} \]
\[ m_w = \text{mass of water in a droplet of the} i^{\text{th}} \text{ size class, gm} \]
\( M_w \) = molecular mass of water, gm/mole

\( N \) = number of particles of radius less than \( r \), cm\(^{-3}\)

\( n_i \) = number of droplets in the \( i \)th size class, cm\(^{-3}\)

\( P_m \) = pressure of moist air, dynes/cm\(^2\)

\( r \) = radius of droplet, cm

\( \bar{r} \) = geometric mean radius

\( r_c \) = critical radius, cm

\( R_d \) = gas constant for dry air, erg/(gm°K)

\( R_m \) = gas constant for moist air, erg/(gm°K)

\( r_n \) = radius of nucleus, cm

\( r_o \) = radius of droplet at cloud base

\( r_{S=1} \) = equilibrium radius of droplet at 100\% relative humidity

\( R_v \) = gas constant for water vapor, erg/(gm°K)

\( r_1 \) = radius of smallest nucleus, cm

\( s \) = maximum solubility of nucleus salt, gm/gm

\( S \) = saturation ratio

\( S_c \) = critical saturation ratio

\( T_a \) = air temperature, °K

\( T_r \) = droplet temperature, °K

\( T^* \) = 4098.03°K

\( V \) = ventilation factor

\( w \) = updraft velocity, cm/sec

\( x \) = water vapor mixing ratio, gm/gm

\( x_1 \) = liquid water mixing ratio, gm/gm

\( y \) = ratio of masses of \((i+1)^{th}\) and \(i^{th}\) size classes

\( z \) = height, cm
\( \beta \) = ratio of molecular masses of water and air
\( \Delta \sigma \) = elevation of surface energy due to dissolved nucleus
\( \epsilon \) = error of droplet growth equation in implicit model
\( \rho_a \) = ambient vapor density, \( \text{gm/cm}^3 \)
\( \rho_l \) = density of liquid water, \( \text{gm/cm}^3 \)
\( \rho_m \) = density of moist air, \( \text{gm/cm}^3 \)
\( \rho_r \) = vapor density at the droplet surface, \( \text{gm/cm}^3 \)
\( \rho' \) = density of droplet, \( \text{gm/cm}^3 \)

\( \sigma \) = standard deviation of droplet distribution
\( \sigma_o \) = surface tension of pure water, dyne/cm
\( \sigma' \) = surface tension of droplet, dyne/cm
List of Symbols, CHAPTER III

A = a dummy variable

$c_1$ = dimensionless constant in the expression of the eddy exchange coefficient

$c_2$ = dimensionless constant for relating the eddy exchange coefficient for momentum to that for heat

$C_p$ = specific heat at constant pressure

$C_v$ = specific heat at constant volume

$F_x, F_y, F_z$ = the x, y, z components of the turbulent dissipation

$F_\theta$ = heat diffusion function

$g$ = gravitational acceleration

$H$ = a sparse matrix of the finite-difference pressure equation

$K_H$ = eddy exchange coefficient for heat

$K_m$ = eddy exchange coefficient for momentum

$L$ = latent heat of condensation

$p$ = pressure

$\hat{p}$ = pressure vector

$q_i$ = specific ice content

$q_w$ = specific liquid water content

$R$ = gas constant for moist air

$S$ = total latent energy source

$S_1$ = latent energy source for condensation/evaporation

$S_2$ = latent energy source for freezing/melting

$S_3$ = latent energy source for deposition/sublimation

$S_p$ = source term in the pressure equation

$\hat{S}$ = source vector
List of Symbols (Concluded)

\( t \) = time

\( T \) = temperature

\( T_e \) = equivalent temperature

\( u \) = \( x \)-component of velocity

\( v \) = \( y \)-component of velocity

\( w \) = \( z \)-component of velocity

\( x_v \) = water vapor mixing ratio

\( x, y \) = horizontal coordinates

\( z \) = vertical coordinate

\( \Delta x, \Delta y \) = horizontal distance increments

\( \Delta z \) = vertical distance increment

\( a \) = a coefficient in the pressure equation

\( \beta \) = a coefficient in the pressure equation

\( \gamma = c_p/c_v \)

\( \Theta \) = potential temperature

\( \Theta_e \) = equivalent potential temperature

\( \rho \) = density of moist air

\( \rho_w \) = density of liquid water

\( \rho_i \) = density of ice

A subscript \( o \) indicates the basic state

A prime indicates a perturbation from the basic state
The modeling of convective storm scavenging processes is going forward on two distinct fronts. The first of these relates to the microphysical processes, particularly to the study of their response to micro-dynamical components of the convective circulation. It is found that the droplet size spectra generated are responsive to variations of vertical velocity due to turbulence within the cloud and to humidity variations due to entrainment processes. Both sets of variations give responses that differ for different amplitudes and frequencies, i.e., rapid, small-amplitude eddies appear to enhance small-droplet development, whereas slow, large-amplitude oscillations tend to enhance large-droplet development. An allied study of the assumptions used in the microphysical equations is also being made. The second modeling effort is addressed to the problem of three-dimensional representation of convective dynamics. Equations are presented, and some of the problems under attack are discussed.
I. INTRODUCTION

In preparing the present report, it is perhaps appropriate to give more or less explicit indication of the relevance of the work we are and have been doing to the objectives of ERDA. We begin with the quotation of ERDA's purpose as stated by Chief Administrator, Dr. Robert C. Seamans, Jr.: "... to guide and promote energy research and development to meet the nation's immediate and long range needs ...".

The roots of the present program of Rain Scavenging Studies lie in the early problems of radioactive fallout from nuclear weapons tests. Growth of the program has come through the period of the nuclear test ban by virtue of broader and deeper concerns, not only with radioactivity but with all substances that man discharges into the air. All of these materials come back to earth in one form or another, as fertilizer, as acid, as poisons, or simply as harmless deposition.

An early concern was that with the concentrated depositions of radioactivity recognized as "hot spots," usually following intensive convective storms, and most frequently impacting upon dairyland pastures, and thence upon the milk produced therefrom. It is apparent that the radioactive hot spot is not unique. Intensive convective rains can be expected to deposit large amounts of whatever contaminants they happen to encounter in relatively localized areas. They will process
atmospheric contaminants more or less similarly whether those contaminants are radioactive, chemical, biological, whether good, bad or indifferent. This work, therefore, is seen as basic to the understanding of the role of convective storms in the entire air cleansing problem: its application is not limited by specification of the contaminant.

Present atmospheric concerns with energy systems, other than solar energy, are largely directed to the problem of atmospheric pollution by waste products from energy production and/or conversion (heat, water, particles; impacts of Power Parks and Nuclear Energy Centers).

The systematic contamination of the surface by industrial wastes has been shown capable of progressive damage to soil and lakes (e.g., in Southern Norway) that cannot readily be reversed or alleviated. This problem is also found within the borders of Canada and the U. S. In terms of the rain scavenging processes, which appear to accomplish the major proportion of the damaging deposition, these systematic situations are relatively simple to simulate by means of models that yield realistic results.

The special interest expressed here in the convective rain contribution has its source in several characteristics of convective storms:
1. Non-predictability
2. Intensity of impact
3. Complexity
4. Fundamental interest as a meteorological problem.
The scenarios of interest are sometimes marshalled in terms of a series of "worst cases." The basic problem is that a contaminant, once emitted to the atmosphere, is subject only to atmospheric processes, not to man's control, and given a particular sequence of atmospheric events, e.g., the inopportune occurrence of strong convective activity which encounters the airborne contaminant, the consequences may be severe. It is our interest in particular to determine the degree of severity in connection with each of several different circumstances, and to fill the gaps in our understanding of the problem so that prediction errors and damaging impacts may be reduced. In the course of this effort, because the processes involved are fundamental, we expect to develop basic scientific insights, the applications of which will be somewhat more broad and general than the presently conceived problems.

The course of our past work in and contribution to this field does not require detailed review here. The early experimental study of the removal and deposition of radioactivity and plant pollens by convective rains led to the formulation of two distinct concepts:

(1) that the input of water and contaminant to a convective system is almost entirely found in the low level convergent air currents which form and feed the convective updraft, with the corollary
(a) that airborne radioactive debris having its atmospheric source in the lower stratosphere likewise enters convective systems mainly by first circulating to low levels where the association with water and condensation nuclei can take place, and

(2) that the atmospheric mixing ratios of the ingested water vapor and the respective contaminants should be roughly proportional to those found in the resulting rain water, i.e., the scavenging of contaminants may be estimated to have about the same efficiency as the conversion of water vapor to rain.

The first of these served as a basis for chemical tracer experiments which we and others have now conducted under field conditions for the purpose of studying scavenging patterns and precipitation processes. The second was tested by Gatz (1966) in his study of the water and contaminant budgets of an Oklahoma storm system during its passage over the Washita River watershed and the Beta network of the National Severe Storms Laboratory.

We have now shifted from the field phase of this research to an emphasis upon the assembly of the viable concepts and theoretical formulations into numerical models of the systems and processes under study.

Progress toward our several objectives is set forth in the present annual report. The work has proceeded on two fronts: (1) the study of the microphysical processes of
nucleation and droplet growth, particularly in relation to dynamical details of cumulus convection, and (2) the study of the dynamics of convection by means of three-dimensional models.

