AN ITERATIVE METHOD FOR FREDHOLM
EQUATIONS OF THE FIRST KIND

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Ph. D. Thesis Submitted to Iowa State University

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An iterative method for Fredholm
equations of the first kind

by

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An iterative method for Fredholm
equations of the first kind

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An iterative method is developed to find an approximation to the least squares solution of minimum norm, f_0 , of an operator equation, $Kf = g$, of the first kind. In applications to integral equations, the superfluous oscillations in the final solution of methods in the literature are not an apparent problem. Let $K: H_1 \rightarrow H_2$ be a completely continuous operator from a real separable Hilbert space H_1 to a real Hilbert space H_2 . Let $g \in R(K) + N(K^*)$. A sequence $\{f_{\lambda_j}\}_{j=1}^{\infty} \subset N(K^\perp)$ converging to f_0 is constructed. f_{λ_j} minimizes the functional $Q_{\lambda_j}(f) = \|Kf - g\| + \lambda_j \|f\|$, $\lambda_j > 0$. Stability and error bounds are obtained.

1. INTRODUCTION AND REVIEW OF THE LITERATURE

1.1. Statement of the Problem

Throughout this paper we will assume that H_1 is a real separable (or finite dimensional) Hilbert space, H_2 is a real Hilbert space and K is a compact (or completely continuous) operator from H_1 to H_2 . The basic definitions used above can be found in Bachman and Narici [3, Chapter 6 and 17].

We will use the notation:

$$R(K) = \text{range of } K,$$

$$N(K) = \{f \in H_1 : Kf = 0\}$$

and

$$M^\perp = \{f \in H_1 : \langle f, h \rangle = 0 \text{ for all } h \in M \subset H_1\}$$

where $\langle, \rangle$ denotes inner product in the Hilbert space.

From Strand [20, page 22] we have that for an operator K

$$H_2 = N(K^*) \oplus \overline{R(K)} \quad \text{where } \overline{R(K)} \text{ is the closure of } R(K),$$

$$H_1 = N(K) \oplus (N(K))^\perp,$$

and

$$\overline{R(K^*)} = [N(K)]^\perp.$$

We are interested in finding an approximation to the least squares solution of minimum norm, f_0 , of the operator equation $Kf = g$ where $g \in H_2$, when such a solution exists. Kammerer and Nashed [11] give the following definitions.

Definition 1.1.1 For $f \in H_1$, $g \in H_2$, an element $\bar{f} \in H_1$ is a least squares solution of $Kf = g$ if and only if

$$(1.1.2) \quad \|K\bar{f} - g\| = \inf\{\|Kf - g\| \mid f \in H_1\}.$$

Definition 1.1.2 For $f \in H_1$, $g \in H_2$, an element $f_0 \in H_1$ is a least squares solution of minimum norm of $Kf = g$ if and only if it is a least squares solution of $Kf = g$ and $\|f_0\| \leq \|\bar{f}\|$ for all least squares solutions $\bar{f}$ of $Kf = g$.

Strand [20, pages 23-24] notes that the least squares solution of minimum norm to $Kf = g$ exists (is unique and in $N(K)^\perp$) if and only if $g \in H_2$ can be written in the form

$$(1.1.3) \quad g = g_1 + g_2$$

where $g_1 \in R(K)$ and $g_2 \in N(K^*)$. For this paper we will assume g is always in the form $g = g_1 + g_2$, with $g_1 \in R(K)$. From Strand [20] and Diaz and Metcalf [5], f is a solution of $\|Kf - g\| = \inf\{\|Kh - g\| \mid h \in H_1\}$ if and only if f is a solution to $Kf = g_1$. They also note that the least square solutions to $K^*Kf = K^*g$ and $Kf = g_1$ are the same, this will become important later on in this paper.

The least squares solution of minimum norm of $Kf = g$ can be written as a series in terms of g and the eigenvalues and eigenvectors of KK^* and K^*K , so is known exactly ([20, page 25], [24, page 143]). However, finding each eigenvector of KK^* is as difficult a problem, numerically, as the original problem.

Twomey [25], Phillips [18], Tikhonov [23] and Strand [20] consider various generalizations of the method of regularization for compact operators K . Strand [20] considers the functional

$$(1.1.4) \quad Q(f) = \|Kf - g\|^2 + \lambda \|f - p\|^2, \quad \lambda > 0,$$

where $p \in H_1$ is an estimate for the least squares solution of minimum norm of $Kf = g$. He proves that (1.1.4) is minimized by

$$(1.1.5) \quad f_\lambda = (K^*K + \lambda I)^{-1} (K^*g + \lambda p).$$

Strand [20] gives a proof for the following theorem.

Theorem 1.1.1. $\bar{f} = \lim_{\lambda \rightarrow 0^+} f_\lambda$ exists and $\bar{f}$ is the unique

solution of $K^*Kf = K^*g$ for which $\|f - p\|^2$ is a minimum.

We will use this theorem in Chapter 2 for the case where $p \equiv 0$, to obtain an approximation to the least squares solution of minimum norm of $Kf = g$. In general a method for determining a good choice for λ in actual applications and construction of an inverse to represent $(K^*K + \lambda I)^{-1}$ is difficult since $(K^*K + \lambda I)^{-1}$ is generally ill-conditioned for a useful choice for λ . In this dissertation, the problem of a choice for λ will be considered and finding $(K^*K + \lambda I)^{-1}$ will be avoided.

1.2. Iterative Methods

Landweber [13], Diaz and Metcalf [5], Kammerer and Nashed [10,11,12] and Nashed [14,15] study iterative methods for finding the least squares solution of minimum norm of $Kf = g$, by constructing a sequence $\{f_i\}_{i=1}^{\infty}$ which converges to the least squares solution of minimum norm. The sequence $\{f_i\}_{i=1}^{\infty}$ is defined by

$$(1.2.1) \quad f_{n+1} = f_n + \alpha_n W_n$$

where α_n and W_n are chosen to give the method of steepest descent, conjugate gradient or weak steepest descent method. These methods converge to the least squares solution of minimum norm of $Kf = g$ for various choices for a starting vector and restrictions on $g \in H_2$. For perturbations of g by a function ϵ in H_2 we have the operator equation $Kf = g + \epsilon$ which, since $g + \epsilon$ might not be an element of $R(K) + N(K^*)$, may fail to have a least squares solution of minimum norm. Strand [20, page 47] shows that these methods are sensitive to small perturbations of g and may fail to converge to a function near

the actual solution. This is a serious problem for a numerical method, since discretization error and roundoff error act like a perturbation of g .

The iterative method considered in this dissertation will depend on operators of the form $(\lambda I + K^*K) : H_1 \rightarrow H_1$, $\lambda > 0$. Since $\lambda I + K^*K$ is bounded and positive definite the inverse $(\lambda I + K^*K)^{-1}$ exists and is defined on all of H_1 for each $\lambda > 0$. For λ not too small the solutions of $(K^*K + \lambda I)f = K^*g$ will not be as sensitive to perturbation of g as the equation $K^*Kf = K^*g$ (see section 2.3), but has a solution close to the least squares solution of minimum norm of $K^*Kf = K^*g$ (or $Kf = g$).

Some examples will be given, together with a discussion of some of the problems of discretization, of finding approximations to least square solutions of minimum norm of integral equations of the first kind.

2. DEVELOPMENT OF AN ITERATIVE METHOD FOR CONVERGENCE TO THE LEAST SQUARES SOLUTION OF MINIMUM NORM

2.1. Introduction

First. An iterative method is obtained for an approximate solution to the least squares solution of minimum norm of $Kf = g$, when such a solution exists.

Second. Several theorems will be given which study the properties of changes in the solution to $Kf = g$ when g is replaced with $g + \epsilon$, ϵ an element of H_2 , to give an indication of stability.

Third. Theorems relating $\lambda > 0$ and $f_\lambda \in H_1$, where f_λ is a solution to $(K^*K + \lambda I)f = K^*g$ will be given, as well as some theorems on error bounds.

2.2. An Iterative Method

The least squares solution of minimum norm, f_0 , is an element of $[N(K)]^\perp$. For any element $f \in N(K)$ we have $K(f_0 + f) = Kf_0 + Kf = Kf_0$, so since $f_0 + f$ and f_0 are mapped to the same element in the range of K it is important for an iterative procedure to stay on $[N(K)]^\perp$. The goal is to construct a sequence of f_λ 's

which satisfy the theorem below, converge to f_0 and stays in $N(K)^\perp$.

Theorem 2.2.1. If $f_\lambda \in H_1$ is the solution to $(K^*K + \lambda I)f = K^*g$, $\lambda > 0$, then $f_\lambda \in R(K^*) \subset N(K)^\perp$.

Proof: Since $(K^*K + \lambda I)f_\lambda = K^*g$ we have

$\lambda f_\lambda = K^*g - K^*K f_\lambda$ which in turn implies that

$$f_\lambda = \frac{K^*(g - K f_\lambda)}{\lambda}, \text{ thus } f_\lambda \in R(K^*) \subset [N(K)]^\perp.$$

Now we develop an iterative method related to a steepest descent method for Tikhonov-Twomey regularization, which converges to the least squares solution of minimum norm, when such a solution exists. The method developed allows one to find a solution of $(K^*K + \lambda I)f = K^*g$ for a $\lambda > 0$. When the convergence fails or becomes slow then the advantage of this method is that a new starting vector can be obtained.

We really wish to implement the procedure in Step 1 and 2 following and in doing so the main result we use is given in the corollary on page 20 for Theorem 2.2.2 which follows.

Step 1: Choose a sequence of positive real numbers

$\{\lambda_i\}_{i=1}^{\infty}$ such that $\lambda_i \rightarrow 0$ as $i \rightarrow \infty$, a starting element $f_{1,0} \in H_1$ for the sequence $\{f_{1,n}\}_{n=0}^{\infty}$ converging to the solution, f_1 , of $(K^*K + \lambda_1 I)f = K^*g$, as required by Theorem 2.2.2.

Step 2: we now proceed inductively. For $j = 2, 3, 4, \dots$

using $f_{j,0} = f_{j-1}$ as a starting vector, obtain the sequence $\{f_{j,n}\}_{n=0}^{\infty}$ converging to f_j , where f_j is the solution to $(K^*K + \lambda_j I)f = K^*g$. Continue until λ_j is less than some pre-assigned value.

Theorem 2.2.2. Let $f_{1,n+1} = f_{1,n} + \alpha_{1,n} W_{1,n}$, $\alpha_{1,n} \in \mathbb{R}$, for each integer $n \geq 0$ and $f_{1,0} \in H_1$ then the sequence $\{f_{1,n}\}_{n=0}^{\infty}$ converges to the function $f = f_1 \in F(K^*)$ that minimizes $Q_{\lambda_1}(f) = \|Kf - g\|^2 + \lambda_1 \|f\|^2$ where

$$W_{1,n} = K^*K f_{1,n} + \lambda f_{1,n} - K^*g,$$

$$\alpha_{1,n} = - \frac{\langle W_{1,n}, W_{1,n} \rangle}{\langle KW_{1,n}, KW_{1,n} \rangle + \lambda_1 \langle W_{1,n}, W_{1,n} \rangle} \quad \text{for } W_{1,n} \neq 0,$$

and

if n_1 is the first n such that $W_{1,n} = 0$

take $\alpha_{1,n} = 0$ and $f_1 = f_{n,1}$ for all $n \geq n_1$.

Proof: From Theorem 2.2.1 we have $f_1 \in R(K^*)$. To choose

$\alpha_{1,n}$ consider $f_{1,n+1} = f_{1,n} + \alpha W_{1,n}$ and

$$\begin{aligned}
 Q_{\lambda_1}(f_{1,n+1}) &= \|K f_{1,n+1} - g\|^2 + \lambda_1 \|f_{1,n+1}\|^2 \\
 &= \langle K^*K f_{1,n+1}, f_{1,n+1} \rangle - 2\langle K^*g, f_{1,n+1} \rangle \\
 (2.2.1) \quad &+ \langle g, g \rangle + \lambda_1 \langle f_{1,n+1}, f_{1,n+1} \rangle \\
 &= \langle K^*K f_{1,n} + \alpha K^*K W_{1,n}, f_{1,n} + \alpha W_{1,n} \rangle \\
 &- 2\langle K^*g, f_{1,n} + \alpha W_{1,n} \rangle + \langle g, g \rangle \\
 &+ \lambda_1 \langle f_{1,n} + \alpha W_{1,n}, f_{1,n} + \alpha W_{1,n} \rangle \\
 &= \langle K^*K f_{1,n}, f_{1,n} \rangle + 2\alpha \langle K^*K f_{1,n}, W_{1,n} \rangle \\
 (2.2.2) \quad &+ \alpha^2 \langle K^*K W_{1,n}, W_{1,n} \rangle - 2\langle K^*g, f_{1,n} \rangle
 \end{aligned}$$

$$\begin{aligned}
& - 2 \alpha \langle K^* g, W_{1,n} \rangle + \langle g, g \rangle + \lambda_1 \langle f_{1,n}, f_{1,n} \rangle \\
& + 2 \lambda_1 \alpha \langle f_{1,n}, W_{1,n} \rangle + \lambda_1 \alpha^2 \langle W_{1,n}, W_{1,n} \rangle.
\end{aligned}$$

We now take the derivative of $Q_{\lambda_1}(f_{1,n+1})$ with respect to α and set the derivative equal to zero:

$$\begin{aligned}
\frac{d}{d\alpha} Q_{\lambda_1}(f_{1,n+1}) &= 2 \langle K^* K f_{1,n}, W_{1,n} \rangle + 2 \alpha \langle K^* K W_{1,n}, W_{1,n} \rangle \\
& - 2 \langle K^* g, W_{1,n} \rangle + 2 \lambda_1 \langle f_{1,n}, W_{1,n} \rangle \\
& + 2 \lambda_1 \alpha \langle W_{1,n}, W_{1,n} \rangle = 0.
\end{aligned}$$

Note also

$$\frac{d^2}{d\alpha^2} Q_{\lambda_1}(f_{1,n+1}) = 2 (\langle K^* K W_{1,n}, W_{1,n} \rangle + \lambda_1 \langle W_{1,n}, W_{1,n} \rangle) > 0$$

for $W_{1,n} \neq 0$. Solving $\frac{d}{d\alpha} Q_{\lambda_1}(f_{1,n+1}) = 0$ for α gives

$$\begin{aligned}
& \alpha (\langle K^* K W_{1,n}, W_{1,n} \rangle + \lambda_1 \langle W_{1,n}, W_{1,n} \rangle) \\
&= \langle K^* g, W_{1,n} \rangle - \langle K^* K f_{1,n}, W_{1,n} \rangle - \lambda_1 \langle f_{1,n}, W_{1,n} \rangle \\
&= - \langle K^* K f_{1,n} + \lambda_1 f_{1,n} - K^* g, W_{1,n} \rangle
\end{aligned}$$

where $W_{1,n} \neq 0$, yielding the result:

$$(2.2.3) \quad \alpha_{1,n} = \alpha = - \frac{\langle K^* K f_{1,n} + \lambda_1 f_{1,n} - K^* g, W_{1,n} \rangle}{\langle K^* K W_{1,n}, W_{1,n} \rangle + \lambda_1 \langle W_{1,n}, W_{1,n} \rangle}.$$

Now we write $f_{1,n+1}$ as

$$f_{1,n+1} = f_{1,n} - \frac{\langle K^* K f_{1,n} + \lambda_1 f_{1,n} - K^* g, W_{1,n} \rangle}{\langle K^* K W_{1,n}, W_{1,n} \rangle + \lambda_1 \langle W_{1,n}, W_{1,n} \rangle} W_{1,n}$$

for $W_{1,n} \neq 0$.

