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DYNAMICS OF A FAST REACTOR WITH IN-CORE THERMIONIC CONVERTERS

Third Annual Report

ENGINEERING EXPERIMENT STATION
COLLEGE OF ENGINEERING
THE UNIVERSITY OF ARIZONA
TUCSON, ARIZONA

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DYNAMICS OF A FAST REACTOR WITH IN-CORE
THERMIONIC CONVERTERS

Third Annual Report

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I. LINEAR STABILITY STUDIES

A. Summary

The linear stability studies reported in the previous annual report⁽¹⁾ are extended to include the consequences of changes in the reactor coolant temperature, previously assumed to remain constant. Two cases are considered: the infinite coolant-time-delay model for which the reactor inlet temperature is assumed to be constant and the zero coolant-time-delay model for which the inlet temperature is assumed to vary simultaneously with variations in the outlet temperature.

The stability requirements for the case of infinite coolant-time-delay are shown to be similar to those obtained previously neglecting coolant temperature changes. By contrast, the requirements for the zero coolant-time-delay model are shown to differ significantly from previous results. The main consequence of these extended results is to indicate that under certain conditions, the possibility of reactor instability, even with both prompt and delayed feedbacks negative, may exist. Further, the results indicate that the influence of system time-delays may be important and should receive appropriate attention in detailed simulation studies.

B. Introduction

As in the previous studies⁽¹⁾ the model utilizes the linearized prompt-jump approximation to represent the neutron dynamics and linearized energy conservation equations for each of the two regions which provide reactivity feedback via constant temperature coefficients. The linearized equations are:

$$\beta \delta \dot{n} = n_0 (\lambda \rho + \dot{\rho}) \quad (1-1)$$

$$\dot{T}_1 = b \delta n - \eta_1 T_1 \quad (1-2)$$

$$\dot{T}_2 = \eta_2 T_1 - \eta_3 T_2 \quad (1-3)$$

$$\rho = -\alpha_1 T_1 - \alpha_2 T_2 \quad (1-4)$$

where δn , T_1 , T_2 represent the deviations from equilibrium values of the reactor power density, prompt region (fuel + emitter) average temperature and delayed region average temperature, respectively.

The remaining nomenclature is standard and identical to that used previously. α_1 is the prompt and α_2 the delayed temperature coefficient of reactivity feedback.

Letting

$$x = \frac{bn_0}{\lambda\beta} \alpha_1 \quad ; \quad y = \frac{\eta_2 bn_0}{\lambda^2 \beta} \alpha_2 \quad (1-5)$$

the linear stability criteria obtained from application of the Routh-Hurwitz criterion are

$$x + \frac{\eta_1}{\lambda} + \frac{\eta_3}{\lambda} > 0 \quad (1-6)$$

$$\frac{\eta_3}{\lambda} x + y > 0 \quad (1-7)$$

$$(1 + \frac{\eta_3}{\lambda})x + y - \frac{\frac{\eta_3}{\lambda} x + y}{x + \frac{\eta_1}{\lambda} + \frac{\eta_3}{\lambda}} + \frac{\eta_1 \eta_3}{\lambda^2} > 0 \quad (1-8)$$

These criteria are identical to those reported previously except that x and y are defined differently in order to non-dimensionalize the equations.

In the previous studies the above model was used to approximate the case of small perturbations to the reactor which were assumed to

cause insignificant changes in the average temperature of the coolant. By appropriate modifications of the reciprocal time constants η_2 and η_3 , the same model may be used to evaluate the stability requirements for perturbations which do cause a change in the average coolant temperature. Two limiting types of changes are considered.

If the thermal time constant of the coolant loop is very large, then changes in the reactor outlet temperature cause a change in the reactor inlet temperature only after the considerable time delay necessary to change the temperature of the primary loop. In the limit of an infinite time delay, only the reactor outlet temperature is assumed to change, the inlet temperature remaining constant. This is the so-called infinite coolant-time-delay model.

The second limiting case arises from assuming that the thermal time constant of the coolant loop is very short relative to the other system time constants. In the limit of a zero time constant, reactor outlet temperature changes are reflected instantaneously in changes in the reactor inlet temperature. This is the so-called zero coolant-time-delay model.

For the thermionic reactor, neither model is strictly applicable. The anticipated time delay may be on the order of five or ten minutes, and hence the infinite coolant-time-delay model is probably most representative for this reactor system. Nonetheless, it is of interest to investigate the consequences of assuming these two extremes in order to determine the importance of time delays for this system and whether or not more detailed examinations of this particular aspect are warranted.

C. Analysis

The evaluation of the reciprocal time constants η_2 and η_3 proceeds by collapsing the multiregion equations given previously. (1, p 55-7) For the collector region

$$t_c \rho_c c_{p_c} \frac{d\bar{T}_c}{dt} = (1 - \eta_e) q_e(t) - U(T_c - T_{co}) \quad (1-9)$$

where t_c is the thickness of the collector, η_e the diode electrical efficiency, and U is an overall heat conduction coefficient for heat transmission through the insulator, cladding sheath, and coolant film drop. The remaining quantities have their usual definitions.

The average coolant temperature in the core is given by

$$\rho_{co} V_{co} c_{p_{co}} \frac{d\bar{T}_{co}}{dt} = N_{DS} A_c U(\bar{T}_c - \bar{T}_{co}) - 2\omega_{co} c_{p_{co}} (\bar{T}_{co} - T_{co,in}) \quad (1-10)$$

where V_{co} and ω_{co} are respectively the volume and mass flow rate of coolant associated with each string of N_{DS} series-connected diodes.

All components outside the reactor (piping, heat exchanger, radiator, coolant mass) are lumped into a single term, and this "system" temperature is represented by

$$M_s c_{p_s} \frac{d\bar{T}_s(t)}{dt} = 2N_{DP} \omega_{co} c_{p_{co}} [\bar{T}_{co}(t - \tau) - T_{co,in}(t - \tau)] - H_s A_s (\bar{T}_s - T_0) \quad (1-11)$$

where $M_s c_{p_s}$ represents the heat capacity of the entire system excluding the reactor, N_{DP} is the number of parallel-connected diodes or fuel elements constituting the reactor, $H_s A_s$ is the effective heat transfer coefficient for heat loss from the system, and T_0 is the (constant) heat rejection temperature. τ is the effective time delay for the transport of heat from the reactor to the rest of the system.

The reactor inlet temperature, expressed in terms of the system and average reactor coolant temperatures, is

$$T_{co,i}(t) = 2[\bar{T}_s(t - \tau) - \bar{T}_{co}(t - 2\tau)] + T_{co,i}(t - 2\tau) \quad (1-12)$$

where it is assumed that heat losses from the piping can be neglected. In the case of zero time delay, the average system temperature becomes equal to the average reactor coolant temperature. As the delay time becomes very long, the time rate of change of the system temperature becomes correspondingly small, i.e., $dT_s/dt \rightarrow 0$.

Let T_1 , T_2 , T_3 , T_4 , and T_5 represent the deviations from equilibrium values of the average emitter, collector, coolant, system, and coolant inlet temperature, respectively. In addition let $\delta q_e(t)$ represent the deviation from steady-state of the emitter heat flux:

$$\begin{aligned} T_1(t) &= \bar{T}_e(t) - \bar{T}_e(0) \\ T_2(t) &= \bar{T}_c(t) - \bar{T}_c(0) \\ T_3(t) &= \bar{T}_{co}(t) - \bar{T}_{co}(0) \\ T_4(t) &= \bar{T}_s(t) - \bar{T}_s(0) \\ T_5(t) &= \bar{T}_{co,in}(t) - T_{co,in}(0) \end{aligned} \quad \} (1-13)$$

$$\delta q_e(t) = q_e(t) - q_e(0) = HT_1(t)$$

where the linear approximation for the relationship between emitter heat flux and emitter temperature has been incorporated. Also define

$$\begin{aligned}
c_2 &= N_{DS} V_c \rho_c c_{p_c} \\
c_3 &= V_{co} \rho_{co} c_{p_{co}} \\
c_4 &= M_s c_{p_s} / N_{DP} \\
U_T &= N_{DS} A_c U \\
W &= 2\omega_{co} c_{p_{co}} \\
K &= H N_{DS} A_c (1 - \eta_e) \\
H_N &= H_s A_s / N_{DP}
\end{aligned} \tag{1-14}$$

With the definitions (1-13) and (1-14), Equations (1-9) - (1-11)

become

$$c_2 \dot{T}_2 = K T_1 - U_T (T_2 - T_3) \tag{1-15}$$

$$c_3 \dot{T}_3 = U_T (T_2 - T_3) - W (T_3 - T_5) \tag{1-16}$$

$$c_4 \dot{T}_4 = W [T_3 (t - \tau) - T_5 (t - \tau)] - H_N T_4(t) \tag{1-17}$$

Considering first the case of infinite time delay,

$$\dot{T}_4 \rightarrow 0$$

$$T_4, T_5 \rightarrow 0$$

and to a reasonable approximation

$$\dot{T}_2 \approx \dot{T}_3$$

Under these simplifications, (1-15) and (1-16) can be combined and T_3 eliminated to give:

$$\dot{T}_2 = \frac{K}{c_2} \left[\frac{1 + \frac{U_T}{W}}{1 + (1 + \frac{c_3}{c_2}) \frac{U_T}{W}} \right] T_1 - \frac{U_T}{c_2} \left[\frac{1}{1 + (1 + \frac{c_3}{c_2}) \frac{U_T}{W}} \right] T_2$$

The factors in the square brackets represent the changes to the reciprocal time constants η_2 and η_3 of this model compared to the previous

model of constant coolant temperature. Both η 's are decreased, η_3 more so than η_2 . For example, the ratio of c_3/c_2 may be on the order of 0.1 for touching fuel elements, NaK coolant, and a 30 mil molybdenum collector. In this case the change in η_2 would be negligible while η_3 would decrease by the factor $(1 + \frac{U_T}{W})^{-1}$. Since only η_3 of these two quantities affects the stability (see inequalities (1-6) to (1-8)), the change in η_3 is of most interest. The general effect of decreasing η_3 on stability will be discussed after consideration of the second limiting case of zero coolant transport time delay.

For this second limiting case, the temperature derivatives are again assumed approximately equal and τ is taken to be zero. From (1-12) $\bar{T}_s = \bar{T}_{co}$ or equivalently $T_3 = T_4$. With these approximations, T_3 , T_4 , and T_5 can be eliminated from Equations (1-15) to (1-17) with the result

$$\dot{T}_2 = \frac{K}{c_2} \left[\frac{1 + \frac{U_T}{H_N}}{1 + \left(\frac{c_2 + c_3 + c_4}{c_2} \right) \frac{U_T}{H_N}} \right] T_1 - \frac{U_T}{c_2} \left[\frac{1}{1 + \left(\frac{c_2 + c_3 + c_4}{c_2} \right) \frac{U_T}{H_N}} \right] T_2$$

Again the factors in square brackets represent the changes to the reciprocal time constants from the case of no coolant temperature change. Since c_4/c_2 is likely to be very large (of order 10), η_3 is reduced significantly. In fact, to good approximation, η_3 may be considered to approach its limiting value of zero.

The general consequences of these changes in η_3 to the stability requirements, obtained from (1-6) to (1-8), are illustrated in Figure 1-1.

Figure 1-1. Regions of Linear Stability for Thermionic Reactor Systems Under Various Modeling Approximations

For the case of no coolant temperature change, stability is controlled by the zero power coefficient line (for thermionic reactor concepts of interest) as noted in the previous work and as indicated in Figure 1-1a. As η_3 is reduced while keeping η_1 constant, the hyperbolic resonance line begins to become part of the determining criteria. However, provided η_3 is sufficiently large, the resonance line remains in the second quadrant, as shown in 1-1b, and some degree of prompt positive feedback can be tolerated. This is the case for situations corresponding to an infinite coolant-transport-time-delay.

For the case of η_3 becoming very small, as indicated for the case of zero coolant-transport-time-delay, two possibilities arise. If $\eta_1 + \eta_3 > \lambda$, then results similar to Figure 1-1b are again obtained. If, however, $\eta_1 + \eta_3 < \lambda$, the stability plot appears as shown in Figure 1-1c with the hyperbolic resonance line moving over into the first quadrant. Under these conditions, the reactor could be unstable even though both the prompt and delayed reactivity feedbacks are negative. This possibility was first noted in Reference 2 for the general two-temperature problem and in Reference 3 specifically for a thermionic reactor system. Here, the transitional nature of the stability criteria, as η_3 is varied from very large values (relative to the other reciprocal time constants of the system) to very small values, is illustrated.

Considering the maximum and minimum values of η_1 and λ reported in Reference 1 (p. 16), the possibility of the stability plot appearing as in Figure 1-1c does exist. Hence, it is concluded that the consequences of time delays in the thermionic reactor system warrants further, more detailed study.

II. NONLINEAR OSCILLATIONS AND STABILITY

A. Introduction

The possibility that a linearly stable system may be unstable for large perturbations has motivated a detailed study of nonlinear oscillations of a two-region reactor model. An important application is the two-region model of a thermionic reactor. This study resulted in the dissertation by Schmidt⁽⁴⁾ which was issued as a topical report under this contract. A condensed version will appear in the journal, Nuclear Science and Engineering.⁽⁵⁾ A significant contribution of this work was the prediction and verification of unstable limit cycles in the two-region reactor with delayed neutrons. The purpose of this section is to present an alternative derivation of Schmidt's Fourier series equations for autonomous oscillations and to note the relation to the "describing-function" method for studying large-amplitude forced oscillations.

Historical reviews of the use of Fourier series in reactor dynamics are given elsewhere.^(1, 4) The following is a new derivation of the reactor describing function. This formulation corrects some inconsistencies in the older published literature that were pointed out in a paper by Tan.⁽⁶⁾ Then, building on this foundation, we derive the magnitudes of the first-harmonic components in reactivity and power oscillations. By starting with an integro-differential equation of reactor dynamics, we bypass the iterative scheme used by Schmidt⁽⁴⁾ and obtain directly his "infinite iteration" result for the case of two harmonics in reactivity and power.

B. Nonlinear Oscillations

The starting point is an integro-differential equation used by Akcasu and Noble⁽⁷⁾ and other writers:

$$\frac{\ell}{\beta} \frac{dn}{dt} = \frac{\rho}{\beta} n(t) + \int_0^{\infty} [n(t-u) - n(t)] D(u) du \quad (2-1)$$

where $n(t)$ is the power or neutron density in the point-reactor model, ρ is the net reactivity, and D is the delayed-neutron kernel (emission probability) given by

$$D(t) = \frac{1}{\beta} \sum_i \beta_i \lambda_i e^{-\lambda_i t} \quad (2-2)$$

The kernel has the property

$$\int_0^{\infty} D(u) du = 1 \quad (2-3)$$

Equation (2-1) may be written as

$$\frac{\ell}{\beta} \frac{dn}{dt} = \frac{\rho}{\beta} n(t) - n(t) + \int_0^{\infty} n(t-u) D(u) du \quad (2-4)$$

We let

$$\frac{\rho}{\beta} = 2b_1 \cos \omega t + \sum_{k=-\infty}^{\infty} A_k e^{jk\omega t} \quad (2-5)$$

$$\frac{n}{n_0} = 1 + \sum_{m=-\infty}^{\infty} A_m e^{jm\omega t} \quad (2-6)$$

where $a_{-k} = a_k^*$ and $A_{-m} = A_m^*$. For a forced oscillation, b_1 and ω are input parameters. We have

$$2b_1 \cos \omega t = b_{-1} e^{-j\omega t} + b_1 e^{j\omega t}$$

$$b_{-1} = b_1$$

where b_1 is real. For an autonomous oscillation, $b_1 = 0$, and ω is to be determined.

