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EXACT FORM FOR SCATTERING AMPLITUDE IN REGGE POLE THEORY*

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Summary

The exact form for the contribution of a single Regge pole is given both for positive and negative angular momentum without making the usual approximations. This exact form involves Legendre functions of the first and second kind with non-integral index and argument greater than one. These functions were calculated using the IBM 7090 computer at Brookhaven and are plotted in this paper. A one-parameter fit is made to the high energy P-P elastic scattering data which gives good agreement with experiment.

Introduction

According to the Regge pole theory elastic scattering at high energies is dominated by the highest angular momentum Regge pole which is called the vacuum pole or Pomeranchuk pole. In Eq. (6) we give an exact form for the contribution of a single pole without making the usual approximations. The form given in this paper is in terms of Legendre functions of non-integral index which we have calculated and which are plotted in Figures 1 to 4. These plots along with Eq. (6) make it a simple task to

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determine the angular momentum values $\alpha(t)$ from the experimental data.

If the usual approximations are made the form given for the scattering cross section is $\frac{d\sigma}{dt} = F(t)(s/2M^2)^{2\alpha-2}$ where $F(t)$ is a completely arbitrary function.[1] The question arises as to whether the theory can say something about the form of $F(t)$. If we make the assumption[2] that the residue $\beta(t)$ of the vacuum pole is constant along its trajectory, then our Eq. (6) gives an explicit prediction for $F(t)$ which as we shall see in Figure 5 happens to fit the high energy P-P scattering data very well.

The Scattering Amplitude

We write the scattering amplitude due to the vacuum pole as

$$A^{+}(s,t) = \beta(t)\alpha(2\alpha+1)(1+e^{-i\pi\alpha})\frac{P_\alpha(z)}{\sin \pi\alpha} \qquad (1)$$

where the factor $(1+e^{-i\pi\alpha})$ provides the even signature, the factor α is to remove the "ghost" singularity at $\alpha = 0$,[3] and where[4]

$$z = \frac{2s}{4M^2-t} - 1 \quad . \qquad (2)$$

Recently S. Mandelstam, by taking the contour in the angular momentum plane to large negative α rather than to $\alpha = -\frac{1}{2}$, has managed to eliminate the usual background integral and has extended the formalism to values of α less than $-\frac{1}{2}$.[5] In his paper Mandelstam finds that $P_\alpha(z)$ must be replaced with

$$M_\alpha(z) = \frac{\tan\pi\alpha}{\pi} Q_{-\alpha-1}(z)$$

where $Q_\ell(z)$ is the Legendre function of the second kind of angular momentum ℓ as defined on page 317 of Whittaker and Watson.[6] Except in the region near $\alpha = \frac{1}{2}$, $P_\alpha(z) \approx M_\alpha(z)$ for positive α. However, for negative α P_α and M_α become quite different and it is clear that the Mandelstam form $M_\alpha(z)$ should then be used.

An apparent difficulty with the Mandelstam form $M_\alpha(z)$ is that as α approaches $\frac{1}{2}$, $M_\alpha(z)$ approaches infinity. However, if we consider the Regge poles for a non-relativistic Schroedinger equation with an ordinary potential, it can be shown that when there is a pole at $\alpha = \frac{1}{2}$, there must also be a conjugate Regge pole having angular momentum $\alpha' = -\frac{3}{2}$.[5] We find this pole contributes an amplitude A' which when added to the amplitude

$$A^{-} = \beta(t)\alpha(2\alpha+1)(1+e^{-i\pi\alpha}) \frac{M_\alpha(z)}{\sin\pi\alpha} \qquad (3)$$

gives a total amplitude

$$A_{total} = \beta(t)\alpha(2\alpha+1)(1+e^{-i\pi\alpha}) \frac{P_\alpha(z)}{\sin\pi\alpha} \quad . \qquad (4)$$

Hence we recommend using Eq. (4) [which happens to be the same as A^+ in Eq. (1)] when computing scattering cross sections corresponding to positive α, and using Eq. (3) when in the region of negative α. We note that at $\alpha = 0$ where we recommend changing over from $P_\alpha(z)$ to $M_\alpha(z)$, both forms, Eqs. (3) and (4), are exactly equal for all z.