Now we show that $\{Q_{\lambda_1}(f_{1,n})\}_{n=0}^{\infty}$ is a decreasing sequence of positive real numbers. From the expression (2.2.2), replacing $n+1$ with n in (2.2.1) and α with $\alpha_{1,n}$ in both (2.2.2) and (2.2.1) we get the following expression.

$$\begin{aligned}
Q_{\lambda_1}(f_{1,n+1}) - Q_{\lambda_1}(f_{1,n}) &= \langle K^*Kf_{1,n}, f_{1,n} \rangle \\
&+ 2 \alpha_{1,n} \langle K^*Kf_{1,n}, W_{1,n} \rangle + \alpha_{1,n}^2 \langle K^*KW_{1,n}, W_{1,n} \rangle \\
&- 2 \langle K^*g, f_{1,n} \rangle - 2 \alpha_{1,n} \langle K^*g, W_{1,n} \rangle + \langle g, g \rangle + \lambda_1 \langle f_{1,n}, f_{1,n} \rangle \\
&+ 2 \lambda_1 \alpha_{1,n} \langle f_{1,n}, W_{1,n} \rangle + \lambda_1 \alpha_{1,n}^2 \langle W_{1,n}, W_{1,n} \rangle \\
&- \langle K^*Kf_{1,n}, f_{1,n} \rangle + 2 \langle K^*g, f_{1,n} \rangle \\
&- \langle g, g \rangle - \lambda_1 \langle f_{1,n}, f_{1,n} \rangle \\
&= 2 \alpha_{1,n} \langle K^*Kf_{1,n}, W_{1,n} \rangle + \alpha_{1,n}^2 \langle K^*KW_{1,n}, W_{1,n} \rangle \\
&- 2 \alpha_{1,n} \langle K^*g, W_{1,n} \rangle + 2 \lambda_1 \alpha_{1,n} \langle f_{1,n}, W_{1,n} \rangle \\
&+ \lambda_1 \alpha_{1,n}^2 \langle W_{1,n}, W_{1,n} \rangle \\
&= 2 \alpha_{1,n} (\langle K^*Kf_{1,n} - K^*g, W_{1,n} \rangle + \lambda_1 \langle f_{1,n}, W_{1,n} \rangle) \\
&+ \alpha_{1,n}^2 (\langle K^*KW_{1,n}, W_{1,n} \rangle + \lambda_1 \langle W_{1,n}, W_{1,n} \rangle).
\end{aligned}$$

So we have from the expression for $\alpha_{1,n}$

$$\begin{aligned} Q_{\lambda_1}(f_{1,n+1}) - Q_{\lambda_1}(f_{1,n}) &= - \frac{2\langle K*Kf_{1,n} + \lambda_1 f_{1,n} - K*g, W_{1,n} \rangle^2}{\langle K*KW_{1,n}, W_{1,n} \rangle + \lambda_1 \langle W_{1,n}, W_{1,n} \rangle} \\ &\quad + \frac{\langle K*Kf_{1,n} + \lambda_1 f_{1,n} - K*g, W_{1,n} \rangle^2}{\langle K*KW_{1,n}, W_{1,n} \rangle + \lambda_1 \langle W_{1,n}, W_{1,n} \rangle} \\ &= - \frac{\langle K*Kf_{1,n} + \lambda_1 f_{1,n} - K*g, W_{1,n} \rangle^2}{\langle K*KW_{1,n}, W_{1,n} \rangle + \lambda_1 \langle W_{1,n}, W_{1,n} \rangle}. \end{aligned}$$

Therefore, we have $Q_{\lambda_1}(f_{1,n+1}) \leq Q_{\lambda_1}(f_{1,n})$ and equality occurs when $\langle K*Kf_{1,n} + \lambda_1 f_{1,n} - K*g, W_{1,n} \rangle = 0$ for some n .

Note that if $K*Kf_{1,n} + \lambda_1 f_{1,n} - K*g = 0$ for some n , say n_1 , then $f_{1,n_1} = (K*K + \lambda_1 I)^{-1} K*g$ from (1.1.5),

where $p \equiv 0$, is the minimizing value for $Q_{\lambda_1}(f)$. Thus

if $K*Kf_{1,n} + \lambda_1 f_{1,n} - K*g \neq 0$ for $n > n_1$ and

$K*Kf_{1,n_1} + \lambda_1 f_{1,n_1} - K*g = 0$ for $n = n_1$ take $\alpha_{1,n} = 0$

for $n \geq n_1$ and $f_{1,n} = f_1$ for $n \geq n_1$. In general take

$W_{1,n} = K*Kf_{1,n} + \lambda_1 f_{1,n} - K*g$. We have discussed the

case $w_{1,n} \neq 0$ for all n , then (2.2.1) becomes

$$\alpha_{1,n} = - \frac{\langle w_{1,n}, w_{1,n} \rangle}{\langle K^* K w_{1,n}, w_{1,n} \rangle + \lambda_1 \langle w_{1,n}, w_{1,n} \rangle}$$

and

$$(2.2.4) \quad Q_{\lambda_1}(f_{1,n+1}) - Q_{\lambda_1}(f_{1,n}) = - \frac{\langle w_{1,n}, w_{1,n} \rangle^2}{\langle K^* K w_{1,n}, w_{1,n} \rangle + \lambda_1 \langle w_{1,n}, w_{1,n} \rangle}$$

So $Q_{\lambda_1}(f_{1,n+1}) < Q_{\lambda_1}(f_{1,n})$ for all n . We now have

$$\begin{aligned} Q_{\lambda_1}(f_{1,n+1}) &= Q_{\lambda_1}(f_{1,n}) - \frac{\langle w_{1,n}, w_{1,n} \rangle^2}{\langle K^* K w_{1,n}, w_{1,n} \rangle + \lambda_1 \langle w_{1,n}, w_{1,n} \rangle} \\ &= Q_{\lambda_1}(f_{1,0}) - \sum_{k=0}^n \frac{\langle w_{1,k}, w_{1,k} \rangle^2}{\langle K^* K w_{1,k}, w_{1,k} \rangle + \lambda_1 \langle w_{1,k}, w_{1,k} \rangle}. \end{aligned}$$

So $\left\{ \sum_{k=0}^n \frac{\langle w_{1,k}, w_{1,k} \rangle^2}{\langle K^* K w_{1,k}, w_{1,k} \rangle + \lambda_1 \langle w_{1,k}, w_{1,k} \rangle} \right\}_{n=0}^{\infty}$ is an increasing

sequence of positive real numbers bounded above by

$Q_{\lambda_1}(f_{1,0})$ and so converges. Now since

$$\begin{aligned}\langle K^*KW_{1,k}, W_{1,k} \rangle &= \langle KW_{1,k}, KW_{1,k} \rangle \\ &\leq \|K\|^2 \langle W_{1,k}, W_{1,k} \rangle\end{aligned}$$

we have

$$\langle K^*KW_{1,k}, W_{1,k} \rangle + \lambda_1 \langle W_{1,k}, W_{1,k} \rangle \leq (\|K\|^2 + \lambda_1) \langle W_{1,k}, W_{1,k} \rangle$$

and

(2.2.5)

$$\frac{1}{(\|K\|^2 + \lambda_1) \langle W_{1,k}, W_{1,k} \rangle} \leq \frac{1}{\langle K^*KW_{1,k}, W_{1,k} \rangle + \lambda_1 \langle W_{1,k}, W_{1,k} \rangle}.$$

Multiply each side of (2.2.5) by $\langle W_{1,k}, W_{1,k} \rangle^2$ and sum over k , yielding

$$\frac{1}{(\|K\|^2 + \lambda_1)} \sum_{k=0}^{\infty} \langle W_{1,k}, W_{1,k} \rangle$$

$$\leq \sum_{k=0}^{\infty} \frac{\langle W_{1,k}, W_{1,k} \rangle^2}{\langle K^* K W_{1,k}, W_{1,k} \rangle + \lambda_1 \langle W_{1,k}, W_{1,k} \rangle} < \infty.$$

So $\sum_{k=0}^{\infty} \langle W_{1,k}, W_{1,k} \rangle$ converges, thus

$$(2.2.6) \quad \langle W_{1,k}, W_{1,k} \rangle \rightarrow 0 \text{ as } k \rightarrow \infty.$$

We now show $\{f_{1,n}\}_{n=0}^{\infty}$ converges to

$f_1 = (K^*K + \lambda_1 I)^{-1} K^*g$. Once this is done the proof will be complete, since this f_1 minimizes $Q_{\lambda_1}(f)$. Since

$(K^*K + \lambda_1 I)$ is invertible, there exists a positive real number $M_{\lambda_1} = \|(K^*K + \lambda_1 I)^{-1}\|^{-1}$ such that for all n

$$\begin{aligned}
M_{\lambda_1} \|f_{1,n} - f_1\| &\leq \| (K^*K + \lambda_1 I) (f_{1,n} - f_1) \| \\
&= \| K^*K f_{1,n} + \lambda_1 f_{1,n} - (K^*K f_1 + \lambda_1 f_1) \| \\
&= \| K^*K f_{1,n} + \lambda_1 f_{1,n} - K^*g \| \\
&= \| w_{1,n} \| .
\end{aligned}$$

Therefore we have

$$(2.2.7) \quad \|f_{1,n} - f_1\| \leq \frac{\|w_{1,n}\|}{M_{\lambda_1}} \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

from (2.2.6).

So we have shown that $f_{1,n}$ converges to f_1 in norm as $n \rightarrow \infty$. This completes the proof of the theorem.

Now take λ_2 from the sequence $\{\lambda_j\}_{j=1}^{\infty}$ and use $f_1 \in R(K^*)$, from Theorem 2.2.1. as a starting element, i.e. take $f_{2,0} = f_1$, and as in the case for λ_1 construct the sequence $\{f_{2,n}\}_{n=0}^{\infty}$ which converges to $f_2 \in R(K^*)$ such that $f_{2,n+1} = f_{2,n} + \alpha_{1,n} w_{2,n}$ where

$$\alpha_{2,n} = - \frac{\langle W_{2,n}, W_{2,n} \rangle}{\langle K^* K W_{2,n}, W_{2,n} \rangle + \lambda_2 \langle W_{2,n}, W_{2,n} \rangle},$$

and

$$W_{2,n} = K^* K f_{2,n} + \lambda_2 f_{2,n} - K^* g$$

for

$$K^* K f_{2,n} + \lambda_2 f_{2,n} - K^* g \neq 0$$

and $\alpha_{2,n} = 0$ whenever

$$K^* K f_{2,n} + \lambda_2 f_{2,n} - K^* g = 0.$$

In general, use f_{j-1} as a starting element in $R(K^*)$, take $f_{j,0} = f_{j-1}$, $j = 2, 3, 4, \dots$ and construct the sequence

$$(2.2.8) \quad \{f_{j,n}\}_{n=0}^{\infty} \text{ which converges to } f_j$$

such that $f_{j,n+1} = f_{j,n} + \alpha_{j,n} W_{j,n}$

$$\text{where } \alpha_{j,n} = - \frac{\langle W_{j,n}, W_{j,n} \rangle}{\langle K^* K W_{j,n}, W_{j,n} \rangle + \lambda_j \langle W_{j,n}, W_{j,n} \rangle},$$

$$W_{j,n} = K^* K f_{j,n} + \lambda_j f_{j,n} - K^* g$$

for

$$K^* K f_{j,n} + \lambda_j f_{j,n} - K^* g \neq 0$$

and $\alpha_{j,n} = 0$ whenever $K^* K f_{j,n} + \lambda_j f_{j,n} - K^* g = 0$.

From the remarks at the beginning of section 2.2, we need that the sequence defined by (2.2.8) stay in $[N(K)]^\perp$. Since for $j > 1$ and $f_{j,0} = f_{j-1} \in R(K^*)$, we have the following corollary to Theorem 2.2.2..

Corollary 2.2.3. If $j > 1$ then the sequence defined by (2.2.8), $\{f_{j,n}\}_{n=0}^\infty \subset R(K^*) \subset [N(K)]^\perp$.

Theorem 2.2.4. Let f_0 be the least squares solution of minimum norm of $Kf = g$, $f_{1,0} \in H_1$, $\{\{f_{j,n}\}_{n=0}^\infty\}_{j=1}^\infty$ constructed as in (2.2.8) and let $\epsilon > 0$, then there exists positive integers n and j such that

$$\|f_{j,n} - f_o\| < \epsilon.$$

Proof: Follows from $\|f_{j,n} - f_o\| \leq \|f_{j,n} - f_j\| + \|f_j - f_o\|.$

2.3. Perturbations of g

We have that $R(K) + N(K^*) \subset H_2$ is dense in H_2 . For a perturbation of g by an element ϵ belonging to H_2 it is possible that $g + \epsilon$ will fail to be an element of $R(K) + N(K^*)$, in which case the least squares solution of minimum norm if $Kf = g + \epsilon$ would fail to exist. The goal is to determine relationships of the solutions of

$$(K^*K + \lambda I)f = K^*g,$$

$$(K^*K + \lambda I)f = K^*(g + \epsilon)$$

and the least squares solution of minimum norm of $Kf = g$. Then an approximate solution of minimum norm of $Kf = g$ can be found, provided the perturbation of g is not too large.

Theorem 2.3.1. If $f_\lambda \in H_1$ is the solution to

$(K^*K + \lambda I)f = K^*g$ and $f_{\lambda,\epsilon} \in H_1$ is the solution to

$(K^*K + \lambda I)f = K^*(g + \epsilon)$ where ϵ belongs to H_2 then

$$\|f_{\lambda, \epsilon} - f_{\lambda}\| \leq \frac{\|K\|}{\|(K^*K + \lambda I)^{-1}\|^{-1}} \|\epsilon\|$$

Proof: We have that

$$(K^*K + \lambda I) f_{\lambda} = K^*g,$$

$$(K^*K + \lambda I) f_{\lambda, \epsilon} = K^*(g + \epsilon).$$

Subtracting the above equations, we get

$$(K^*K + \lambda I) f_{\lambda, \epsilon} - (K^*K + \lambda I) f_{\lambda} = K^*(g + \epsilon) - K^*g,$$

$$(K^*K + \lambda I) (f_{\lambda, \epsilon} - f_{\lambda}) = K^*\epsilon.$$

Now since $K^*K + \lambda I$ has a bounded inverse, there exists an $M_{\lambda} = \|(K^*K + \lambda I)^{-1}\|^{-1}$ such that

$$M_{\lambda} \|f_{\lambda, \epsilon} - f_{\lambda}\| \leq \|(K^*K + \lambda I) (f_{\lambda, \epsilon} - f_{\lambda})\|$$

$$= \|K^*\epsilon\|$$

$$\leq \|K^*\| \|\epsilon\|$$

$$= \|K\| \|\epsilon\|,$$

so
$$\|f_{\lambda, \epsilon} - f_{\lambda}\| \leq \frac{\|K\|}{M_{\lambda}} \|\epsilon\|.$$

Remark: With reference to Theorem 2.3.1, if we let $\underline{\mu}$ be the minimum (or inf of the) non-negative eigenvalues of K^*K then

$$\|(K^*K + \lambda I)^{-1}\|^{-1} = \underline{\mu} + \lambda.$$

Theorem 2.3.2. If given $\delta > 0$ and

1. f_0 is the least squares solution of minimum norm of $Kf = g$,
2. $f_{j, \epsilon}$ is the solution of $(K^*K + \lambda I)f = K^*(g + \epsilon)$ for each $j, j = 1, 2, \dots$,
3. $f_{j, n, \epsilon}$ defined by (2.2.8),
4. ϵ is an element of H_2 then

$$\|f_{\bar{j}, \bar{n}, \epsilon} - f_0\| \leq \frac{\|K\| \|\epsilon\|}{\|(K^*K + \lambda_{\bar{j}} I)^{-1}\|^{-1}} + \delta \text{ for some } \bar{j} \text{ and } \bar{n}.$$

$$\begin{aligned}
\text{Proof: } \|f_{j,n,\epsilon} - f_0\| &= \|f_{j,n,\epsilon} - f_{j,\epsilon} + f_{j,\epsilon} - f_j + f_j - f_0\| \\
&\leq \|f_{j,n,\epsilon} - f_{j,\epsilon}\| + \|f_{j,\epsilon} - f_j\| + \|f_j - f_0\| \\
&\leq \frac{\|K\|\|\epsilon\|}{\|(K^*K + \lambda_j I)^{-1}\|^{-1}} + \|f_{j,n,\epsilon} - f_{j,\epsilon}\| + \|f_j - f_0\|.
\end{aligned}$$

Now there exists a j and n such that

$$\|f_{j,n,\epsilon} - f_{j,\epsilon}\| < \frac{\delta}{2} \quad \text{and} \quad \|f_j - f_0\| < \frac{\delta}{2}, \quad \text{say for } \bar{j} = j$$

$\bar{n} = n$ we have

$$\|f_{\bar{j},\bar{n},\epsilon} - f_0\| \leq \frac{\|K\|\|\epsilon\|}{\|(K^*K + \lambda I)^{-1}\|^{-1}} + \delta.$$

2.4. Some Error Bounds and Properties of λ

Now we will prove some theorems which will yield some bounds on $\|f_\lambda - f_0\|$ in terms of $\lambda > 0$ where $f_\lambda \in H_1$ is a solution to $(K^*K + \lambda I)f = K^*g$ and f_0 is the least squares solution of minimum norm of $Kf = g$. The bound on $\|f_\lambda - f_0\|$ will depend on knowledge of the singular system

$(U_n, V_n; \mu_n)$ for the compact operator K which is described below. The notation and results follow that of Strand

[20, page 19], [21] and Tricomi [24]. Define

$\sigma_0(K^*K) = \sigma_0(KK^*) = \{\gamma_n : \gamma_n \text{ is an eigenvalue of } K^*K, \gamma_n > 0, n \in N_k\}$ where $N_k = \{1, 2, \dots, k\}$, (k may be ∞).