Reference

Gatz, D. F., 1966: Deposition of atmospheric particulate matter by convective storms: The role of the convective updraft as an input mechanism. Scientific Report No. 1, publication COO-1407-6, Department of Meteorology and Oceanography, University of Michigan, Ann Arbor. xiv + 216pp.
II. MICROPHYSICAL MODELING

A. INTRODUCTION

All contaminants placed in the atmosphere by natural and/or artificial means are subject to the cleansing processes that serve to renew and recycle the air. The artificially introduced air contaminants are mainly, but not quite exclusively, released from energy-conversion processes, e.g., the burning of a fossil fuel, or nuclear fission. Clearly, therefore, the rates and efficiencies of the atmospheric cleansing processes determine the capacity of the air to dispose of the wastes of energy-conversion systems at the same time that it continues to support living systems.

Of the air cleansing processes available, those associated with cloud and precipitation are believed to be the most effective; and among these wet processes, the nucleation of cloud droplets upon condensation nuclei is the most prominent. At the same time, this same nucleation is the starting point for the generation of rain/snow precipitation. It is therefore apparent that a cloud microphysics model must form the basis upon which the study of atmospheric recycling and precipitation chemistry is built.

Howell (1949) was the first to calculate the growth of droplets by condensation. He described the growth of an individual droplet with an ordinary differential equation...
which depended on radius, saturation ratio, and nucleus size. His model consisted of a parcel of air, containing a discrete distribution of sodium chloride particles, rising adiabatically within a steady updraft. Because of the mutual relationship between cooling rate, vapor consumption, and saturation ratio, he was able to perform hand calculations for only three cases; his updraft velocities ranged from 0.05 to 0.6 m/sec.

He found that as adiabatic cooling continues, the saturation ratio increases until the aerosol reaches a height of some tens of meters above the cloud base. At this point, the vapor gradient increase as a result of the cooling is balanced by the consumption of vapor by the growing droplets. This maximum saturation ratio determines the smallest particle size which can surpass its critical radius (Köhler, 1926) and become an activated droplet.

Continued cooling results in the further growth of the activated droplets, as the saturation ratio falls back toward 1.0. Those which fail to nucleate begin to evaporate and are carried along as mist droplets.

Hovell discovered that because the growth rate of an individual droplet decreases with droplet size, the distribution of active droplets narrows with age. He postulated that condensation could be the dominating process creating homogeneous distributions found in some young clouds. Also, he suspected that this process may be the cause of the
brilliant irridescence seen in altocumulus clouds of uniform composition. Finally, he concluded that the updraft speed is most important in determining the spectral shape, while the particle distribution has a minor effect. Although no further calculations were performed, he suggested that evaporation or turbulent mixing could create broader distributions.

Following the increase in the number of particle measurements and the development of computers, Mordy (1959) made further calculations. He included droplet sedimentation, but concluded the effect is important only if the updraft velocity is less than 10 cm/sec or if there are extremely high concentrations of giant particles.

By including large particles in his spectrum, Mordy recognized that the particle distribution is as important as the updraft speed in determining the final droplet population. Because of the slow radial growth of the large droplets, they will lag behind their equilibrium sizes, while the small droplets are able to react quickly to environmental changes. In a fast updraft, then, the droplet spectrum is narrow; the spectrum is broader in a slow updraft, in which the large droplets can approach further their equilibrium sizes.

Mason and Ghosh (1957) showed that large droplets may be formed on giant salt nuclei. This may be acceptable for maritime clouds, but can not hold for continental clouds where giant salt nuclei are rare or comprise a small proportion of the total aerosol mass.
Kornfeld (1970) considered growth upon a particle distribution which included both salt nuclei and insoluble discs. The particle distributions used were not representative of those found in the atmosphere. In several cases, the salt particles had only one size ($r = 1\mu$). At most, they were spread out over three sizes ($r = 0.25$, $0.5$, and $1.0\mu$). After 800 seconds, the inclusion of insoluble particles broadened the distribution somewhat. While the presence of insoluble particles may be instrumental in broadening the range of sizes, Kornfeld's particle distributions were insufficiently realistic to properly evaluate their effect. Paluch (1971) showed that by replacing the insoluble discs with small salt particles ($\sim 10^{-16}$ moles), she could approximate the result of Kornfeld.

Fitzgerald (1972) measured droplet distributions at the 200 to 300 meter level for fifteen clouds in Minnesota and Florida and compared them to computed droplet distributions. Nucleus distributions were implied from the measured saturation ratio spectra. He found fairly good agreement between the measured and computed droplet distributions, but his efforts were limited to small or medium sized nonprecipitating cumulus clouds of near uniform composition. An extension of his study to cumulus congestus clouds is desirable.

Other authors have used extensions to the classical model in laminar conditions to explain the broad spectrum. Most of them considered turbulent mixing.
Instead of assuming a steady updraft speed, Neiburger and Chien (1960) employed vertical velocities based on the results of the Thunderstorm Project (Byers and Braham, 1949). By choosing a suitable particle distribution, the authors were able to produce, after 2400 seconds, 20 μ radius droplets at a concentration of 1/cm³. Thus they claimed that coalescence could take place after that time. Even so, as with those of Mordy and Howell, their droplet distributions narrowed with age and did not resemble measured droplet spectra (e.g., Weickmann and aufm Kampe, 1953; Warner, 1969a; Spyers-Duran, 1972). One discrepancy was the elimination of droplets less than 5 μ radius (Warner, 1969b).

Warner (1969a) measured droplet spectra at various levels in several cumulus clouds near the Australian coast. While most of these had maritime origins, some were likely influenced by continental sources. Average droplet concentrations may differ from those of North American continental clouds, but the behavior of the droplet distributions is of interest.

Warner found that when the environment was unstable, the droplet distribution became bimodal higher in the cloud, with modes near 5 μ and 15 μ radius. During more stable conditions, the bimodal nature was not as common. It was suggested that mixing of cloud and environmental air can produce the observed bimodal distributions.

When a distribution of cloud droplets with a single mode is mixed with drier, polluted environmental air, the
saturation ratio is reduced and the small droplets evaporate. Large droplets may evaporate or continue growing depending on the mixed saturation ratio. As the parcel continues to rise, the small droplets are not as likely to become reactivated. Instead, water will condense on the smaller numbers of large droplets. If this aerosol is subsequently mixed with cloud air, a bimodal distribution is produced.

During horizontal passes through the cloud, Warner found that the bimodal nature was not confined strictly to the edges of the cloud. If entrainment through the sides of the cloud were the principal mixing mechanism, this horizontal uniformity would not be expected. Warner suggests that the primary mixing occurs at the top of the growing cloud.

Mason and Chien (1962) developed a model assuming turbulent mixing between cloud droplets and unactivated particles at the cloud edge. Squires (1958) suggested a similar model in which dry air entered the cloud top and produced regions of low humidity. Both of these models predict that the process is most efficient near the cloud boundaries, and neither accounts for the broad spectra and abundance of small droplets within the interior.

Warner (1973) improved somewhat the model of Mason and Chien. His turbulent model produced, at 150 meters above the cloud base, droplet distributions that characteristically had a mode near 6μ radius, and a plateau, or flat concentration level, of smaller droplets. The mode is a
result of the growth of droplets which formed at the cloud base, while the much lower plateau is caused by the introduction of fresh nuclei into the cloud edges. However, the plateau is not normally found in cloud measurements. He concluded that while entrainment of polluted air results in a broader spectrum, it does not yield realistic distributions. Moreover, he presented evidence that the amount of mixing, as indicated by the ratio of the measured and adiabatic liquid water contents, does not have an obvious direct relation to the dispersion (standard deviation divided by the mean radius) of the measured distribution.

Srivastava and Roy (1962) investigated the effects of turbulence within the cloud. Droplets at a given height were assumed to have experienced different trajectories and hence different growth times. Their distributions were broader than those in a uniform updraft, but it was assumed that the saturation ratio was constant. Fitzgerald (1972) has claimed that if the saturation ratio is not assumed constant, then the distribution would be much narrower than that predicted by Srivastava and Roy.

Belyaev (1961) and Sedunov (1965) studied the growth in an updraft with fluctuating speed and saturation ratio. Particles entering the cloud scattered about and experienced different growth times and growth rates. Although the distributions were broader, they cannot be related to those at a fixed height.
Paluch (1971) developed a model which included small humidity variations along the vertical axis of an updraft. Droplets within each region of the updraft grow at different rates. Large droplets are allowed to settle into regions of low humidity occupied by small droplets, thus retarding the growth rate. The high humidity region, now void of large droplets, can lead to rapid growth of the remaining small droplets. This process is able to maintain variations of droplet concentration and relative humidity. Although turbulence will counter this effect somewhat, observations indicate that turbulence is not extensive enough to completely homogenize a cloud.

Nearly all of the models which include turbulence have not considered one important aspect of the eddy motion. That is, while an aerosol is swirling around, there is a downward component of motion.

Storebø and Dingle (1974) have calculated the growth and washout of particles moving up and over a land barrier. Under adiabatic compression of an air parcel, the relative humidity decreased enough to initiate evaporation from small activated droplets, while larger ones continued to grow as long as the humidity was high enough. These findings led them to speculate that if the parcel were now lifted again, the droplet size spectrum could be broadened.