Substitute Equations (2-5) and (2-6) into Equation (2-4) and use

$$\int_0^\infty n(t-u) D(u) du = n_0 + n_0 \sum_m A_m \bar{D}(jm\omega) e^{jm\omega t}$$

where $\bar{D}(s)$ is the Laplace transform of $D(t)$ and where $\sum_m$ signifies the infinite summation. By equating terms in like harmonics and noting that the double sum

$$\sum_r \sum_s a_r A_s e^{j(r+s)\omega t}$$

contributes to the m -th harmonic only if $r+s = m$, we find

$$[1 - \bar{D}(jm\omega) + jm\omega\ell/\beta] A_m = a_m + \sum_k a_k A_{m-k} \quad (2-7)$$

when $b_1 = 0$. If $b \neq 0$, we simply replace $a_{\pm 1}$ by $a_{\pm 1} + b_{\pm 1}$.

Define Z_m by

$$Z_m = Z(jm\omega) = 1 - \bar{D}(jm\omega) + jm\omega\ell/\beta \quad (2-8)$$

This is related to the conventional zero-power transfer function

$G(s) = \delta n(s)/\rho(s)$ by

$$G(jm\omega) = \frac{n_0}{\beta} Z_m^{-1} \quad (2-9)$$

From Eq. (2-7), dropping third and higher harmonics and noting that $Z(0) = 0$, we find:

$$m = 0$$

$$0 = a_0 + 2\text{Re } a_1 A_1^* + 2b_1 \text{Re } A_1 + a_0 A_0 + 2\text{Re } a_2 A_2^* \quad (2-10)$$

$$m = 1$$

$$A_1 Z_1 = a_1 + b_1 + a_1^* A_2 + a_0 A_1 + a_1 A_0 + a_2 A_1^* + b_1 (a_0 + A_2) \quad (2-11)$$

$$m = 2$$

$$A_2 Z_2 = a_2 + a_1 A_1 + b_1 A_1 + a_0 A_2 + a_2 A_0 \quad (2-12)$$

Next, we consider the likely orders of magnitude of the various terms. It is reasonable to expect that the first-harmonic amplitudes a_1 , b_1 , and A_1 are of the same order. The shifts in average reactivity and power, a_0 and A_0 , which are induced by nonlinear effects, are assumed to be of order a_1^2 . Further, it is assumed that a_2 and A_2 are of order a_1^2 . All attempts to make a further breakdown than this have led to inconsistencies, such as some contradictory formulas reported by Smets.⁽⁸⁾

From Equations (2-10) through (2-12), we obtain the consistent approximations

$$0 = a_0 + 2\text{Re } a_1 A_1^* + 2b_1 \text{Re } A_1 \quad (2-13)$$

correct to order a_1^2 ,

$$A_1 Z_1 = a_1 + b_1 \quad (2-14)$$

correct to order a_1 , and

$$A_2 Z_2 = a_2 + a_1 A_1 + b_1 A_1 \quad (2-15)$$

correct to order a_1^2 . Equations (2-13) through (2-15) are then used to approximate the higher-order terms in Equation (2-11).

An important result is obtained by substituting Equations (2-14) and (2-15) into Equation (2-13):

$$a_0 = -2|A_1|^2 \text{Re } Z_1 \quad (2-16)$$

correct to order a_1^2 . The zero-power transfer function has a real part only if delayed neutrons are included; therefore, the negative reactivity shift is absent unless delayed neutrons are included. This result has been derived by several writers. This derivation is quite simple and straightforward.

Next, it is found that higher-order terms in Equation (2-11) may be approximated in such a way that a_0 , A_1 , and A_2 are eliminated from the right-hand side. The approximate relations used are

$$a_0 = -2[|a_1|^2 + b_1(b_1 + 2\text{Re } a_1)]\text{Re } Z_1^{-1} \quad (2-17)$$

$$A_1 = (a_1 + b_1)Z_1^{-1} \quad (2-18)$$

$$A_2 = [a_2 + (a_1 + b_1)^2 Z_1^{-1}]Z_2^{-1} \quad (2-19)$$

Equation (2-11) may then be written as

$$\begin{aligned} A_1 Z_1 = (a_1 + b_1) \{ & 1 + A_0 - [2|a_1|^2 + 2b_1(b_1 + 2\text{Re } a_1)]Z_1^{-1} \text{Re } Z_1^{-1} \} \\ & + (a_1^* + b_1)[a_1 + b_1]^2 Z_1^{-1} Z_2^{-1} + a_2(Z_1^{-1} + Z_2^{-1}) \} \end{aligned} \quad (2-20)$$

This complete expression, with all first and second harmonics properly included, appears to be a new result.

C. The Describing Function

The goal is to express the right-hand side of Equation (2-24) in terms of b_1 and the reactor transfer functions. This requires the elimination of a_1 , a_2 , and A_0 . The reactivity may be written as

$$\rho = \rho_0 - \rho_f = \rho_0 - \int_{-\infty}^t h(t-u) \delta n(u) du$$

where $h(t)$ is the linear feedback kernel whose Laplace transform is

$$H(s) = \rho_f(s)/\delta n(s)$$

It is convenient to define

$$K_m = K(jm\omega) = \frac{n_0}{\beta} H(jm\omega) \quad (2-21)$$

Noting the sign change in the feedback, we have

$$a_m = -K_m A_m \quad (m = 0, 1, 2) \quad (2-22)$$

It turns out to be simpler to return to Equation (2-11), and, using Equations (2-13) through (2-15) for approximating the higher-order terms, to introduce the feedback at that point. Some of the intermediate results are

$$A_0 \cong 2|A_1|^2 K_0^{-1} \operatorname{Re} Z_1 \quad (2-23)$$

$$A_1 \cong \frac{b_1}{Z_1 + K_1} \quad (2-24)$$

$$A_2 \cong \frac{b_1 - A_1 K_1}{Z_2 + K_2} A_1 \quad (2-25)$$

Note that the shift in the time-average power disappears if there are no delayed neutrons, according to Equation (2-23). Note also that Equation (2-24) is the linear approximation, giving the closed-loop transfer function

$$\frac{A_1}{b_1} = \frac{Z_1^{-1}}{1 + Z_1^{-1} K_1} \quad (2-26)$$

Equation (2-11) becomes

$$A_1 Z_1 = (b_1 - A_1 K_1) Q \quad (2-27)$$

where

$$Q = 1 + \frac{b_1^2}{|Z_1 + K_1|^2} [2(K_0^{-1} - Z_1^{-1}) \operatorname{Re} Z_1 + \frac{Z_{-1} - K_2}{Z_2 + K_2}] \quad (2-28)$$

This is the correct reactor describing function for a linear feedback. (The procedure could also be carried out with a nonlinear feedback, but Equation (2-22) would then contain additional terms involving summations.)

$$\frac{A_1}{b_1} = \frac{Z_1^{-1} Q}{1 + Z_1^{-1} K_1 Q} \quad (2-29)$$

which corresponds to the block diagram in Figure 2-1. Conventional describing-function analysis can be used from this point on, except

Figure 2-1. Block Diagram for the Describing-Function Representation
(the Linear System Corresponds to $Q = 1$)

that the nonlinear element is represented by Equation (2-28). Of course, such analysis is limited to small amplitudes and two harmonics. It can be shown that a third harmonic makes no contribution of order a_1^2 , so the amplitude limitation is dominant.

Previous studies have generally assumed that the second harmonic is negligible when it returns through the feedback loop, in the spirit of a conventional describing-function analysis. However, that assumption can lead to difficulties when the nonlinearity involves more than one state variable. The present derivation avoids this pitfall, which appears to have caused some confusion in the existing literature of reactor dynamics.

D. Autonomous Oscillations

Setting $b_1 = 0$, we find from Equation (2-20):

$$\begin{aligned} A_1 Z_1 = a_1 [1 + A_0 - 2|a_1|^2 Z_1^{-1} \operatorname{Re} Z_1^{-1} + |a_1|^2 Z_1^{-1} Z_2^{-1}] \\ + a_1^* a_2 (Z_1^{-1} + Z_2^{-1}) \end{aligned} \quad (2-30)$$

From Equations (2-17) and (2-22):

$$A_0 = 2|a_1|^2 K_0^{-1} \operatorname{Re} Z_1^{-1} \quad (2-31)$$

From Equations (2-19) and (2-22):

$$a_2 = - \frac{Z_1^{-1} Z_2^{-1}}{Z_2^{-1} + K_2^{-1}} a_1^2 \quad (2-32)$$

Use Equation (2-22) to eliminate A_1 . Equation (2-30) then becomes an equation that may be solved for $|a_1|^2$, with the result

$$|a_1|^2 = \frac{1 + Z_1^{-1} K_1}{Z_1^{-1} K_1 [2(Z_1^{-1} - K_0^{-1}) \operatorname{Re} Z_1^{-1} - Z_1^{-1} Z_2^{-1} (1 - Z_1^{-1} K_2)(1 + Z_2^{-1} K_2)^{-1}]}$$

(2-33)

An analogous procedure leads to

$$|A_1|^2 = \frac{1 + Z_1^{-1} K_1}{Z_1^{-1} K_1 [-2(K_1^{-1} + K_0^{-1}) \operatorname{Re} Z_1 + K_1^{-1} Z_2^{-1} (1 + K_1^{-1} K_2)(1 + Z_2^{-1} K_2)^{-1}]}$$

(2-34)

Equations (2-33) and (2-34) are identical to Schmidt's results in the "infinite iteration" for the case of two harmonics in both reactivity and power.⁽⁴⁾ Here the equations have been obtained without iteration and without any questions of convergence of an infinite series. Moreover, the present formulation is intimately related to the describing function, and the confirmation of Schmidt's work lends additional support to the new describing function presented in Equation (2-28).

This treatment of nonlinear oscillations will be published, as part of a chapter on nonlinear system stability, in a forthcoming textbook on reactor dynamics.⁽⁹⁾

III. OSCILLATION BOUNDARIES IN PARAMETER SPACE

A. Introduction

In the linear stability studies reported for the two-feedback thermionic reactor model,^(1,10) the stable and unstable regions were plotted in a parameter space. The space consists of a plane whose coordinate axes are the two reactivity coefficients, each multiplied by the equilibrium power level. The stability boundary of greatest interest is called the resonance line (otherwise known as the critical line or the dynamic stability boundary).

A radial line through the origin in this plane corresponds to a fixed ratio of the two reactivity coefficients. Increasing the equilibrium power then corresponds to moving outward along one such radial line. The interesting cases are stable at low power and unstable at high power, and the value of equilibrium power that corresponds to crossing the resonance line is called the critical power. Just below the resonance line there are two complex conjugate characteristic roots with negative real parts (damped oscillations), and just above the resonance line the real parts are positive (growing oscillations); the resonance line itself corresponds to pure imaginary roots.

Stability boundaries in a parameter space, such as shown in Figures 2, 3, and 4 of Reference 10, contain much information about the dynamic behavior of a system. However, except for the above observations about the neighborhood of the resonance line, these diagrams do not indicate whether the system response is oscillatory or not.

This information is easily added to the parameter plane in the form of a boundary line between oscillatory and non-oscillatory regions. For

the control designer, this information is an important supplement to the stability information. In our work with nonlinear oscillations, it has proved very useful to know in advance whether the corresponding linear systems were oscillatory or not.

B. Oscillation Boundaries and Departure Points

An oscillation boundary is a locus of points in the parameter plane where the system characteristic equation has two equal real roots. Near this boundary, there are two complex roots with small imaginary parts on one side of the boundary and two nearly equal real roots on the other side. One way to locate the boundary is by trial and error, interpolating among numerical roots as functions of the parameters.

Another procedure is to seek algebraic conditions for the existence of two equal real roots. This is most efficiently done by solving for the points at which the root locus departs from the real axis (departure points).

Consider the two-region system of Equations (3-10) through (3-27) and Figure 1 of Reference 10. The open-loop transfer function is

$$G(s)H(s) = K \frac{(s + \lambda)(s + \xi)}{s(s + \eta_1)(s + \eta_3)} \quad , \quad (3-1)$$

where the symbols are defined in what follows. K and ξ may be either positive or negative, depending on the signs and magnitudes of the two reactivity coefficients.

The characteristic polynomial is a cubic in s :

$$D(s) = s^3 + (K + \eta_1 + \eta_3)s^2 + [\eta_1\eta_3 + K(\lambda + \xi)]s + K\lambda\xi \quad (3-2)$$

Stability may be determined by the Routh-Hurwitz criterion. Using the parameters

$$x = \frac{\alpha_1 b n_0}{\beta} = K \quad (3-3)$$

$$y = \frac{\eta_2}{\eta_3} \frac{\alpha_2 b n_0}{\beta} \quad (3-4)$$

and the relation

$$\xi = \eta_3 \left(1 + \frac{y}{x}\right) \quad (3-5)$$

the vanishing of each Routh number yields a stability boundary in the x, y plane.

The symbols are

λ = delayed-neutron decay constant

η_1 = reciprocal time constant (Region 1)

η_2 = coupling coefficient

η_3 = reciprocal time constant (Region 2)

α_1 = reactivity coefficient (Region 1)

α_2 = reactivity coefficient (Region 2)

b = reciprocal heat capacity (Region 1)

β = delayed-neutron fraction

n_0 = equilibrium power

Heat is generated only in Region 1 (the prompt region). The sign convention is such that positive α means negative reactivity feedback.

Note that K , x , and y are proportional to the equilibrium power n_0 and that $\xi = \text{constant}$ (fixed zero-location) corresponds to a fixed radial line in the parameter plane.

The departure points are the real solutions of

$$\frac{1}{s} + \frac{1}{s + \eta_1} + \frac{1}{s + \eta_3} - \frac{1}{s + \lambda} - \frac{1}{s + \xi} = 0 \quad (3-6)$$

Eq. (3-6) yields a quartic in s :

$$\begin{aligned} s^4 + 2(\lambda + \xi)s^3 + [(\eta_1 + \eta_3)(\lambda + \xi) - \eta_1\eta_3 + 3\lambda\xi]s^2 \\ + 2(\eta_1 + \eta_3)\lambda\xi s + \eta_1\eta_3\lambda\xi = 0 \end{aligned} \quad (3-7)$$

For each real solution of Eq. (3-7), K is computed from Eq. (3-1) with $GH = -1$, and the point (x, y) on the oscillation boundary is obtained from Eqs. (3-3) and (3-5).

C. Numerical Results

Cases I and III of Reference 10 were chosen for illustration, along with one variation of each, denoted as Case IA or IIIA. The fixed parameters are

$$\begin{aligned} \lambda &= 0.1 \text{ sec}^{-1} \\ \eta_2 &= 0.25 \text{ sec}^{-1} \\ \alpha_1 &= -\frac{1}{3} \times 10^{-6} \text{ per } ^\circ\text{C} \\ \alpha_2 &= 10^{-6} \text{ per } ^\circ\text{C} \\ b &= 100 \text{ } ^\circ\text{C per mw-sec} \\ \beta &= 10^{-2} \end{aligned}$$

The four cases are as follows:

Case	η_1 (sec ⁻¹)	η_3 (sec ⁻¹)
I	0.5	0.2
IA	0.3	0.12
III	0.5	0.05
IIIA	0.15	0.05

The resonance line in each case is in the second quadrant of the x,y plane, corresponding to prompt-positive and delayed-negative reactivities. The systems that possess critical power levels are non-minimum-phase, negative-gain, conditionally stable systems.