Let

$$U = \{U_n : KK^*U_n = \gamma_n U_n, n \in N_k\}$$

and

$$V = \{V_n : K^*KV_n = \gamma_n V_n, n \in N_k\}.$$

Since K^*K is compact, symmetric and non-negative definite, assume that the γ_n are ordered such that

$\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n \geq \dots > 0$. Define $\mu_n = \gamma_n^{-1/2}$, $n \in N_k$.

Then we have $0 < \mu_1 \leq \mu_2 \leq \dots$,

$$U_n = \mu_n KV_n$$

and

$$V_n = \mu_n K^* U_n.$$

Now in this notation we can state Picard's Theorem, a proof of which is given in Strand [20, page 25].

Theorem 2.4.1. (Picard). Let $(U_n, V_n; \mu_n)$ be a singular system for the compact operator $K: H_1 \rightarrow H_2$ and $\bar{g} \in H_2$.

Then the equation $Kf = \bar{g}$ has a solution $f_0 \in H_1$ if

and only if $\sum_{n \in N_m} \mu_n^2 |\langle \bar{g}, U_n \rangle|^2 < \infty$ and $\bar{g} \in \overline{R(K)}$. Also

for $\bar{g} \in R(K)$ we have $f_0 = \sum_{n \in N_m} \langle \bar{g}, U_n \rangle \mu_n V_n$.

Strand [20, page 26] notes that for

$g = g_1 + g_2$, $g_1 \in R(K)$ and $g_2 \in N(K^*)$ then

$f_0 = \sum_{n \in N_m} \langle g, U_n \rangle \mu_n V_n$ is the least squares solution of

minimum norm of the equation $Kf = g$.

Theorem 2.4.2. Let

1. K^*K have finite non-zero spectrum σ_0 , $N_K = m$,
2. $f_\lambda \in H_1$ be the least squares solution of minimum norm of $Kf = g$ then

$$\|f_\lambda - f_0\| \leq \frac{\lambda}{\lambda + \gamma_m} \sqrt{\sum_{i=1}^m |\langle g, u_i \rangle|^2 \mu_i^2}.$$

Proof:

$$\begin{aligned} \|f_\lambda - f_0\|^2 &= \|(K^*K + \lambda I)^{-1} K^*g - f_0\|^2 \\ &= \|(K^*K + \lambda I)^{-1} K^*K f_0 - f_0\|^2 \\ &= \|(K^*K + \lambda I)^{-1} (K^*K - (K^*K + \lambda I)) f_0\|^2 \\ &= \lambda^2 \|(K^*K + \lambda I)^{-1} f_0\|^2 \\ &= \lambda^2 \|(K^*K + \lambda I)^{-1} \sum_{n=1}^m \langle g, u_n \rangle \mu_n v_n\|^2 \\ &= \lambda^2 \left\| \sum_{n=1}^m \langle g, u_n \rangle \mu_n (K^*K + \lambda I)^{-1} v_n \right\|^2, \end{aligned}$$

and since $(K^*K + \lambda I)^{-1} V_n = \frac{V_n}{\gamma_n + \lambda}$ we have

$$\begin{aligned} \|f_\lambda - f_0\|^2 &= \lambda^2 \left\| \sum_{n=1}^m \langle g, U_n \rangle \mu_n \frac{V_n}{\gamma_n + \lambda} \right\|^2 \\ &= \lambda^2 \sum_{n=1}^m \frac{|\langle g, U_n \rangle|^2 \mu_n^2}{(\gamma_n + \lambda)^2} \\ &\leq \left(\frac{\lambda}{\gamma_m + \lambda} \right)^2 \sum_{n=1}^m |\langle g, U_n \rangle|^2 \mu_n^2. \end{aligned}$$

Now we take the square root of each side of the above inequality, giving

$$\|f_\lambda - f_0\| \leq \frac{\lambda}{\gamma_m + \lambda} \sqrt{\sum_{n=1}^m |\langle g, U_n \rangle|^2 \mu_n^2}.$$

This completes the proof of the theorem.

Theorem 2.4.2 can be extended to the case where K^*K does not have a finite non-zero spectrum.

Theorem 2.4.3. If under the same hypotheses as Theorem 2.4.2, except that K^*K has non-finite spectrum and an integer $N > 0$, a real number $\delta > 0$ are known such that

$$\sum_{n=N+1}^{\infty} |\langle g, u_n \rangle|^2 \mu_n^2 \leq \delta$$

then

$$\|f_\lambda - f_0\| \leq \sqrt{\left(\frac{\lambda}{\lambda + \gamma_N}\right)^2 \sum_{n=1}^N |\langle g, u_n \rangle|^2 \mu_n^2 + \delta}.$$

Proof: Similar to the proof of Theorem 2.4.2 we have

$$\begin{aligned} \|f_\lambda - f_0\|^2 &= \lambda^2 \left\| \sum_{n=1}^{\infty} \langle g, u_n \rangle \mu_n \frac{v_n}{\gamma_n + \lambda} \right\|^2 \\ &= \lambda^2 \sum_{n=1}^{\infty} |\langle g, u_n \rangle|^2 \frac{\mu_n^2}{(\gamma_n + \lambda)^2} \end{aligned}$$

$$\begin{aligned}
&= \lambda^2 \sum_{n=1}^N |\langle g, u_n \rangle|^2 \frac{\mu_n^2}{(\gamma_n + \lambda)^2} \\
&\quad + \sum_{n=N+1}^{\infty} |\langle g, u_n \rangle|^2 \mu_n^2 \left(\frac{\lambda}{\gamma_n + \lambda} \right)^2 \\
&\leq \left(\frac{\lambda}{\gamma_N + \lambda} \right)^2 \sum_{n=1}^N |\langle g, u_n \rangle|^2 \mu_n^2 + \sum_{n=N+1}^{\infty} |\langle g, u_n \rangle|^2 \mu_n^2 \\
&\leq \left(\frac{\lambda}{\gamma_N + \lambda} \right)^2 \sum_{n=1}^N |\langle g, u_n \rangle|^2 \mu_n^2 + \delta.
\end{aligned}$$

Now we take the square root of each side of the above inequality, giving

$$\|f_\lambda - f_0\| \leq \sqrt{\left(\frac{\lambda}{\gamma_N + \lambda} \right)^2 \sum_{n=1}^N |\langle g, u_n \rangle|^2 \mu_n^2 + \delta}.$$

Remark: Let

$$A_\lambda = \begin{cases} \frac{\lambda}{\gamma_n + \lambda} \sqrt{\sum_{n=1}^m |\langle g, u_n \rangle|^2 \mu_n^2} & \text{if } m < \infty \\ \sqrt{\left(\frac{\lambda}{\gamma_n + \lambda}\right)^2 \sum_{n=1}^N |\langle g, u_n \rangle|^2 + \delta} & \text{otherwise.} \end{cases}$$

From (2.2.7) and (2.2.8) we have

$$\begin{aligned} (2.4.1) \quad \|f_{j,n} - f_0\| &\leq \|f_{j,n} - f_j\| + \|f_j - f_0\| \\ &\leq \frac{\|w_{j,n}\|}{\|(K^*K + \lambda_j I)^{-1}\|^{-1}} + A_{\lambda_j} \end{aligned}$$

Remark: Since the eigenvectors of K^*K associated with its non-zero eigenvalues span $N(K)^\perp$ and $f_0, f_j, f_{j,n}$ belong to $N(K)^\perp$ for $j > 1$, we have

$$\gamma \|f_j - f_0\| \leq \|K^*K(f_j - f_0)\|$$

and

$$(\gamma + \lambda_j) \|f_{j,n} - f_j\| \leq \| (K^*K + \lambda I) (f_{j,n} - f_j) \|$$

where $\gamma = \inf_{\gamma_n \in \sigma_0(K^*K)} \gamma_n$ (see Helmborg [7, page 225]).

Since $K^*K(f_j - f_0) = \lambda_j f_j$ and
 $(K^*K + \lambda I) (f_{j,n} - f_j) = w_{j,n}$ we have

$$\|f_j - f_0\| \leq \frac{\lambda_j \|f_j\|}{\gamma}$$

and

$$\|f_{j,n} - f_j\| \leq \frac{\|w_{j,n}\|}{\gamma + \lambda_j}.$$

Thus (2.4.1) can be rewritten as

$$(2.4.2) \quad \|f_{j,n} - f_0\| \leq \frac{\|w_{j,n}\|}{\gamma + \lambda_j} + \frac{\lambda_j \|f_j\|}{\gamma}$$

giving non- a priori error bounds when $\gamma \neq 0$.

Using the same techniques as in the above remark, we can replace $\|(K^*K + \lambda I)^{-1}\|^{-1}$ with $\gamma + \lambda$ in Theorems 2.3.1 and 2.3.2.

Theorem 2.4.4. If f_λ is the solution to $(K^*K + \lambda I)f = K^*g$ then $\|f_\lambda\|$ is strictly decreasing in $\lambda > 0$.

Proof: Since $K^*K: H_1 \rightarrow H_1$ is a symmetric, non-negative definite and compact linear operator, there is an orthonormal bases $\{e_i\}$ for H_1 such that $(K^*K + \lambda I)e_i = (\gamma_i + \lambda)e_i$ for all i . Since K^*K is compact and non-negative we may assume $\gamma_i \geq \gamma_{i+1} \geq 0$ for all i . So we have that $K^*K + \lambda I \simeq \text{diag}(\gamma_1 + \lambda, \gamma_2 + \lambda, \dots)$ relative to $\{e_i\}$. Thus

$$(K^*K + \lambda I)^{-1} \simeq \text{diag}\left(\frac{1}{\gamma_1 + \lambda}, \frac{1}{\gamma_2 + \lambda}, \dots\right).$$

Since $f_\lambda = (K^*K + \lambda I)^{-1}K^*g$, it follows that

$$f_\lambda = \sum_i \frac{\langle K^*g, e_i \rangle}{\gamma_i + \lambda} e_i$$

and

$$\|f_\lambda\|^2 = \sum_i \frac{|\langle K^*g, e_i \rangle|^2}{(\gamma_i + \lambda)^2}.$$

The theorem follows immediately.

We now prove that $\|Kf_\lambda\|$ is increasing as $\lambda > 0$ is decreasing. The theorem following is of interest, in its own right, when compared to Theorem 2.4.4.

Lemma 2.4.5. If $\alpha > 0$, $\beta > 0$, $f_\alpha \in R(K^*)$ is the solution to $(K^*K + \alpha I)f = K^*g$ and $f_\beta \in R(K^*)$ is the solution to $(K^*K + \beta I)f = K^*g$ then

$$\langle f_\alpha, f_\beta \rangle \geq \frac{\alpha \langle f_\alpha, f_\alpha \rangle + \beta \langle f_\beta, f_\beta \rangle}{\alpha + \beta}$$

and

$$\langle Kf_\alpha, Kf_\beta \rangle \geq \frac{\alpha \langle Kf_\alpha, Kf_\alpha \rangle + \beta \langle Kf_\beta, Kf_\beta \rangle}{\alpha + \beta}$$

Proof: We have the result that

$$K^*Kf_\alpha + \alpha f_\alpha = K^*Kf_\beta + \beta f_\beta$$

which implies

$$(2.4.3) \quad K^*K(f_\alpha - f_\beta) = \beta f_\beta - \alpha f_\alpha.$$

Now taking the inner product of each side of (2.4.3) with $f_\alpha - f_\beta$ we obtain the following result.

$$\begin{aligned} 0 &\leq \langle K^*K(f_\alpha - f_\beta), f_\alpha - f_\beta \rangle \\ &= \langle \beta f_\beta - \alpha f_\alpha, f_\alpha - f_\beta \rangle \\ &= \beta \langle f_\beta, f_\alpha - f_\beta \rangle - \alpha \langle f_\alpha, f_\alpha - f_\beta \rangle \\ &= \beta \langle f_\beta, f_\alpha \rangle - \beta \langle f_\beta, f_\beta \rangle - \alpha \langle f_\alpha, f_\alpha \rangle + \alpha \langle f_\alpha, f_\beta \rangle \\ &= (\alpha + \beta) \langle f_\alpha, f_\beta \rangle - (\beta \langle f_\beta, f_\beta \rangle + \alpha \langle f_\alpha, f_\alpha \rangle). \end{aligned}$$

So we have

$$\frac{\beta \langle f_\beta, f_\beta \rangle + \alpha \langle f_\alpha, f_\alpha \rangle}{\alpha + \beta} \leq \langle f_\beta, f_\alpha \rangle.$$

The result

$$\frac{\beta \langle Kf_\beta, Kf_\beta \rangle + \alpha \langle Kf_\alpha, Kf_\alpha \rangle}{\alpha + \beta} \leq \langle Kf_\alpha, Kf_\beta \rangle$$

has a similar proof, starting with

$$K^*K(K^*K(f_\alpha - f_\beta)) = K^*K(\beta f_\beta - \alpha f_\alpha)$$

and then taking the inner product of each side with

$$f_\alpha - f_\beta.$$

Theorem 2.4.5. If $0 < \beta < \alpha$, $f_\alpha \in R(K^*)$ is the solution to $(K^*K + \alpha I)f = K^*g$ and $f_\beta \in R(K^*)$ is the solution to $(K^*K + \beta I)f = K^*g$ then $\|Kf_\beta\| \geq \|Kf_\alpha\|$.

Proof: Since we have

$$(K^*K + \alpha I)f_\alpha = (K^*K + \beta I)f_\beta,$$

take the inner product of each side with f_α yielding the following result.

$$\langle K^*Kf_\alpha + \alpha f_\alpha, f_\alpha \rangle = \langle K^*Kf_\beta + \beta f_\beta, f_\alpha \rangle,$$

$$\langle K^*Kf_\alpha, f_\alpha \rangle + \alpha \langle f_\alpha, f_\alpha \rangle = \langle K^*Kf_\beta, f_\alpha \rangle + \beta \langle f_\beta, f_\alpha \rangle,$$

$$\langle Kf_\alpha, Kf_\alpha \rangle + \alpha \langle f_\alpha, f_\alpha \rangle = \langle Kf_\beta, Kf_\alpha \rangle + \beta \langle f_\beta, f_\alpha \rangle.$$

In a similar way, taking the inner product with f_β yields:

$$\langle Kf_\alpha, Kf_\beta \rangle + \alpha \langle f_\alpha, f_\beta \rangle = \langle Kf_\beta, Kf_\beta \rangle + \beta \langle f_\beta, f_\beta \rangle.$$

Now using the previous lemma we have

$$\begin{aligned} \langle Kf_\alpha, Kf_\alpha \rangle + \alpha \langle f_\alpha, f_\alpha \rangle &= \langle Kf_\beta, Kf_\alpha \rangle + \beta \langle f_\beta, f_\alpha \rangle \\ &\geq \frac{\beta \langle Kf_\beta, Kf_\beta \rangle + \alpha \langle Kf_\alpha, Kf_\alpha \rangle}{\alpha + \beta} + \beta \frac{(\beta \langle f_\beta, f_\beta \rangle + \alpha \langle f_\alpha, f_\alpha \rangle)}{\alpha + \beta} \end{aligned}$$

When we multiply each side of the previous result by $\alpha + \beta$ and simplify the algebraic expression, we obtain

(2.4.4)

$$\beta \langle Kf_{\alpha}, Kf_{\alpha} \rangle + \alpha^2 \langle f_{\alpha}, f_{\alpha} \rangle \geq \beta \langle Kf_{\beta}, Kf_{\beta} \rangle + \beta^2 \langle f_{\beta}, f_{\beta} \rangle.$$

In a similar way from

$$\langle Kf_{\alpha}, Kf_{\beta} \rangle + \alpha \langle f_{\alpha}, f_{\beta} \rangle = \langle Kf_{\beta}, Kf_{\alpha} \rangle + \beta \langle f_{\beta}, f_{\alpha} \rangle$$

we obtain:

(2.4.5)

$$\alpha \langle Kf_{\beta}, Kf_{\beta} \rangle + \beta^2 \langle f_{\beta}, f_{\beta} \rangle \geq \alpha \langle Kf_{\alpha}, Kf_{\alpha} \rangle + \alpha^2 \langle f_{\alpha}, f_{\alpha} \rangle.$$

Now add equations (2.4.4) and (2.4.5), obtaining the inequality:

$$(2.4.6) \quad \beta \langle Kf_{\alpha}, Kf_{\alpha} \rangle + \alpha \langle Kf_{\beta}, Kf_{\beta} \rangle \geq$$

$$\beta \langle Kf_{\beta}, Kf_{\beta} \rangle + \alpha \langle Kf_{\alpha}, Kf_{\alpha} \rangle.$$

So (2.4.6) yields the result:

$$(\alpha - \beta) \langle Kf_\beta, Kf_\beta \rangle \geq (\alpha - \beta) \langle Kf_\alpha, Kf_\alpha \rangle.$$

Since $0 < \beta < \alpha$ we have

$$\langle Kf_\beta, Kf_\beta \rangle \geq \langle Kf_\alpha, Kf_\alpha \rangle.$$

Thus we have $\|Kf_\beta\| \geq \|Kf_\alpha\|$ and the theorem.