Kornfeld (1970) included in her model a nonsteady updraft velocity. Periodic fluctuations about a mean velocity did not produce a spectrum significantly different from that within
a steady updraft. Paluch (1973), however, pointed out that the results depend to a large degree on the amplitude and frequencies of the variations.

B. OBJECTIVES OF THE RESEARCH

Eddies are a result of the natural turbulence in a cumulus congestus cloud. They are associated with internal turbulence and entrainment along the cloud edges and in the cauliflower tops of the growing cloud. An aerosol traveling within one of the eddies experiences fluctuations of the saturation ratio produced by the alternating expansion and compression, as well as by the intermittent entrainment of air and nuclei from outside the cloud.

As a parcel of air starts upward from the cloud base, each droplet has a radial growth rate approximately proportional to the vapor gradient at the droplet surface and inversely proportional to the radius. Initially, the large droplets are far below their respective equilibrium sizes and grow faster than the small droplets. Even with a low activation supersaturation, the largest particles may never become activated because of the long time constants of growth. On the other end of the scale, the small droplets require a high supersaturation to become activated. Thus it is a medium size droplet that is first activated. Eventually, the small droplets, by virtue of their size, grow faster than the large ones and the distribution narrows.
During downward motion, the saturation ratio decreases and the small droplets evaporate. The large droplets, being smaller than their equilibrium radii, continue to grow at the expense of the smaller droplets, at least until the saturation ratio goes below their equilibrium value.

If the aerosol now goes upward again, the droplets reenter the growth cycle, but the distribution starts out broader than it was before. Continued cycling should further broaden the distribution.

If an eddy is too small, or the frequency is too high, there will be insufficient time for a significant broadening of the spectrum. If the eddy is too large, or the frequency too low, the large droplets will evaporate and the broadening is nullified. Then there should exist a regime wherein certain combinations of eddy size and frequency will efficiently broaden the droplet distribution. If entrainment is included, it is expected that the important critical eddy sizes could be somewhat smaller, and the frequencies higher.

A well-developed cumulus cloud is envisioned as having large eddies near the cloud core, with decreasing eddy sizes toward the edge. It is possible that entrainment into small eddies at the cloud edge could produce a distribution similar to that created by the large nonentraining eddies near the core.

Most comprehensive cloud models detail the dynamics and parameterize at least some of the thermodynamics to accommodate to the limitations of computers. In this model, as
in others discussed in the introduction, the thermodynamics is carefully modeled and the dynamics is parameterized. Minute study of the thermodynamics is necessary in order to have a firm basis for its parameterization in the larger dynamic cloud models.

By arbitrarily choosing several oscillatory updraft velocities, one can determine what eddy sizes and frequencies are necessary for significant broadening of the droplet spectrum. Whether such eddies actually exist within a cloud must be determined by future measurements.
C. MODELS OF DROPLET GROWTH BY CONDENSATION

Two models of droplet growth are presented here. The implicit model is based on the works of several previous authors (e.g., Mason, 1957; Mordy, 1959; Fitzgerald, 1972). The explicit model was suggested by Storebjø and Dingle (1974). The models differ essentially in the treatment of the heat budget of the droplet and of the droplet growth equation.

Both models begin with a parcel of air that is allowed to rise or fall according to an imposed updraft velocity. If no mixing is allowed with other cloud elements or with the outside air, the parcel will expand or contract adiabatically, but the mass will remain constant. Thus all equations are written in terms of unit mass, rather than volume, to ensure that the number of particles and the total mass of water will be unchanged throughout the integration.

Each model has five equations. They are expressions of

1. the change of pressure within the parcel,
2. the change of air temperature,
3. the condensation of water,
4. the heat balance of a droplet, and
5. the rate of growth of an individual droplet.

Only the first three equations are common to both models. The treatment of the last two equations forms the basis of the difference between the two models.
1. The Basic Equations

a. Pressure equation

The air parcel is assumed to have an unsteady updraft velocity, \( w \). The pressure equation, then, is

\[
\frac{dp_m}{dz} = -\rho_m \left( g + \frac{dw}{dt} \right)
\]

where
- \( p_m \) = pressure of moist air,
- \( \rho_m \) = density of moist air,
- \( g \) = acceleration of gravity, and
- \( z \) = height.

Using the ideal gas law and multiplying by \( \frac{dz}{dt} \), this transforms to

\[
\frac{dp_m}{dt} = -\frac{p_m}{R_m} \frac{R_m}{T_a} \left( g + \frac{dw}{dt} \right) w
\]

where \( T_a \) = air temperature.

The gas constant for moist air is (Fleagle and Businger, 1963)

\[
R_m = \left[ 1 + \left( \frac{1}{\beta} - 1 \right) \frac{x}{1 + x} \right] R_d
\]

where
- \( x \) = water vapor mixing ratio,
- \( \beta \) = ratio of the molecular masses of water and air, and
- \( R_d \) = gas constant for dry air.
b. Temperature equation

The parcel is assumed to contain one gram of air, x grams of water vapor, and \( x_1 \) grams of liquid water. Then the first law is

\[
\frac{dq}{dt} = (1 + x) \left[ c_{pm} \frac{dT_a}{dt} + \frac{1}{\rho_m} \frac{dp_m}{dt} \right],
\]

where \( q \) = heat, and

\( c_{pm} \) = specific heat of moist air.

There are two sources of heat:

1. the heat released by condensation

\[
\frac{dq_1}{dt} = -L \frac{dx}{dt}
\]

2. the sensible heat due to the cooling of the droplets

\[
\frac{dq_2}{dt} = -x_1 c_w \frac{dT_a}{dt}
\]

where \( c_w \) = specific heat of liquid water, and

\( L \) = latent heat of condensation. In units of ergs/gm it is

\[
L = 2.503 \times 10^{10} - 2.425 \times 10^7 (T_a - 273.16).
\]

Combining, and substituting the pressure equation (1), yields

\[
\frac{dT_a}{dt} = - \frac{(1 + x) \left[ (g + \frac{dw}{dt}) + (w + L \frac{dx}{dt}) \right]}{c_{pd} + x c_{pv} + x_1 c_w}. \tag{2}
\]

Here, \( c_{pm} \) was replaced by the equivalent expression \((c_{pd} + x c_{pv})/(1 + x)\), in which \( c_{pv} \) is the specific heat of water vapor, and \( c_{pd} \) is the specific heat of dry air.
c. Conservation of mass

Since the total mass of water in the parcel is constant,

\[ dx = -dx_1. \]

For a discrete distribution of droplets,

\[ \frac{dx}{dt} = - \sum_i n_i \frac{dm_{wi}}{dt}, \]

where \( n_i = \) number of droplets in the \( i^{th} \) radius interval and,

\[ m_{wi} = \text{mass of water in a droplet of the } i^{th} \text{ radius interval.} \]

d. Heat budget equation

A complete expression for the heat balance of a growing droplet requires the consideration of the energy:

1. gained from condensation of the vapor,
2. lost by conduction to the environment,
3. stored by the droplet,
4. lost to the added liquid water,
5. gained through friction of the air,
6. required for expansion of the surface area, and
7. lost by radiation.

Further consideration could be given to the conduction of heat within the droplet, but it is assumed here that the temperature of the droplet is everywhere equal to that at the surface.

Following the lead of Neiburger and Chien (1960), the
energy exchange due to friction and expansion of the surface is assumed negligible. Radiative exchange is ignored. The resulting heat budget equation is

\[
L \frac{dm}{dt} + c_w \frac{dm}{dt} (T_a - T_r) + 4 \pi r K F_h (T_a - T_r) = m_w c_w \frac{dT_r}{dt} \tag{3}
\]

where \( K = \text{coefficient of conduction. In units of ergs/(cm}^0\text{k sec}) \)

\[ K = 2395. + 8.08375 (T_a - 273.16) \]

\( F_h = \text{Fuchs' (1959) correction for heat diffusion, and} \)

\( T_r = \text{droplet temperature.} \)

e. Droplet growth equation

Consider a spherical droplet of radius \( r \). The flux of vapor along the radial distance is

\[
\frac{dm_w}{dt} = 4 \pi r D F_v (\rho_a - \rho_r) V,
\]

where \( \rho_a = \text{ambient vapor density} \)

\( \rho_r = \text{vapor density at the droplet surface,} \)

\( F_v = \text{Fuchs' (1959) correction for vapor diffusion,} \)

\( V = \text{ventilation factor (Squires, 1952), and} \)

\( D = \text{coefficient of diffusion of water vapor in air. In units of cm}^3/\text{sec it is} \)

\[ D = 0.2264 + 0.00139375 (T_a - 273.16) \times 1.013 \times 10^6/P_m \text{ (Storekø & Dingle, 1972)}. \]
Using the ideal gas law and the transformation

\[ \frac{\partial m_w}{\partial t} = 4\pi \rho_1 r^2 \, dr, \]

where \( \rho_1 \) = density of liquid water, this becomes

\[ \frac{dr}{dt} = \frac{D F_v}{r \rho_1 R_v} \left( \frac{e_a}{T_a} - \frac{e_r}{T_r} \right) V, \tag{4} \]

where \( R_v \) = gas constant for water vapor, \( e_a \) = ambient vapor pressure, and \( e_r \) = vapor pressure at the droplet surface.