Case I, which is from Figure 2 of Reference 10, is shown in Figure 3-1. The stable region is bounded by the resonance line (second last Routh number is zero) and the line $\xi = 0$ (last Routh number is zero). The radial line labeled $\xi = -0.2$ is illustrated by a root locus for $K < 0$ in Figure 3-2. At low gain (small n_0) there is no oscillation. As the gain is raised, the system encounters the first departure point and enters the region of damped oscillations. The stability boundary is crossed when the root locus enters the right half plane. Oscillations eventually cease, but at high gain, far inside the unstable region.

Below the line $\xi = 0$ in the second quadrant of Figure 3-1, the system is unstable; the zero is in the left half plane but the gain is still negative. The oscillation boundary recedes to infinity when $\xi = \lambda$, because the root locus returns to the real axis at the double zero only for $|K| \rightarrow \infty$.

Figure 3-1. Oscillation Boundary in Parameter Space for Case I.

Figure 3-2. Root Locus for Case I ($\xi = -0.2$).

Just below the line $\xi = 0$, an interesting phenomenon appears. This is shown in Figure 3-3, which is a magnified view of a portion of Figure 3-1. The oscillation boundary has a sharp point extending below the line $\xi = 0$. Thus for small positive ξ , a system could experience two distinct oscillatory regimes as n_0 is increased. This possibility is illustrated in Figures 3-4, 3-5, and 3-6, which is a sequence of root-locus plots for successively smaller positive values of ξ . When ξ is sufficiently small, the oscillation boundary is crossed four times (four real roots of Equation (3-7), corresponding to four departure points). This type of root locus (two concentric loops) was also noted in Figure 7 of Reference 11.

The cases shown in Figures 3-4 through 3-6 are unstable for all n_0 . To illustrate the analogous behavior for conditionally stable systems, we contrived Case IA (see Figs. 3-7 and 3-8). Here the oscillation boundary has its sharp point in the stable region. A root locus plot for small negative ξ is shown in Figure 3-9. The system enters the oscillatory region, leaves it, and reenters, remaining stable throughout this range of n_0 .

Case III, which is from Figure 4 of Reference 10, is shown in Figure 3-10. Here the unusual systems are in the first quadrant (stable at all power levels n_0). The points for $\xi = \eta_1$ are easily obtained, because the cancellation of a pole and a zero makes the system quite simple. (The oscillation boundary is not asymptotic to the line $\xi = \eta_1$, but is tangent to it at a point not shown in the figure). For ξ slightly greater than η_1 (a radial line slightly closer to the y-axis), the root locus has two loops as shown in Figure 3-11. Note that $K > 0$ in this case.

Figure 3-3. Oscillation Boundary in Parameter Space for Case I.

Figure 3-4, Root Locus for Case I ($\xi = 0.0125$)

Figure 3-5. Root Locus for Case I ($\xi = 0.08$).

Figure 3-6. Root Locus for Case I ($\xi = 0.002$).

Figure 3-7. Oscillation Boundary in Parameter Space for Case IA.

Figure 3-8. Oscillation Boundary in Parameter Space for Case IA.

Figure 3-9. Root Locus for Case IA ($\xi = -6 \times 10^{-4}$).

Figure 3-10. Oscillation Boundary in Parameter Space for Case III.

Figure 3-11. Root Locus for Case III ($\xi = 0.51$)

Case IIIA, shown in Figure 3-12, is similar, except that the oscillation boundary has its sharp point below the line $\xi = \eta_1$. A magnified portion of the parameter plane is shown in Figure 3-13, and a root locus for ξ slightly less than η_1 is shown in Figure 3-14.

Third-order systems with four crossings of the oscillation boundary (four departure points with the same sign of K) are not encountered very frequently. In general, the oscillation boundary may be roughly located by making a few root-locus sketches and correlating them with the x, y plane. The peculiar systems we used for illustration are not very important in practice. However, when more detailed information is desired, it can be obtained qualitatively. A double-loop root locus such as Figure 3-14 can be anticipated by sketching limiting cases with the zero very close to the pole. The two loops then have very little effect on each other, and the smaller loop must exist for a zero very near, and to the right of, the pole of $-\eta_1$. Once this is established, the details of the oscillation boundary are easily obtained.

Figure 3-12. Oscillation Boundary in Parameter Space for Case IIIA.

Figure 3-13. Oscillation Boundary in Parameter Space for Case IIIA.

Figure 3-14. Root Locus for Case IIIA ($\xi = 0.14986$).

IV. STABILITY ANALYSIS OF SYSTEMS WITH TIME DELAY

A. Introduction

In the study of nuclear reactor systems, it is often necessary to predict the stability of systems whose mathematical models include terms that are delayed in time. These terms usually occur in the description of transport phenomena such as coolant lags.

Upon perturbation of a system from equilibrium, a term with "pure" or "ideal" time delay will remain at its equilibrium value for some finite time, τ , (the delay time) after which changes will occur continuously but with lag. These effects are represented as state variables with delayed arguments

$$x(t - \tau). \quad (4-1)$$

Pure time delays appear in the description of distributed systems to characterize material and signal transport. Probably the most common of this kind is the description of coolant transport. In a thermionic reactor model,⁽¹⁾ four delays are necessary in modeling the coolant transport between the reactor and the heat exchanger and between the heat exchanger and the radiator. Similar lags appear in models of desalination plants⁽¹²⁾ and nuclear power plants.⁽¹³⁾

The coupled-core reactor concept is another example of nuclear systems containing time delays.⁽¹⁴⁾ In these models the source of neutrons in one core, which were born in another, are represented by a pure time delay of the form

$$S(t - \tau) \quad (4-2)$$

where τ , the delay time, represents the neutron transport time between cores.

Not all of the time delay terms appear in the description of transport phenomena. They also occur between sensing and reacting of control devices or in any instance where a human decision is necessary. Köhler⁽¹⁵⁾ uses a pure time delay to account for the delayed reactivity effect of the time required for a neutron to slow from fission to resonance energies.

Many so-called delayed effects which are prevalent in the modeling of nuclear systems are in fact not delayed in the ideal or pure sense. Upon perturbation of a system which is in equilibrium, deviations in these terms occur instantaneously although any appreciable change may develop much later. The mathematical description of delayed neutrons falls into this category. Since these effects present no special problems in stability analysis, our interest will be directed to systems which include pure time delays.

It should be remembered that in many cases characterization by a pure time delay may be an approximation to a system more accurately described by a more complex set of equations. This is often necessary to make the system tractable. Occasionally this fact can be used advantageously when approximations to the time delay are necessary.

Ordinary differential equations which include terms that are delayed in time are members of a class termed differential-difference equations⁽¹⁶⁾ (also called difference-differential, differentials with deviating argument, and others). Although this class includes equations with variable time leads as well as lags, our interest here will be centered only on constant time delays.

We will study time invariant, autonomous equations written in state variable notation. In matrix form a system with m different time delays will be written

$$\dot{\vec{x}}(t) = A(\vec{x})\vec{x}(t) + \sum_{i=1}^m B_i(\vec{x})\vec{x}(t - \tau_i) \quad (4-3)$$

where $\vec{x}$ is the vector of deviations from equilibrium of the state variables, and A and B_i are matrices. Further

$$\tau_m \geq \tau_{m-1} \geq \dots \geq \tau_1 > 0 \quad (4-4)$$

Generally, there will be many zero terms in A with the non-zero terms clustering around the diagonal exhibiting a band structure. This can be convenient in stability applications. Also, the B_i matrices are normally very sparse.

Whereas the solution of ordinary differential equations displays a continuous dependence on the initial values of the variables at $t=0$, solutions of equations with delayed arguments continuously depend on initial function conditions

$$\vec{x}(t) = \vec{g}(t) \quad -\tau_m \leq t \leq 0 \quad (4-5)$$

Since we are only interested in deviations from equilibrium, our initial functions will always be assumed to be

$$\vec{x}(t) = \vec{0} \quad -\tau_m \leq t \leq 0 \quad (4-6)$$

The addition of time delays can often result in serious alterations to the stability of a system. Weaver⁽¹⁷⁾ gives a simple example of a lag term driving a stable system unstable. Heath⁽³⁾

discusses a thermionic system which as a result of effective lags may be unstable in spite of negative temperature feedbacks.

A number of different techniques have been used for studying the stability of systems with delayed arguments. They range from poor approximations to exact treatments, from necessary to sufficient criteria. In the following section we will discuss and attempt to evaluate the usefulness of the more common techniques. We will subsume them under three headings: 1) approximations to the time delay, 2) linear methods, and 3) nonlinear methods. In all we will emphasize the applicability of these techniques to systems of practical order--say ten to fifty. Suggestions for new and improved methods will be discussed in the last section.

B. Approximations to the Time Delay

In some situations certain approximations to the pure time delay terms may be justified. These approximations generally transform the system into a set of differential equations which have no delayed terms and hence can be studied by conventional, more easily applied methods such as Routh or Root-Locus, or by simply solving for the roots of the characteristic polynomial and checking for positive real parts. Some of the techniques which are normally used to approximate time delays are discussed below.

Since we are considering only time delays in this work and since for stability we are considering deviations from equilibrium, probably the simplest way of approximating the delayed terms is by letting them be zero. This in fact assumes that the time delay is much larger than the time constants of the system, implying that the

delayed terms never differ significantly from their equilibrium values. When these assumptions are justified, this technique can be very useful.

In his discussion of this particular approximation, El'sgol'ts⁽¹⁸⁾ proves the following theorem for linear systems: if the characteristic polynomial of the equation in which the delayed terms have been set to zero has at least one root with a positive real part, then for a sufficiently large value of the smallest time delay, the system with delay is unstable. Unfortunately, similar assertions to asymptotic stability cannot be made without additional assumptions.⁽¹⁸⁾

Another simplifying approximation which can be used is to neglect the delay constant thus leaving the system without delayed terms. This approach is justified when the delay is very short compared to the time constants of the system (e.g., very long prompt-neutron generation times and very short delays).⁽¹⁹⁾ In his discussion of this approximation, El'sgol'ts proves the following theorem: if the set of equations in which the time delays have been set to zero is asymptotically stable, then for a sufficiently small value of the largest time delay, the unapproximated system is asymptotically stable. Similar conditions are true for unstable systems.

Unfortunately, in both of the above approximations, one must have knowledge about the size of the parameters before the simplifications can be made; and one can often be limited in the range of certain parameters if he has chosen to use one of these approximations.⁽¹⁵⁾ Clearly, these approximations are frequently too restrictive for a complete system stability analysis.

A method which has been used with some success is approximating the delayed terms by a Taylor series expanded about t .^(1,13,19,20,21)

$$x(t - \tau) = x(t) - \tau \dot{x}(t)/1! + \tau^2 \ddot{x}(t)/2! + \cdots + (-\tau)^n x^{(n)}(t)/n! + \cdots \quad (4-7)$$

Because carrying more than two terms of the series will increase the order of the system, the expansion is usually truncated after two terms, thus reducing the system model to a set of equations without delayed terms but with the important advantage that the time delay constants remain among the parameters of the system. This allows a rough analysis of how the stability depends on the delay constants. Although the Taylor series is still an approximation which can lead to invalid results, it is nevertheless an important technique if used with care.

In cases where the delayed terms are themselves approximations, it is possible that their further approximation by the methods discussed above may be just as representative of the physical nature of the system. In these cases the approximations are wholly justified.

C. Linear Methods

Of the methods normally applied to linear systems without delay, few can be extended to include delayed terms. This is due primarily to the change in the form of the characteristic equation. This problem as well as approximations which alleviate it will be discussed in this section.

Taking the Laplace transform of an n^{th} order system without delays, performing the necessary algebra, and forming the integral for

the inverse Laplace transform, all lead to a characteristic polynomial of order n . The zeros of this polynomial are the poles of the integrand of the inverse Laplace transform closed at infinity. From Cauchy's Residue Theorem, it follows that the roots of the characteristic polynomial having positive real parts will result in growing exponentials in the solution of the equations, thus signifying an unstable system. Most of the methods applied to linear systems without delay are in some manner based on discovering whether or not the characteristic equation has roots which have positive real parts. These methods can be subsumed under three categories:⁽²²⁾ 1) methods which determine the eigenvalues of the system matrix (roots of the characteristic polynomial), such as the reduction of the matrix to Hessenberg form and then using the QR transformation to determine the eigenvalues,⁽²³⁾ 2) methods which operate on the coefficients of the characteristic polynomial such as the Routh criterion,⁽¹⁷⁾ and 3) geometric methods such as the Mikhailov criterion⁽²⁴⁾ and the Nyquist criterion.⁽¹⁷⁾

As with ordinary systems, the stability of linear systems with delayed terms requires that all roots of the characteristic equation have non-positive real parts.⁽¹⁶⁾ However, for delayed systems, the characteristic equation contains exponentials resulting from the Laplace transforms of the delayed terms,

$$L[x(t - \tau)] = e^{-\tau s} x(s) \quad (4-8)$$

Therefore, the characteristic equation of an n^{th} order system is no longer a polynomial with at most n distinct roots but rather a transcendental equation with an infinite number of roots. This equation is normally termed a "quasi" or "exponential" polynomial. Because of

the transcendental nature of the exponential polynomial, most of the above methods are precluded. A few, for example the geometric criteria, can be extended to include time delays. These techniques will be discussed later.

The characteristic equation for a system with one constant delay, τ , may be written in the form

$$M(s) = Q(s) + \sum_{k=1}^n R_k(s)e^{-k\tau s} \quad (4-9)$$

where $Q(s)$ and $R_k(s)$ represent ordinary polynomials and $Q(s)$ and $R_n(s)$ cannot both be zero nor both nonzero for the same system.

Before a technique based on manipulations of the characteristic equation may be utilized, the equation must be determined from the system matrices. Although this presents no problems for systems without delay or for those of low order with delay, the determination quickly becomes ponderous for systems of practical order with delay. Part of the current work is directed toward the development of a computational technique for determining the coefficients of the exponential polynomial from the system matrices.

1. Approximation to the Exponentials in the Characteristic Equation

One class of approximations which is used with time delay systems is forming some approximation to the exponential terms in the characteristic equation. Doing this converts the equation again to a polynomial which can then be studied by the conventional procedures. A number of such techniques are discussed below.

Since the exponential function can be expressed by the limit⁽²⁵⁾

$$e^{-\tau s} = \lim_{n \rightarrow \infty} \left(\frac{1}{1 - \tau s/n} \right)^n \quad (4-10)$$

one approximation is to substitute this expression with a finite n .

A similar approach is to approximate the exponential function by the first few terms of the Maclaurin series.^(24,25) The series for either the positive or negative exponential may be used to give either

$$e^{-\tau s} = 1 - \tau s + (\tau s)^2/2! - (\tau s)^3/3! + \dots \quad (4-11)$$

or

$$e^{-\tau s} = \frac{1}{1 + \tau s + (\tau s)^2/2! + (\tau s)^3/4! + \dots} \quad (4-12)$$

Clearly, the approximations are valid only for small values of τs .

Another of this type is the more sophisticated Padé approximation.⁽²⁵⁾ This is a rational algebraic function with numerator polynomial of degree n and denominator polynomial of degree m , such that a maximum number of terms in the Maclaurin expansion of the approximating function (obtained by long division) agree with similar terms in the expansion of the exponential function.

Another approach which uses the Padé approximation is the simulation of the phase function of the time frequency response delay.⁽²⁵⁾ Since the time delay has a transfer function whose magnitude is unity at all frequencies and whose phase angle response is $-\omega\tau$, the approach begins by selecting a rational transfer function with a magnitude of unity and using the Padé approximation to simulate the phase response.