3. IMPLEMENTING THE ALGORITHM

3.1. Calculation of a Starting Element

For the basic algorithm described in section 3.2 we require an initial choice for λ , λ_1 , and any starting element $f_{1,0} \in H_1$.

In practice we will choose any initial starting element $f \in H_1$ such that $f \neq 0$ and $\|K*Kf\| \neq 0$. Then we determine an actual starting vector $f_{1,0} \in H_1$ and a initial λ , λ_1 , such that $f_{1,0}$ is, in some sense, close to a solution of $K*Kf + \lambda_1 f = K*g$.

Now to choose the actual $f_{1,0}$, consider the element

$$f_s = \frac{K*Kf}{\|K*Kf\|} \in R(K^*).$$

The element f_s is chosen normalized so as to control the size of the norm of the actual starting element $f_{1,0}$, which will be either f_s or $-f_s$.

Choose $\lambda_1 \geq 0$ so that

$$\hat{Q}_\lambda(f_s) = \|(K*K + \lambda I)f_s - K*g\|^2$$

is minimized. Note that if $\hat{Q}_{\lambda_1}(f_s) = 0$ for some choice for $\lambda_1 > 0$ then we would have obtained an equation

$$(K^*K + \lambda_1 I)f = K^*g$$

which has f_s as its solution.

To minimize $\hat{Q}_\lambda(f_s)$ with respect to λ , we write $\hat{Q}_\lambda(f_s)$ as

$$\begin{aligned}\hat{Q}_\lambda(f_s) &= \|K^*Kf_s - K^*g\|^2 \\ &+ 2\lambda \langle f_s, K^*Kf_s - K^*g \rangle + \lambda^2 \|f_s\|^2.\end{aligned}$$

Taking the derivative $\hat{Q}_\lambda(f_s)$ with respect to λ and setting $\frac{d\hat{Q}_\lambda(f_s)}{d\lambda} = 0$ we get the result

$$\begin{aligned}\frac{d\hat{Q}_\lambda(f_s)}{d\lambda} &= 2\langle f_s, K^*Kf_s - K^*g \rangle + 2\lambda \|f_s\|^2 \\ &= 2\langle f_s, K^*Kf_s - K^*g \rangle + 2\lambda = 0.\end{aligned}$$

Note also that $\frac{d^2}{d\lambda} \hat{Q}(f_s) = 2\langle f_s, f_s \rangle > 0$, so that the

solution to $\frac{d}{d\lambda} \hat{Q}(f_s) = 0$ yields a minimum.

By solving $\frac{d}{d\lambda} \hat{Q}_\lambda(f_s) = 0$ for $\lambda = \bar{\lambda}$, the minimum,

we have the result:

$$\begin{aligned}
 (3.1.1) \quad \bar{\lambda} &= - \langle f_s, K^*Kf_s - K^*g \rangle \\
 &= \langle g - Kf_s, Kf_s \rangle \\
 &= \langle Kf_s, g \rangle - \langle Kf_s, Kf_s \rangle.
 \end{aligned}$$

By substituting $\lambda = \bar{\lambda}$ in $\hat{Q}_\lambda(f_s)$ we obtain

$$\hat{Q}_{\bar{\lambda}}(f_s) = \|K^*Kf_s - K^*g\|^2 - \langle Kf_s, g - Kf_s \rangle^2.$$

For the choice for λ_1 we need $\lambda_1 \geq 0$, so consider the following three cases:

Case 1. If $\bar{\lambda} = 0$, take $\lambda_1 = \bar{\lambda}$ and we have computed an exact solution, $f_s \in R(K^*) \subset N(K)^\perp$, to $K^*Kf = K^*g$.

Since K^* is one-to-one on the range of K we have that

$$K^*(Kf_s - g_1) = 0,$$

so $Kf_s = g_1$. Thus we have computed the least squares solution of minimum norm, $f_s = f_o$, of $Kf = g$.

Case 2. If $\bar{\lambda} > 0$ then take $\lambda_1 = \bar{\lambda}$ and $f_{1,o} = f_s$ for a starting element.

Case 3. If $\bar{\lambda} < 0$ then replace f_s with $-f_s$ in (3.1.1), since then the new value of $\bar{\lambda}$ is

$$\begin{aligned} (3.1.2) \quad \bar{\lambda} &= \langle K(-f_s), g - K(-f_s) \rangle \\ &= \langle -Kf_s, g + Kf_s \rangle \\ &= -\langle Kf_o, g \rangle - \langle Kf_o, Kf_o \rangle, \end{aligned}$$

as compared with equation (3.1.1). So if $\langle Kf, g - Kf \rangle$

is negative for f_o and positive for $-f_o$ take

$$f_{1,o} = -f_s \quad \text{and} \quad \lambda_1 = \bar{\lambda} = -\langle Kf_o, g \rangle - \langle Kf_o, Kf_o \rangle.$$

If $\langle Kf_o, g \rangle = 0$ then a new starting element f must be chosen.

Now from here on suppose that $\langle Kf_s, g \rangle \neq 0$.

If $\langle Kf, g - Kf \rangle$ is negative for both f_s and $-f_s$ then from (3.1.1) and (3.1.2) there exists a real number $a > 1$ such that for the scaled equation

$$Kf = ag$$

we have $\bar{\lambda} = \langle Kf, ag - Kf \rangle \geq 0$ for f one of the elements f_s or $-f_s$. In the above computation it was critical that $\langle Kf_s, g \rangle \neq 0$. So we take $\lambda_1 = \bar{\lambda}$ and $f_{1,0}$ to be the element f_s or $-f_s$ which makes $\bar{\lambda} \geq 0$. The new equation $Kf = ag$ is then solved for af_0 , the least squares solution of minimum norm of $Kf = ag$, and scaled back to the least squares solution of minimum norm, f_0 , of the original equation $Kf = g$ at the end of the algorithm.

In summary, for each $f \neq 0$ where $\|K^*Kf\| \neq 0$ and $f \in H_1$ a starting element $f_{1,0} \in R(K^*)$ and a $\lambda_1 \geq 0$ can be chosen so that $\|(K^*K + \lambda I)f_s - aK^*g\|$ is a minimum (here $a \geq 1$). Thus we start as close as possible (in norm) to a solution of $(K^*K + \lambda I)f - aK^*g = 0$.

Once λ_1 is chosen, a sequence $\{\lambda_j\}_{j=1}^{\infty}$ can be constructed such that $\lambda_j > 0$, $\lambda_j \rightarrow 0$ as $j \rightarrow \infty$, by taking $\lambda_{j+1} = x\lambda_j$ where $0 < x < 1$. In practice the sequence $\{\lambda_j\}_{j=1}^{\infty}$ is terminated when λ_j is less than some pre-assigned value μ . For the first λ_j such that $\lambda_j \leq \mu$ take $\lambda_j = \mu$.

x is determined by numerical experimentation.

3.2. The Sequence $\{f_{j,n}\}_{n=0}^{\infty}$

The sequence $\{f_{j,n}\}_{n=0}^{\infty}$, for j fixed, is constructed by taking

$$f_{j,n+1} = f_{j,n} + \alpha_{j,n} W_{j,n}$$

where $\alpha_{j,n}$ and $W_{j,n}$ are given by (2.2.8). The construction of the sequence $\{f_{j,n}\}_{n=0}^{\infty}$ is terminated when a measure of convergence fails, due to rounding errors in computation, to give an approximate solution f_j to the equation $(K^*K + \lambda_j I)f = K^*g$ (possibly scaled). The measure of convergence used is to note that from (2.2.4),

$$Q_{\lambda_j}(f_{j,n}) - Q_{\lambda_j}(f_{j,n+1}) = \frac{\langle W_{j,n}, W_{j,n} \rangle^2}{\langle KW_{j,n}, KW_{j,n} \rangle + \lambda_j \langle W_{j,n}, W_{j,n} \rangle}$$

for $W_{j,n} \neq 0$, i.e. $Q_{\lambda_j}(f_{j,n+1}) < Q_{\lambda_j}(f_{j,n})$, an approximation to f_j is chosen to be the least value of n for which $Q_{\lambda_j}(f_{j,n+1}) < Q(f_{j,n})$ fails in the computations, or $\|W_{j,n}\|$ is less than some pre-assigned value; whichever occurs first.

We use $\|W_{j,n}\|$ as a measure of $\|f_{j,n} - f_o\|$, since

$$\|f_{j,n} - f_o\| \leq \frac{\|W_{j,n}\|}{\|(K^*K + \lambda I)^{-1}\|^{-1}} + \|f_j - f_o\|$$

(see 2.4.1). Ideally we first choose λ so that $\|f_j - f_o\|$ is small and then make $\|W_{j,n}\|$ small. In practice it is difficult to make $\|f_j - f_o\|$ small without some knowledge of the solution.

3.3. A Flow Diagram of the Basic Algorithm

The following diagram, given in Figure 3.3.1, is a basic description of the algorithm to find the least squares solution of minimum norm of the operator equation $Kf = g$. The flow diagram describes the program in Appendix A for matrix equations and the subroutine BITER for discretized integral equations in Appendix B.

Figure 3.3.1. Flow Chart of the Basic Algorithm

Figure 3.3.1. (Continued)

Figure 3.3.1. (Continued)

Figure 3.3.1. (Continued)

Figure 3.3.1. (Continued)

Figure 3.3.1. (Continued)

4. EXAMPLES

4.1. Matrix Examples

Let the real Hilbert spaces $H_1 = E^n$, $H_2 = E^m$ where $E^n = \{(x_1, \dots, x_n)^T : x_i \text{ real}\}$ with the inner product

$$\langle x, y \rangle = \sum_{i=1}^n x_i y_i, \quad x, y \in E^n$$

be given. In this finite dimensional case each linear transformation $A: H_1 \rightarrow H_2$ has a unique matrix representation

$$A = (a_{i,j})_{\substack{i=1,m \\ j=1,n}}$$

with respect to each bases $\{v_1, v_2, \dots, v_n\}$ in H_1 and $\{w_1, w_2, \dots, w_m\}$ in H_2 . We shall always take our bases vectors to be the canonical bases. Since H_1 and H_2 are finite dimensional, all linear transformations are bounded and compact. Since A is of finite rank we have $R(A) = \overline{R(A)}$, so the algorithm (see section 2.2) will

converge to the least squares solution of minimum norm, x_0 , of $Ax = g$ for all $g \in H_2$.

Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 3 & 3 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 1 & 1 \end{bmatrix}.$$

Then A has rank three.

Consider $Ax = g$ for the following two cases.

Case 1. $g_1 = (10, 12, 13, 22, 43, 35)^T \in R(A)$.

Case 2.

$$\begin{aligned} g_2 &= g_1 + (-5, -2, -2, 1, 1, 1)^T \\ &= (5, 10, 11, 23, 24, 36)^T \end{aligned}$$

where $(-5, -2, -2, 1, 1, 1)^T \in N(A)$.

In both Case 1 and Case 2 the least squares solution of minimum norm, x_0 , of $Ax = g_i$, $i = 1, 2$ is.

$$\vec{x}_0 = (17/6, 43/12, 43/12, 29/6, 49/12, 49/12) \in N(A)^\perp.$$

$\vec{x}_0$ was constructed by finding a basis

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

for $N(A)$. Then a basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{is found for } N(A)^\perp.$$

Since $\vec{x}_0$ is contained in $N(A)^\perp$, $\vec{x}_0$ must be a linear combination of the above three vectors. Therefore to solve $Ax = g_i$ we need only solve

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_2 \\ -x_1 + x_2 + x_3 \\ x_3 \\ x_3 \end{bmatrix} = g_i, \quad i = 1, 2 \quad .$$

The algorithm in Appendix A was used to find the approximate least squares solution of minimum norm for Case 1 and Case 2 with the results given in Table 1. Comparisons with the actual solution were computed separately.

For Run 1 Case 1 we will satisfy the error bounds of Statement (2.4.1) so that $\|f_{j,n} - f_o\| < 10^{-6}$. We will assume that we know the eigenvalues of $A^T A$, which were computed to be 56.96, 4.72, 2.32, 0, 0, 0. Also we will assume that we have an approximation for $\|f_o\|$, the norm of the least squares solution of minimum norm, say 9.5. Now from (2.4.1) we have

$$\|f_{j,n} - f_o\| \leq \frac{\|w_{\lambda_j, n}\|}{\lambda_j + 2.32} + \frac{9.5\lambda_j}{\lambda_j + 2.32}$$

We will choose terminal lamda, $\mu = \lambda_j$ and $\|w_{j,n}\|$ so that

$$\frac{9.5\mu}{\mu + 2.32} < \frac{10^{-6}}{2}$$

and

$$\frac{\|w_{j,n}\|}{\mu + 2.32} < \frac{10^{-6}}{2} .$$

We actually used $\mu = 1. \times 10^{-7}$ and made $\|w_{j,n}\|^2 \leq 10^{-12}$.

The results of several runs of Case 1 and Case 2 are given in Table 1. For each run we used the initial starting vector $F = (1., 1., 1., 0, 0, 0)^T$. We computed

$$\|A^T g_i\| \approx 470.14.$$

Table 1. Matrix Example

	Run 1	Run 2	Run 3	Run 4
g	g_1	g_1	g_1	g_2
M	400	400	400	400
Tλ	1.(-7)	1.(-16)	1.(-16)	1.(-16)
W1	1.(-12)	1.(-10)	1.(-20)	1.(-10)
λm	1.(-5)	1.(-5)	1.(-5)	1.(-5)
Iλ	411.68	411.68	411.68	411.68
EN	4.2(-7)	3.3(-6)	3.3(-11)	3.3(-6)
E1	1.(-6)	4.4(-6)	4.4(-11)	4.4(-6)
E2	1.(-6)	4.4(-6)	4.4(-11)	4.4(-6)
E3	2.3(-7)	1.9(-6)	1.8(-11)	1.9(-6)
W2	8.8(-7)	8.6(-6)	8.7(-11)	8.6(-6)
Re	6.5(-7)	5.(-6)	5.1(-11)	6. + 2.(-12)
TI	132	104	293	104
Ex	3.19	2.60	6.93	1.66

M = maximum number of allowable iterations for a given step.

Tλ = terminal Lambda.

W1 = terminal choice for $\|w_{j,n}\|^2$.

λm = Lambda multiplier.

Iλ = computed initial λ_1 , approximately.

EN = computed $\|f_{T\lambda,n} - f_o\|$.

E1 = error bound computed from (2.4.1).

E2 = error bound computed from (2.4.2).

E3 = computed $\max |f_{T\lambda,n} - f_o|$.

W2 = computed $\|(A^T A + \lambda I) f_{T\lambda,n} - A^T g\|$.

Re = computed $\|(A f_{T\lambda,n} - g)\|$.

TI = total iterations.

Ex = execution time, seconds, WATFIVE.

4.2. An Inner Product Space

We will consider the real n -dimensional space R^n with scalar field the real numbers. Define a mapping from $R^n \times R^n$ to R by

$$(u, v) \rightarrow \langle u, v \rangle = \sum_{j=1}^n T_j u_j v_j$$

where $u = (u_1, u_2, \dots, u_n)^T$, $v = (v_1, v_2, \dots, v_n)^T$ and $T_j > 0$, $T_j \in R$ for all $j = 1, \dots, n$. Thus we have a Hilbert space, which will be denoted by H_T^n . Let $\{e_1^n, e_2^n, \dots, e_n^n\}$ be the orthogonal (not necessarily normalized) bases for H_T^n where $e_j^n = (0, 0, \dots, 1_j, \dots, 0)^T$, $j = 1, n$.

If $A : H_T^m \rightarrow H_S^n$ then $A^* : H_S^n \rightarrow H_T^m$ will denote the adjoint.

Theorem 4.2.1. If $A : H_T^m \rightarrow H_S^n$ is $(T_j a_{i,j})_{\substack{i=1,n \\ j=1,m}}$ then

$$A^* = (S_i a_{i,j})_{\substack{j=1,m \\ i=1,n}}$$

Proof: We will show that the above A^* is the adjoint of A .

$$\begin{aligned}
 \langle Ae_k^m, e_l^n \rangle &= \left\langle \sum_{i=1}^n T_{k i, k} a_{i, k}^n e_l^n \right\rangle \\
 &= T_{k l, k} a_{l, k}^n \langle e_l^n, e_l^n \rangle \\
 &= T_{k l} S_{l, k} a_{l, k}^n
 \end{aligned}$$

and

$$\begin{aligned}
 \langle e_k^m, A^* e_l^n \rangle &= \left\langle e_k^m, \sum_{j=1}^m S_{l j, j} a_{l, j}^m e_j^m \right\rangle \\
 &= S_{l l, k} a_{l, k}^m \langle e_k^m, e_k^m \rangle \\
 &= S_{l k} T_{k l} a_{l, k}^m \\
 &= \langle Ae_k^m, e_l^n \rangle
 \end{aligned}$$

for all k and l , therefore A^* is the adjoint of A .