Using the relations $\frac{d\sigma}{d\omega} = \frac{|A|^2}{s}$ and $\frac{d\sigma}{dt} = \frac{\pi}{p^2}\frac{d\sigma}{d\omega}$, we have

$$\frac{d\sigma}{dt} = \frac{4\pi|A|^2}{s(s-4M^2)} \quad . \tag{5}$$

At $t = 0$ the vacuum pole has angular momentum $\alpha = 1$, and then

$$\frac{d\sigma}{dt} = 36\pi\beta^2(0)\,\frac{P_1^{\,2}(z)}{s(s-4M^2)} \qquad \text{for } t = 0 \; .$$

If we divide Eq. (5) by this, we obtain the quantity X which is defined as the ratio of $\frac{d\sigma}{dt}$ to its value at the same energy, but at 0^o or $t = 0$. Using $P_1(z) = z$, we obtain the result

$$X^+ = \frac{\beta^2(t)}{\beta^2(0)}\,\frac{\alpha^2(2\alpha+1)^2}{9\sin^2\pi\frac{\alpha}{2}}\,\frac{P_\alpha^{\,2}(z)}{(\frac{s}{2M^2}-1)^2} \qquad \text{for } \alpha > 0$$

$$X^- = \frac{\beta^2(t)}{\beta^2(0)}\,\frac{4\alpha^2(2\alpha+1)^2\cos^2\pi\frac{\alpha}{2}}{9\pi^2\cos^2\pi\alpha}\,\frac{Q_{-\alpha-1}^{\;2}(z)}{(\frac{s}{2M^2}-1)^2} \qquad \text{for } \alpha \leqslant 0 \tag{6}$$

where $z = (\frac{2s}{4M^2-t} - 1)$.

Using the IBM 7090 at Brookhaven National Laboratory we have calculated the P and Q functions needed for use in Eq. (6).[7] The P functions for the region $-\frac{1}{2} < \alpha < 1$ are plotted in Figures 1 and 2. Other P functions may be easily obtained from Figures 1 and 2 using the recursion relations

$$P_{-\alpha-1} = P_\alpha$$

$$(2\alpha+1)P_\alpha = (\alpha+1)P_{\alpha+1} + \alpha P_{\alpha-1} \quad .$$

The Q functions $Q_{-\alpha-1}(z)$ for the region $-2 < \alpha < 0$ are plotted in Figures 3 and 4. Other Q functions may be obtained by using the recursion relation

$$P_\alpha = \frac{1}{\pi} \tan\pi\alpha(Q_\alpha - Q_{-\alpha-1}) \quad . \qquad (7)$$

For $z > 5$, Figures 3 and 4 agree within 1% with the asymptotic form

$$Q_{-\alpha-1}(z) \simeq \sqrt{\pi} \frac{\Gamma(-\alpha)}{\Gamma(-\alpha+\frac{1}{2})}(2z)^\alpha \quad .$$

The asymptotic form for $P_\alpha(z)$ is obtained by substituting the above into Eq. (7).

Comparison With Experiment

We have tried fitting all the existing P-P scattering data above 10 Gev[8] with Eq. (6) by setting $\frac{\beta(t)}{\beta(0)} = 1$. This gives us an exact form for F(t) and permits an accurate determination of α for each individual value of X.[2] Individual solutions for α so obtained from 16 different experimental cross sections are plotted in Figure 5. It is encouraging to see that in the several cases where two different points have about the same value of t, the same value of α is obtained even though the beam energies may be a factor two apart. The usual method[8] of calculating α by comparing two different cross sections at the same t, but different s, therefore gives solutions for α about the same as ours, but with considerably less accuracy. A smooth curve is drawn through our 16 values of α in Figure 5 and it is seen that Eq. (6) gives a good one-parameter fit to all the high energy P-P scattering data.

References

1. Frautschi, Gell-Mann, and Zachariasen, Phys. Rev. 126, 2204 (1962).
2. Hadjioannou, Phillips, and Rarita in Phys. Rev. Letters 9, 183 (1962) also assume $\beta(t)$ is constant in order to get a form for $F(t)$ in the region of the diffraction peak ($-t < 1.0\ (\mathrm{Gev}/c)^2$) using a 3 pole fit to the data. They do not extend their formalism to the region of negative α.
3. M. Gell-Mann, Proceedings of the 1962 International Conference on High Energy Physics at CERN.
4. In our calculations we have used z as given by Eq. (2). In the previous reference it is suggested that t should be omitted from the denominator in order to give the correct threshold dependence in the t channel. We have not done so; this makes very little difference in the quantitative results.
5. S. Mandelstam, Ann. Phys. 19, 254 (1962).
6. E. T. Whittaker and G. N. Watson, Modern Analysis (Cambridge University Press, 1927).
7. Detailed tables of these Legendre functions are available as a Brookhaven National Laboratory report BNL-0000.
8. Baker, Jenkins, Read, Cocconi, Cocconi, and Orear, Phys. Rev. Letters 9, 221 (1962); Diddens, Lillethun, Manning, Taylor, Walker, and Wetherell, Phys. Rev. Letters 9, 111 (1962).