All methods of this class which seek to approximate the exponential terms in the characteristic equation suffer from the same disadvantages. They are as follows: 1) the required accuracy of the approximation can only be determined after the approximate expression has been used in the analysis, and 2) the approximations do not necessarily lead to conservative stability results. To exemplify this second disadvantage, Murray and Weaver⁽²⁶⁾ plotted the stability zones in the parameter space of a simple system obtained by an exact analysis and by an analysis based on the Padé approximation. The approximate analysis gave a significantly larger stability zone than was the actual case, thus leading to the erroneous prediction of stability in regions where in fact the system was unstable. Clearly, this represents a hazardous situation. Although they considered only the Padé approximation for their example, their results are typical for all the approximations of this class.⁽²⁴⁾

2. Methods of Pontryagin

Several algebraic stability criteria for determining the stability of linear systems with delay were developed by L. S. Pontryagin. Although practical applications remain complicated,⁽²⁴⁾ these theorems, directed to determining the sign of the real parts of the roots of the exponential polynomial, are summarized along with related theorems by Bellman and Cooke.⁽¹⁶⁾ Several of his results will be discussed below.

One result which has application to practical system models is based on the notion of a principal term. If $M(y,w)$ is a polynomial in two variables

$$M(y,w) = \sum_{m,n} a_{m,n} y^m w^n \quad m,n \text{ non-negative} \quad (4-13)$$

then we call $a_{rs} y^r w^s$ the principal term of the polynomial if $a_{rs} \neq 0$ and, if for each other term $a_{mn} y^m w^n$ with $a_{mn} \neq 0$, we have either 1) $r > m, s > n$, 2) $r = m, s > n$, or 3) $r > m, s = n$.

In application to delayed systems, we will be interested in exponential polynomials of the form $M(s, e^s)$ which would result from a system which has one constant delay. A useful result is given in the following theorem: if the polynomial $M(y,w)$ has no principal term, the function $M(s, e^s)$ has an unbounded number of zeros with arbitrarily large positive real parts. An application of this theorem will be made in the discussion of the Nyquist criterion.

The main result of Pontryagin is given in the following theorem.⁽¹⁶⁾ Let $M(z) = M(z, e^z)$ be a polynomial with a principal term and let $M(j\omega) = F(\omega) + jG(\omega)$. If all the zeros of the polynomial lie to the left of the imaginary axis thus implying stability, the vector $M(j\omega)$ for real ω ranging from $-\infty$ to $+\infty$ continually revolves in the positive direction with a positive velocity. If we let primes denote derivatives, this criterion can be expressed analytically by the condition

$$G'(\omega)F(\omega) - F'(\omega)G(\omega) > 0 \quad (4-14)$$

for all ω .

As indicated previously, the analytic criterion is very difficult to apply to systems of practical size. However, the geometric notion may possibly find application in a computer technique, since it is very similar to the Mikhailov criterion (see the discussion on the Mikhailov criterion).

3. The Nyquist Criterion

The Nyquist criterion can be extended to study the stability of systems with time delay.^(17,18,21,27) To indicate this procedure, we will consider a general n^{th} order system with one constant delay, τ . The characteristic equation of this system is

$$M(s) = Q(s) + \sum_{k=1}^n R_k(s)e^{-k\tau s} = 0 \quad (4-15)$$

If we apply Pontryagin's theorem on principal terms to this equation, it is evident that for stability the order of $Q(s)$ must be greater than or equal to the largest order of the $R_k(s)$.

Dividing $M(s)$ by $Q(s)$ gives

$$W(s) = M(s)/Q(s) = 1 + 1/Q(s) \sum_{k=1}^n R_k(s)e^{-k\tau s} \quad (4-16)$$

If we assume that $Q(s)$ has no purely imaginary zeros and if we map a Γ -contour (see Fig. 4-1) by $W(s) - 1$ from the complex plane of s to the complex plane of $W(s) - 1$, then stability requires that the number of clockwise encirclements of the point $(-1,0)$ in the plane of $W(s) - 1$ must equal the number of poles of W in the right-half s -plane. If $Q(s)$ does have roots on the imaginary axis of the s -plane, then similar results can be obtained by a slight alteration of the Γ -contour.

To demonstrate the use of this criterion, consider the following characteristic equation:

$$M(s) = s^2 + 2s - 1 + 4e^{-s\tau} \quad (4-17)$$

Then

Figure 4-1. Γ -Contour in s-Plane

Figure 4-2. Nyquist Diagram, $\tau = 0.4$

$$W(s) - 1 = 4e^{-s\tau} / (s^2 + 2s - 1) \quad (4-18)$$

Figure 4-2 shows the Nyquist plot for this equation for $\tau = 0.4$, and Figure 4-3 shows the same curve for $\tau = 0.8$. Since $Q(s)$ has one pole in the right-half plane of s , stability requires one clockwise encirclement of the $(-1,0)$ point. Therefore, it is clear from Figures 4-2 and 4-3 that the system is stable for $\tau = 0.4$ but unstable for $\tau = 0.8$.

Although this method is not tractable for analytic studies of systems of practical order, it may be possible to develop a computer technique which will increase its applications, and thus it is worthy of further study.

4. The Mikhailov Criterion

A method which has been used with some success to study delayed systems in the Mikhailov criterion^(22,24) (also called the Leonhard criterion).⁽²⁸⁾ Like the Nyquist, this is a geometric stability criterion and is based on the idea of a Mikhailov curve. If $M(s)$ is the characteristic equation of a system, then the Mikhailov curve is the image of the positive imaginary axis of s in the complex plane of $M(j\omega)$.

The Mikhailov theorem is stated as follows. For stability of an n^{th} order linear system, it is necessary and sufficient that the vector $M(j\omega)$ complete a rotation of the angle $\phi = \frac{n\pi}{2}$ as ω ranges from zero to infinity. For systems without delay, this implies that stability of an n^{th} order system requires $M(j\omega)$ for $0 \leq \omega \leq \infty$ to pass through n quadrants in succession circling the origin of coordinates in a counterclockwise manner.

Figure 4-3. Nyquist Diagram, $\tau = 0.8$

Figure 4-4. Mikhailov Curves

To exemplify the use of this criterion, consider again the characteristic equation

$$M(s) = s^2 + 2s - 1 + 4e^{-s\tau}. \quad (4-19)$$

As with the Nyquist, we will plot the Mikhailov curve for two different values of τ . If we let

$$M(j\omega) = X(\omega) + jY(\omega) \quad (4-20)$$

then

$$X(\omega) = -\omega^2 - 1 + 4 \cos(\omega\tau) \quad (4-21)$$

$$Y(\omega) = 2\omega - 4 \sin(\omega\tau). \quad (4-22)$$

Table 4-1 gives the values of X and Y versus ω for $\tau = 0.4$ and for $\tau = 0.8$.

Table 4-1. Mikhailov Curves

ω	$\tau = 0.8$		$\tau = 0.4$	
	$X(\omega)$	$Y(\omega)$	$X(\omega)$	$Y(\omega)$
0	3.00	0.00	3.00	0.00
0.4	2.64	-0.46	2.79	0.16
0.8	1.57	-0.79	2.16	0.34
1.2	-1.46	-0.88	1.11	0.55
1.6	-2.41	-0.63	-0.35	0.81
2.0	-5.12	0.00	-2.21	1.13
3.0	-13.0	3.30	-8.55	2.27
4.0	-21.0	8.23	117.1	4.00
10.0	-102.	16.0	-104.	23.0
∞	$-\infty$	∞	$-\infty$	∞

Since the order of this system is two, stability requires the Mikhailov curve to pass through a net angle of π as ω proceeds to infinity. Clearly, from Figure 4-4 the example system is stable for a time delay 0.4 but unstable for a delay of 0.8.

It is evident from the above exercise that a computer technique would be necessary to utilize the Mikhailov method for systems of practical size. Some of the difficulties encountered with this approach will be discussed in the next section.

5. The Modified Mikhailov Criterion

In numerical applications of the Mikhailov criterion, the major shortcoming arising from the study of large systems is the necessity of obtaining a large enough number of points to place the Mikhailov curve in each successive quadrant. This problem is amplified by the ignorance of the values of ω at which points should be calculated. These difficulties pervade the Nyquist and Pontryagin results as well.

In an effort to overcome these shortcomings, Wright and Kerlin^(22,29) have modified the Mikhailov criterion to state that the linear system is stable if and only if the locus of $L(j\omega)$ for $-\infty \leq \omega \leq +\infty$ does not have any net clockwise encirclements of the origin. $L(j\omega)$ is given by

$$L(j\omega) = \frac{M(j\omega)}{\prod_{i=1}^n (j\omega + |a_{ii}|)} \quad (4-23)$$

where a_{ii} represents the diagonal elements of A and $M(j\omega)$ is the characteristic equation of the system. Wright and Kerlin use a very

efficient algorithm for calculating the determinants of $L(j\omega)$ for each value of ω . The technique has been used successfully for systems of order over 100.⁽²²⁾

To demonstrate the use of this criterion, consider again the characteristic equation given earlier.

$$M(s) = s^2 + 2s - 1 + 4 e^{-s\tau} \quad (4-24)$$

Figure 4-5 shows the modified Mikhailov curves for $\tau = 0.4$ and $\tau = 0.8$. As expected, the stability results are the same as given by the Mikhailov criterion. Figure 4-5 shows that the modified plots approach the point (1,0) as a limit rather than infinity. This offers a definite advantage in the computer applications to large systems where otherwise scaling might be necessary. The modification may be thought of as a kind of normalization of both magnitude and phase. (This technique is similar to dividing the characteristic equation by $Q(s)$ when employing the Nyquist criterion.)

D. Nonlinear Methods

In an earlier contract report,⁽¹⁾ Liapunov's second method was used to study the stability of nonlinear nuclear systems, and several different methods of constructing Liapunov functions were compared. The theorems of Liapunov as given earlier can be applied directly to systems with time delay^(18,21) with the inclusion of delay implying the time derivative of the Liapunov function will include the delayed terms. Nevertheless, if the derivative can be shown to be negative definite, the stability results are the same. Unfortunately, this

Figure 4-5. Modified Mikhailov Diagrams

approach to nonlinear stability is specious, being applicable only to a few contrived examples.^(18,21)

The theorems of Liapunov's direct method have been generalized in order to be more useful for studying time delayed systems,^(30,31) and these generalizations have been used only with limited success.^(26,32) They do not lead to changes of the fundamental concepts of the second method; the idea is still to find a positive-definite Liapunov function and to draw stability conclusions from the negative definiteness of its time derivative. Nevertheless, certain modifications are necessary. Driver⁽³¹⁾ states and proves these more general theorems. They take into account the fact that with delayed systems the Liapunov function may exhibit an oscillatory nature (alternately positive and negative time derivative) rather than a strict monotonically decreasing nature (negative time derivative) and still represent a stable system. Hence, the stability results have been generalized to include time derivatives which are alternately positive and negative.

Murray and Weaver⁽²⁶⁾ applied Liapunov's direct method to a delayed model describing coupled-core reactor systems. In this study quadratic V-functions of the form

$$V(\vec{x}) = \sum_{i=1}^n x_i^2 \quad (4-25)$$

were employed. They discovered that this method quickly lost its tractability as the order of the system increased.

Others have suggested different forms of the V-functions. Krasovskii⁽³⁰⁾ suggests the use of the form

$$V(\vec{x}) = v(\vec{x}) + \sum_{j=1}^n \int_{t-\tau}^t x_j^2(s) ds \quad (4-26)$$

Although this form possesses certain advantages for special problems, it is not generally useful.⁽²⁶⁾ Razumikhin⁽³²⁾ has suggested a substitution which provides stability results as a function of τ by using a mean-value approach to the delayed terms.

Although Liapunov's direct method is powerful in its range of applicability to both delayed and regular systems, it is not practical to apply the method in analytic form to any but very low order systems. Some work has been done in the area of numerical applications of the direct method,^(33,34) but to this author's knowledge none have been extended to include time delays or even applied to practical systems without delay. Liapunov's direct method does not appear promising as a technique for studying the stability of large order systems.

The well-known Poincare-Liapunov theorem, which states that if a system is linearly stable, then it is also nonlinearly stable in a sufficiently small region of the equilibrium, has direct extensions to models with time delay.⁽¹⁶⁾ Use of this knowledge may lead to a practicable method for studying delayed, nonlinear system stability based on the linear stability results.

E. Direction of Future Work

Although we have seen that there is no paucity of methods for determining the stability of delayed systems, no method in its current form appears amenable to a time delay parameter study (i.e., the determination of the critical values of time delay at which a

stable system becomes unstable). The modified Mikhailov criterion as developed by Wright and Kerlin⁽²²⁾ appears to be very workable for determining the stability given a specific set of parameters, but the complete computer algorithm must be repeated for each change because the characteristic polynomial is never developed. Therefore work is currently being conducted in two areas: 1) the development of a computational technique which will calculate the coefficients of the exponential polynomial, and 2) the development of a method which will enable the calculation of critical time delays, thereby permitting a parametric stability study.

Although methods for determining the characteristic equation of systems without delay are numerous, the addition of delayed terms seriously complicates the process. One approach to solving this problem is based on performing certain operations on the delayed determinant which will transform it into a form amenable to current methods.

By expanding the characteristic determinant into a specific sequence of minors, it is possible to bring all the exponentials into the coefficients of the remaining minors and to leave these minors in a standard form from which the polynomials can be easily developed. The technique, although simple in principle, has proven to be quite difficult to program for computer applications. However, it now appears promising.

To illustrate the procedure we will consider the following example.

$$M(s) = \begin{bmatrix} a_{11} - s & a_{12} & a_{13} + b_{13}e^{-\tau s} & a_{14} \\ a_{21} & a_{22} - s & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} - s & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} - s \end{bmatrix} \quad (4-27)$$

Following rules for determinant addition, we can write

$$M(s) = \begin{bmatrix} a_{11} - s & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} - s & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} - s & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} - s \end{bmatrix} + \begin{bmatrix} 0 & 0 & b_{13}e^{-\tau s} & 0 \\ a_{21} & a_{22} - s & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} - s & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} - s \end{bmatrix} \quad (4-28)$$

The first determinant is now in standard form, and current methods may be used to develop its polynomial, $Q(s)$. Doing this, and taking the obvious minor in the second determinant, gives

$$M(s) = Q(s) - b_{13}e^{-\tau s} \begin{bmatrix} a_{21} & a_{22} - s & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} - s \end{bmatrix} \quad (4-29)$$

Now by the elementary operations on the remaining minor (assume $a_{31} \neq 0$) of multiplying Column One by the appropriate factors and adding it to Columns Two and Three to form zeros in the (3,2) and (3,4) position,

$$M(s) = Q(s) - b_{13} e^{-s\tau} \begin{bmatrix} a_{21} & a_{22}^* - s & a_{24}^* \\ a_{31} & 0 & 0 \\ a_{41} & a_{42}^* & a_{44}^* - s \end{bmatrix} \quad (4-30)$$

Taking the indicated minor gives

$$M(s) = Q(s) + a_{31} b_{13} e^{-s\tau} \begin{bmatrix} a_{22}^* - s & a_{24}^* \\ a_{42}^* & a_{44}^* - s \end{bmatrix} \quad (4-31)$$

The remaining minor is now in standard form, and its polynomial may be calculated giving

$$M(s) = Q(s) + a_{31} b_{13} e^{-\tau s} R(s) \quad (4-32)$$

$$M(s) = Q(s) + R_1(s) e^{-\tau s} \quad (4-33)$$

If we return once again to the example used to illustrate the Mikhailov and modified Mikhailov criteria, we remember that for a delay of 0.4 the system was stable and for 0.8 unstable. Clearly, some value for τ between these two will represent the critical value--the boundary between stability and instability. Popov⁽²⁴⁾ suggests a technique for calculating the critical delays. He indicates that the critical delay is one which permits the Mikhailov curve to pass through the origin, but he fails to point out that this represents only a candidate for the critical value.