4.3. Discretizing Integral Equations

We will obtain a discretized form of Fredholm integral equations of the first kind,

$$(4.3.1) \quad Kf(y) = \int_0^1 k(y,x) f(x) dx = g(y), \quad 0 \leq y \leq 1$$

where $g \in R(K) + N(K^*) \subset L_2[0,1]$, $f \in L_2[0,1]$ and $k(y,x)$ is square integrable with respect to the product Lebesgue measure on $[0,1] \times [0,1]$.

Akhiezer and Glazman [1] prove that if

$$\int_0^1 \int_0^1 |k(y,x)|^2 dy dx < \infty \quad \text{then } K \text{ is compact.}$$

$$\text{For } Kf(s) = \int_0^1 k(s,x) f(x) dx \quad \text{Bachman and Narici}$$

[3, page 403] note that the adjoint to K is K^* , given by

$$K^*f(s) = \int_0^1 k(x,s) f(x) dx.$$

Of particular interest is the equation

(4.3.2)

$$\int_0^1 k(y,s) \int_0^1 k(y,x) f(x) dx dy + \lambda f(s) = \int_0^1 k(y,s) g(y) dy$$

for $\lambda > 0$.

For numerical purposes, we will restrict the discussion to the functions $k(y,x)$, g and f_λ where we have Riemann integration.

The methods of discretization of the integrals will initially parallel that found in Anselone [2, page 13] and Isaacson and Keller [9].

We have that

(4.3.3)

$$\int_0^1 k(y,s) g(y) dy = \sum_{j=1}^m S_j k(y_j, s) g(y_j) + R_m(s)$$

where $R_m(s)$ is the discretization error. Here $S_j > 0$

$j = 1, m$, $\sum_{j=1}^m S_j = 1$ are the quadrature weights for

$0 = y_1 < y_2 < \dots < y_m = 1$, a partition of the interval $[0,1]$, chosen equally spaced for the Newton-Cotes closed

integration formulae.

For $\int_0^1 k(y,s) \int_0^1 k(y,x) f(x) dx dy$ we have

$$\begin{aligned}
 (4.3.4) \quad & \int_0^1 k(y,s) \int_0^1 k(y,x) f(x) dx dy \\
 &= \sum_{j=1}^m S_j k(y_j, s) \sum_{i=1}^n T_i k(y_j, x_i) f(x_i) + \bar{R}(s)
 \end{aligned}$$

where $\bar{R}(s)$ is the discretization error. Here $T_i > 0$

for $i = 1, n$, $\sum_{i=1}^n T_i = 1$ are the quadrature weights for

$0 = x_1 < x_2 < \dots < x_n = 1$, a partition of the interval $[0,1]$ chosen equally spaced for the Newton-Cotes closed integration formulae. The partition and weights for the y_j 's are chosen as for (4.3.3).

Thus from (4.3.3) and (4.3.4), (4.3.2) becomes

$$\begin{aligned}
 (4.3.5) \quad & \sum_{j=1}^m S_j k(y_j, s) \sum_{i=1}^n T_i k(y_j, x_i) f(x_i) + \lambda f(s) + \bar{R}(s) \\
 &= \sum_{j=1}^m S_j k(y_j, s) g(y_j) + R_m(s).
 \end{aligned}$$

Now, (4.3.5) can be written in the equivalent form

$$\begin{aligned} & (S_j^k(y_j, s))_{j=1, m}^T (T_i^k(y_j, x_i))_{j=1, m} (f(x_i))_{i=1, n} + \lambda f(s) + \bar{R}(s) \\ &= \sum_{j=1}^m S_j^k(s, y_j) g(y_j) + R_m(s). \quad \text{Where } (v_i)_{i=1, \ell} \text{ denotes an} \\ & \ell\text{-dimensional column vector.} \end{aligned}$$

ℓ -dimensional column vector.

Now partition the interval $[0, 1]$ for s such that $s_i = x_i$ for $i = 1, n$. Thus we obtain the linear system

(4.3.6)

$$\begin{aligned} & (S_j^k(y_j, x_i))_{i=1, n} \cdot (T_i^k(y_j, x_i))_{j=1, m} (f(x_i))_{i=1, n} \\ & + \lambda (f(x_i))_{i=1, n} = (S_j^k(y_j, x_i))_{i=1, n} (g(y_j))_{j=1, m} \\ & + (R_m(x_i) - \bar{R}(x_i))_{i=1, n}. \end{aligned}$$

Let

$$K = (T_i^k(y_j, x_i))_{\substack{j=1,m \\ i=1,n}}, \quad K : H_T^n \rightarrow H_S^m;$$

$$K^* = (S_j^k(y_j, x_i))_{\substack{i=1,n \\ j=1,m}}, \quad K^* : H_S^m \rightarrow H_T^n;$$

$$\vec{f} = (f(x_i))_{i=1,n}, \quad \vec{f} \in H_T^n;$$

$$\vec{g} = (g(x_j))_{j=1,m}, \quad \vec{g} \in H_S^m$$

and

$$\vec{R}_1 = (R_n(x_i) - \bar{R}(x_i))_{i=1,n}, \quad \vec{R} \in H_T^n.$$

Thus (4.3.6) can be written in the form

$$(4.3.7) \quad K^* K \vec{f} + \lambda \vec{f} = K^* \vec{g} + \vec{R}_1.$$

Similar to (4.3.5) we have the integral equation

$$(4.3.8) \quad \int_0^1 k(y, s) \int_0^1 k(y, x) f(x) dx dy = \int_0^1 k(y, s) g(y) dy$$

and its discretized form

$$(4.3.9) \quad K^*K\vec{f} = K^*\vec{g} + \vec{R}_1,$$

where $\vec{R}_1$ is the discretization error vector. To solve equation (4.3.8) numerically we consider the system

$$(4.3.10) \quad K^*K\vec{f} + \lambda\vec{f} = K^*\vec{g}$$

for $\lambda > 0$ and small.

4.4. Error Bounds

Throughout this section we will assume that

$\gamma = \inf_{\gamma_n \in \sigma_0(K^*K)} \gamma_n$. For $\lambda > 0$, first we will develop

relationships between the solutions of the following equations

$$K^*K\vec{f}_\lambda + \lambda\vec{f}_\lambda = K^*\vec{g} + \vec{R}_\lambda,$$

$$K^*K\vec{f}_0 = K^*\vec{g} + \vec{R}_0$$

and

$$K^*K\vec{f}_\lambda + \lambda\vec{f}_\lambda = K^*\vec{g}$$

where $\vec{R}_\lambda, \vec{R}_0$ are the discretization error vector.

$\vec{f}_\lambda, \vec{f}_0$ and $\vec{\bar{f}}_\lambda$ are the respective solutions to the above equations. These relationships will depend on an estimate for $\|\vec{R}_\lambda\|, \|\vec{R}_0\|$ and γ .

Assuming the above notation we have the following two theorems.

Theorem 4.4.1.

$$(4.4.1) \quad \|\vec{f}_\lambda - \vec{\bar{f}}_\lambda\| \leq \frac{\|\vec{R}_\lambda\|}{\underline{\mu} + \lambda}$$

where $\underline{\mu} = \inf_{\gamma_n \in \sigma(K^*K)} \gamma_n$.

Proof: Since $(K^*K + \lambda I)\vec{f}_\lambda = K^*\vec{g} + \vec{R}_\lambda$ and $(K^*K + \lambda I)\vec{\bar{f}}_\lambda = K^*\vec{g}$ we have $(K^*K + \lambda I)(\vec{f}_\lambda - \vec{\bar{f}}_\lambda) = \vec{R}_\lambda$. Thus

$$\begin{aligned} (\underline{\mu} + \lambda)\|\vec{f}_\lambda - \vec{\bar{f}}_\lambda\| &= \|(K^*K + \lambda I)^{-1}\|^{-1}\|\vec{f}_\lambda - \vec{\bar{f}}_\lambda\| \\ &\leq \|(K^*K + \lambda I)(\vec{f}_\lambda - \vec{\bar{f}}_\lambda)\| \\ &= \|\vec{R}_\lambda\|. \end{aligned}$$

Thus we have the theorem.

Remark: Since $K^*K\vec{f}_0 = K^*\vec{g} + \vec{R}_0$ we have $K^*(K\vec{f}_0 - \vec{g}) = \vec{R}_0$.
Thus $\vec{R}_0 \in R(K^*) \subset N(K)^\perp$. Therefore $\vec{f}_0 \in N(K^*K)^\perp$.

Theorem 4.4.2.

$$(4.4.2) \quad \|K^*K(\vec{f}_0 - \vec{f}_\lambda)\| \leq \|\vec{R}_0\| + \lambda\|\vec{f}_\lambda\|,$$

$$(4.4.3) \quad \|\vec{f}_0 - \vec{f}_\lambda\| \leq \frac{\|\vec{R}_0\| + \lambda\|\vec{f}_\lambda\|}{\gamma} \quad \text{for } \gamma \neq 0,$$

$$(4.4.4) \quad \|\vec{f}_0 - \vec{f}_\lambda\| \leq \frac{\|\vec{R}_0\| + \lambda\|\vec{f}_0\|}{\lambda + \gamma}$$

and

$$(4.4.5) \quad \|\vec{f}_0 - \vec{f}_\lambda\| \leq \frac{(\lambda + \gamma)\|\vec{R}_0\| + \lambda\|K^*\vec{g}\|}{\gamma(\lambda + \gamma)} \quad \text{for } \gamma \neq 0.$$

Proof: For (4.4.2): Since $K^*K\vec{f}_\lambda + \lambda\vec{f}_\lambda = K^*\vec{g}$ and $K^*K\vec{f}_0 = K^*\vec{g} + \vec{R}_0$ we have $K^*K\vec{f}_\lambda + \lambda\vec{f}_\lambda = -\vec{R}_0 + K^*K\vec{f}_0$.
Thus $K^*K(\vec{f}_0 - \vec{f}_\lambda) = \vec{R}_0 + \lambda\vec{f}_\lambda$. Taking the norm of each side and using the triangular inequality we have

$$\|K^*K(\vec{f}_0 - \vec{f}_\lambda)\| \leq \|\vec{R}_0\| + \lambda\|\vec{f}_\lambda\|.$$

For (4.4.3): Since $\gamma\|\vec{f}_0 - \vec{f}_\lambda\| \leq \|K^*K(\vec{f}_0 - \vec{f}_\lambda)\|$, the result follows from (4.4.2).

For (4.4.4): Since $K^*K(\vec{f}_0 - \vec{f}_\lambda) = \vec{R}_0 + \lambda\vec{f}_\lambda$ we have $(K^*K + \lambda I)(\vec{f}_0 - \vec{f}_\lambda) = \vec{R}_0 + \lambda\vec{f}_0$. Taking norms of each side of the above and using the triangular inequality yields $\|(K^*K + \lambda I)(\vec{f}_0 - \vec{f}_\lambda)\| \leq \|\vec{R}_0\| + \lambda\|\vec{f}_0\|$. Since $(\lambda + \gamma)\|\vec{f}_0 - \vec{f}_\lambda\| \leq \|(K^*K + \lambda I)(\vec{f}_0 - \vec{f}_\lambda)\|$ (4.4.4) follows.

For (4.4.5): Since $\|(K^*K + \lambda I)\vec{f}_\lambda\| = \|K^*g\|$ and

$$(\lambda + \gamma)\|\vec{f}_\lambda\| \leq \|(K^*K + \lambda I)\vec{f}_\lambda\| \text{ we have } \|\vec{f}_\lambda\| \leq \frac{\|K^*g\|}{\lambda + \gamma}. \text{ Thus}$$

from (4.4.3) the result follows.

Now take $\vec{f}_{j,n}$ as a term in the sequence (2.2.8) converging to the least squares solution of minimum norm, $\vec{f}_0$, of $K^*K\vec{f} = K^*g + \vec{R}_0$. The following error bounds on $\|\vec{f}_0 - \vec{f}_{j,n}\|$ can be obtained.

Theorem 4.4.3.

(4.4.6)

$$\|\vec{f}_0 - \vec{f}_{j,n}\| \leq \frac{\|\vec{R}_0\| + \|K^*K\vec{f}_{j,n} - K^*\vec{g}\|}{\gamma} \quad \text{if } \gamma \neq 0,$$

(4.4.7)

$$\|\vec{f}_0 - \vec{f}_{j,n}\| \leq \frac{\|w_{j,n}\|}{\lambda_j + \gamma} + \frac{\|\vec{R}_0\| + \lambda_j \|\vec{f}_j\|}{\gamma} \quad \text{if } \gamma \neq 0,$$

$$(4.4.8) \quad \|\vec{f}_0 - \vec{f}_{j,n}\| \leq \frac{\|w_{j,n}\|}{\lambda_j + \gamma} + \frac{\|\vec{R}_0\| + \lambda_j \|\vec{f}_0\|}{\lambda_j + \gamma},$$

and

(4.4.9)

$$\|\vec{f}_0 - \vec{f}_{j,n}\| \leq \frac{\|w_{j,n}\|}{\lambda_j + \gamma} + \frac{(\lambda_j + \gamma) \|\vec{R}_0\| + \lambda_j \|K^*\vec{g}\|}{\gamma(\lambda_j + \gamma)} \quad \text{if } \gamma \neq 0.$$

Proof: For (4.4.6): Since $K^*K\vec{f}_0 = K^*\vec{g} + \vec{R}_0$, we have $K^*K(\vec{f}_0 - \vec{f}_{j,n}) = K^*\vec{g} - K^*K\vec{f}_{j,n} + \vec{R}_0$. Take the norm of each side of the above result and use the triangle inequality, yielding $\|K^*K(\vec{f}_0 - \vec{f}_{j,n})\| \leq \|\vec{R}_0\| + \|K^*\vec{g} - K^*K\vec{f}_{j,n}\|$. The result (4.4.6) follows from $\gamma\|\vec{f}_0 - \vec{f}_{j,n}\| \leq \|K^*K(\vec{f}_0 - \vec{f}_{j,n})\|$.

For (4.4.7), (4.4.8), (4.4.9): These bounds on $\|\vec{f}_0 - \vec{f}_{j,n}\|$ follow from the triangle inequality.

$$\begin{aligned} \|\vec{f}_0 - \vec{f}_{j,n}\| &\leq \|\vec{f}_{j,n} - \vec{f}_j\| + \|\vec{f}_j - \vec{f}_0\|, \quad \text{Theorem 4.4.2 and} \\ (\lambda_j + \gamma) \|\vec{f}_{j,n} - \vec{f}_j\| &\leq \|(K^*K + \lambda_j I)(\vec{f}_{j,n} - \vec{f}_j)\| = \|w_{j,n}\|. \end{aligned}$$

4.5. Perturbations of g

In the case where g is not known exactly, we have the integral equation

$$(4.5.1) \quad \int_0^1 k(y, x) f(x) dx = g(y) + \epsilon(y),$$

where $\epsilon(y) \in L_2[0,1]$. (4.4.10) may or may not have a solution (see 2.3). We follow the procedure and notation of section 4.3 and obtain the matrix representation for (4.5.1). Thus

$$(4.5.2) \quad K^*K\vec{f} + \lambda\vec{f} = K^*\vec{g} + K^*\vec{\epsilon} + \vec{R}_2$$

where $\vec{R}_2$ is the discretization error vector. The system of equations actually solved is

$$(4.5.3) \quad K^*K\vec{f} + \lambda\vec{f} = K^*\vec{g} + K^*\vec{\epsilon}.$$

Let $\vec{f}_{\lambda, \epsilon}$ be the solution to (4.5.3).

$$\|\vec{f}_{\lambda} - \vec{f}_{\lambda, \epsilon}\| \leq \frac{\|K\|\|\vec{\epsilon}\|}{\gamma + \lambda} \text{ from Theorem 2.3.1 and statement}$$

(4.3.10).

Let $\vec{f}_{j, n, \epsilon}$ be a term in the sequence defined by (2.2.8). If a bound for $\|K^*\vec{\epsilon}\|$ or $\|\vec{\epsilon}\|$ can be found, we can obtain bounds on $\|\vec{f}_0 - \vec{f}_{j, \epsilon}\|$ and $\|\vec{f}_0 - \vec{f}_{j, n, \epsilon}\|$. Theorems 4.4.2 and 4.4.3 can be extended, with similar proof, to the following theorems.