Figure Captions

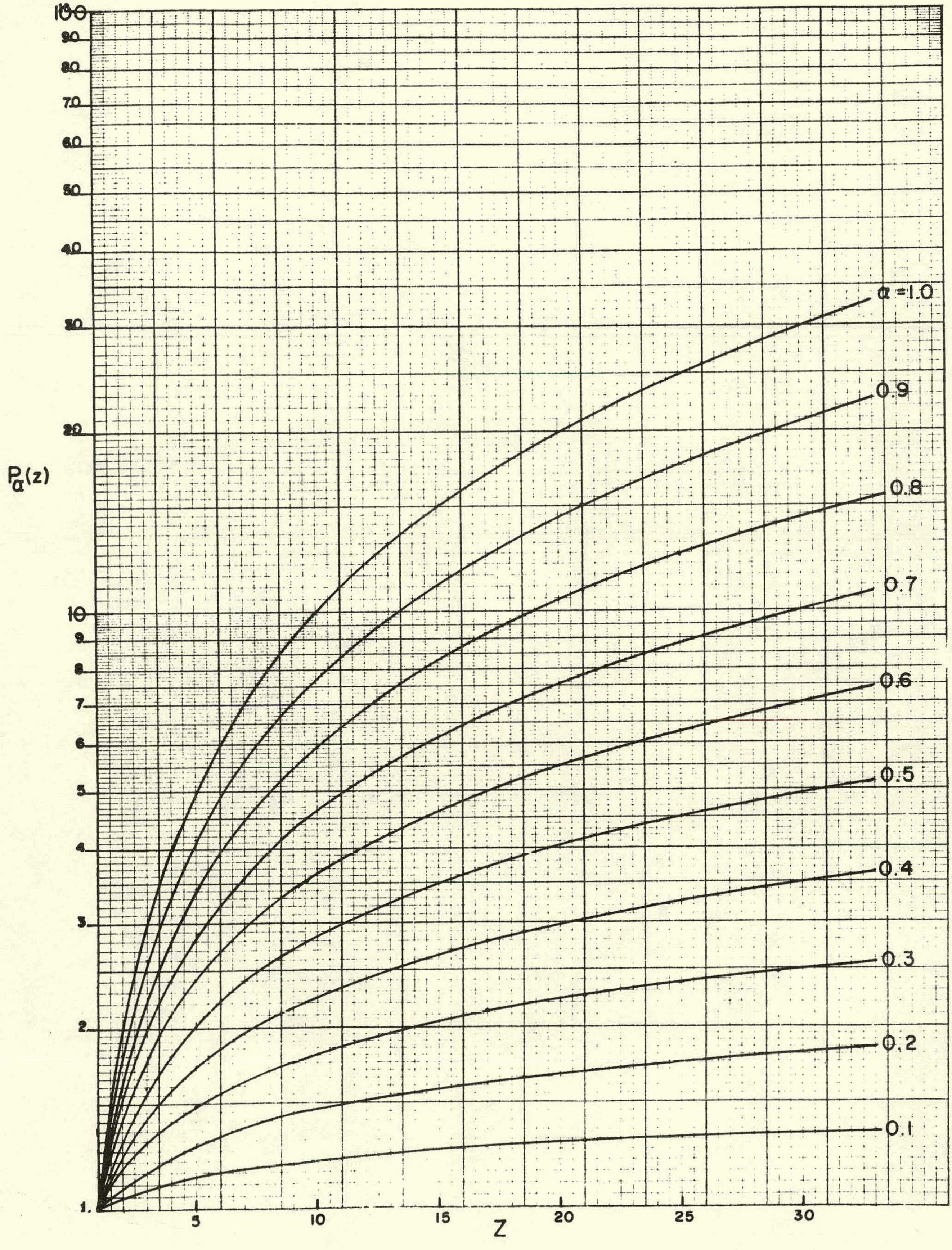
Figure 1. The Legendre function $P_{\alpha}(z)$ vs. z for values of α from 0.1 to 1.0.