The ambient vapor pressure is related to the vapor mixing ratio with

\[ e_a = \frac{x}{\beta + x} \, P_m. \]

The vapor pressure in equilibrium with a droplet containing a soluble particle is

\[ e_r = e_s (T_r) \, a \exp \left( \frac{2a'}{x \rho \, R_v \, T_r} \right), \tag{5} \]

where \( \sigma' \) = surface tension of the droplet and \( a \) = water activity (Low, 1969a).

Low's tables of activity are presented as a function of molality. The activities are insensitive enough to temperature and pressure (Harned and Owens, 1958), that the published values are used without modification.
The saturation vapor pressure for pure water is computed from the Murray (1967) formulation

\[ e_s(T) = 6107.8 \exp \left( \frac{17.2693882 (T - 273.16)}{T - 35.86} \right), \]  

with \( T \) in °K and \( e_s \) in dynes/cm\(^2\). At 283.16 K, it predicts a value only 0.057% greater than that given by List (1968).

To determine the solution surface tension it is assumed here that temperature and molality act independently on the surface tension, viz.

\[ \sigma'(T, m) = \sigma_0(T) + \Delta\sigma(m), \]

where \( m \) = molality,

\( \sigma_0 \) = surface energy of pure water, and

\( \Delta\sigma \) = correction due to the presence of an electrolyte.

Values of \( \sigma_0 \) are interpolated between those given by Weast (1968). The correction term is approximated from Low's (1969b) curves. For ammonium sulfate it is

\[ \Delta\sigma = 2.18 \text{ m}. \]

The solution density, \( \rho' \), is taken from tabulated values of Hodgeman (1961). Between the listings, linear interpolation is used.
2. Equations for Implicit Model

a. Heat balance equation

The right hand side of the general heat balance equation (3) represents the lag of heat loss as a result of the heat storage within the droplet. Mordy (1959) calculated that for the maximum cooling rate of air (10°C/km) and an updraft speed of 3 m/sec, the error introduced by ignoring heat storage is less than \( \pm 5 \times 10^{-6} \) °C for 10μ droplets. Thus the heat storage term is dropped. Also, the sensible heat transfer to the added water is ignored. For the implicit model, the latent heat is balanced by the conduction of heat to the air:

\[
L \frac{dm_w}{dt} = 4\pi r K \rho K (T_r - T_a) .
\]  

b. Droplet growth equation

The implicit model develops an analytical expression for the rate of growth of an individual droplet. To begin, it is assumed that the droplet and air temperatures are similar enough to approximate (4) with

\[
\frac{dr}{dt} = \frac{D \rho y}{\rho_h a V T_a} \left( e_a - e_r \right) .
\]
The vapor pressure over the droplet (5) can be substituted into this last equation. Before this can be done, however, one must have an expression for the saturation vapor pressure at the droplet temperature, $e_s(T_r)$. This is obtained by integrating, from $T_a$ to $T_r$, the derivative of the Murray formulation (6). From this,

$$e_s(T_r) = e_s(T_a) \exp \left( \frac{4098.03}{(T_a - 35.86)^2} (T_r - T_a) \right).$$

Since the difference between the droplet and air temperatures is small, one can use

$$e_s(T_r) = e_s(T_a) \left[ 1 + \frac{4098.03}{(T_a - 35.86)^2} (T_r - T_a) \right]. \quad (9)$$

From the implicit model heat budget (7),

$$T_r - T_a = \frac{L}{K} \frac{\rho_1}{T_a} \frac{dr}{dt}. \quad (10)$$

Combining equations (5), (8), (9), and (10), one arrives at the rate of growth of an individual droplet,

$$\frac{dr}{dt} = \frac{V}{r \rho_1} \left( \frac{e_a - e_s(T_a) a K_a}{R_v T_a} + \frac{T^* L e_s(T_a) a K_a}{D} + \frac{T^* L e_s(T_a) a K_a}{K (T_a - 35.86)^2} \right). \quad (11)$$

where $T^* = 4098.03$ K, and

$K_a$ = the Kelvin curvature term. It is assumed here that the curvature effect can be calculated based on the air temperature.
c. Droplet temperature equation

Once the growth rate of a droplet is determined, the droplet temperature elevation can be estimated from the heat budget (10).

3. Errors Introduced by Approximation in the Implicit Model

The growth of a cloud droplet by condensation of the vapor is properly given by (4), rewritten as

\[
\frac{dr}{dt} = \frac{D}{r} \frac{F_v}{\rho_l R_v T_a} \left( e_a - e_r \frac{T_a}{T_r} \right). \tag{4}
\]

The implicit model assumes this can be adequately approximated by

\[
\frac{dr}{dt} = \frac{D}{r} \frac{F_v}{\rho_l R_v T_a} \left( e_a - e_r \right). \tag{8}
\]

Further, in the implicit model it is the droplet temperature which is implicit, so the curvature contribution to the equilibrium vapor pressure is calculated according to the air temperature.

Inasmuch as the approximations are made within a difference term, the error introduced should not be assumed negligible simply on the maxim that the droplet and ambient air temperatures are nearly equal. When vapor condenses onto a droplet, \( T_r > T_a \), and (8) underestimates the true growth rate. Similarly, the evaporation rate is underestimated in a downdraft. It is of interest to examine the magnitude of the error.

The vapor pressure at the droplet surface is related to the air temperature by (5) and (9), such that
\[ e_r = e_s(T_a) a K_r \exp \left[ \frac{T^* (T_r - T_a)}{(T_a - 35.86)^2} \right] \]

where

\[ K_r = \exp \left[ \frac{2\sigma'}{r \rho' R_v T_r} \right] \]

is the Kelvin curvature term at the droplet temperature.

The error, \( e \), effected by (8) is then

\[ e = e_s(T_a) a K_r \exp \left[ \frac{T^* (T_r - T_a)}{(T_a - 35.86)^2} \right] \left[ \frac{K_r}{K_a} \frac{T_a}{T_r} - 1 \right] \]

in which \( K_a \) is the curvature term at the air temperature.

This error function depends on the saturation ratio, \( e_a/e_s(T_a) \). To circumvent this problem, the heat budget (7) from the implicit model is used to give the error as

\[ e = e_s(T_a) a K_r \exp \left[ \frac{T^* (T_r - T_a)}{(T_a - 35.86)^2} \right] \left[ \frac{K_r}{K_a} \frac{T_a}{T_r} - 1 \right] \]

The Fuchs and ventilation factors have been ignored here.

The error is represented in Figure 1, for several air temperatures from 253.16 to 293.16 K. Droplet temperature elevations range from -0.5 to 0.5 K, although in a cloud they probably are less than \( \pm 0.1 \) K (Storebø and Dingle, 1974). The effect of the droplet size is shown for 283.16 K.
Figure 1. Error introduced by the implicit model for various air temperatures. Solid lines apply to a droplet of 10μ radius. Effect of droplet size is shown for Ta = 283.16 K.
To facilitate computations, it was assumed here that $a = 1$.

It is clear that the error can be quite sizeable, running as high as 15% and more. The computations show that the error is nearly independent of the temperature elevation.

Since the two approximations leading to (8) complement each other, two other computations, setting either $K_a/K_r$ or $T_a/T_r$ to unity, were made to determine the effect of each assumption. These efforts revealed that the result of computing the curvature function based on the air temperature gives a minor error, being everywhere less than 1.6%. The bulk of the error is caused by factoring out the temperature in the growth equation.

Since the error caused by approximations within the implicit model appear to be more substantial than previously considered, the explicit model was devised wherein the droplet temperatures are accounted for throughout the integration.

4. Equations for Explicit Model

The explicit model does not derive an analytical expression for the growth rate of a droplet, so that the equations used are essentially the basic equations derived in section 1.

a. Heat balance equation

The basic heat balance equation (3) is used without elimination of further terms. As a function of radius it is
\[
L \frac{dr}{dt} + (c_w \frac{dr}{dt} + \frac{K F h}{r \rho_1}) (T_a - T_r) = \frac{r}{3} c_w \frac{dT_r}{dt}.
\]

(12)

b. Droplet growth equation

The basic droplet growth equation (4), is used without modification.

c. Droplet temperature elevation

The explicit model differs from the implicit model in that the droplet temperature is computed separately and is part of the set of equations to be integrated. A differential equation for the droplet temperature can be written because the heat storage of the droplet is not ignored. By substituting the growth equation (4) into the heat budget (12), one obtains

\[
\frac{dT_r}{dt} = \frac{3 D F_v}{r^2 \rho_1 R_v} \left[ \frac{e_a}{T_a} - \frac{e_f}{T_r} \right] \left[ \frac{L}{C_w} - (T_r - T_a) \right]
- \frac{3 K}{r^2 \rho_1 C_w} (T_r - T_a).
\]

(13)

D. INITIAL CONDITIONS

The model is envisioned in the following manner. A parcel of air with a population of wet particles starts at the base of a cloud at 900 mb and 283.16 °K. In nature, the amount of water associated with each nucleus depends on the particular history of each particle before entering the
cloud. In this model, the precloud histories are not specified; rather, dry particles are placed artificially into the cloud base and allowed to approach equilibrium with the environment. During this adjustment period, the temperature and pressure are held constant and the relative humidity is held steady at 100%. This procedure is not intended to duplicate nature exactly, but to give a representative distribution of droplets at the cloud base.