Theorem 4.5.1.

$$(4.5.3) \quad \|K^*K(\vec{f}_0 - \vec{f}_{\lambda, \epsilon})\| \leq \|\vec{R}_0\| + \|K^*\vec{\epsilon}\| + \lambda\|\vec{f}_{\lambda, \epsilon}\|$$

(4.5.4)

$$\|\vec{f}_0 - \vec{f}_{\lambda, \epsilon}\| \leq \frac{\|\vec{R}_0\| + \|K^*\vec{\epsilon}\| + \lambda\|\vec{f}_{\lambda, \epsilon}\|}{\gamma} \quad \text{if } \gamma \neq 0$$

(4.5.6)

$$\|\vec{f}_0 - \vec{f}_{\lambda, \epsilon}\| \leq \frac{\|\vec{R}_0\| + \|K^*\vec{\epsilon}\| + \lambda\|\vec{f}_0\|}{\lambda + \gamma}$$

(4.5.7)

$$\|\vec{f}_0 - \vec{f}_{\lambda, \epsilon}\| \leq \frac{(\lambda + \gamma) \|\vec{R}_0\| + \lambda \|K^*(\vec{g} + \vec{\epsilon})\|}{\gamma(\lambda + \gamma)} \quad \text{if } \gamma \neq 0.$$

Theorem 4.5.2.

(4.5.8)

$$\|\vec{f}_0 - \vec{f}_{j, n, \epsilon}\| \leq \frac{\|\vec{R}_0\| + \|K^* K \vec{f}_{j, n, \epsilon} - K^*(\vec{g} + \vec{\epsilon})\|}{\gamma} \quad \text{if } \gamma \neq 0,$$

(4.5.9)

$$\|\vec{f}_0 - \vec{f}_{j, n, \epsilon}\| \leq \frac{\|w_{j, n}\|}{\lambda_j + \gamma} + \frac{\|\vec{R}_0\| + \|K^* \vec{\epsilon}\| + \lambda_j \|\vec{f}_{j, \epsilon}\|}{\gamma} \quad \text{if } \gamma \neq 0,$$

(4.5.10)

$$\|\vec{f}_0 - \vec{f}_{j, n, \epsilon}\| \leq \frac{\|w_{j, n}\|}{\lambda_j + \gamma} + \frac{\|\vec{R}_0\| + \|K^* \vec{\epsilon}\| + \lambda_j \|\vec{f}_0\|}{\lambda_j + \gamma},$$

(4.5.11)

$$\|\vec{f}_0 - \vec{f}_{j, n, \epsilon}\| \leq \frac{\|w_{j, n}\|}{\lambda_j + \gamma} + \frac{(\lambda + \gamma) \|\vec{R}_0\| + \lambda_j \|K^*(\vec{g} + \vec{\epsilon})\|}{\gamma(\lambda_j + \gamma)} \quad \text{if } \gamma \neq 0.$$

It should be noted that the error bounds given by (4.4.6) and (4.5.8) are numerically not as nice as they look. Even for $\|\vec{R}_0\| = \|K^* \vec{\epsilon}\| = 0$ and γ fairly large,

making $\|K^*K\vec{f}_{i,n} - K^*\vec{g}\|$ small is not necessarily good measure of convergence for the following reason. The methods of numerical quadrature used to obtain the matrix representation of the integral equation are essentially piecewise polynomial approximations to the integrand. From numerical experimentation, making $\|K^*K\vec{f}_{j,n} - K^*\vec{g}\|$ as small as possible, will cause oscillations in the final result. This is due to computer convergence. An apparent measure of this effect can be noted in the output column AF-G of the algorithm in Appendix B.

4.6. Integral Equation Examples

For a first example we will consider the integral equation with symmetric kernel $k(x,y) = x + y$,

(4.6.1)

$$Kf(y) = \int_0^1 (x+y)f(x)dx = \frac{1}{3} + \frac{y}{2} \in R(K), \quad y \in [0,1].$$

K^*K has non-zero eigenvalues

$$\frac{7}{12} + \frac{\sqrt{3}}{3} = 1.160683 \quad \text{and} \quad \frac{7}{12} - \frac{\sqrt{3}}{3} \approx .005983.$$

For (4.6.1) the least squares solution of minimum norm is $f_0(x) = x$.

The algorithm described in Chapter 2 and listed in Appendix B was used to obtain the following approximate solutions. Simpson's rule was used to determine the quadrature weights. In this example $\|\vec{R}_0\| = \|\vec{R}_1\| = 0$, the norm of the discretization error vectors. The interval $[0,1]$ was partitioned:

$$\vec{y} = \vec{x} = \begin{bmatrix} 0.00 \\ 0.25 \\ 0.30 \\ 0.75 \\ 1.00 \end{bmatrix} .$$

A 5×5 matrix representation for the integral equation (4.4.1) gives

$$K = \begin{bmatrix} 0.0000 & 0.0833 & 0.0833 & 0.2500 & 0.0833 \\ 0.0208 & 0.1667 & 0.1250 & 0.3333 & 0.1042 \\ 0.0417 & 0.2500 & 0.1667 & 0.4167 & 0.1250 \\ 0.0625 & 0.3333 & 0.2083 & 0.5000 & 0.1458 \\ 0.0833 & 0.4167 & 0.2500 & 0.5833 & 0.1667 \end{bmatrix}$$

and

$$K^*K = \begin{bmatrix} 0.0278 & 0.1528 & 0.0972 & 0.2361 & 0.0694 \\ 0.0382 & 0.2153 & 0.1389 & 0.3403 & 0.1007 \\ 0.0486 & 0.2778 & 0.1806 & 0.4444 & 0.1319 \\ 0.0590 & 0.3403 & 0.2222 & 0.5486 & 0.1632 \\ 0.0694 & 0.4028 & 0.2639 & 0.6528 & 0.1944 \end{bmatrix} .$$

The non-zero eigenvalues of $K^*K: H_T^5 \rightarrow H_T^5$ were computed to be 1.160677 and 0.005983. We assumed that an estimate for $\|\vec{f}_0\|$ is known, $\|\vec{f}_0\| \approx .577$ from (4.4.8)

$$\|\vec{f}_0 - \vec{f}_{j,n}\| \leq \frac{\|w_{j,n}\|}{\lambda_j + 0.005983} + \frac{.577\lambda_j}{\lambda_j + 0.005983} .$$

$\lambda_j = \text{terminal } \lambda = 1. \times 10^{-9}$ and $\|w_{j,n}\|^2 = 10^{-8}$ are determined so that $\|\vec{f}_0 - \vec{f}_{j,n}\| < 10^{-6}$. We used $M = 600$, maximum iterations per change in λ , $\lambda_{i+1} = .0001\lambda_i$, and initial starting vector $(f_i)_{i=1,5}$, $f_i = 1$, $i = 1,5$. The algorithm computed an initial starting λ , $\lambda_1 = .133889238$. After 225 iterations, we found the $\vec{f}_j$ approximate least squares solution of minimum norm to be

$$\begin{bmatrix} \bar{f}_j(0) \\ \bar{f}_j(.25) \\ \bar{f}_j(.50) \\ \bar{f}_j(.75) \\ \bar{f}_j(1.0) \end{bmatrix} = \begin{bmatrix} .000000176 \\ .250000097 \\ .500000023 \\ .749999948 \\ .999999873 \end{bmatrix}.$$

Also the following information was computed:

$$\|\vec{f}_{j,n} - \vec{f}_o\| < 9 \times 10^{-8} < 10^{-6},$$

$$\|\vec{f}_{j,n} - \vec{f}_o\| \leq \frac{\|K^*K\vec{f}_{j,n} - K^*\vec{g}\|}{\gamma} < 10^{-6},$$

and

$$\max_{i=1,5} |\bar{f}_{j,n}(x_i) - f_o(x_i)| < 1.8 \times 10^{-7}.$$

The same problem was run a second time using the same initial information, except to terminate when $\lambda_j = 10^{-15}$ and $\|w_{j,n}\|^2 < 10^{-28}$ obtaining, after 745 iterations,

$$\|\vec{f}_{j,n} - \vec{f}_o\| < 8. \times 10^{-13}$$

and

$$\max_{i=1,5} |\bar{f}_{j,n}(x_i) - f_o(x_i)| < 1.6 \times 10^{-12}.$$

As a second example consider the integral equation

(4.6.2)

$$Kf(y) = \int_0^1 (y-x)^2 f(x) dx = \frac{y^2}{2} - \frac{2y}{3} + \frac{1}{4} \in R(K)$$

which Bellman et al. [4, page 159] used for testing purposes of an algorithm he developed for finding the least squares solution of minimum norm. (4.6.2) has least squares solution of minimum norm $f_o(x) = x$. The quadrature method Bellman used was Simpson's rule, with 11 equally spaced points, as we will use.

A bound for the norm of the quadrature error was found to be $\|\vec{R}_0\| = 0$, since Simpson's rule integrates $\int_0^1 (y-x)^2 f(x) dx$ for $f(x) = x$ exactly and the errors for

$$\int_0^1 (y-x)^2 \int_0^1 (y-x)^2 f(x) dx dy = \int_0^1 (y-x)^2 g(y) dy$$

subtract identically when $f(x) = x$.

In Bellman's example, also based on a variation of regularization, he used $\lambda = 10^{-7}$ and required a good initial approximation to the solution. To obtain somewhat comparable results, the algorithm in Appendix B with his λ was used.

The matrix operator K^*K ; $K^*K: H_T^{11} \rightarrow H_T^{11}$, has computed non-zero eigenvalues .038101, .027778 and .000814. To compare the results of the algorithm in Appendix B with his we used $\lambda_j = \text{Terminal Lambda} = 10^{-7}$, control on $\|(K^*K + \lambda_i I)f - K^*g\|^2 = 10^{-16}$, lambda multiplier = 10^{-3} , maximum steps for each $\lambda_i = 300$ and initial starting vector $(f_i)_{i=1,11}$ where $f_i = 0$ for $i = 1, 10$, $f_{11} = 1$.

The algorithm given in Appendix B chooses a new starting vector and gave the results given in Table 2 for Run 1. The bound for $\|\vec{f}_{j,n} - \vec{f}_0\|$ was computed using $\gamma = .000814$ and .577 as an estimate for $\|\vec{f}_0\|$ in (4.4.8). The results were about the same as Bellman obtained.

Now Bellman also looked at the same example (4.6.2), but rounds the data representing $\vec{g}$ correct to three places. We ran the same example with the same starting

information as for Run 1, to obtain two place accuracy compared to none for Bellman (see Table 2, Run 2). For Run 2, the bound for $\|\vec{f}_{j,n,\epsilon} - \vec{f}_0\|$ was computed using $\gamma = .000814, .577$ as an estimate for $\|\vec{f}_0\|$ and

$$\|K^*\vec{\epsilon}\| \leq \|K^*\| \|\vec{\epsilon}\| \approx \sqrt{.038101} \cdot .0005 \approx .0001$$

in (4.5.10).

For a third run of a discretized version of (4.6.2) the 11-point trapezoidal rule was used as the quadrature method. In this case, the matrix operator K^*K has computed non-zero eigenvalues .040182, .028900 and .000926. An upper bound on the norm of the discretization error vector was computed to be $\|\vec{R}_0\| \leq .056$. To compare with Run 1, the same starting information was used and the computed error determined by using (4.4.8).

Table 2. Integral Equation Example 2

	Run 1	Run 2	Bellman	Run 3
Total iterations	339	339		341
$\ \vec{f}_{j,n} - \vec{f}_0\ $	2.7(-5)	1.9(-3)		1.2(-2)
$\max_i \vec{f}_{j,n} - \vec{f}_0 $	4.6(-5)	3.(-3)	.9	3.1(-2)
Computed error	1.(-4)	.11		61.

For a last example we will consider the integral equation of the first kind with kernel

$$k(x,y) = \begin{cases} (1-y)x & \text{if } 0 \leq x \leq y \leq 1 \\ (1-x)y & \text{if } 0 \leq y \leq x \leq 1. \end{cases}$$

Tricomi [24, page 116] notes that the eigenvalues associated with

$$K^*K(f)(x) = \int_0^1 k(x,y) \int_0^1 k(x,y) f(x) dx dy = \gamma f(x)$$

$$\text{are } \gamma_1 = \frac{1}{\pi^4}, \gamma_2 = \frac{1}{(2\pi)^4}, \dots, \gamma_n = \frac{1}{n^4 \pi^4}, \dots$$

Strand [20, page 71] shows that for

$$g(y) = y(3 - 5y^2 + 3y^4 - y^5)/30 \in R(K),$$

$$Kf(y) = \int_0^1 k(x,y) f(x) dx = g(y) \quad \text{has least squares solution}$$

$$\text{of minimum norm } f_0(x) = x - 2x^3 + x^4.$$

For the discretization of the integral equation

$$\int_0^1 k(x,y) \int_0^1 k(x,y) f(x) dx dy = \int_0^1 k(x,y) g(y) dy$$

the error for a 51 point Simpson's rule was estimated to be (upper bound) 1.4×10^{-7} .

The matrix representation for K^*K has minimum positive non-zero eigenvalue $\gamma = 7.915 \times 10^{-9}$ (computed). Since γ is small a choice for the terminal λ_j , from the error bounds is not practical (see Theorem 4.4.3). Considering the expression

$$\|\vec{f}_j - \vec{f}_0\| \leq \frac{\lambda_j \|\vec{f}_j\| + \|\vec{R}_0\|}{\gamma},$$

numerical experimentation indicates that a reasonable choice for terminal λ_j is for the order of magnitude of $\lambda_j \|\vec{f}_j\|$ and $\|\vec{R}_0\|$ be about the same. Assuming $\|\vec{f}_j\| \approx .22 \approx \|\vec{f}_0\|$ we get a choice for terminal λ_j , $\lambda_j = 6.4 \times 10^{-7}$. Using $\|K^*K\vec{f}_{j,n} + \lambda\vec{f}_{j,n} - K^*\vec{g}\|^2 = 10^{-16}$ we get a computed error bound: $\|\vec{f}_{j,n} - \vec{f}_0\| < .46$ from (4.4.8). We allowed a maximum of 500 iterations per step and set the lambda multiplier = .001. The algorithm in Appendix B generated its own starting vector from the initial starting vector

$$\vec{f}(x_i) = \begin{cases} 1 & \text{if } 17 \leq i \leq 35 \\ 0 & \text{otherwise} \end{cases}$$

and initial $\lambda_1 \approx .00795$. After 52 iterations we obtained actual results:

$$\|\vec{f}_{j,n} - \vec{f}_0\| < 1.332 \times 10^{-4}$$

$$\max_{i=1,52} |f_{j,n}(x_i) - f_0(x_i)| < 3.024 \times 10^{-4},$$

and

$$\|K^*(K\vec{f}_{j,n} - \vec{f}_0)\| < 7.121 \times 10^{-6}.$$

5. SUMMARY AND FUTURE RESEARCH

5.1. Summary

The advantage of the iterative method developed in the paper is primarily that it avoids the calculation of $(K*K + \lambda I)^{-1}$ directly and that the iterative process is stable. The unwanted oscillations in the final solution that often appear in solutions by other methods in the literature do not occur. In some applications the error bounds obtained become large numerically when an upper bound on $\|\vec{R}_0\|$ or $\|\vec{\epsilon}\|$ is large or when γ is near zero. An upper bound for $\|\vec{R}_0\|$ can be difficult to obtain unless some information about the solution is known.

When comparing with other iterative methods in the literature, we generally have at least two additional significant figures. The iterative method here converges for examples where other methods in the literature fail to converge to a solution.

5.2. Future Research

It would be desirable to modify the algorithms in Appendix A and B so that γ and the parameters for convergence are computed within the algorithm.

A method of discretization for integral equations with non-uniform mesh size should be developed that better describes the properties of a particular kernel and g .

It would be nice to find a method of determining the weights for quadrature in representing the kernel, independent of any information about the form of the solution.

6. BIBLIOGRAPHY

1. Akhiezer, N. I. and Glazman, I. M. Theory of Linear Operators in Hilbert Space. Vol.1. New York: Frederick Ungar Publishing Co., 1961.
2. Anselone, P. M. Collectively Compact Operator Approximation Theory and Applications to Integral Equations. Englewood Cliffs, New Jersey: Prentice Hall, Inc., 1971.
3. Bachman, G. and Narici, L. Functional Analysis. New York: Academic Press Inc., 1966.
4. Bellman, R., Kalaba, R. and Lockett, J. A. Numerical Inversion of the Laplace Transform. New York: American Elsevier Publishing C., Inc., 1966.
5. Diaz, J. B. and Metcalf, F. T. "On iteration procedures for equations of the first kind, $Ax = y$, and Picard's criterion for the existence of a solution". Math. Comp. 24 (1970): 923-935.
6. Hanson, R. J. "A numerical method for solving Fredholm integral equations of the first kind using singular values". SIAM J. Numer. Anal. 8 (1971): 616-622.
7. Helmberg, G. Introduction to Spectral Theory in Hilbert Space. New York: John Wiley and Sons, Inc., 1969.
8. Herr, D. G. "On a statistical model of Strand and Westwater for numerical solution of a Fredholm integral equation of the first kind". J. Assoc. Comput. Mach. 21 (1974): 1-5.
9. Isaacson, E. and Keller, H. Analysis of Numerical Methods. New York: John Wiley and Sons, Inc., 1966.