Figure 2. The Legendre function $P_{\alpha}(z)$ vs. z for values of α from -0.5 to 0.

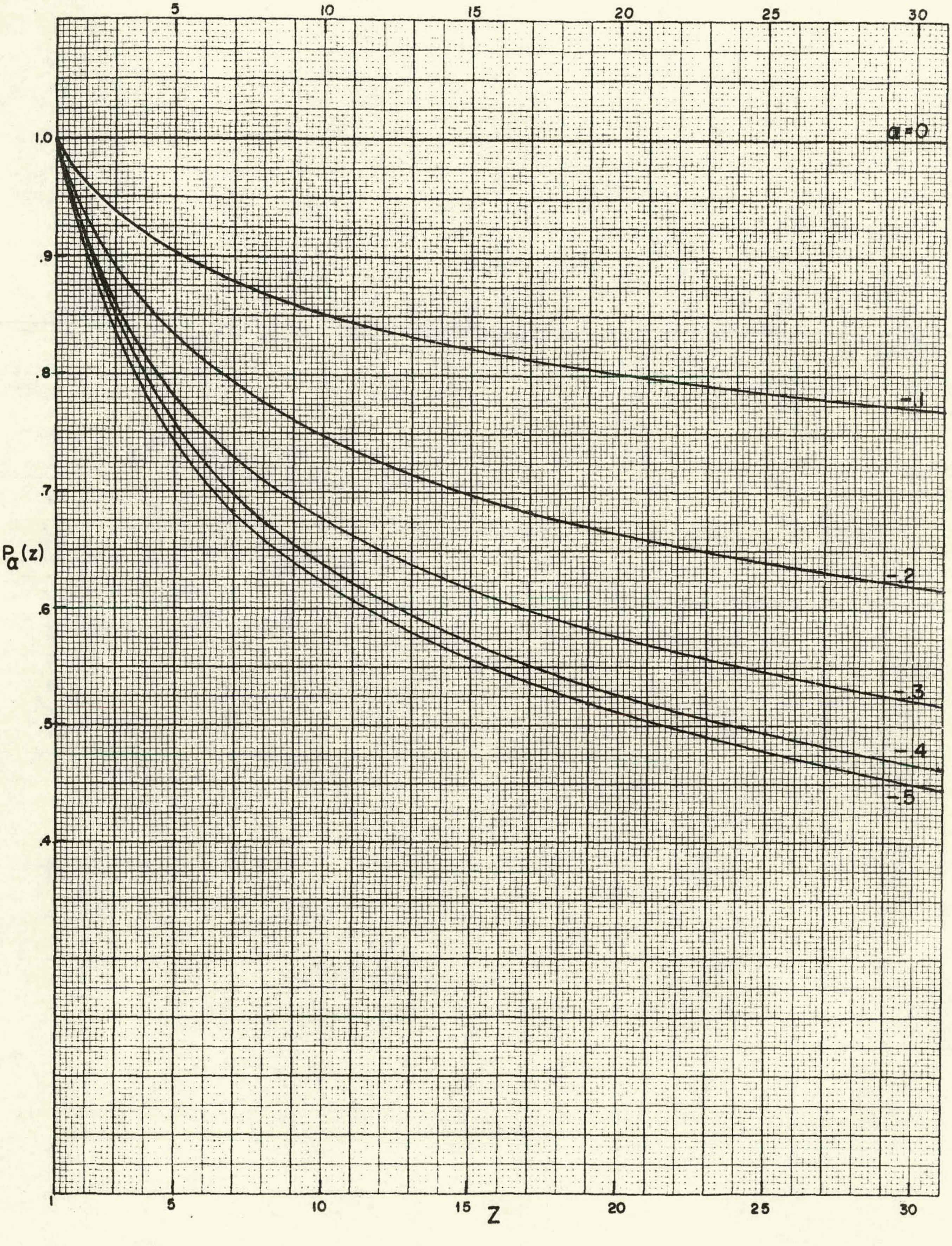
Figure 3. The Legendre function of the second kind $Q_{-\alpha-1}(z)$ vs. z for values of α from -0.1 to -1.5.