The aerosol containing one gram of dry air and its associated vapor and droplets is now raised according to a predetermined velocity to a height of 100 meters. The distribution resulting from a steady updraft of 1 m/sec is used as a standard to compare those created by the turbulent motion.

1. Dry Particle Distribution

There is evidence (e.g., Twomey, 1971; Junge, 1963) that ammonium sulfate, \( (NH_4)_2SO_4 \), is a common constituent of atmospheric particles. In this study, spherical particles of pure ammonium sulfate are used.

The effects of including insoluble matter within an aerosol and of using another pure salt, sodium chloride \( NaCl \), are discussed in reports by Junge and McLaren (1971) and Fitzgerald (1974). Junge and McLaren indicate that mixed particles containing both soluble and insoluble matter behave
essentially as soluble particles as long as they have at least 10% by volume soluble material.

Fitzgerald performed further numerical computations to study the effect of the chemical composition. Solubilities ranged from 1% to 90% throughout a particle distribution. He also used distributions of pure ammonium sulfate and pure sodium chloride. His results showed that increasing the proportion of insoluble matter lowered the concentration of droplets and correspondingly raised the mean droplet size. However, a change in the chemical composition of the particles did not significantly alter the breadth of the resultant droplet distribution. Also, both pure salt distributions produced similar droplet spectra. His conclusion reinforced that of Junge and McLaren that a change of shape of the particle distribution has a greater effect than the chemical composition on the dispersion of the droplet spectrum.

In theoretical models it is assumed there is no energy barrier to the initial condensation of vapor onto the dry crystal surface. Experiments by Knight (1971) indicated that early diffusion of vapor to sodium chloride particles results in formation of droplets at the surface. The crystals used were apparently quite large (tens and hundreds of microns) and it is uncertain how the behavior he reported is to be applied to the nucleation of particles found in the atmosphere.
Katz and Kocmond (1973) concluded from their experiments with sodium chloride that particles must be two to three times larger than predicted by theory to become activated at a particular supersaturation. This may be related to the non-zero contact angle between the liquid water and the crystal, evident in Knight's work.

However, there were no claims by either author of purity of the salt surface. Adsorption of organic or other matter onto the surface of the crystals can lead to an energy barrier to the initial condensation. In nature there is even less guarantee that nuclei are uncontaminated. Impurities can either increase or reduce the surface energy. Also, atmospheric nuclei are likely to be irregular agglomerates rather than spherical particles. This agglomeration creates cracks and crevices in the surface which can contain concave water surfaces that accept condensing vapor readily due to a negative curvature effect.

Because of all the uncertainties of the physical and chemical properties of particles within an aerosol, it is assumed for this study that the nuclei are pure spherical particles of ammonium sulphate with zero contact angle.

Particle sizes are continuous in a natural aerosol, but must be distributed in a discrete manner for modeling. The particle sizes used here are defined such that the mass of a particle increases by a factor $y$ from one size to the next. Then if $i$ is the size class number,
\[
\frac{m_i}{m_{i+1}} = y, \text{ or }
\]

\[
d(ln r) = \ln r_{i+1} - \ln r_i = \frac{1}{3} \ln y,
\]

With a constant \(d(ln r)\),

\[
r_i = \exp (\ln r_1 + (i-1) d(ln r)).
\]

Particle sizes can be readily calculated by specifying \(r_1\), the smallest particle radius, and \(y\). In this study \(r_1 = 0.015\mu\) and \(y = 1.9\). The distribution is represented by 27 size classes so that the largest particle has a radius of 3.908\(\mu\).

The particle numbers are distributed according to a formulation similar to that found by Junge (1963),

\[
n_i = \text{const.} \quad r \leq 0.08\mu
\]

\[
n_i = c r_i^{-3} d(ln r) \quad r \geq 0.08\mu.
\]

If the total particulate mass is \(M\), then

\[
c = \frac{M}{4/3 \pi \rho_n \sum r_i^3 n_i^3}.
\]
where \( \rho_n \) is the density of the nucleus salt. For the aerosol modeled, the total particulate loading is 3 \( \mu g/m^3 \). Figure 2 represents the distribution of dry particles.

2. Distribution of Droplets at the Cloud Base

Not all particles at the cloud base are at their respective equilibrium sizes because of the slow rate of condensation onto the largest particles. For dilute droplets, the saturation ratio, \( S \), in equilibrium with a droplet is (Fletcher, 1966)

\[
S = 1 + \frac{2 \rho'}{\rho} \frac{T_v}{T_a} \frac{1}{r} - \frac{i M_v \rho_n}{M_n \rho} \frac{r_n^3}{r^3},
\]

where the subscript \( n \) refers to the nucleus. The van't Hoff factor is \( i \). To find the equilibrium droplet size at 100% relative humidity, set \( S = 1 \). Then

\[
r_{a=1} \propto r_n^{3/2}.
\]

Thus the droplet distribution at the cloud base is broader than the dry particle distribution if the droplets are at or near their equilibrium sizes.

In order to estimate how long the dry particles should be adjusted to the cloud base environment, a distribution
Figure 2. Cumulative distribution of dry particles.
of sodium chloride particles was adjusted and compared periodically to the cloud base droplet distributions used by Mordy (1959) and Fitzgerald (1972). From this it was judged that an adjustment time of 60 seconds would give a representative distribution. During the adjustment period the temperature and relative humidity are constant, so the adjustment time can not be compared directly to a pre-cloud history of the particles.

The distribution of droplets at the cloud base is shown graphically in Figure 3. Also shown are the equilibrium sizes. Nuclei of radius less than 0.13μ are at their equilibrium sizes while the largest particles are at their 99% relative humidity equilibrium size.
Figure 3. Droplet radii at the cloud base. Also shown are the radii in equilibrium with an environment of 100% relative humidity.
E. SOME PRELIMINARY RESULTS OF THE INVESTIGATION

The main thrust of the investigation to date has been the examination of the effect of turbulent motion on the distribution of droplets, using the implicit model. The explicit model is in its final debugging stage and should be ready to be run soon.

The first run (Case I) of the implicit model was done under the condition of a steady updraft of 1 m/s to 100 meters. The resulting droplet distribution is to be used as a standard, against which the distributions in the turbulent updraft can be compared.

Figure 4 describes the course of the temperature and liquid water content. Initially, the temperature follows closely the dry adiabatic cooling rate, and the liquid water content increases only slightly, until the supersaturation reaches its maximum value of 0.71% at 19 meters. After that, the temperature profile resembles that of a moist adiabat as the liquid water content increases sharply and then slows as the supersaturation decreases.

Figure 5 shows how the supersaturation and droplet radii change with height. The two smallest nuclei are not activated and thus evaporate as the supersaturation decreases. The droplets of all other size classes grow throughout the entire 100 meters. Near the supersaturation maximum, the small droplets are growing quite fast because
Figure 4. Variation of temperature and of liquid water content for a steady updraft of $w = 1$ m/s to 100 meters (Case I).
Figure 5. Droplet radii and supersaturation as functions of height. Case I.
of their small size. With time, the distribution narrows. Even though the difference between the largest and the smallest (activated) droplet is slightly larger at 100 meters than at the cloud base (14.88\(\mu\) vs. 14.74\(\mu\)), the dispersion is considerably less (0.059 vs. 0.387).

Within the first few meters from the cloud base, the fastest growing droplets are not the smallest, but somewhat larger droplets. The radial growth rate is inversely proportional to the radius and proportional to the vapor gradient. The larger the nucleus size, the further the droplet lags behind its equilibrium radius, thereby increasing the vapor gradient to it. The effect, then, is to maximize the growth rate for the first few seconds for size classes 5-8 \((r_n = 0.035 \text{ to } 0.067\mu)\).

In addition, even though the largest nuclei have the lowest critical supersaturation values, they are not the first to become activated. There are two conditions to be met before a droplet can become activated. One is that the ambient supersaturation exceed the critical value, and the other is that the radius of the droplet reach its critical value. A large droplet, by virtue of its slow radial growth rate, does not reach its critical radius and hence is not activated even though the ambient supersaturation far exceeds the critical supersaturation of the droplet.

Listed below in Table 1 are some important points of the equilibrium (Köhler) curves for some ammonium sulfate
particles at 10°C. Values indicated are $r_{s=1}$, the equilibrium radius at 100% relative humidity; $r_c$, the critical radius; and $S_c$, the critical supersaturation. For this study, the largest nucleus used is $r_n = 3.9\mu$. Clearly, it is not going to surpass even its $S=1$ equilibrium radius, much less become activated. It would take much too long to reach that size, and would surely fall out before it did.