If we consider the characteristic equation given above, we can write

$$M(j\omega) = M_1(\omega) \exp[j\phi_1(\omega)] + M_2(\omega) \exp[j(\phi_2(\omega) - \tau\omega)] \quad (4-34)$$

where $M_1(\omega)$, $\phi_1(\omega)$, and $M_2(\omega)$, $\phi_2(\omega)$ represent the modulus and phase of $Q(s)$ and $R_1(s)$ respectively. For the curve to pass through the origin, we must have

$$M_1(\omega_k) = M_2(\omega_k) \quad \omega_k > 0 \quad (4-35)$$

and the phase directly opposed at these values of ω_k

$$\phi_1(\omega_k) = \pi + \phi_2(\omega_k) - \tau\omega_k + 2q\pi \quad q = 0, 1, 2, \dots \quad (4-36)$$

Hence the critical value of the time delay is given by

$$\tau_k = \frac{\pi + \phi_2(\omega_k) - \phi_1(\omega_k) + 2q\pi}{\omega_k} \quad q = 0, 1, 2, \dots \quad (4-37)$$

In the example, this procedure predicts a critical value of 0.55. If a procedure similar to that shown above is carried out on the modified Mikhaiov equation or the Nyquist, the correct critical value is predicted.

Work is currently being directed to developing computer techniques which will calculate the critical values of the time delay. The Mikhailov, Nyquist, and Modified Mikhailov criteria as well as other modifications of these will be investigated.

V. START-UP ANALYSIS

A. Summary

A procedure to calculate the various thermionic quantities of interest in dynamic studies has been developed. The basis of this work is a digital computer program for determining steady-state thermionic converter performance characteristics. Although this steady-state program cannot be used directly in the system dynamics model, it provides an excellent starting point from which to devise a usable scheme. This converter analysis is extended to allow applications to diodes undergoing transients. The resulting method is then incorporated into the system dynamic model, and investigations are conducted to determine the relative worth of this more complicated approach over a simpler thermionic description.

B. Introduction

The thermionic quantities of prime importance in system dynamics studies are the heat removed by the diode, the output current, and the output voltage. Various complex phenomena such as volume ionization, surface physics, and Schottky effects contribute to the determination of these quantities. Thus, any comprehensive model to describe converter performance over wide operating ranges, such as experienced during system start-up, must correctly account for these processes.

A computational model capable of representing a diode's performance characteristics over a wide range of operating variables has been developed by Wilkins.⁽³⁵⁾ This model forms the basis for the digital

computer code SIMCON, which is now in extensive use at several laboratories. SIMCON is utilized here to calculate the steady-state converter parameters of interest. The problem at hand, however, entails the description of a thermionic diode suitable for transient analysis.

Using SIMCON as a starting point, this analysis is extended to allow applications to a diode experiencing transients. Modifications to enhance convergence, as well as usage of this method with the system dynamics equations, are discussed. Finally, comparisons are drawn between this present model and the Richardson-Dushman approach for a diode description. Since this latter method is simpler, it is of interest to determine whether the more complex analysis is indeed warranted to depict adequately the transient converter performance.

C. Transient Converter Analysis

The description of a gas-filled thermionic diode requires the complex intermeshing of three main phenomena; i.e., interelectrode plasma, electrostatic sheaths, and surface effects. The only thermionic model developed to date which successfully incorporates all these requirements and is applicable over wide ranges of normal diode operation is the SIMCON digital code by Wilkins.

The plasma analysis, as modeled by Wilkins and Gyftopoulos,⁽³⁶⁾ considers a low energy, three component, two temperature plasma consisting of electrons, ions, and neutral particles. The electrostatic sheath analysis serves as a set of boundary conditions which allows solution to the plasma differential equations. The surface effects are treated separately, but must be correctly accounted for in the electrostatic sheath boundary conditions.

The prime function of SIMCON as written by Wilkins is to calculate the converter output voltage for a given set of operating conditions. This operating point is determined by the quantities, T_E , T_C , T_R , J , and d , where

T_E = emitter temperature ($^{\circ}\text{K}$)

T_C = collector temperature ($^{\circ}\text{K}$)

T_R = interelectrode gas temperature, as given by its reservoir temperature ($^{\circ}\text{K}$)

J = net output current density (amps/cm²)

d = interelectrode spacing (mil)

The output voltage is set by

$$V_0 = \phi'_E + V_E + V_P + V_C - \phi'_C \quad (5-1)$$

(See Fig. 5-1). The other important parameter determined by this analysis is the total heat transported across the diode. This quantity (denoted q''_E) is divided into radiation, cesium conduction, electron cooling, and ion cooling contributions and may be written

$$q''_E = Q_r + Q_{cs} + Q_e + Q_i \quad (5-2)$$

where

$$Q_r = \sigma \left[\frac{T_E^4}{1/\epsilon_E(T_E) + 1/\epsilon_c(T_E, T_C) - 1} - \frac{T_C^4}{1/\epsilon_E(T_E, T_C) + 1/\epsilon_c(T_C) - 1} \right] \quad (5-3)$$

$$Q_{cs} = \frac{1.1 \times 10^{-6} (T_E^{3/2} - T_C^{3/2})}{d + (.15 \times 10^{-5} (T_E + T_C)/P_g)} \quad (5-4)$$

- ϕ_E = zero field emitter work function
- $\Delta\phi_E$ = change in emitter work function due to Schottky effect
- ϕ'_E = effective emitter work function
- V_E = emitter sheath voltage change
- V_P = plasma voltage change
- V_C = collector sheath voltage change
- ϕ'_C = effective collector work function
- $\Delta\phi_C$ = change in collector work function due to Schottky effect
- ϕ_C = zero field collector work function
- V_O = diode output voltage

Figure 5-1. Typical Voltage Profile Across Thermionic Diode

$$Q_e = J_e(0)(\phi_E' + V_E) + q_E(0) \quad (5-5)$$

$$Q_i = -J_i(0)(\phi_E' + V_E) + J_i(0)V_i \quad (5-6)$$

and

σ = Stefan-Boltzmann constant (watts/cm²-°K⁴)

ϵ_E = temperature dependent emitter emissivity

ϵ_C = temperature dependent collector emissivity

p_g = interelectrode gas pressure (torr)

$J_e(0)$ = electron current density at emitter sheath (amps/cm²)

$J_i(0)$ = ion current density at emitter sheath (amps/cm²)

$q_e(0)$ = electron kinetic energy flux at emitter sheath (watts/cm²)

V_i = ionization potential of interelectrode gas (eV)

These quantities (i.e., V_0 and q_E'') may be calculated for the specified diode operating conditions once the plasma differential equations, electrostatic sheath boundary conditions, and surface contributions are determined.

Through the use of SIMCON, J may be varied and V_0 calculated while holding the other converter conditions constant. When the resulting points are plotted, a typical current-voltage (I-V) curve is generated as shown in Figure 5-2. While such information is important when analyzing steady-state thermionic converter performance, it is not applicable as such to a diode undergoing a transient.

In order to pinpoint the proper diode quantities for transient studies, an additional condition is needed. Any operating diode will be working across some resistive load. The output voltage is then linearly related to the output current through Ohm's Law

Figure 5-2. Typical Steady-State I-V Curve With Superimposed Load Line

$$V_0 = J R_L \quad (5-7)$$

where R_L is the load resistance. A typical load line is superimposed on the I-V plot shown in Figure 5-2. Thus, this added relationship permits the unique determination of the operating point on any I-V curve.

The incorporation of the load line condition into the SIMCON program necessitates an additional iteration on the net output current density (J). The technique is analogous to shifting the current along the appropriate I-V curve as depicted in Figure 5-2. J is continually incremented and V_0 calculated until the conditions of Equations (5-7) are satisfied. Consequently, the values of J, V_0 , and q_E'' are pinpointed.

The required set of diode conditions to determine the thermionic parameters of interest for transient analysis becomes T_E , T_C , T_R , d, and R_L . The variables T_E and T_C are given by their corresponding thermal describing equations. T_R , d, and R_L are specified separately, but may be time varying. Thus, the SIMCON program, modified to include the given load resistance, can now yield the proper diode current, voltage, and emitter heat flux values as the emitter and collector temperatures fluctuate.

Since the entire thermionic analysis deals with an array of non-linear equations, the calculation of many diode parameters requires an iterative procedure. Inherent with any iterative process is the possibility of convergence failure. When this repetitive thermionic scheme is coupled with the transient system equations, multiple use of the analysis is necessary. To minimize the possibility of non-convergence, several internal refinements are made in the plasma analysis.

Several critical variables are not allowed to fluctuate beyond set deviation limits from iteration to iteration. This insures smooth transitions as the particular converged solution is sought. Since solutions to the plasma diffusion equation may assume any of three forms, as determined by the prevailing values of certain parameters, a nonconvergence can occur as the crossover points are approached. Whenever such a failure is detected, an added check is made to see whether the solution lies near a crossover point. If it does, the calculations are diverted to the proper analysis, and the solution continues. Otherwise, the problem terminates.

All iterations are checked by an iteration counter. If the specified maximum number is exceeded, a nonconvergence has occurred and a proper indication is made. However, provided the variable in question is known to within two or three times the normal error tolerance, the solution is allowed to proceed.

In addition to these internal iterations, the entire diode description is continually being used as the surface thermal conditions or other system parameters vary during transients. In order to insure that smooth transitions are made from case to case, previous solutions may be employed. The final converged values of all iterative parameters are stored and subsequently reused as starting points for the succeeding converter calculation.

Special attention is paid to the initial estimate of the current, as it constitutes the variable for the outside iteration. By using previous values to indicate the direction and slope with which it is presently changing, an initial corrected guess may be made. This

allows the thermionic routine to converge usually within two iterations on J , rarely more than three.

The one remaining problem in utilizing the transient thermionic routine is the specification of initial conditions. Certain critical parameters within the plasma routine must be set at reasonable values for the original call to the converter analysis at the start of any system transient case. An excellent way to accomplish this is to use the steady-state analysis for which SIMCON was originally designed, i.e., the generation of I-V characteristic curves.

Since initial estimates to the emitter and collector temperatures can usually be made, and the reservoir temperature and diode spacing are set, the whole range of possible diode performance at these conditions may be investigated with SIMCON. When the initial load is specified, the current and voltage will be set, and appropriate values for the remainder of the critical converter parameters may be selected.

Due to the necessity of iterative methods for solution, the resulting converged values are known only to within certain pre-specified error limits. Most important of these is the accuracy to which the output voltage is determined. Following Wilkins,⁽³⁵⁾ the diode voltage is calculated to within five millivolts (± 0.005 V). Although this limit may be changed, the plasma model used does not warrant any higher degree of accuracy.

The convergence of V_0 reflects the error of J through Equation (5-7). Since typical values of the load resistance are around five hundredths ohm-centimeter squared ($0.05 \Omega\text{-cm}^2$). J will be known to approximately one-tenth of an ampere per centimeter squared ($\pm 0.1 \text{ amp/cm}^2$).

All other iterative parameters are calculated to within one percent of their relative magnitude, or to within an absolute value of 0.01 if their absolute value is less than unity. It should be noted that in all transient cases attempted, the thermionic analysis never failed to reach a converged solution, providing the input converter specifications remained within the model's region of validity.

D. Overall System Dynamic Analysis

With the incorporation of the thermionic analysis into the system dynamic equations, a complete model capable of describing the transient response of a single diode, thermionic reactor system has been formulated. The dynamic equations used to characterize the system's thermal response have been discussed in the previous annual report.⁽¹⁾

The solution of this set of differential equations describing the system transient response necessitates use of an integration scheme. Preliminary work was done with a routine obtained from Los Alamos.⁽³⁷⁾ However, a more advanced and readily usable routine was later secured from the Jet Propulsion Laboratory.⁽³⁵⁾ This scheme consists of a very fast predictor-corrector method and easily handles a set of nonlinear first-order differential equations.

The entire thermionic routine as well as the integration scheme are written so that they may be completely isolated from the remainder of the system description. Thus, these portions of the digital program may be converted into machine language, consequently saving extensive compilation time. The overall schematic of this computer program used for the dynamic studies is shown in Figure 5-3.

Figure 5-3. Program Structure for Transient System Analysis

MAIN handles the required input and initial condition specifications and contains the differential equations describing the transient thermal response. It institutes calls to the integrator (GSVDQ) and to the thermionic analysis (PLASMA). The calculations conducted by the various thermionic subroutines are basically the same as described in the SIMCON report.⁽³⁵⁾

The emitter and collector surface temperatures are determined from the transient thermal equations, and it is their fluctuations that mainly dictate when the thermionic analysis should be used. Converter calculations at every integration step would lead to prohibitive computational time and, indeed, are not required that frequently. Investigations show that diode computations need only be made whenever either surface temperature changes by five degrees Kelvin (5°K). The results obtained are of sufficient accuracy to match transient responses of calculations made for deviations as low as 1°K .

Also, as the system variables approach their new equilibria following a perturbation, the situation is reached where the surface temperatures vary less than the prescribed limit of 5°K . Consequently, the additional criterion is included that diode calculations be made if a specified time interval has elapsed since the preceding thermionic computation. Thus, the converter variables are adjusted to the new prevailing conditions whenever a surface temperature change of 5°K or an elapsed time of five seconds has occurred, whichever comes first.

While these limits are flexible, the choice of 5°K or 5 seconds has been shown to yield good results. In addition, computational time directly reflects the frequency at which the thermionic analysis is

employed. It is therefore advantageous to use the criterion having the widest range while still retaining the desired accuracy.

For typical single diode reactor model transient cases considered, the ratio of simulation time to computer time required on a CDC 6400 was almost 10:1. Thus, a simulated transient of 300 seconds duration required a corresponding computational time of approximately 30 seconds.

E. Comparison to Richardson-Dushman Approach

A complex analog computer simulation was used by Gronroos⁽³⁹⁾ to investigate the open loop response of a single diode thermionic reactor system. While the reactor description was reasonably comprehensive, the thermionic approximation used limited severely the region of application of the model.

The thermionic analysis employed by Gronroos assumes that a Richardson-Dushman approach (R-D) is valid. In order to compare the R-D method with the SIMCON-based (S-b) treatment, the present analysis must first be "calibrated" to steady-state R-D results. The set of equilibrium reference conditions used is listed in Table 5-1.

The S-b analysis has the capability of being "calibrated" to reproduce any given experimental data to a good degree of accuracy. By adjusting certain surface and plasma parameters, as discussed by Wilkins and McCandless,⁽⁴⁰⁾ the I-V curves produced with the S-b analysis may be altered somewhat. They may be shifted either right or left, and the slope at higher current values can be varied. These corrections do not change the resulting values of q_E'' (Equation (5-2)) by any appreciable amount, but rather lead to variations in the output voltage for a specified diode current.

TABLE 5-1
Reference Operating Conditions

Parameter	Value
Emitter Temperature	2050°K
Collector Temperature	1300°K
Diode Spacing	10 mil
Collector Work Function	2.1 electron volts
Total Emitter Heat Flux	58.25 watts/cm ²
Diode Current	10 amps/cm ²
Output Voltage	0.6 volts
Load Resistance	0.06 Ω -cm ²

The "data" to be duplicated are the I-V curves produced by the R-D approach. The resulting I-V curve for the specified reference conditions of Table 5-1 is shown by the solid line #2 in Figure 5-4. With the adjustment of the appropriate parameters as well as the reservoir temperature (which the R-D approach cannot set or vary) the I-V curve resulting from the S-b analysis is shown by the dashed line #2 in Figure 5-4. The agreement through the mid-range, where load lines of interest lie, is excellent.