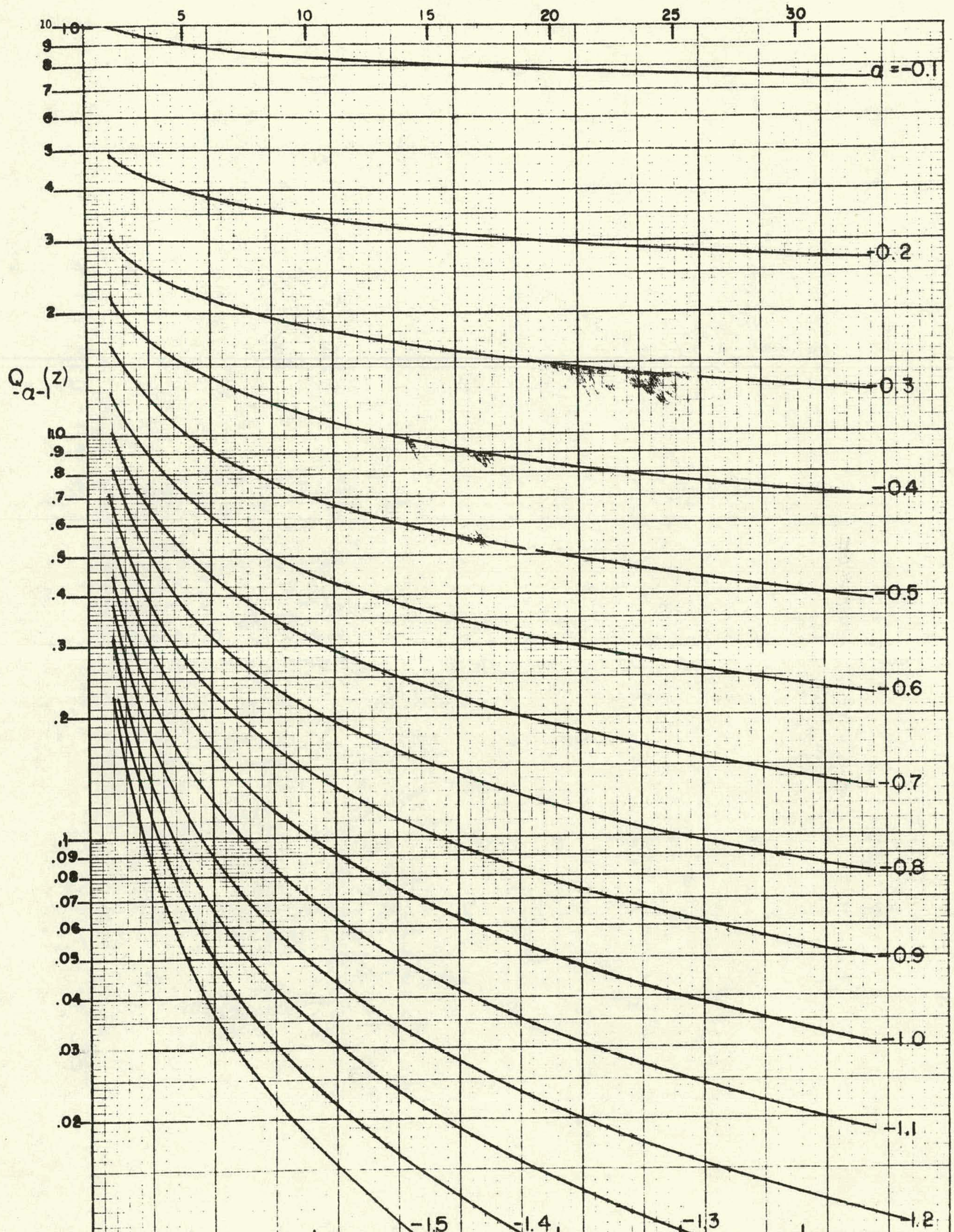
Figure 4. The Legendre function of the second kind $Q_{-\alpha-1}(z)$ vs. z for values of α from -1.3 to -2.0.