### TABLE 1

**Köhler Curve Points**

**Ammonium Sulfate**

$t = 10^\circ C$

<table>
<thead>
<tr>
<th>$r_n$ (µ)</th>
<th>$r_{s=1}$ (µ)</th>
<th>$r_c$ (µ)</th>
<th>$(S_c-1) \times 10^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01µ</td>
<td>0.0217</td>
<td>0.0375</td>
<td>197.6</td>
</tr>
<tr>
<td>0.03</td>
<td>0.113</td>
<td>0.201</td>
<td>37.83</td>
</tr>
<tr>
<td>0.05</td>
<td>0.248</td>
<td>0.431</td>
<td>17.64</td>
</tr>
<tr>
<td>0.1</td>
<td>0.701</td>
<td>1.21</td>
<td>6.25</td>
</tr>
<tr>
<td>0.3</td>
<td>3.63</td>
<td>6.29</td>
<td>1.20</td>
</tr>
<tr>
<td>0.5</td>
<td>7.82</td>
<td>13.5</td>
<td>0.56</td>
</tr>
<tr>
<td>1.0</td>
<td>22.1</td>
<td>38.5</td>
<td>0.20</td>
</tr>
<tr>
<td>3.0</td>
<td>115</td>
<td>190.</td>
<td>0.038</td>
</tr>
<tr>
<td>5.0</td>
<td>247</td>
<td>448.</td>
<td>0.018</td>
</tr>
<tr>
<td>10.</td>
<td>~700</td>
<td>~1200</td>
<td>0.007</td>
</tr>
</tbody>
</table>
Figure 6. Height, velocity, supersaturation, change in droplet radius, and temperature. Case I.
Figure 7. Droplet distribution at 100 meters. Case I.
In contrast, the smallest droplets, even though they have a small and easily attainable critical radius, are not activated until their critical supersaturation value is reached. As a result, the first droplets to become activated are those in size class 7 ($r_n = 0.054\mu$) after 7 seconds. With time, the threshold of activated droplets spreads outward to larger and smaller sizes. At the point of maximum supersaturation, droplets in size classes 3 to 13 ($r_n = 0.023$ to $0.195\mu$) are activated. Even as the supersaturation decreases, size classes 14 and 15 ($r_n = 0.24$ and $0.30\mu$) become activated.

The Effects of Turbulence

To assess the effects that cyclical motion has upon the growth of the droplets, there are several parameters to investigate:

1. the upward speed
2. the downward speed
3. the size of the eddy
4. the number of cycles, and
5. the timing, or placement of the eddies, especially in relation to the location of the maximum supersaturation.

The first approach to the simulation of turbulent motion was to impose a sinusoidal velocity function on the aerosol. The one reported here (Case II) is for
\[ w (\text{cm/sec}) = 100 + 300 \sin (0.15 \ t). \]

This represents an eddy of 40 meters diameter being translated upward at a speed of 1 m/s. The velocity of the parcel itself, then, ranges from -2 to 4 m/s. There is only one cycle before the parcel reaches 100 meters, and it takes only 61 sec to get there so that the average velocity is actually higher than 1 m/s. The parcel goes up for 62 meters (Figure 8), travels down to the 41 meter level, and goes back up to 100 meters.

For the steady updraft case the maximum supersaturation attained was 0.71%. But here, because of the high updraft speed, the droplets have less time to react to environmental changes and the supersaturation goes as high as 1.47% \((t = 13 \ s, z = 39 \ m)\). As a consequence, droplets of the smallest size class \((r_n = 0.015\mu)\) become activated.

The first particles to become activated are those of the size class 6\((r = 0.044\mu)\). At the point of maximum supersaturation, size classes 1 to 13 \((r_n = 0.015 \text{ to } 0.195\mu)\) are activated. Before the parcel reaches the 100 meter level, the droplets of size class 14 are activated. Here, droplets of size class 15 do not have time to become activated, as they did in the steady updraft case.

In the downdraft the supersaturation decreases to a low of -0.26%. This minimum occurs before the lowest
Figure 8. Height, velocity, supersaturation, change in droplet radius, and temperature. Case II. Note change of scale from Figure 6.
altitude is reached because, as the velocity decreases, the droplets replenish the vapor supply by evaporating. After 2 sec within the downdraft, 17 size classes \( r_n \) up to 0.46\( \mu \) are evaporating. After 8 sec, 24 size classes \( r_n \) up to 2.06\( \mu \) are evaporating. At the bottom of the loop, only 22 size classes are still evaporating. As the parcel starts upward again, the supersaturation sharply increases and the droplets soon start growing again. During the course of the loop, none of the particles has time to deactivate.

Graphs of \( r - r_o \) in Figure 8 indicate the change of radius for a few droplets; \( r_o \) is the radius at the cloud base. The curves are marked according to the nucleus size. The smallest droplet, \( r_n = 0.015 \mu \) starts growing slowly, maintaining equilibrium with the environment, until it becomes activated, and then increases in size rapidly. Its rate of radial growth slows as the supersaturation decreases, and reverses as soon as the supersaturation goes below its equilibrium value. It resumes growing shortly after the parcel again begins to rise.

The droplet for \( r_n = 0.1 \mu \) follows a similar pattern, except that it increases its radius by a greater amount since it is one of the first to become activated. The largest droplet, \( r_n = 0.39 \mu \), initially is the fastest growing droplet, but soon becomes the slowest growing droplet by virtue of its large size. Unlike the others shown here, it continues to grow throughout the entire cycle indicating
that it is always below its equilibrium value. Because of the short time to reach 100 meters, the largest droplet does not grow as much in this case as in the steady updraft case (3.57µ vs. 4.74µ).

The dispersion for this case (not illustrated) decreases to 0.059 at the top of the loop, increases to 0.069 in the downdraft, and decreases again to 0.046 at 100 meters. Inasmuch as the dispersion for the steady updraft case was 0.059 at 100 meters, the velocity chosen for this case does not change the dispersion in the expected direction. There are two principle reasons for this. One is the high speed and frequency of the assumed velocity, resulting in an exaggerated maximum supersaturation. The other is the fact that the downdraft begins some 10 to 11 seconds after the supersaturation has peaked. The droplets increase their mass quickly in the region of the high supersaturation. If the evaporation is delayed much beyond that level, the droplets require a long time to become deactivated.

The results here may give an insight to the development of clouds with a violent updraft. These clouds may be conserving their water by having regions of high liquid water content concentrated on large numbers of small droplets.

The unexpected result above led to a review of the basis of our early ideas. The finding of Storebø and Dingle (1974) was that the large droplet end of the spectrum was
favored by its evaporative lag, and should be enhanced by a further resumption of growth. The evaporation phase of their experiment was a great deal slower than that assumed above, taking some 400 sec at which time all the droplets were evaporating.

The droplet spectrum produced after 100 sec in Case I was lowered at a speed of -0.2 m/s for 250 sec. None of the droplets was deactivated in that time, while droplets of size classes 20 to 27 grew throughout.

It is apparent, then, that high frequency eddies are not responsible for the broadening of a droplet spectrum. Rather, processes which allow droplets more time to evaporate must be considered. A cloud has regions of both high and low frequency eddies; it is not unreasonable to suggest that broadening of the droplet distribution is restricted mainly to the areas of low frequency turbulence. If droplets are removed from the updraft, they likely are blown downward gently or settle at a rate dictated by their terminal velocities.

In view of these considerations, a new approach was tried (Case III). Here, the parcel is raised at a steady rate of 1 m/s for 20 meters, lowered to the cloud base at -0.2 m/s, and raised 100 m at 1 m/s. The downward speed may still be somewhat fast, but it represents the maximum speed we anticipate using.
Case III, Note change of scale from Figure 6, 8.

Figures 6, Height, Velocity, Super saturation,
Change in droplet radius, and temperature.
Figure 9 shows the course of some of the parameters for Case III. In contrast to the sinusoidal velocity, the stepwise velocity pattern results in a maximum supersaturation the same (0.71%) as that attained in the steady updraft case. This is not coincidental since the downward motion was assumed to begin (at t = 20 s) shortly after the supersaturation peaked (at t = 19 s). The supersaturation drops to below 0% and abruptly begins to increase when the aerosol reverses itself and goes back up. Another 21 sec after leaving the cloud base a second time, the supersaturation reaches a new, slightly lower maximum of 0.64%.

As in the steady updraft case, the two smallest nuclei do not become activated. The smallest activated particle \( r_n = 0.023 \mu \) grows slowly, despite its small radius, until it becomes activated, and then grows quickly. Since after activation it is not in equilibrium with the environment, it continues to grow for about 15 sec after it begins its downward trajectory. At that point, the environmental humidity drops below the droplet's equilibrium humidity, and the droplet evaporates in about 45 sec to a value close to its original cloud base size. When it reenters the updraft, it again becomes activated and grows to a slightly smaller size than it did in the steady updraft case.

The largest particle \( r_n = 3.9 \mu \) does at first grow faster than the others because it is so far below its equilibrium size. It grows during both the upward and downward trajectories and finally increases its size by more than
In all, 13 size classes are activated in the first upward passage, and all of these become deactivated in the downward passage.

Table 2, below, compares some of the parameters at 100 meters for cases I and III.