While the comparison to the reference equilibrium I-V curve is good, the agreement worsens as the surface temperatures deviate from these steady-state values. Figure 5-4 shows the comparison of corresponding I-V plots between the two methods for 100°K deviations in emitter temperature, while holding the collector constant. Figure 5-5 shows the comparison for 50°K changes from equilibrium, while maintaining a constant emitter temperature. The differences between the corresponding I-V plots are seen to be much more pronounced for collector temperature deviations.

However, a 1300°K collector temperature is much higher than those of normal operating diodes. Since a major contribution to changes in output voltage during a transient arises from the temperature dependence of the surface work functions, it would be expected that fluctuations in this high collector temperature would be strongly felt. This important effect on the output voltage is not accounted for in the R-D model, but is in the present treatment. Thus, the I-V curves of the S-b analysis change more than those predicted by the R-D approach for corresponding deviations in both the emitter and collector temperatures.

Figure 5.4. Steady-State Comparisons of R-D and Present Thermionic Analysis for Variable $\bar{T}_E$

Figure 5-5. Steady-State Comparisons of R-D and Present Thermionic Analysis for Variable $\bar{T}_C$

The transient cases studied depict a positive step in power and a decrease in resistive load. Since the results of a digital computer simulation of the R-D reactor model have been shown to be of better accuracy than those of the corresponding analog computer analysis,⁽³⁸⁾ the digital results are used for the corresponding comparisons.

The system of transient equations used is a duplicate of that reported by Gronroos.⁽³⁹⁾ The difference lies in the determination of q_E'' , J , and V_0 .

With the reactor system operating at equilibrium power, a 2¢ step in reactivity is inserted. The resulting transient responses of major parameters are shown in Figure 5-6. While power and temperature traces are seen to be quite close, the thermionic properties are substantially different. The somewhat stepwise nature of the thermionic responses of the present method follow because this analysis is not used at every integration point, as discussed earlier.

With the system again at its reference equilibrium, a 33% step decrease in resistive load ($-0.02 \Omega\text{-cm}^2$) is made. Transient results are shown in Figure 5-7. Since no correction is made to maintain equilibrium power, it consequently fluctuates slightly due to thermal reactivity feedback. The temperature responses are similar qualitatively, but the final emitter values differ by close to 20°K. Again, the thermionic traces vary significantly. The resulting decrease in emitter temperature and increase in collector temperature are due to the increased effect of electron cooling following the decrease in resistance.

Figure 5-6. Comparison of Present Thermionic Analysis to R-D Approach for +2c Step

Figure 5-7. Comparison of Present Thermionic Analysis to R-D Approach for Change in Load

It becomes evident that significant discrepancies exist between the two diode treatments as well might be expected. They are a direct consequence of the inadequacies inherent in the R-D approach, as indicated by the steady state comparisons with the present analysis. The validity of the present approach has been well established by Wilkins⁽³⁵⁾ through comparisons with experimental results of static I-V characteristics.

The R-D model is very easy to use and does produce a reasonable approximation for the emitter heat flux (q_E'') as witnessed by the relative closeness of the power and temperature comparisons. However, the inaccuracies in the calculation of current and voltage values limits the validity of the results to small deviations from equilibrium. Also, the R-D model cannot easily account for changes in reservoir temperature or readily handle variations in diode materials.

Revisions in the R-D approach, notably improvements to account for temperature-dependent work functions and collector effects, would be necessary for any large operating condition changes, such as experienced during start-up. In view of the stronger theoretical foundation of the present analysis, this would hardly be warranted and modification of the R-D model is not recommended.

VI. MULTIPLE INPUT MULTIPLE OUTPUT CONTROL STUDIES

A. Introduction

A linear multivariable system is defined in the general sense as a system which has a multiplicity of inputs and outputs and whose impulse response can be described by a set of linear differential equations. The differential equations which describe a physical dynamic system result from the Newtonian approach to relate mathematically the causal-effect relationships observed in the real world. The mathematical structure of the multivariable system is further restricted to have the same number of system outputs as control inputs, which in turn must be less than or equal to the number of system state variables. The number of system state variables depends upon the order and number of the differential equations describing the system and represents the minimum amount of information that we need to know about a system at any time t_0 , such that its future behavior can be determined without reference to the input before t_0 .⁽⁴¹⁾

By definition, uncompensated multivariable systems are inherently interacting due to the fact that each state-variable in a multivariable physical system may depend on the previous history in time of some of the other system state variables and on present and future external forces on the system. Most of the effort to date in this area has consisted of attempts to formulate methods to decouple the multivariable system into single-input, single-output subsystems which can then be controlled independently. However, in this case the requirement of non-interaction restricts the controlled behavior of the system in the

sense that an impulse applied to any one of the inputs will cause a response only at one of the system outputs.

The result of these decoupling efforts, starting with the frequency domain techniques of Kavanagh⁽⁴⁴⁾ and proceeding to the latest studies by Falb and Wolovich,⁽⁵⁰⁾ and Gilbert,⁽⁵³⁾ is that techniques have been developed 1) for determining whether or not state variable feedback can decouple the multivariable system, 2) for accomplishing the decoupling of the multivariable system, and 3) for identifying the single-input, single-output subsystems resulting from the decoupling. All the multivariable system studies cited from Kavanagh⁽⁴⁴⁾ to Gilbert⁽⁵³⁾ employ the modern control theory approach of state variable feedback.

Since each of these techniques will be used in the design of an interacting multivariable system using state variable feedback, the necessary procedures required to obtain a decoupled multivariable system will be reviewed briefly below.

B. System Description, Notation, and a Brief Review on Decoupling

In order to apply modern control theory, the linear differential equations which describe the dynamic system must be represented in matrix notation as follows:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (6-1)$$

$$y(t) = Cx(t) \quad (6-2)$$

Here the (') indicates differentiation with respect to time t and

$x(t)$ is an n -dimensional vector, representing the states of the system

$u(t)$ is an m -dimensional vector, the control input to the system
 $y(t)$ is an m -dimensional vector, the output variables of the
 system

A is an $n \times n$ system state matrix

B is an $n \times m$ control input matrix

C is an $m \times n$ output matrix

n is a scalar and denotes the number of state variables representing
 the system

m is a scalar and denotes the number of control inputs or system
 outputs where $m \leq n$ in general.

The matrices A , B , and C are constant with no time dependence. Therefore, only the linear representation of the system is mathematically expressed.

The multivariable system must first be tested to ensure that it is controllable and observable.^(42,43) If a system is uncontrollable, it has state variables which depend only on initial conditions or disturbance inputs and cannot be affected by the control input vector $u(t)$. If a system is unobservable, it has state variables which are not detectable in the output vector $y(t)$ and therefore will have dynamic transitions in certain state variables which cannot be determined by measurements at the available outputs. The methodology of testing, for controllability and observability, is straightforward and given in the above-cited reference.

The closed-loop controlled system is obtained by using the control law

$$u(t) = Fx(t) + Gr(t) \quad (6-3)$$

where

$r(t)$ is an m -dimensional vector, representing the external control inputs to the system

G is an $m \times m$ input gain matrix

F is an $m \times n$ state feedback matrix

The control law of Eq. (6-3) is defined⁽⁴⁶⁾ as a state-variable feedback control law since the states of the system are "fed back" to the control inputs $u(t)$. The closed-loop transfer matrix is obtained in the following manner: substitute Eq. (6-3) into Eq. (6-1) for $u(t)$:

$$\dot{x}(t) = Ax(t) + BF x(t) + BG r(t) \quad (6-4)$$

Take the Laplace transform of Eq. (6-4) assume zero initial conditions on the state vector $x(0)$, and solve for $x(s)$,

$$sI_n x(s) = Ax(s) + BF x(s) + BG r(s) \quad (6-5)$$

$$x(s) = [sI_n - A - BF]^{-1} BG r(s) \quad (6-6)$$

where I_n is the $n \times n$ identity matrix. Therefore

$$y(s) = Cx(s) = C[sI_n - A - BF]^{-1} BG r(s) \quad (6-7)$$

or, the transfer matrix is:

$$H(s, F, G) = C[sI_n - A - BF]^{-1} BG \quad (6-8)$$

If the transfer matrix $H(s, F, G)$ can be diagonalized by an appropriate choice of F and G , the closed-loop multivariable system is said to be decoupled. The system then possesses non-interaction because each component of the vector input $r(t)$ will affect only one component of the vector output $y(t)$ due to the diagonal nature of $H(s, F, G)$.

The diagonalization of the transfer matrix first requires a test to determine whether the multivariable system can be decoupled. Next, one pair of F and G matrices from the set of all decoupling matrices is then determined. This necessarily forces a specification of the diagonalized transfer matrix and means that the specification of each transfer function for the single input-output subsystems is represented by the diagonal entries in the transfer matrix, $H(s,F,G)$. In order to be able to treat each subsystem independently, each of the n state variables of a multiple input-output system must appear once and only once in a particular subsystem, i.e.,

$$\sum_{i=1}^m n_i = n \quad (6-9)$$

where n_i is the number of states in the i^{th} subsystem, and m is the number of inputs or outputs to the multivariable system.

The number of states in any subsystem will correspond to the number of frequency domain poles in that same subsystem, since each state equation is first order, and there are n_i states (i.e., the subsystem is of order n_i). A desired subsystem response is consequently obtained by appropriate location of each pole in the subsystem through the choice of the elements of the F and G matrices in the control-law equation (6-3).⁽⁴¹⁾

To determine one pair of F and G matrices from the set of all decoupling matrices, suppose each transfer function for the subsystems of the decoupled multivariable system is specified in the following manner: Let z_i represent the number of zeros in the i^{th} subsystem.

Then use z_i of the n_i poles in the i^{th} subsystem to cancel each of the z_i zeros in the i^{th} subsystem, and place the remaining $(n_i - z_i)$ poles at the origin. If (d_i+1) represents the pole-zero excess of the i^{th} subsystem, i.e.,

$$d_i + 1 = n_i - z_i, \quad i = 1, 2, 3, \dots, m \quad (6-10)$$

then the transfer function for the i^{th} subsystem will be,

$$y_i(s)/r_i(s) = s^{-(d_i+1)}, \quad i = 1, 2, 3, \dots, m \quad (6-11)$$

where s is the Laplace variable, and $r_i(s)$ and $y_i(s)$ are the Laplace transforms of the i^{th} elements in the input and output vectors respectively for the i^{th} subsystem. The F and G matrices, which will decouple the multivariable system into independent subsystems possessing transfer functions indicated by Eq. (6-11) and the necessary and sufficient condition to determine if the system represented by Eqs. (6-1) and (6-2) can be decoupled into independent subsystems, have been derived by Falb and Wolovich.⁽⁵⁰⁾

The F and G matrices, denoted as F^* and G^* by Falb and Wolovich, are obtained as follows. If the transfer function between the i^{th} output, $y_i(s)$, and the i^{th} external input, $r_i(s)$ has no zeros there must not be any time derivatives of $r_i(t)$ appearing in the time-domain relationship of $r_i(t)$ and $y_i(t)$. This fact yields the mathematical definition of d_i in terms of the originally defined system matrices, A , B , and C . Substituting the control law of Eq. (6-3) into Eq. (6-1):

$$\dot{x}(t) = (A + BF) x(t) + BG r(t) \quad (6-12)$$

Then, repeatedly differentiating with respect to time, the output for the i^{th} subsystem is

$$\dot{y}_i(t) = C_i \dot{x}(t) \quad (6-13)$$

where C_i is the i^{th} row of the output matrix C . After each differentiation of $y_i(t)$, substitute Eq. (6-12) for $\dot{x}(t)$ until the $(d_i + 1)^{\text{th}}$ derivative for $y_i(t)$ is obtained. The result is that,

$$\begin{aligned} \dot{y}_i^{(d_i+1)} &= C_i (A + BF)^{(d_i+1)} x(t) + C_i (A + BF)^{d_i} BGr + \\ &\quad + C_i (A + BF)^{(d_i-1)} BGr^{(2)} + C_i (A + BF)^{(d_i-2)} BGr^{(2)} + \dots \\ &\quad + C_i (A + BF) BGr^{(d_i-1)} + C_i BGr^{(d_i)} \end{aligned} \quad (6-14)$$

where $\dot{y}_i^{(n)}$ and $\dot{r}^{(n)}$ denote the n^{th} derivative of the particular variable. To eliminate the derivative terms of $r(t)$ in Eq. (6-14), Falb and Wolovich require that d_i be set to the minimum value of a power j such that the product $C_i A^j B \neq 0$, i.e.,

$$d_i = \min [j; C_i A^j B \neq 0, \quad j = 0, 1, \dots, n-1] \quad (6-15)$$

or

$$d_i = n-1 \quad \text{if } C_i A^j B = 0 \text{ for all } j.$$

Then, using Eq. (6-15), Eq. (6-14) becomes

$$\dot{y}_i^{(d_i+1)} = C_i A^{(d_i+1)} x(t) + C_i A^{d_i} B F x(t) + C_i A^{d_i} B Gr(t) \quad (6-16)$$

Since there are m outputs, $y_i(t)$, m scalar equations with the above form of Eq. (6-16) can be written.

Now define the m -dimensional vector $y^*(t)$, the $m \times m$ matrix B^* , and the $m \times n$ matrix A^* as follows:

$$y^*(t) = \begin{bmatrix} \dot{y}_1^{(d_1+1)} \\ \dot{y}_2^{(d_2+1)} \\ \vdots \\ \dot{y}_m^{(d_m+1)} \end{bmatrix}, \quad B^* = \begin{bmatrix} C_1 A^{d_1} B \\ C_2 A^{d_2} B \\ \vdots \\ C_m A^{d_m} B \end{bmatrix}, \quad A^* = \begin{bmatrix} C_1 A^{(d_1+1)} \\ C_2 A^{(d_2+1)} \\ \vdots \\ C_m A^{(d_m+1)} \end{bmatrix} \quad (6-17)$$

where $y^*(t)$ is proportional to $y(t)$ by way of the Hamiltonian operator for differentiation. Using these definitions and all m scalar equations of the form of Eq. (6-16), the vector-matrix equation for $y^*(t)$ is,

$$y^*(t) = [A^* + B^* F] x(t) + B^* G r(t) \quad (6-18)$$

To decouple the multivariable system, define the compensation matrices F and G to be

$$F \equiv F^* = -B^{*-1} A^* \quad (6-19)$$

$$G \equiv G^* = B^{*-1} \Omega \quad (6-20)$$

where Ω is any $m \times m$ matrix of constant elements which is diagonal and nonsingular. Substitution of Eqs. (6-19) and (6-20) into Eq. (6-18) yields

$$\begin{aligned} y^*(t) &= [A^* + B^* (-B^{*-1} A^*)] x(t) + B^* B^{*-1} \Omega r(t) \\ y^*(t) &= [A^* - A^*] x(t) + \Omega r(t) \\ y^*(t) &= \Omega r(t) \end{aligned} \quad (6-21)$$

Since Ω is diagonal and nonsingular, the system is decoupled, and each $y_i^*(t)$ affects only $r_i(t)$. Now, what is left is to show that the

definitions of F^* and G^* will decouple the originally defined system equations (6-1), (6-2), and (6-3). The i^{th} element $y_i^*(t)$ of the output vector $y^*(t)$ is the $(d_i + 1)^{\text{th}}$ derivative of the i^{th} element $y_i(t)$ of the output vector $y(t)$. For zero initial conditions, the Laplace transform of $y^*(t)$ in terms of $y(t)$ is

$$y^*(s) = \begin{bmatrix} s^{(d_1+1)} & y_1(s) \\ s^{(d_2+1)} & y_2(s) \\ \vdots & \vdots \\ s^{(d_m+1)} & y_m(s) \end{bmatrix} = L(s) y(s) \quad (6-22)$$

where

$$L(s) = \begin{bmatrix} s^{(d_1+1)} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & s^{(d_2+1)} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & s^{(d_3+1)} & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & s^{(d_m+1)} \end{bmatrix} \quad (6-23)$$

Since $L^{-1}(s) = \text{inverse } L(s) =$

$$= \begin{bmatrix} -s^{(d_1+1)} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & -s^{(d_2+1)} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & -s^{(d_3+1)} & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & s^{(d_m+1)} \end{bmatrix} \quad (6-24)$$

the output vector $y(s)$ written in terms of the input vector, $r(s)$ is

$$y(s) = L^{-1}(s) y^*(s) = L^{-1}(s) \Omega r(s) \quad (6-25)$$

But from Equation (6-8),

$$H(s, F, G) = L^{-1}(s) \Omega = \begin{bmatrix} \lambda_1 s^{-(d_1+1)} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \lambda_2 s^{-(d_2+1)} & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdot & \cdot & \cdot & \lambda_m s^{-(d_m+1)} \end{bmatrix} \quad (6-26)$$

Therefore, Eqs. (6-19) and (6-20) represent definitions for a decoupling pair of matrices F and G , since $H(s, F, G)$ is diagonalized. The decoupled system, upon application of F^* and G^* , is said to be integrator decoupled (ID), since to synthesize $y_i(s)/r_i(s)$ on an analog computer a $(d_i+1)^{th}$ order integrator is required. The i^{th} subsystem will have a transfer function

$$y_i(s)/r_i(s) = h_i(s, F, G) = \lambda_i s^{-(d_i+1)} \quad (6-27)$$

The second main contribution to the decoupling problem consists of the test to determine if the originally given multivariable system can be decoupled. An examination of Eqs. (6-19) and (6-20) reveals that to use F^* and G^* as a decoupling pair, the nonsingularity of B^* is explicitly assumed. The nonsingularity of B^* represents the necessary and sufficient condition for determining whether a given multivariable system can be decoupled into subsystems possessing non-interaction.