10. Kammerer, W. J. and Nashed, M. Z. "Steepest descent for singular linear operators with nonclosed range". Appl. Anal. 1 (1971): 143-159.
11. Kammerer, W. J. and Nashed, M. Z. "Iterative methods for best approximate solutions of linear integral equations of the first and second kinds". J. Math. Anal. Appl. 40 (1972): 547-573.
12. Kammerer, W. J. and Nashed, M. Z. "On the convergence of the conjugate gradient method for singular linear operator equations". SIAM J. Numer. Anal. 9 (1972): 165-181.
13. Landweber, L. "An iteration formula for Fredholm integral equations of the first kind". Amer. J. Math. 73 (1951): 615-624.
14. Nashed, M. Z. "Steepest descent for singular linear operator equations". SIAM J. Numer. Anal. 7 (1970): 358-362.
15. Nashed, M. Z. "Generalized inverses, normal solvability, and iteration for singular operator equations". In Nonlinear Functional Analysis and Applications, pp.311-359. Edited by L. B. Rall. New York: Academic Press, Inc., 1971.
16. Nashed, M. Z. and Wahba, G. "Convergence rates of approximate least squares solutions of linear integral and operator equations of the first kind". Math. Comp. 28 (1974): 69-80
17. Peterson, W. C. "An elementary characterization of Fredholm equations of the first kind". Document Library, Ames Laboratory, USAEC, Ames, Iowa, Report No. IS-3497, 1974.
18. Phillips, D. L. "A Technique for the numerical solution of certain integral equations of the first kind". J. Assoc. Comput. Mach. 9 (1962): 84-97.

19. Smithies, F. "Integral equations". Cambridge Tracts in Mathematics and Mathematical Physics 49 (1958): 1-169.
20. Strand, O. N. Theory and Methods for Operator Equations of the First Kind. Ph.D. Thesis, Colorado State Univ., 1972. Ann Arbor, Michigan: University Microfilms (Diss. Abstr. 73: 2820).
21. Strand, O. N. "Theory and methods related to the singular-function expansion and Landweber's iteration of integral equations of the first kind". SIAM J. Numer. Anal. 11 (1974): 798-825.
22. Strand, O. N. and Westwater, E. R. "Statistical estimation of the numerical solution of a Fredholm integral equation of the first kind". J. Assoc. Comput. Mach. 1 (1968): 100-114.
23. Tikhonov, A. N. "Solution of incorrectly formulated problems and the regularization method". Soviet Math. Dokl. 4 (1963): 1035-1038.
24. Tricomi, F. G. Integral Equations. New York: Interscience Publishers, Inc., 1957.
25. Twomey, S. "On the numerical solution of Fredholm integral equations of the first kind by inversion of the linear system produced by quadrature". J. Assoc. Comput. Mach. 10 (1963): 97-101.

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8. APPENDIX

8.1. Appendix A, Linear System

The following program is written to find the least squares solution of minimum norm of an NN by N system of equations, $Ax = g$. The basic flow chart for the program is given in Figure 3.2.1.

The user must input the following information in the order given:

1. NN = the number of rows of the matrix A . The format for reading NN is given in line SYST0280.
2. N = the number of columns of the matrix A . The format for reading N is given in line SYST0280.
3. M = the maximum number of iterations per change in λ . The format for reading M is given in line SYST0280.
4. BB = terminal λ . The format for reading M is given in line SYST0320.
5. DD = control on $\|ATAF + LAMBDA * F - ATG\|^2$. The format for reading DD is given in line SYST0320.

6. The matrix A is read by rows. The format for reading A is given in line SYST0340.
7. Read the vector g . The format for reading g is given in line SYST0340.
8. Read the initial starting vector F . The format is given in line SYST0340.
9. $XLAM$ = lambda multiplier. The format for reading $XLAM$ is given in line SYST0340.

The output of the starting information begins at line SYST1440. The output of the final solution begins at line SYST2370.

```

C
C THE FOLLOWING IS A PROGRAM TO SOLVE AN NN BY N SYSTEM OF
C EQUATIONS,  $AF=G$ , FOR THE LEAST SQUARES SOLUTION OF MINIMUM
C NORM. IF N OR NN IS GREATER THAN 10, THE DIMENSION
C STATEMENT MUST BE CHANGED TO AT LEAST  $A(NN,N), G(NN), F(N),$ 
C  $AT(N,NN), ATA(N,N), R(NN), T(N), V(NN), W(N), S(NN),$  AND  $WW(N)$ .
C
C INPUT:
C
C     N= THE NUMBER OF COLUMNS OF THE MATRIX A .
C
C     NN=THE NUMBER OF ROWS OF THE MATRIX A.
C
C     M=THE MAXIMUM NUMBER OF ITERATIONS PER CHANGE IN
C     LAMBDA.
C
C     BB=TERMINAL LAMBDA.
C
C     DD=CONTROL ON  $||ATAF+LAMBDA*F-ATG||^{**2}$ .
C
C     XLAM=LAMBDA MULTIPLIER.
C
C     IMPLICIT REAL*8 (A-H,O-Z)
C     DIMENSION A(10,10), G(10), F(10), AT(10,10), ATA(10,10), R(10)
C     DIMENSION T(10), V(10), W(10), S(10), WW(10)
C     READ(5,11) NN
C     READ(5,11) N
11  FORMAT(I5)
C     READ(5,11) M
C     READ(5,21) BB
C     READ(5,21) DD
21  FORMAT(D10.5)
31  FORMAT(8F10.5)
C     DO 10 I=1,NN
C
C READ MATRIX A BY ROWS.
C

```

```

SYST0010
SYST0020
SYST0030
SYST0040
SYST0050
SYST0060
SYST0070
SYST0080
SYST0090
SYST0100
SYST0110
SYST0120
SYST0130
SYST0140
SYST0150
SYST0160
SYST0170
SYST0180
SYST0190
SYST0200
SYST0210
SYST0220
SYST0230
SYST0240
SYST0250
SYST0260
SYST0270
SYST0280
SYST0290
SYST0300
SYST0310
SYST0320
SYST0330
SYST0340
SYST0350
SYST0360
SYST0370

```

10 READ(5,31) (A(I,J),J=1,N)	SYST0380
C	SYST0390
C READ VECTOR G OF AF=G.	SYST0400
C	SYST0410
READ(5,31) (G(I),I=1,NN)	SYST0420
C	SYST0430
C READ INITIAL STARTING VECTOR F.	SYST0440
C	SYST0450
READ(5,31) (F(I),I=1,N)	SYST0460
READ(5,31) XLAM	SYST0470
WRITE(6,12) NN,N,M	SYST0480
12 FORMAT('0','NN=',I5,5X,'N=',I5,5X,'M=',I5)	SYST0490
WRITE(6,26) DD	SYST0500
26 FORMAT('0','CONTROL ON ATAF+LAMBDA*F-ATG **2=',D24.16)	SYST0510
WRITE(6,22) BB	SYST0520
22 FORMAT('0','CONTROL ON LAMBDA=',D24.16)	SYST0530
WRITE(6,24) XLAM	SYST0540
24 FORMAT('0','LAMBDA MULTIPLIER=',D24.16)	SYST0550
WRITE(6,32)	SYST0560
32 FORMAT('0','MATRIX A=',/)	SYST0570
DO 20 I=1,NN	SYST0580
20 WRITE(6,42) (A(I,J),J=1,N)	SYST0590
42 FORMAT(' ',10(F10.5,1X))	SYST0600
WRITE(6,52)	SYST0610
52 FORMAT('0','G=',/)	SYST0620
WRITE(6,42) (G(I),I=1,NN)	SYST0630
WRITE(6,62)	SYST0640
62 FORMAT('0','STARTING VECTOR F=',/)	SYST0650
WRITE(6,42) (F(I),I=1,N)	SYST0660
LL=0	SYST0670
DO 30 I=1,N	SYST0680
DO 30 J=1,NN	SYST0690
30 AT(I,J)=A(J,I)	SYST0700
DO 410 I=1,N	SYST0710
W(I)=0.0D0	SYST0720
DO 410 J=1,NN	SYST0730
410 W(I)=W(I)+AT(I,J)*G(J)	SYST0740

```

      ZZ=0.0D0
      DO 420 I=1,N
420  ZZ=ZZ+W(I)*W(I)
      ZZ=DSQRT(ZZ)
      WRITE(6,29) ZZ
29  FORMAT('0',*,|| ATG ||=' ',D24.16)
      IF(ZZ.GE.1.D-40) GO TO 430
      WRITE(6,421)
421  FORMAT('0',6X,'SOLUTION IS THE ZERO VECTOR',/)
      GO TO 600

C
C  OBTAIN A STARTING VECTOR AND AN INITIAL LAMBDA.
C
430  DO 40 I=1,N
      DO 40 J=1,N
      ATA(I,J)=0.0D0
      DO 40 L=1,NN
40  ATA(I,J)=ATA(I,J)+AT(I,L)*A(L,J)
      DO 50 I=1,N
      T(I)=0.0D0
      DO 50 J=1,N
50  T(I)=T(I)+ATA(I,J)*F(J)
      ZZ=0.0D0
      DO 60 I=1,N
60  ZZ=ZZ+T(I)*T(I)
      ZZ=DSQRT(ZZ)
      DO 70 I=1,N
70  T(I)=T(I)/ZZ
      DO 80 I=1,NN
      R(I)=0.0D0
      DO 80 J=1,N
80  R(I)=R(I)+A(I,J)*T(J)
      CR=1.0D0
      U=2.0D0
      J=1
90  X=0.0D0
      DO 100 I=1,NN

```

```

SYST0750
SYST0760
SYST0770
SYST0780
SYST0790
SYST0800
SYST0810
SYST0820
SYST0830
SYST0840
SYST0850
SYST0860
SYST0870
SYST0880
SYST0890
SYST0900
SYST0910
SYST0920
SYST0930
SYST0940
SYST0950
SYST0960
SYST0970
SYST0980
SYST0990
SYST1000
SYST1010
SYST1020
SYST1030
SYST1040
SYST1050
SYST1060
SYST1070
SYST1080
SYST1090
SYST1100
SYST1110

```

```

100 X=X+(G(I)-R(I))*R(I)
    IF(X.GE.0.000) GO TO 1000
    IF(J.EQ.2) GO TO 1010
    DO 110 I=1,N
110 T(I)=-T(I)
    DO 115 I=1,NN
115 R(I)=-R(I)
    GO TO 50
1010 DO 120 I=1,NN
120 G(I)=U*G(I)
    J=1
    CR=CR*U
    GO TO 50
1000 DO 130 I=1,NN
130 R(I)=R(I)-G(I)
    RR=0.000
    DO 140 I=1,NN
140 RR=RR+R(I)*R(I)
    DO 150 I=1,N
    W(I)=0.000
    DO 150 J=1,NN
150 W(I)=W(I)+AT(I,J)*R(J)
    ZZ=0.000
    DO 160 I=1,N
160 ZZ=ZZ+W(I)*W(I)
    YY=ZZ-X*X
    AB=RR+X
    DO 170 I=1,N
170 W(I)=W(I)+X*T(I)

```

C

C OUTPUT STARTING INFORMATION.

C

```

    WRITE(6,72) X
72 FORMAT('0',' INITIAL LAMBDA=',D24.16)
    WRITE(6,82)
82 FORMAT('0',.3X,' STARTING VECTOR F',.10X,' ATAF+LAMBDA*F-ATG',/)
    DO 180 I=1,N

```

```

SYST1120
SYST1130
SYST1140
SYST1150
SYST1160
SYST1170
SYST1180
SYST1190
SYST1200
SYST1210
SYST1220
SYST1230
SYST1240
SYST1250
SYST1260
SYST1270
SYST1280
SYST1290
SYST1300
SYST1310
SYST1320
SYST1330
SYST1340
SYST1350
SYST1360
SYST1370
SYST1380
SYST1390
SYST1400
SYST1410
SYST1420
SYST1430
SYST1440
SYST1450
SYST1460
SYST1470
SYST1480

```


180 WRITE(6,92) T(I),W(I)	SYST1490
92 FORMAT(' ',3D24.16)	SYST1500
96 FORMAT(' ',D24.16)	SYST1510
WRITE(6,84)	SYST1520
84 FORMAT('0',10X,'AF-G',/)	SYST1530
DO 192 I=1,NN	SYST1540
182 WRITE(6,96) R(I)	SYST1550
WRITE(6,94) CR	SYST1560
94 FORMAT('0',' SCALING FACTOR=',D24.16)	SYST1570
WRITE(6,102) YY	SYST1580
102 FORMAT('0',' MINIMUM VALUE=',D24.16)	SYST1590
XX=X	SYST1600
C	SYST1610
C MAIN ITERATIONS OF THE ALGORITHM BEGIN.	SYST1620
C	SYST1630
1020 X=XX	SYST1640
K=1	SYST1650
1030 AA=AB	SYST1660
YY=ZZ	SYST1670
DO 190 I=1,N	SYST1680
WW(I)=W(I)	SYST1690
W(I)=0.0D0	SYST1700
DO 190 J=1,NN	SYST1710
190 W(I)=W(I)+AT(I,J)*R(J)	SYST1720
DO 200 I=1,N	SYST1730
200 W(I)=W(I)+X*T(I)	SYST1740
ZZ=0.0D0	SYST1750
DO 210 I=1,N	SYST1760
210 ZZ=ZZ+W(I)*W(I)	SYST1770
IF(ZZ.LT.DD) GO TO 1040	SYST1780
DO 220 I=1,NN	SYST1790
220 V(I)=R(I)	SYST1800
DO 225 I=1,N	SYST1810
225 F(I)=T(I)	SYST1820
VV=RR	SYST1830
DO 230 I=1,NN	SYST1840
S(I)=0.0D0	SYST1850

```

DO 230 J=1,N
230 S(I)=S(I)+A(I,J)*W(J)
Y=0.0D0
DO 240 I=1,NN
240 Y=Y+S(I)*S(I)
Y=Y+X*ZZ
E=ZZ/Y
DO 250 I=1,N
250 T(I)=T(I)-B*W(I)
DO 260 I=1,NN
R(I)=0.0D0
DO 260 J=1,N
260 R(I)=R(I)+A(I,J)*T(J)
DO 270 I=1,NN
270 R(I)=R(I)-G(I)
RR=0.0D0
DO 280 I=1,NN
280 RR=RR+R(I)*R(I)
AB=0.0D0
DO 290 I=1,N
290 AB=AB+T(I)*T(I)
AB=RR+X*AB
IF(K.EQ.1) GO TO 300
IF(AB.LT.AA) GO TO 300
DO 310 I=1,N
T(I)=F(I)
310 W(I)=WW(I)
DO 295 I=1,NN
295 R(I)=V(I)
RR=VV
ZZ=YY
K=K-1
GO TO 330
300 IF(M-K) 600,330,340
340 K=K+1
GO TO 1030
1040 K=K-1

```

```

SYST1860
SYST1870
SYST1880
SYST1890
SYST1900
SYST1910
SYST1920
SYST1930
SYST1940
SYST1950
SYST1960
SYST1970
SYST1980
SYST1990
SYST2000
SYST2010
SYST2020
SYST2030
SYST2040
SYST2050
SYST2060
SYST2070
SYST2080
SYST2090
SYST2100
SYST2110
SYST2120
SYST2130
SYST2140
SYST2150
SYST2160
SYST2170
SYST2180
SYST2190
SYST2200
SYST2210
SYST2220