<table>
<thead>
<tr>
<th>Case</th>
<th>Temperature</th>
<th>Liquid Water Content</th>
<th>X</th>
<th>Vel.</th>
<th>Super. Saturation</th>
<th>Radius of Smallest Activated Droplet</th>
<th>Radius of Largest Activated Droplet</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>282.637</td>
<td>0.200</td>
<td>0.059</td>
<td>0.171</td>
<td>4.68</td>
<td>19.56</td>
<td>27</td>
</tr>
<tr>
<td>III</td>
<td>282.631 K</td>
<td>0.195 gm/m^3</td>
<td>0.086</td>
<td>0.174%</td>
<td>4.58 μ</td>
<td>21.98μ</td>
<td>27</td>
</tr>
</tbody>
</table>

The temperature, liquid water content, and super-saturation differ little at the 100 meter height. Apparently, the hysteretic effect for them is small for the velocity pattern assumed here. The dispersion, on the other hand, is increased by 46%. The radius of the smallest activated droplet, r_3 (the subscript refers to the size class), is somewhat smaller, and that of the largest droplet, r_{27}, is 2.42% (12.4%) larger for Case III. Its radius increased 51.1% more than it did in the steady updraft case.
An even more marked change in Case III is the distribution of particles, illustrated in Figure 10. Shown here are the initial distribution and several others along the path of the parcel. For reference, the point for \( r_n = 0.075 \mu \) is indicated on each graph by an open circle.

At the top of the first updraft, \((t = 20 \text{ s}, z = 20 \text{ m})\), the distribution of droplets has a single mode; the higher numbers of unactivated particles are not shown here. When the aerosol is lowered to the cloud base \((t = 120 \text{ s}, z = 0 \text{ m})\), the geometric mean radius is visibly smaller.

After the parcel reenters the updraft, the droplet distribution develops an interesting characteristic. There is a large mode representing the droplets which had become deactivated during the evaporation phase and were subsequently reactivated. For comparison, the two distributions at 20 meters \((t = 20 \text{ and } 140 \text{ s})\) are indicated. Also shown are the distributions at 100 meters for the turbulent updraft \((t = 220 \text{ s})\) and for the steady updraft \((t = 100 \text{ s})\). The velocity pattern assumed in Case III results in larger numbers of both large and small droplets at 100 meters than does the steady updraft.

This is after only one cycle, between 0 and 20 meters. It is expected that more cycles would accentuate the effect. It remains to be seen what effect this and other changes to the updraft velocity pattern will create.
Figure 10. Droplet distributions at various times. Case III. Dashed line is the distribution at 100 meters for Case I.
F. REFERENCES


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III.

Modeling of Convection in 3-D.
A. A SYSTEM OF EQUATIONS FOR DEEP MOIST CONVECTION

The theoretical developments of Ogura and Phillips (1962) and of Dutton and Fichtl (1969) provide the basis for the system of equations used to express the dynamics of deep moist convection.

Beginning with the ideal gas assumption for a mixture of dry air and water vapor, i.e., the equation of state

\[ p = \rho RT \]  

(1)

and Poisson's equation

\[ \frac{\theta_e}{T_e} = \frac{\theta}{T} = \frac{(1000)}{p} \frac{R}{c_p} \]  

(2)

may be written. Here \( \theta_e \) and \( T_e \) are respectively the equivalent potential and equivalent temperatures, related to \( \theta \) and \( T \), the potential and actual temperatures, by

\[ T_e = T \exp \left( \frac{L x_1}{c_p T} \right) \]

Also \( p \), \( c_p \), \( \rho \) and \( R \) are respectively the pressure, specific heat at constant pressure, density, and gas constant, all for moist air; \( L \) is the latent heat of condensation, and \( x_1 \) is the water vapor mixing ratio.

Small perturbations, \( p', \rho', T' \) and \( \theta' \) are now superimposed upon the basic state (zero subscript) giving

\[ \frac{\rho'}{\rho} = \frac{\rho'}{\rho} - \frac{T'}{T} \]  

(3)

\[ \frac{T'}{T} = \frac{\theta'}{\theta} + \frac{R}{c_p} \frac{\rho'}{\rho} \]  

(4)

from (1) and (2), resp. The perturbation expressions used above are of the form \( A = A_0 + A' \), and the basic state,
λ_o depends only upon z whereas A and A' depend upon x, y, z and t.

In the particular case of the density at level z_c in a cloud, the total perturbed density includes the liquid water and/or ice, thus

\[ \rho(z_c) = \rho_o(z_c) + \rho'(z_c) \]

where

\[ \rho'(z_c) = \rho'(x,y,z,t) + \rho_w + \rho_i \]

and

\[ \frac{\rho'(z_c)}{\rho_o(z_c)} = \frac{\rho'(x,y,z,t)}{\rho_o(z_c)} + q_w + q_i \] (5)

Here \( \rho'(x,y,z,t) \) represents the departure of the local cloud air density from that of moist air in its basic state, \( \rho_o(z_c) \), and \( q_w \) and \( q_i \) are respectively the specific liquid water and ice contents. Finally by substitution of (3) and (4) into (5)

\[ \frac{\rho'}{\rho_o} = \frac{\theta'}{\theta_o} + \frac{1}{\gamma} \frac{p'}{p_o} + q_w + q_i, \quad \gamma = \frac{c_p}{c_v} \] (6)

The formulation of the dynamical system is based upon the following assumptions:

1. the basic density, \( \rho_o \), is a function of height only;
2. density variations occur only in the vertical equations of motion in terms where they are multiplied by g (Boussinesq approx.);
3. the basic wind shear is expressed by a polynomial of first degree.
The equations of motion are expressed in scalar form:

\[
\frac{3u}{3t} + u\frac{3u}{3x} + v\frac{3v}{3y} + w\frac{3w}{3z} = -\frac{1}{\rho} \frac{3p}{3x} + F_x \tag{7}
\]

\[
\frac{3v}{3t} + u\frac{3v}{3x} + v\frac{3v}{3y} + w\frac{3w}{3z} = -\frac{1}{\rho} \frac{3p}{3y} + F_y \tag{8}
\]

\[
\frac{3w}{3t} + u\frac{3w}{3x} + v\frac{3w}{3y} + w\frac{3w}{3z} = -\frac{1}{\rho} \frac{3p}{3z} + F_z - g \tag{9}
\]

The continuity equation is

\[
\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{10}
\]

and the thermodynamic equation is

\[
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta}{\partial z} = S + F_\theta \tag{11}
\]

where \( u, v, w \) and \( F_x, F_y, F_z \) are respectively the \( x-, y-, \) and \( z- \) components of the air velocity and of the turbulent dissipation effect, \( F_\theta \) is the thermal diffusion effect, and \( S \) is the latent energy contribution.

By orienting the \( x- \) axis parallel with the mean flow, the perturbed equations take the following form, respectively:

\[
\frac{3u'}{3t} + (u_o + u') \frac{3u'}{3x} + v' \frac{3v'}{3y} + w' \frac{3w'}{3z} = -\frac{1}{\rho_o} \frac{3p'}{3x} + F'_x \tag{12}
\]

\[
\frac{3v'}{3t} + (u_o + u') \frac{3v'}{3x} + v' \frac{3v'}{3y} + w' \frac{3w'}{3z} = -\frac{1}{\rho_o} \frac{3p'}{3y} + F'_y \tag{13}
\]

\[
\frac{3w'}{3t} + (u_o + u') \frac{3w'}{3x} + v' \frac{3w'}{3y} + w' \frac{3w'}{3z} = -\frac{1}{\rho_o} \frac{3p'}{3z} + F'_z - \frac{g'}{\rho_o} \tag{14}
\]

\[
\frac{3u'}{3x} + \frac{3v'}{3y} + \frac{3w'}{3z} = -\frac{w'}{\rho_o} \frac{\partial \theta}{\partial z} \tag{15}
\]

\[
\frac{3\theta'}{3y} + (u_o + u') \frac{3\theta'}{3x} + v' \frac{3\theta'}{3y} + w' \frac{3\theta'}{3z}(\theta_o + \theta') = S + F'_\theta \tag{16}
\]
The stress terms may be expressed as follows:

\[ F'_{x} = \frac{3}{\partial x} [K_m (2\frac{\partial u'}{\partial x} + \frac{2}{3} \frac{w'}{\rho_o} \frac{\partial \rho_o}{\partial z})] + \frac{3}{\partial y} [K_m (\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial y})] \]

+ \frac{3}{\partial z} [K_m (\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial z})],

\[ F'_{y} = \frac{3}{\partial x} [K_m (\frac{\partial v'}{\partial x} + \frac{\partial u'}{\partial y})] + \frac{3}{\partial y} [K_m (\frac{\partial v'}{\partial y} + \frac{2}{3} \frac{w'}{\rho_o} \frac{\partial \rho_o}{\partial z})] \]

+ \frac{3}{\partial z} [K_m (\frac{\partial v'}{\partial z} + \frac{\partial w'}{\partial z})],

\[ F'_{z} = \frac{3}{\partial x} [K_m (\frac{\partial w'}{\partial x} + \frac{\partial u'}{\partial z})] + \frac{3}{\partial y} [K_m (\frac{\partial w'}{\partial y} + \frac{\partial v'}{\partial y})] \]

+ \frac{3}{\partial z} [K_m (\frac{\partial w'}{\partial z} + \frac{2}{3} \frac{w'}{\rho_o} \frac{\partial \rho_o}{\partial z})],

\[ F'_{\theta} = \frac{3}{\partial x} [K_H \frac{\partial \theta'}{\partial x}] + \frac{3}{\partial y} [K_H \frac{\partial \theta'}{\partial y}] - \frac{3}{\partial z} [K_H (\frac{\partial \theta'}{\partial z} + \frac{\partial \theta'}{\partial z})]

K_m = c_1 (\Delta x \Delta y \Delta z)^{2/3} \left[ \left( \frac{\partial u'}{\partial x} \right)^2 + \left( \frac{\partial v'}{\partial y} \right)^2 + \left( \frac{\partial w'}{\partial z} \right)^2 \right] + \left( \left( \frac{\partial u'}{\partial x} \right)^2 + \left( \frac{\partial v'}{\partial y} \right)^2 + \left( \frac{\partial w'}{\partial z} \right)^2 \left( \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} \right)^2 \right) \right]^{1/2}

and \( K_H = c_2 K_m \).