A more complete and rigorous proof of the two contributions of Falb and Wolovich to the decoupling problem are given in detail in Reference 50.

The use of the pair of matrices, F^* and G^* results in a decoupled multivariable system which has several disadvantages. Namely, each subsystem has one or more poles only at the origin and no zeros. These shortcomings cause highly impractical features from a design standpoint since zeros are required to obtain a zero velocity-error coefficient⁽⁵⁴⁾ and only closed-loop time-responses of the form,

$$y_i(t) = a_i t^{d_i} r_i(t) \quad (6-28)$$

exist. Also, since the cancellation of all subsystem zeros with some of the subsystem poles is an inherent consequence of using F^* and G^* , not all of the original poles of the open-loop system can be arbitrarily positioned using state-variable feedback. This fact, as shown in Reference 41, always results in a sub-optimally designed closed-loop system.

With the ability to determine whether a multivariable system can be decoupled by state-variable feedback and the form for the decoupling pair of matrices which integrator decouple, Gilbert⁽⁵³⁾ proceeded to formulate the remainder of the decoupling problem with state variable feedback. His contributions describe the restrictions of state-variable feedback with explicit formulas, used to determine the specific form for each subsystem of the compensated decoupled system. He also provides an analytical formula to compute the feedback matrix F in Eq. (6-3) which will allow freedom in the design of a decoupled closed-loop response. Therefore the closed-loop multivariable system need not possess the limitations of the integrator decoupled response of Falb and Wolovich.⁽⁵⁰⁾

Gilbert first integrator-decoupled the system using the F^* and G^* matrices defined by Equations (6-19) and (6-20) and then proceeded to find a canonical transformation which would place the ID system in a unique form from which each of the m subsystems could be identified. This form was denoted as the canonically decoupled (CD) system. A brief informal development of these ideas now follows.

Let $S(A,B,C)$ represent the multivariable system defined by Equations (6-1) and (6-2), and define F and G to be the matrices for the state-variable control law of Equation (6-3) used to compensate $S(A,B,C)$. Also, let $\bar{S}(\bar{A},\bar{B},\bar{C})$ represent the ID system, found by compensating $S(A,B,C)$ with F^* and G^* , and define $\bar{F}$ and $\bar{G}$ to be the matrices used to compensate $\bar{S}(\bar{A},\bar{B},\bar{C})$ to obtain a desired closed-loop response. The multivariable systems $S(A,B,C)$ and $\bar{S}(\bar{A},\bar{B},\bar{C})$ are control-law equivalent if a one-to-one correspondence between F and $\bar{F}$, G and $\bar{G}$ can be found such that for this correspondence $H(s,F,G)$ equals $\bar{H}(s,\bar{F},\bar{G})$, where $H(s,F,G)$ and $\bar{H}(s,\bar{F},\bar{G})$ are the transmission matrices (Equation (6-8)) relating the output vector $y(t)$ to the input vector $r(t)$ for $S(A,B,C)$ and $\bar{S}(\bar{A},\bar{B},\bar{C})$ respectively.

The state equations for $S(A,B,C)$ are the familiar ones of Equations (6-1) and (6-2). The state equations for $\bar{S}(\bar{A},\bar{B},\bar{C})$ are

$$\dot{\bar{x}} = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) \quad (6-29)$$

$$y(t) = \bar{C}\bar{x}(t) \quad (6-30)$$

where

$$\bar{A} = A + BF^* \quad (6-31)$$

$$\bar{B} = BG^* \quad (6-32)$$

$$\bar{C} = C \quad (6-33)$$

But, F^* and G^* are defined by Equations (6-19) and (6-20) so that upon substitution,

$$\dot{x} = (A - BB^{*-1}A^*)x(t) + BB^{*-1}\bar{u} \quad (6-34)$$

Using the previously defined F and G matrices, the control law input for $\bar{S}(\bar{A}, \bar{B}, \bar{C})$ is therefore,

$$\bar{u}(t) = \bar{F}x(t) + \bar{G}r(t) \quad (6-35)$$

and Equation (6-34) becomes

$$\dot{x} = (A - BB^{*-1}A^* + BB^{*-1}\bar{F})x(t) + BB^{*-1}\bar{G}r(t) \quad (6-36)$$

The corresponding expression for $S(A, B, C)$ using Equation (6-3) is

$$\dot{x} = (A + BF)x(t) + BGr(t) \quad (6-37)$$

The systems $S(A, B, C)$ and $\bar{S}(\bar{A}, \bar{B}, \bar{C})$ will have identical responses if Equations (6-36) and (6-37) are identical. This requires that

$$F = B^{*-1}(\bar{F} - A^*)$$

$$F = B^{*-1}\bar{F} + F^* \quad (6-38)$$

$$G = B^{*-1}\bar{G} \quad (6-39)$$

By this development it is established that one can work with either the original system or the ID system and still retain the same design freedom. Gilbert chose to work with the ID system because it is decoupled.

The canonical form to which the ID system can be transformed and in which it is apparent that the system is decoupled is discussed next. The ID system and the CD system are related by a linear change

of variables using the transformation matrix T , where

$$\hat{x}(t) = Tx(t) \quad (6-40)$$

The $\hat{x}(t)$ represents the state-variable vector for the CD system, and $x(t)$ is the state-variable vector for both the original and the ID system. The transformation matrix T is calculated in the following manner.

Let T denote the n -dimensional space of n -element row matrices, and for $i = 1, 2, \dots, m$, define

$$T_i = [\tau; \tau \in T; \tau \bar{A}^j \bar{B}_k = 0, \quad \text{for } k = 1, 2, \dots, m, \quad k \neq i] \quad (6-41)$$

and $j = 0, 1, \dots, n-1$

where $\bar{A}$ and $\bar{B}$ are defined by Equations (6-31) and (6-32). The $\bar{B}_k$ denotes the k^{th} column of the matrix $\bar{B}$. Two scalars p_i and ℓ_i are needed in the definition of the CD representation and are defined by

$$p_i = \text{dimension of } T_i, \quad i = 1, 2, \dots, m, m+1 \quad (6-42)$$

$$p = \sum_{i=1}^m p_i \quad \text{and} \quad p_{m+1} = n - p \quad (6-43)$$

$$\ell_i = p_i - d_i - 1 \quad (6-44)$$

where d_i is defined by Equation (6-15). The scalar p_i is the number of poles of the i^{th} subsystem which can be controlled by state-variable feedback and ℓ_i is the number of fixed zeros of the i^{th} subsystem.

The transformation matrix T is written as

$$T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{m+1} \end{bmatrix} \quad (6-45)$$

where the rows of the $(p_i \times n)$ matrix T_i , $i = 1, 2, \dots, m$, are a basis for T_i . The first $d_i + 1$ rows of each of these T_i are chosen to be $C_i, C_i A, \dots, C_i A^{d_i}$ and the last $p_i - d_i - 1$ rows are any other linearly independent row vectors which make the set of rows of T_i a basis for T_i . For T_{m+1} , the rows are chosen so that the collection of rows of T form a basis for T .

Substitution of Equation (6-40) into Equations (6-29) and (6-30) yields

$$\hat{\dot{x}} = T \bar{A} T^{-1} \hat{x}(t) + T \bar{B} u(t) \quad (6-46)$$

$$y(t) = C T^{-1} \hat{x}(t) \quad (6-47)$$

Define:

$$\hat{A} \equiv T \bar{A} T^{-1} \quad (6-48)$$

$$\hat{B} \equiv T \bar{B} \quad (6-49)$$

$$\hat{C} = C T^{-1} \quad (6-50)$$

$$\hat{F} = \bar{F} Q^{-1} \quad (6-51)$$

$$\hat{G} = \bar{G} \quad (6-52)$$

Then, the canonical transformation is completed if the $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{F}$, and $\hat{G}$ matrices can be partitioned into specific forms. The $\hat{A}$, $\hat{B}$, and $\hat{C}$

matrices must have the partitioned standard decoupled form as shown in Figure 6-1, and the subsystems $\hat{A}_{ii}$, $\hat{B}_{ii}$, and $\hat{C}_{ii}$ must exhibit the canonically decoupled partitioned form illustrated in Figure 6-2. The diagonal nature of the upper portions of the $\hat{A}$ and $\hat{B}$ matrices and the left portion of the $\hat{C}$ matrix permit the identification of m decoupled subsystems. Gilbert⁽⁵³⁾ proved that upon using the correctly constructed transformation matrix T , the matrices $\hat{A}$, $\hat{B}$, and $\hat{C}$ have the structure required for the CD system representation.

To summarize, after the original system has been integrator-decoupled (ID), the resulting ID system is considered to be an open-loop system with control-input vector $\bar{u}(t)$. This ID system is to be further compensated with $\bar{F}$ and $\bar{G}$ to obtain the desired diagonal transmission matrix. This is accomplished by applying state-variable feedback again as given by the control law specified by Equation (6-35). But, before this control law can be used as compensation, the canonical transformation must be performed to identify the m subsystems.

Gilbert has also shown that $\bar{F}$ and $\bar{G}$ will preserve noninteraction if and only if the matrices $\hat{F}$ and $\hat{G}$ related by Equations (6-51) and (6-52) have the partitioned forms shown in Figure 6-3.⁽⁵³⁾

With his contributions to the decoupling problem, Gilbert has established the important point that when compensating the multivariable plant, all that is required is to integrator-decouple the system, change variables to get the canonically decoupled form and identify the subsystems, then finally design each subsystem for a particular response using well-known single-input, single-output state-variable feedback design techniques.⁽⁴¹⁾

$$\hat{A} = \left| \begin{array}{cccccc} \hat{A}_{11} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \hat{A}_{22} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \hat{A}_{33} & \cdot & \cdot & \cdot & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \hat{A}_{mm} & 0 \\ \hat{A}_1^C & \hat{A}_2^C & \hat{A}_3^C & \cdot & \cdot & \cdot & \hat{A}_m^C & \hat{A}_{m+1}^C \end{array} \right|$$

where $\hat{A}_{ii}$ is a $p_i \times p_i$ matrix
 $\hat{A}_i^C$ is a $p_{m+1} \times p_i$ matrix

$$\hat{B} = \left| \begin{array}{cccccc} \hat{B}_{11} & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \hat{B}_{22} & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \hat{B}_{33} & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \hat{B}_{mm} \\ \hat{B}_1^C & \hat{B}_2^C & \hat{B}_3^C & \cdot & \cdot & \cdot & \hat{B}_m^C \end{array} \right|$$

where $\hat{B}_{ii}$ is a $p_i \times 1$ matrix
 $\hat{B}_i^C$ is a $p_{m+1} \times 1$ matrix

$$C = \left| \begin{array}{cccccc} \hat{C}_{11} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & \hat{C}_{22} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \hat{C}_{33} & \cdot & \cdot & \cdot & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & \hat{C}_{mm} & 0 \end{array} \right|$$

where $\hat{C}_{ii}$ is a $1 \times p_i$ matrix

$p_i \geq d_i + 1, \quad i = 1, 2, \dots, m$
 $p_i = \ell_i + d_i + 1$

Figure 6-1. Standard Decoupled Form.

$$\hat{A}_{ii} = \begin{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} & 0 \\ \gamma_i & \phi_i \end{bmatrix} \quad \begin{array}{l} \text{where } I \text{ is a } d_i \times d_i \text{ sub-matrix} \\ \gamma_i \text{ is an } \ell_i \times (d_i + 1) \text{ sub-matrix} \\ \phi_i \text{ is an } \ell_i \times \ell_i \text{ sub-matrix} \end{array}$$

$$\hat{B}_{ii} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \beta_i \end{bmatrix} \quad \text{where } \beta_i \text{ is an } \ell_i \times 1 \text{ sub-matrix}$$

$$\hat{C}_{ii} = [1 \quad 0 \quad 0 \quad \dots \quad 0]$$

Figure 6-2. Canonically Decoupled Form.

$$\hat{F} = \begin{bmatrix} \hat{f}_{11} & 0 & 0 & \dots & 0 & 0 \\ 0 & \hat{f}_{22} & 0 & \dots & 0 & 0 \\ 0 & 0 & \hat{f}_{33} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \hat{f}_{mm} & 0 \end{bmatrix} \quad \text{where } \hat{f}_{ii} \text{ is a } 1 \times p_i \text{ sub-matrix}$$

$$\hat{G} = \begin{bmatrix} \lambda_{11} & 0 & 0 & \dots & 0 \\ 0 & \lambda_{22} & 0 & \dots & 0 \\ 0 & 0 & \lambda_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{mm} \end{bmatrix} \quad \begin{array}{l} \text{where } \lambda_{ii} \text{ is a } 1 \times 1 \text{ sub-matrix} \\ \lambda_{ii} \neq 0, i=1,2,\dots,m \end{array}$$

Figure 6-3. Feedback Decoupling Forms.

Several other contributions should be noted to make this review of the decoupling of multivariable systems with state-variable feedback complete. Morgan⁽⁴⁶⁾ was the first author to formulate the state-variable feedback control law of Equation (6-3). Slivinsky⁽⁵⁵⁾ showed how to add dynamic compensation to the subsystems when the number of subsystem poles and zeros are not adequate to achieve a desired system response. He has proven that when dynamics in the form of lead-lag compensators are added to the decoupled subsystems, it does not cause a loss of decoupling when state-variable feedback is added and that state-variable feedback still permits the arbitrary positioning of all the poles in the augmented subsystem. All other contributions in References 44 to 51, which represent special cases of the noninteraction design problem, are superceded by the contributions of Falb, Wolovich, and Gilbert.

C. An Example

An example of a thermionic reactor system utilizing the simple model previously employed in the stability studies will illustrate the above basic calculations involved in decoupling a multivariable system. All required calculations are performed using the procedures outlined in the theory discussed above. It should be emphasized that this is merely an example of procedure, not a control design study of the thermionic reactor system.