```

330 LL=LL+K	SYST2230
IF(K.LT.M) GO TO 440	SYST2240
WRITE(6,112) X	SYST2250
112 FORMAT('0',3X,'AT LAMBDA=',D24.16,3X,'THE MAXIMUM VALUE OF M HAS	SYST2260
CBEEEN ATTAINED')	SYST2270
440 IF(X.EQ.BB) GO TO 360	SYST2280
XX=XLAM*X	SYST2290
IF(XX.LT.BB) GO TO 350	SYST2300
GO TC 1020	SYST2310
350 XX=BB	SYST2320
GO TO 1020	SYST2330
C	SYST2340
C RE-SCALE THE COMPUTATIONS AND OUTPUT.	SYST2350
C	SYST2360
360 WRITE(6,122)	SYST2370
122 FORMAT('0',3X,'FINAL SOLUTION F',10X,'ATAF+LAMBDA*F-ATG',/)	SYST2380
DO 370 I=1,N	SYST2390
T(I)=T(I)/CR	SYST2400
W(I)=W(I)/CR	SYST2410
370 WRITE(6,92) T(I),W(I)	SYST2420
WRITE(6,84)	SYST2430
DO 375 I=1,NN	SYST2440
R(I)=R(I)/CR	SYST2450
375 WRITE(6,96) R(I)	SYST2460
ZZ=ZZ/(CR*CR)	SYST2470
RR=RR/(CR*CR)	SYST2480
ZZ=DSQRT(ZZ)	SYST2490
RR=DSQRT(RR)	SYST2500
TT=0.0D0	SYST2510
DO 380 I=1,N	SYST2520
380 TT=TT+T(I)*T(I)	SYST2530
TT=DSQRT(TT)	SYST2540
WRITE(6,132)	SYST2550
132 FORMAT('0',8X,' F ',12X,' ATAF+LAMBDA*F-ATG ',10X,	SYST2560
C' AF-G ',/)	SYST2570
WRITE(6,92) TT,ZZ,RR	SYST2580
DO 390 I=1,N	SYST2590

```

      V(I)=0.000
      DO 390 J=1,NN
390   V(I)=V(I)+AT(I,J)*R(J)
      VVV=0.000
      DO 400 I=1,N
400   VVV=VVV+V(I)*V(I)
      VVV=DSQRT(VVV)
      WRITE(6,144) VVV
144   FORMAT('0',8X,'|| AT(AF-G) ||=',D24,16)
      WRITE(6,142) LL
142   FORMAT('0','TOTAL ITERATIONS=',I6)
C
C   IF THE ACTUAL SOLUTION IS KNOWN, A PROGRAM TO COMPARE THE
C   ACTUAL SOLUTION WITH THE COMPUTED SOLUTION CAN BE PLACED
C   HERE. THE COMPUTED SOLUTION IS GIVEN BY THE VECTOR T. THE
C   VECTOR F CAN BE ASSIGNED THE ACTUAL SOLUTION.
C
600   STOP
      END

```

```

SYST2600
SYST2610
SYST2620
SYST2630
SYST2640
SYST2650
SYST2660
SYST2670
SYST2680
SYST2690
SYST2700
SYST2710
SYST2720
SYST2730
SYST2740
SYST2750
SYST2760
SYST2770
SYST2780

```

8.2. Appendix B, Subroutine BITER

Subroutine BITER is a program written to find a λ -approximate least squares solution of minimum norm of an integral equation of the first kind. The subroutine BITER assumes that the integral equation is of the form

$$\int_0^1 k(y,x)f(x)dx = g(y), \quad y \in [0,1].$$

A driver program must be written which obtains the following information:

1. Partitions the interval integrated over, $[0,1]$, into $N-1$ equal subintervals for an N -point closed Newton-Cotes integration formula. The N -points constructed are stored in the vector XV .
2. Partitions the domain of g , $[0,1]$, into $NN-1$ equal subintervals for an NN -point closed Newton-Cotes integration formula. The NN -points constructed are stored in the vector YV .
3. Construct the vector $GV(I) = g(YV(I))$ $I = 1, NN$.

4. Construct the matrix representing the kernel

$$k(x,y), A(k(XV(I),YV(j)))_{\substack{j=1,NN \\ I=1,N}}$$
5. Construct the vector WTK of weights for the kernel using a closed N-point Newton-Cotes formula.
6. Construct the vector WTKT of weights for integration of the adjoint of the kernel using a closed NN-point Newton-Cotes formula.
7. The driver program must be dimensioned as the dimension statement for the subroutine BITER.
8. The driver program must supply the controls on convergence given by

XLMDA = Lambda multiplier,

BB = Terminal lambda,

and

DD = control on $\|ATAF + LAMBDA * F - ATG\|^2$.

The output section of the starting vector and initial lambda begins at line BITR1080. The output section for the final results begins at line BITR1990.

```

SUBROUTINE BITER(N,NN,A,GV,WTk,WTkT,XV,YV,F,XLMDA,M,B3,
CAT,ATA,R,T,V,W,WW,S,DD)
  IMPLICIT REAL*8 (A-H,O-Z)
  DIMENSION A(NN,N),AT(N,NN),WTk(N),WTkT(NN),XV(N),YV(NN),F(N)
  DIMENSION ATA(N,N),GV(NN),R(NN),T(N),V(NN),W(N),WW(N),S(NN)
  LL=C
C
C   CONSTRUCT THE ADJOINT=AT TO KERNAL=A.
C
C   TRANSPOSE THE UNWEIGHTED KERNEL.
C
      DO 510 I=1,NN
      DO 510 J=1,N
510  AT(J,I)=A(I,J)
C
C   CONSTRUCT THE WEIGHTED KERNEL.
C
      DO 520 I=1,NN
      DO 520 J=1,N
520  A(I,J)=A(I,J)*WTk(J)
C
C   CONSTRUCT THE WEIGHTED KERNEL TRANSPOSE.
C
      DO 530 I=1,N
      DO 530 J=1,NN
530  AT(I,J)=AT(I,J)*WTkT(J)
C
C   TEST FOR A ZERO SOLUTION.
C
      DO 810 I=1,N
      W(I)=0.0D0
      DO 810 J=1,NN
810  W(I)=W(I)+AT(I,J)*GV(J)
      ZZ=0.0D0
      DO 820 I=1,N
820  ZZ=ZZ+WTk(I)*(W(I)*W(I))

```

```

BITR0010
BITR0020
BITR0030
BITR0040
BITR0050
BITR0060
BITR0070
BITR0080
BITR0090
BITR0100
BITR0110
BITR0120
BITR0130
BITR0140
BITR0150
BITR0160
BITR0170
BITR0180
BITR0190
BITR0200
BITR0210
BITR0220
BITR0230
BITR0240
BITR0250
BITR0260
BITR0270
BITR0280
BITR0290
BITR0300
BITR0310
BITR0320
BITR0330
BITR0340
BITR0350
BITR0360
BITR0370

```

```

      ZZ=DSGRT(ZZ)
      WRITE(6,29) ZZ
29  FORMAT('0','|| ATG ||=' ,D24.16)
      IF(ZZ.GE.1.D-40) GO TO 830
      WRITE(6,821)
821  FORMAT('0',6X,'SOLUTION IS THE ZERO FUNCTION',/)
      GO TO 620
C
C   CONSTRUCT THE WEIGHTED KERNEL TRANSPOSE*WEIGHTED KERNEL.
C
830  DO 540 I=1,N
      DO 540 J=1,N
      ATA(I,J)=0.0D0
      DO 540 L=1,NN
540  ATA(I,J)=ATA(I,J)+AT(I,L)*A(L,J)
C
C   OBTAIN AN ACTUAL STARTING VECTOR AND INITIAL LAMBDA.
C
      DO 20 I=1,N
      T(I)=0.0D0
      DO 20 J=1,N
20  T(I)=T(I)+ATA(I,J)*F(J)
      ZZ=0.0D0
      DO 30 I=1,N
30  ZZ=ZZ+WTK(I)*(T(I)*T(I))
      ZZ=DSGRT(ZZ)
      DO 40 I=1,N
40  T(I)=T(I)/ZZ
      DO 50 I=1,NN
      R(I)=0.0D0
      DO 50 J=1,N
50  R(I)=R(I)+A(I,J)*T(J)
      CR=1.0D0
      U=2.0D0
      J=1
S2  X=0.0D0
      DO 60 I=1,NN

```

```

BITR0380
BITR0390
BITR0400
BITR0410
BITR0420
BITR0430
BITR0440
BITR0450
BITR0460
BITR0470
BITR0480
BITR0490
BITR0500
BITR0510
BITR0520
BITR0530
BITR0540
BITR0550
BITR0560
BITR0570
BITR0580
BITR0590
BITR0600
BITR0610
BITR0620
BITR0630
BITR0640
BITR0650
BITR0660
BITR0670
BITR0680
BITR0690
BITR0700
BITR0710
BITR0720
BITR0730
BITR0740

```



```

60 X=X+(GV(I)-R(I))*(R(I)*WTKT(I))
   IF(X.GE.0.0D0) GO TO 1000
   IF(J.EQ.2) GO TO 1001
   J=J+1
   DO 70 I=1,N
70 T(I)=-T(I)
   DO 80 I=1,NN
80 R(I)=-R(I)
   GO TO 92
1001 DO 450 I=1,NN
450 GV(I)=U*GV(I)
   J=1
   CR=CR*U
   GO TO 92
1000 DO 90 I=1,NN
90 R(I)=R(I)-GV(I)
   RR=0.0D0
   DO 100 I=1,NN
100 RR=RR+WTKT(I)*(R(I)*R(I))
   DO 110 I=1,N
   W(I)=0.0D0
   DO 110 J=1,NN
110 W(I)=W(I)+AT(I,J)*R(J)
   ZZ=0.0D0
   DO 120 I=1,N
120 ZZ=ZZ+WTK(I)*(W(I)*W(I))
   YY=ZZ-X*X
   AB=RR+X
   DO 130 I=1,N
130 W(I)=W(I)+X*T(I)

```

```

C
C   OUTPUT SECTION FOR THE STARTING INFORMATION.
C

```

```

   WRITE(6,2) X
2  FORMAT('0',' INITIAL LAMBDA=',D24.16)
   WRITE(6,3)
3  FORMAT('0',12X,'X=',18X,' STARTING=',12X,' ATAF+LAMBDA*F-ATG=',/)

```

```

BITR0750
BITF0760
BITR0770
BITR0780
BITR0790
BITR0800
BITF0810
BITR0820
BITR0830
BITR0840
BITR0850
BITF0860
BITR0870
BITR0880
BITF0890
BITR0900
BITF0910
BITR0920
BITR0930
BITF0940
BITR0950
BITF0960
BITR0970
BITF0980
BITF0990
BITR1000
BITF1010
BITR1020
BITF1030
BITF1040
BITR1050
BITF1060
BITR1070
BITF1080
BITR1090
BITR1100
BITF1110

```

```

      DO 140 I=1,N
140  WRITE(6,4) XV(I),T(I),W(I)
      4  FORMAT(' ',3D24.16)
      WRITE(6,5)
      5  FORMAT('0',12X,'Y=',18X,'AF-G=',/)
      DO 150 I=1,NN
150  WRITE(6,6) YV(I),R(I)
      6  FORMAT(' ',2D24.16)
      WRITE(6,7) CR
      7  FORMAT('0','SCALING FACTOR=',D24.16)
      WRITE(6,8) YY
      8  FORMAT('0','MINIMUM VALUE=',D24.16)
      XX=X
C
C  THE MAIN ITERATIONS OF THE ALGORITHM.
C
171  X=XX
      K=1
      41  AA=AB
      YY=ZZ
      DO 150 I=1,N
      WW(I)=W(I)
      W(I)=0.000
      DO 160 J=1,NN
160  W(I)=W(I)+AT(I,J)*R(J)
      DO 170 I=1,N
170  W(I)=W(I)+X*T(I)
      ZZ=0.000
      DO 180 I=1,N
180  ZZ=ZZ+WTK(I)*(W(I)*W(I))
      IF(ZZ.LT.DD) GO TO 403
      DO 190 I=1,NN
190  V(I)=R(I)
      DO 200 I=1,N
200  F(I)=T(I)
      VV=RR
      DO 210 I=1,NN

```

```

BITR1120
BITR1130
BITR1140
BITR1150
BITF1160
BITR1170
BITR1180
BITF1190
BITR1200
BITF1210
BITR1220
BITF1230
BITF1240
BITR1250
BITF1260
BITR1270
BITF1280
BITF1290
BITR1300
BITF1310
BITR1320
BITR1330
BITF1340
BITR1350
BITF1360
BITR1370
BITF1380
BITF1390
BITR1400
BITF1410
BITR1420
BITF1430
BITR1440
BITR1450
BITF1460
BITR1470
BITF1480

```

```

      S(I)=0.000
      DO 210 J=1,N
210   S(I)=S(I)+A(I,J)*W(J)
      Y=0.000
      DO 220 I=1,NN
220   Y=Y+WTKT(I)*(S(I)*S(I))
      Y=Y+X*ZZ
      B=ZZ/Y
      DO 230 I=1,N
230   T(I)=T(I)-B*W(I)
      DO 240 I=1,NN
      R(I)=0.000
      DO 240 J=1,N
240   R(I)=R(I)+A(I,J)*T(J)
      DO 250 I=1,NN
250   R(I)=R(I)-GV(I)
      RR=0.000
      DO 260 I=1,NN
260   RR=RR+WTKT(I)*(R(I)*R(I))
      AB=0.000
      DO 270 I=1,N
270   AB=AB+WTK(I)*(T(I)*T(I))
      AB=RR+X*AB
      IF(K.EQ.1) GO TO 201
      IF(AB.LT.AA) GO TO 201
      DO 280 I=1,N
      T(I)=F(I)
280   W(I)=WW(I)
      DO 290 I=1,NN
290   R(I)=V(I)
      RR=VV
      ZZ=YY
      K=K-1
      GO TO 401
201   IF(M-K) 620,401,301
301   K=K+1
      GO TO 41

```

```

      BITF1490
      BITR1500
      BITF1510
      BITR1520
      BITR1530
      BITF1540
      BITR1550
      BITF1560
      BITR1570
      BITF1580
      BITF1590
      BITR1600
      BITF1610
      BITR1620
      BITR1630
      BITF1640
      BITR1650
      BITF1660
      BITR1670
      BITF1680
      BITF1690
      BITR1700
      BITF1710
      BITR1720
      BITF1730
      BITF1740
      BITR1750
      BITF1760
      BITR1770
      BITF1780
      BITR1790
      BITR1800
      BITF1810
      BITR1820
      BITF1830
      BITF1840
      BITR1850

```

403 K=K-1	BITR1860
401 LL=LL+K	BITR1870
IF(K.LT.M) GO TO 440	BITR1880
WRITE(6,1) X	BITR1890
1 FORMAT('0',3X,'AT LAMBDA=',D24.16,3X,'THE MAXIMUM VALUE OF M HAS	BITR1900
CBEEN ATTAINED')	BITR1910
440 IF(X.EQ.BB) GO TO 600	BITR1920
XX=XLMDA*X	BITR1930
IF(XX.LT.BB) GO TO 610	BITR1940
GO TO 171	BITR1950
610 XX=BB	BITR1960
GO TO 171	BITR1970
C	BITR1980
C OUTPUT SECTION FOR THE FINAL RESULTS.	BITR1990
C	BITR2000
600 WRITE(6,23)	BITR2010
23 FORMAT('0',12X,'X=',14X,'FINAL SOLUTION F=',10X,	BITR2020
C'ATAF+LAMBDA*F-ATG=',/)	BITR2030
DO 320 I=1,N	BITR2040
T(I)=T(I)/CR	BITR2050
W(I)=W(I)/CR	BITR2060
320 WRITE(6,4) XV(I),T(I),W(I)	BITR2070
WRITE(6,5)	BITR2080
DO 330 I=1,NN	BITR2090
R(I)=R(I)/CR	BITR2100
330 WRITE(6,6) YV(I),R(I)	BITR2110
ZZ=ZZ/(CR*CR)	BITR2120
RR=RR/(CR*CR)	BITR2130
ZZ=DSQRT(ZZ)	BITR2140
RR=DSQRT(RR)	BITR2150
WRITE(6,26) ZZ,RR	BITR2160
26 FORMAT('0',', ATAF+LAMBDA*F-ATG =',D24.16,5X,', AF-G =',	BITR2170
CD24.16)	BITR2180
WRITE(6,28) LL	BITR2190
28 FORMAT('0',', TOTAL ITERATIONS=',I6)	BITR2200
CC=0.0D0	BITR2210
DO 340 I=1,N	BITR2220

```

340 CC=CC+WTK(I)*(T(I)*T(I))
    CC=DSQRT(CC)
    WRITE(6,9) CC
    9 FORMAT('0', ' NORM OF THE SOLUTION=', D24.16)
    DO 390 I=1,N
    V(I)=0.000
    DO 390 J=1,NN
390 V(I)=V(I)+AT(I,J)*R(J)
    VVV=0.000
    DO 400 I=1,N
400 VVV=VVV+WTKT(I)*(V(I)*V(I))
    VVV=DSQRT(VVV)
    WRITE(6,144) VVV
144 FORMAT('0', 8X, '|| AT(AF-G) ||=', D24.16)
620 RETURN
    END

```

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BITF2230
BITF2240
BITR2250
BITF2260
BITR2270
BITF2280
BITF2290
BITR2300
BITF2310
BITR2320
BITF2330
BITR2340
BITR2350
BITF2360
BITR2370
BITF2380

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