The coefficients \( c_1 \) and \( c_2 \) are pure constants, \( c_2 \) having

a range between 2 and 4, usually specified at 3 (Deardorff, 1972; Steiner, 1973).

The latent energy source term has three components:

\[ S = S_1 + S_2 + S_3 \]

for condensation/evaporation, freezing/melting, and deposition/sublimation, respectively.
The equations for moist deep convection follow by substitution from (6) into (14), and omitting the primes:

\[
\frac{\partial u}{\partial t} + (u_o + u) \frac{\partial u}{\partial x} + v \frac{\partial (u_o + u)}{\partial y} + w \frac{\partial (u_o + u)}{\partial z} = \frac{1}{\rho_o} \frac{\partial P}{\partial x} + F_x \tag{17}
\]

\[
\frac{\partial v}{\partial t} + (u_o + u) \frac{\partial v}{\partial x} + v \frac{\partial (u_o + u)}{\partial y} + w \frac{\partial (u_o + u)}{\partial z} = -\frac{1}{\rho_o} \frac{\partial P}{\partial y} + F_y \tag{18}
\]

\[
\frac{\partial w}{\partial t} + (u_o + u) \frac{\partial w}{\partial x} + v \frac{\partial (u_o + u)}{\partial y} + w \frac{\partial (u_o + u)}{\partial z} = -\frac{1}{\rho_o} \frac{\partial P}{\partial z} + F_z + g \left( \frac{\theta}{\theta_o} - \frac{1}{\gamma} \frac{\partial P}{\partial o} - q_w - q_d \right) \tag{19}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{w}{\rho_o} \frac{\partial \rho_o}{\partial z} \tag{20}
\]

and

\[
\frac{\partial \theta}{\partial t} + (u_o + u) \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} + w \frac{\partial \theta (\theta_o + \theta)}{\partial z} = S + F_\theta \tag{21}
\]

The time-independent pressure equation is derived by differentiating (17), (18), and (19) with respect to \(x, y, \) and \(z, \) respectively and combining terms with the aid of (20):

\[
\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} + \alpha(z) \frac{\partial P}{\partial z} + \beta(z) P = S_p(x, y, z) \tag{22}
\]

where

\[
\alpha(z) = \frac{g}{\gamma R T_o(z)}
\]

\[
\beta(z) = \frac{\partial}{\partial z} \left( \frac{g}{\gamma R T_o(z)} \right)
\]
and

\[ S_p(x, y, z) = \rho_0 \left[ w^2 \left( \frac{\partial}{\partial z} \left( \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} \right) \right) - \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 + \right. \]

\[ 2 \left( \frac{\partial}{\partial y} (u_0 + u) \right) \frac{\partial v}{\partial x} + \frac{\partial u}{\partial z} \left( \frac{\partial u_0}{\partial z} + u \right) \right] + \]

\[ \rho_0 \left( \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right) - \frac{1}{\rho_0} \frac{\partial \rho_0}{\partial z} F_z \]

\[ + \sigma \frac{\partial}{\partial z} \left[ \rho_0 \left( \frac{\partial}{\partial y} (\phi - q_w - q_i) \right) \right] \]

Equations (17), (18), (19), (21) and (22) form the set of expressions for the dynamical problem.

The system of equations for the dry shallow convection case is obviously a special simplified form of the above equations derived by adding the assumptions:

1. that the air is incompressible, i.e., \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \),

2. that the pressure perturbation is negligible, i.e., \( \frac{p}{p_0} \ll \frac{T}{T_0} \) (or \( \frac{\partial p}{\partial \rho} \)),

and (3) that there is no water present, i.e., \( p = p_d \)

B. A COMPUTATIONAL PLAN FOR MODELING CONVECTION

The formulation of the equations for deep moist convection in three dimensions and time is the first step, and perhaps the easiest one, in the problem of modeling convective rain-generating systems. It remains to devise and/or adapt a suitable algorithm whereby the necessary
model solutions can be produced. In approaching this part of the problem we have chosen to attack first the relatively simpler problem of modeling the shallow dry convective system which is derived as shown above by imposing three simplifying assumptions upon the system of equations for deep moist convection. This step is deemed useful partly because others have done 3-D shallow dry convection models (e.g., Steiner, 1973) which may serve as a basis for comparison, but principally because there is a serious need to develop more exact and efficient algorithms for the 3-D convective models. Inasmuch as this process must be costly of computer time, it is our plan to hold the cost aspect down by testing the numerical procedures and algorithms on the relatively limited shallow convection grid without wet processes. The exploratory nature of this aspect of the research, and the importance of such exploration are emphatic realities.

In the shallow dry convection model, eddy dissipation of heat and momentum is included, and energy conservation over the closed model domain is invoked as a check on the efficiency of the numerical methods used. A cubic spatial grid is specified, being composed of 17 grid points in each dimension, spaced at 200 m intervals.

Programming has been done for experiments in which a thermal impulse is introduced at the center in x and y, and at the bottom in z, of the spatial domain. Initially
the leap-frog difference scheme with smoothing (Shuman, 1957) will be used for the advective equation, and successive over-relaxation (SOR) will be used for the pressure equation.

Inasmuch as other solution methods, e.g., alternating direction implicit (ADI), fast Fourier transformation (FFT), and direct cyclic reduction (DCR) have been shown in two-dimensional models to have advantages over successive over-relaxation (SOR) in terms of both computing efficiency and accuracy, it is our intention to work on the adaptation of one or more of these improved methods to the three-dimensional problem. The most immediately promising method appears to be DCR. One possible approach has been suggested for solution of the three-dimensional Poisson equation (elliptic). This is to use the FFT method in one dimension, and then solve the uncoupled two-dimensional Helmholtz equations using NCAR Subroutines (DCR). The complete answer to our problem is not yet available.

Another promising alternative is noted for the pressure equation. Use of a centered finite-differencing scheme yields a sparse matrix

$$H_p^+ = \tilde{g}.$$  

It appears that this matrix may be solved directly by means of DCR with mixed Dirichlet- and Neumann-type boundary conditions in the regular domain of the present problem.

The present program has not yet yielded satisfactory
Figure 11. Flow diagram for 3 dimensional dry model.
results, mainly because of debugging problems. A diagrammatic summary of the system is given in Figure 11. Explanations of the code designations are as follows:

1. RCVY: for recovery. It permits the continuation of a run after a temporary stop.
2. INPUT: for data input. It enters all initial data input from either external files or decks.
3. CONST: for calculating all physical constants.
4. INFLD: for printing the title and status of the run.
5. TOUT: for printing the potential temperature fields in the x-y and x-z planes.
6. VOUT: for printing the velocity fields in the x-y and x-z planes.
7. POUT: for printing the dynamic pressure fields in x-y and the x-z planes.
8. FRICM: for calculating the turbulent dissipation of momentum.
10. RELAX: for calculating dynamic pressure by SOR scheme.
11. ENER: for calculating the energy budget.
12. TINT: for predicting the potential temperature for the first time step by a forward difference scheme.
13. VINT: for predicting the three components of the velocity for the first time step by a forward difference scheme.
(14) TFCST: for predicting the potential temperature after the first time step.

(15) VFCST: for predicting the three components of the velocity after the first time step.

(16) CNGTV: for transferring data between internal and external memory storage.

This system affords the advantage of a useful degree of flexibility. The respective subroutines can be modified individually as the need arises.

For the dry shallow convection case, the following variables are required at each of the grid points:

- \( U \): x-component of velocity,
- \( V \): y-component of velocity,
- \( W \): z-component of velocity,
- \( TH \): potential temperature,
- \( XK \): eddy-exchange coefficient,
- \( FRICX \): x-component of turbulent dissipation of momentum,
- \( FRICY \): y-component of turbulent dissipation of momentum,
- \( FRICZ \): z-component of turbulent dissipation of momentum,
- \( FRICH \): thermal diffusion, and
- \( P \): dynamic pressure.

Because \( U, V, W, \) and \( TH \) involve time integration, each of them needs computing space for three successive time steps.

By generating a sequential file, the internal storage requirement is reduced to \((8 \times 4,913) = 39,304\) words, and use of the IBM virtual memory system, which is indeed costly, is minimized.
C. References


IV.

Administration
A. Publications


C00-1407-54 Abstract of "Scavenging and dispersal of tracer by a self-propagating convective shower system" by A.N. Dingle. Precipitation Scavenging Symposium, 14-18 October 1974, Champaign, Ill.


Personnel

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