The control inputs to the system are assumed to be to the neutron power and the collector surface temperature. The state-variables representing the output variables of the system are the neutron power

level and the collector temperature. The following first-order differential equations, taken from Reference 1 represent the multi-variable system.

$$\dot{x}_1 = -n_0 \xi_1 x_2 - n_0 \xi_2 x_3 + u_1(t) \quad y_1(t) = x_1(t)$$

$$\dot{x}_2 = b x_1 - \eta_1 x_2 \quad y_2(t) = x_3(t)$$

$$x_3 = \eta_2 x_3 - \eta_3 x_3 + u_2(t)$$

where

x_1 = state-variable of neutron power density

x_2 = state-variable of emitter-surface temperature

x_3 = state-variable of collector-surface temperature

n_0 = equilibrium neutron power density

ξ_1, ξ_2 are reactivity-temperature feedback coefficients

$b, \eta_1, \eta_2, \eta_3$ are reciprocal time constants of the system and are defined explicitly in Reference 1.

The state, control, and output vectors are respectively:

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad u(t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad y(t) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

and the associated system matrices are:

$$A = \begin{bmatrix} 0 & -\eta_0 \xi_1 & -\eta_0 \xi_2 \\ b & -\eta_1 & 0 \\ 0 & -\eta_2 & -\eta_3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The decoupling procedure now consists of the following steps:

a) Determine the scalars d_1 defined by Equation (6-15)

For this example, $m=2$, $n=3$. Therefore, only d_1 and d_2 are needed because there are only two rows to the output matrix C , namely

$$C_1 = [1 \quad 0 \quad 0]$$

$$C_2 = [0 \quad 0 \quad 1]$$

Consider first d_1 and $j=0$; $i=1$.

$$C_1 B = [1 \quad 0 \quad 0] \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = [1 \quad 0] \neq \text{null matrix}$$

so, $d_1 = 0$.

Consider d_2 , and $j=0$; $i=1$.

$$C B = [0 \quad 0 \quad 1] \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = [0 \quad 1] \neq \text{null matrix}$$

so, $d_2 = 0$.

b) Determine if the system can be decoupled by calculating B^* as defined by Equation (6-17).

Since d_1 and d_2 are both zero,

$$B^* = \begin{bmatrix} C_1 & B \\ C_2 & B \end{bmatrix} = CB$$

or

$$B^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since B^* equals the $m \times m$ identity matrix, the inverse of B^* , B^{*-1} , does exist and is

$$B^{*-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, the multivariable system can be decoupled.

c) Determine the A^* matrix as defined by Equation (6-17).

Since d_1 and d_2 are both zero,

$$A^* = \begin{bmatrix} C_1 & A \\ C_2 & A \end{bmatrix} = CA$$

or,

$$A^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -n_0\xi_1 & -n_0\xi_2 \\ b & -\eta_1 & 0 \\ 0 & \eta_2 & -\eta_3 \end{bmatrix} = \begin{bmatrix} 0 & -n_0\xi_1 & -n_0\xi_2 \\ 0 & \eta_2 & -\eta_3 \end{bmatrix}$$

d) Determine F^* and G^* , the compensation matrices which put the system in integrator decoupled form, defined by Equations (6-19) and (6-20).

Since B^{*-1} equals the (2x2) identity matrix,

$$F^* = -B^{*-1}A^* = -A^* = \begin{bmatrix} 0 & \eta_0 \xi_1 & \eta_0 \xi_2 \\ 0 & -\eta_2 & \eta_3 \end{bmatrix}$$

and

$$G^* = B^{*-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

e) Determine the $\bar{A}$ and $\bar{B}$ matrices defined by Equations (6-31) and (6-32).

Now, $\bar{A} = A + BF^*$

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0 \\ b & -\eta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$\bar{B} = BG^*$

$$\bar{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

f) Determine the matrix T , used to change variables and to find the canonically decoupled representation of the (ID) system, defined by Equation (6-40).

From Equation (6-41), the vector spaces T_1 and T_2 must be calculated since $m=2$. So,

$$T_1 = [\tau: \tau \in T; \tau \bar{A}^j \bar{B}_2 = 0, \quad j = 0, 1, 2]$$

$$T_2 = [\tau: \tau \in T_2; \tau \bar{A}^j \bar{B}_1 = 0, \quad j = 0, 1, 2]$$

where, since $n=3$,

$$\tau = [\tau_1 \tau_2 \tau_3]$$

For $j=0$,

$$\tau \bar{B}_2 = \tau_3 = 0$$

$$\tau \bar{B}_1 = \tau_1 = 0$$

For $j=1$,

$$\tau \bar{A} \bar{B}_2 = 0$$

$$\tau \bar{A} \bar{B}_1 = b \tau_2 = 0, \quad \text{so } \tau_2 = 0$$

For $j=2$,

$$\tau \bar{A}^2 \bar{B}_2 = 0$$

$$\tau \bar{A}^2 \bar{B}_1 = 0$$

So,

$$T_1 = [\tau, \tau = (\tau_1 \tau_2 0)]$$

$$T_2 = [\tau, \tau = (0 \ 0 \ \tau_3)]$$

By definition, the dimension of T_1 is p_1 and

$$p_1 = 2 = \dim T_1$$

Also, the dimension of T_2 is p_2 and

$$p_2 = 1 = \dim T_2$$

The dimensions of T_1 and T_2 add to 3, the number of system states, so $p_{m+1} = 0$, and only T_1 and T_2 are needed to construct T . Since d_1 and d_2 are both zero, the first row of T_1 is C_1 and the first row of T_2 is C_2 . The second row of T_1 is any row vector which is independent of C_1 and belongs to the subspace of T_1 . Choose the row vector $(0 \ 1 \ 0)$ because if $T_1 = 0$, $T_2 = 1$, then

$$T_1 = [\tau, \tau = (0 \ 1 \ 0)]$$

so $(0 \ 1 \ 0)$ belongs to the subspace of T_1 .

Therefore,

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$T^{-1} = \text{inverse } T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

g) Determine $\hat{A}$, $\hat{B}$, and $\hat{C}$ defined by Equations (6-48) (6-49) and (6-50).

$$\hat{A} = T \bar{A} T^{-1} = \bar{A} \quad \text{since } T = T^{-1} = I$$

$$\hat{B} = T\bar{B} = \bar{B} \quad \text{since } T = I$$

$$\hat{C} = CT^{-1} = C \quad \text{since } T^{-1} = I$$

So,

$$\hat{A} = \begin{bmatrix} 0 & 0 & 0 \\ b & -\eta_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

h) Determine the integer, ℓ_i , for each subsystem in order to partition $\hat{A}$, $\hat{B}$, and $\hat{C}$.

From step f), $p_1 = 2$ and $p_2 = 1$, so using Equation (6-44),

$$\ell_1 = p_1 - d_1 - 1 = 2 - 1 = 1 \quad d_1 = 0$$

$$\ell_2 = p_2 - d_2 - 1 = 1 - 1 = 0 \quad d_2 = 0$$

i) Partition $\hat{A}$, $\hat{B}$, and $\hat{C}$, and identify the subsystems.

$$\hat{A} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ b & -\eta_1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right] \quad \hat{B} = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 0 \\ \hline 0 & 1 \end{array} \right]$$

$$\hat{C} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array} \right]$$

So,

$$\hat{A}_{11} = \begin{bmatrix} 0 & 0 \\ b & -\eta_1 \end{bmatrix} \quad \hat{A}_{22} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\hat{B}_{11} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \hat{B}_{22} = \begin{bmatrix} 1 \end{bmatrix}$$

$$\hat{C}_{11} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \hat{C}_{22} = \begin{bmatrix} 1 \end{bmatrix}$$

j) Determine the permissible forms of F and G matrices which will preserve decoupling when the closed-loop system is calculated.

Since $p_1 = 2$ and $p_2 = 1$, f_{11} must be a (1x2) submatrix, and f_{22} must be a (1x1) submatrix. Take $\hat{f}_{11}$ and $\hat{f}_{22}$ to be

$$\hat{f}_{11} = \begin{bmatrix} f_{11} & f_{12} \end{bmatrix}$$

$$\hat{f}_{22} = \begin{bmatrix} f_{23} \end{bmatrix}$$

So,

$$\hat{F} = \begin{bmatrix} f_{11} & f_{12} & 0 \\ 0 & 0 & f_{23} \end{bmatrix}$$

and

$$\hat{G} = \begin{bmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{bmatrix}$$

k) Determine the compensation matrices for the (ID) system, $\bar{F}$ and $\bar{G}$, from Equations (6-51) and (6-52).

$$\bar{F} = \hat{F}T = \begin{bmatrix} f_{11} & f_{12} & 0 \\ 0 & 0 & f_{23} \end{bmatrix}$$

$$\bar{G} = \hat{G} = \begin{bmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{bmatrix}$$

ℓ) Determine the control-law-equivalent matrices F and G used to compensate the original system equations.

Using Equations (6-38) and (6-39),

$$F = B^{*-1}\bar{F} + F^* = \begin{bmatrix} f_{11} & n_0\xi_1 + f_{12} & n_0\xi_2 \\ 0 & -\eta_2 & \eta_3 + f_{23} \end{bmatrix}$$

$$G = B^{*-1}\bar{G} = \begin{bmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{bmatrix}$$

The application of the F and G matrices in the control law of Equation (6-3) should result in a decoupled closed-loop multivariable system.

m) Determine the closed-loop system response to check the transfer matrix $H(s, F, G)$ for noninteraction.

Using Equation (6-7),

$$y(s) = C[sI_3 - A - BF]^{-1}BG r(s) = H(s, F, G) r(s)$$

$$y(s) \begin{bmatrix} \frac{\lambda_{11}(s - \eta_1)}{(s - f_{11})(s - \eta_1) - bf_{12}} & 0 \\ 0 & \frac{\lambda_2}{s - f_{23}} \end{bmatrix} r(s)$$

where

$$H(s, F, G) = \begin{bmatrix} \frac{\lambda_{11}(s - \eta_1)}{(s - f_{11})(s - \eta_1) - bf_{12}} & 0 \\ 0 & \frac{\lambda_{23}}{s - f_{23}} \end{bmatrix}$$

$H(s, F, G)$ is diagonalized and hence the compensated system possesses noninteraction. This completes the example, since the objective of obtaining a decoupled closed-loop response has been achieved. Note that there is one fixed zero in subsystem 1 and none in subsystem 2, which agrees with $\ell_1 = 1$ and $\ell_2 = 0$. Also, note that two poles in subsystem 1 and only one pole in subsystem 2 can be controlled by state-variable feedback, which agrees with $p_1 = 2$ and $p_2 = 1$. Also, note that one of the poles of subsystem 1 can be used to cancel the zero by choosing $f_{12} = 0$. The state variables which comprise subsystem 1 are the neutron power density and the emitter surface temperature while the remaining state variable, the collector surface temperature, has been isolated and comprises subsystem 2.

D. Future Work

The question of whether a closed-loop multivariable system can be designed such that noninteraction is not a requirement will be answered in future work. The design engineer would like to be able to choose a non-zero, specific transfer function between the input $r_i(t)$ and the output $y_j(t)$ ($i \neq j$) and realize it by state-variable feedback. In this way, a transfer matrix could be envisioned which, instead of being just diagonalized for noninteraction, would consist of off-diagonal entries which result in an interacting closed-loop system. The system would be interacting or coupled as desired because each input (r_i) in general would affect all outputs (y_i , $i = 1, 2, \dots, m$) but would affect all outputs in a manner prescribed by the designer.

At present, a computational procedure for the feedback synthesis of a coupled multivariable system seems possible by using the results of Falb, Wolovich,⁽⁵⁰⁾ and Gilbert⁽⁵³⁾ for the state-variable feedback design of noninteracting multivariable systems. The possibility rests on the fact that for a set of n -first order differential equations, there are many possible matrix representations in the form of Equations (6-1) and (6-2), depending upon the definitions of the vector quantities. Each representation will decompose to the same set of differential equations but will not obey the equivalency requirements of elementary matrix algebra.

Now m different matrix notations for the same initially given set of linear first-order differential equations could be obtained in the subsequent manner. Define m output vectors $y^1, y^2, \dots, y^m$ where

$$y^1 \equiv \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix}, \quad y^2 \equiv \begin{bmatrix} y_2(t) \\ y_3(t) \\ \vdots \\ y_m(t) \\ y_1(t) \end{bmatrix}, \dots, \quad y^m \equiv \begin{bmatrix} y_m(t) \\ y_1(t) \\ y_2(t) \\ \vdots \\ y_{m-1}(t) \end{bmatrix}$$

Again $\underline{m}$ is the number of system inputs or outputs. Also, define $\underline{m}$ output matrices, $C^1, C^2, \dots, C^m$, where C^i is the matrix which satisfies the matrix equation,

$$y^i(t) = C^i x(t), \quad i = 1, 2, \dots, m$$

where $x \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, the state-variable vector

Then, each matrix output equation (6-2) along with the state-variable plant matrix equation (6-1) would represent one separate matrix notation for the original multivariable system. If each F^i and G^i , matrices from the state-variable feedback control law,

$$u(t) = F^i x(t) + G^i r(t)$$

are computed such that each transfer matrix,

$$H^i(s, F^i, G^i) = C^i [sI_n - A - BF^i]^{-1} BG^i$$

is diagonalized for noninteraction, it may be possible to combine all $\underline{m}$ F^i and G^i matrices into one state feedback matrix, F , and one gain matrix, G . The result would then be a procedure for the feedback synthesis of a coupled multivariable system since the resulting transmission matrix $H(s, F, G)$ would have off-diagonal terms.

Also, since the given multivariable system could be decoupled in all possible ways, every transfer function between the i^{th} output and the j^{th} input, $y_i(s)/r_j(s)$, could be determined for all possible combinations of i and j , $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m$. Then, it would be hypothetically possible to determine a "best way" to decouple the original system if the designer wanted to impose the non-interaction requirement. This "best way" could be determined by minimizing a sensitivity function of the form $S_{\lambda}^T = \frac{\lambda}{T} \frac{dT}{d\lambda}$ where T would equal $y_i(s)/r_j(s)$ and λ would be any variable parameter. For example, λ might be the coefficients in the matrix $\bar{F}$, system parameters that change with time, or some variation at the input $r_j(s)$ such as a unit step, $r_j(s) = 1/s$.

The mathematical formulation of these ideas will constitute the future work in the area of multivariable control systems. It is intended that after the theory is extended to allow interaction between inputs and outputs as a design requirement, a practical example utilizing the equations pertaining to the thermionic reactor system can be worked out in detail.

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Students	1966			1967				1968				1969				1970				1971				Dissertation Topic		
	6	9	12	3	6	9	12	3	6	9	12	3	6	9	12	3	6	9	12	3	6	9	12			
T. Schmidt	—	↑	—	—	—	—	A P	P	P	—	A	—	D T J P P ↓	—	A	J	—	—	—	—	—	—	—	Nonlinear Oscillations and Stability		
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K. Kearns	—	—	—	—	—	—	—	—	—	—	A	—	↑	—	—	—	A	TP D ↓	—	J	—	—	—	Nonlinear Stability and Time Delay Systems		
D. Ferg	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	A	TP D ↓	—	J	—	—	—	Multiple-Input, Multiple-Output Control Studies		
J. Kendall	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	↑	—	—	—	T D	—	P	J	—	Nonlinear Stability
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R. L. Brehm	↑	—	—	—	—	—	A P	—	—	—	A P	—	—	J	P P	P	—	A P	—	—	—	—	—	—		

Symbols: D Dissertation
 T Topical
 J Journal Article
 P Paper Presented
 A Annual Report
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