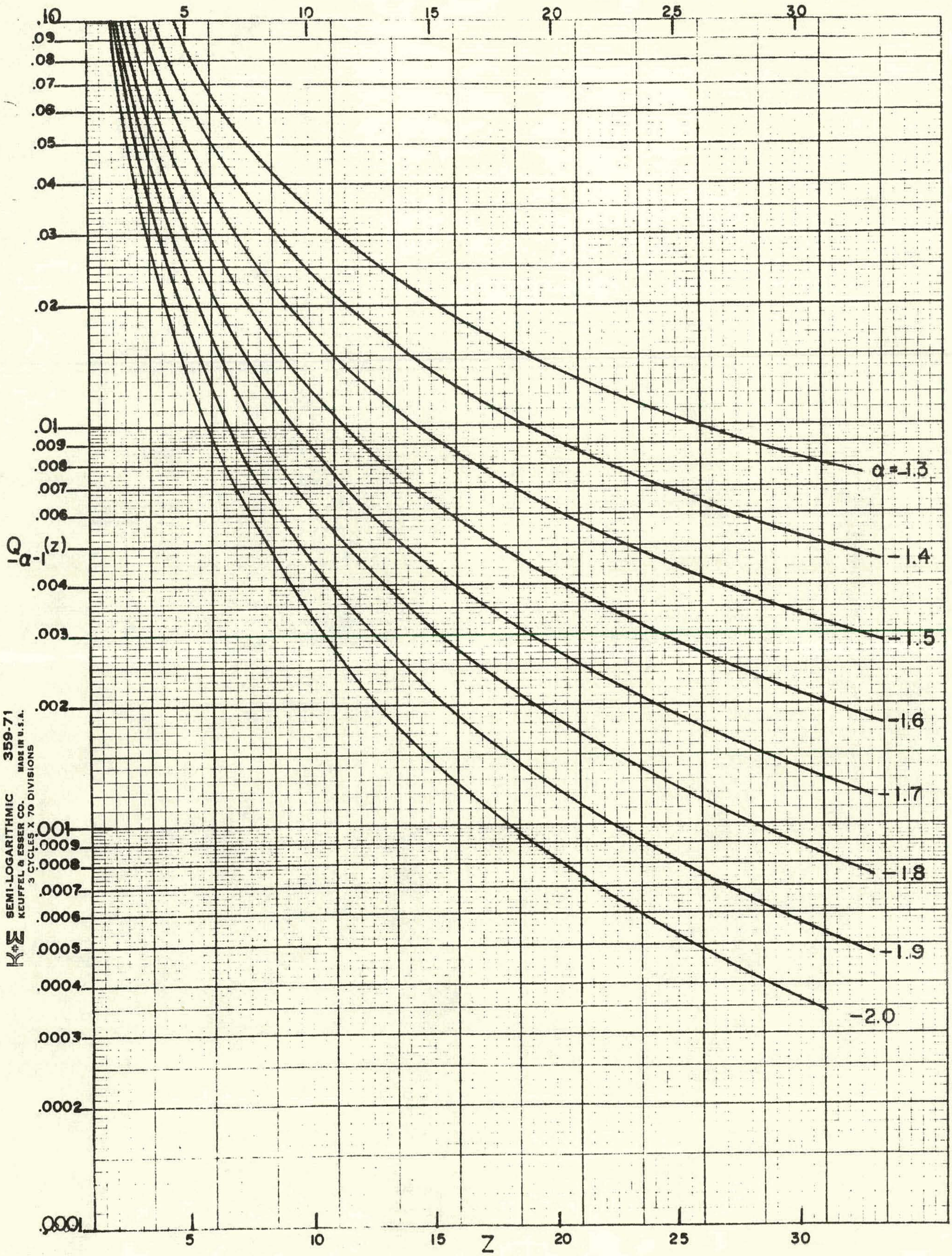
Figure 5. The angular momentum α of the vacuum trajectory vs. t, the 4-momentum transfer squared. The experimental points are solutions of Eq. (6) (assuming $\frac{\beta(t)}{\beta(0)} = 1$) corresponding to the proton-proton scattering cross sections given in Reference 8.



100
90
80
70
60
50
40
30
20
$P_\alpha(z)$
10
9
8
7
6
5
4
3
2
1.
5
10
15
20
25
30
Z
α = 1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1


5
10
15
20
25
30
1.0
.9
.8
.7
.6
.5
.4
α=0
-.1
-.2
-.3
-.4
-.5
$P_\alpha(z)$
1
5
10
15
20
25
30
Z


$Q_{-a-1}(z)$
Z
a = -0.1
-0.2
-0.3
-0.4
-0.5
-0.6
-0.7
-0.8
-0.9
-1.0
-1.1
-1.2
-1.3
-1.4
-1.5



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.09
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$Q_{-\alpha-1}(z)$
.004
.003
.002
.001
.0009
.0008
.0007
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.0005
.0004
.0003
.0002
.0001
5
10
15
20
25
30
Z
α = -1.3
-1.4
-1.5
-1.6
-1.7
-1.8
-1.9
-2.0
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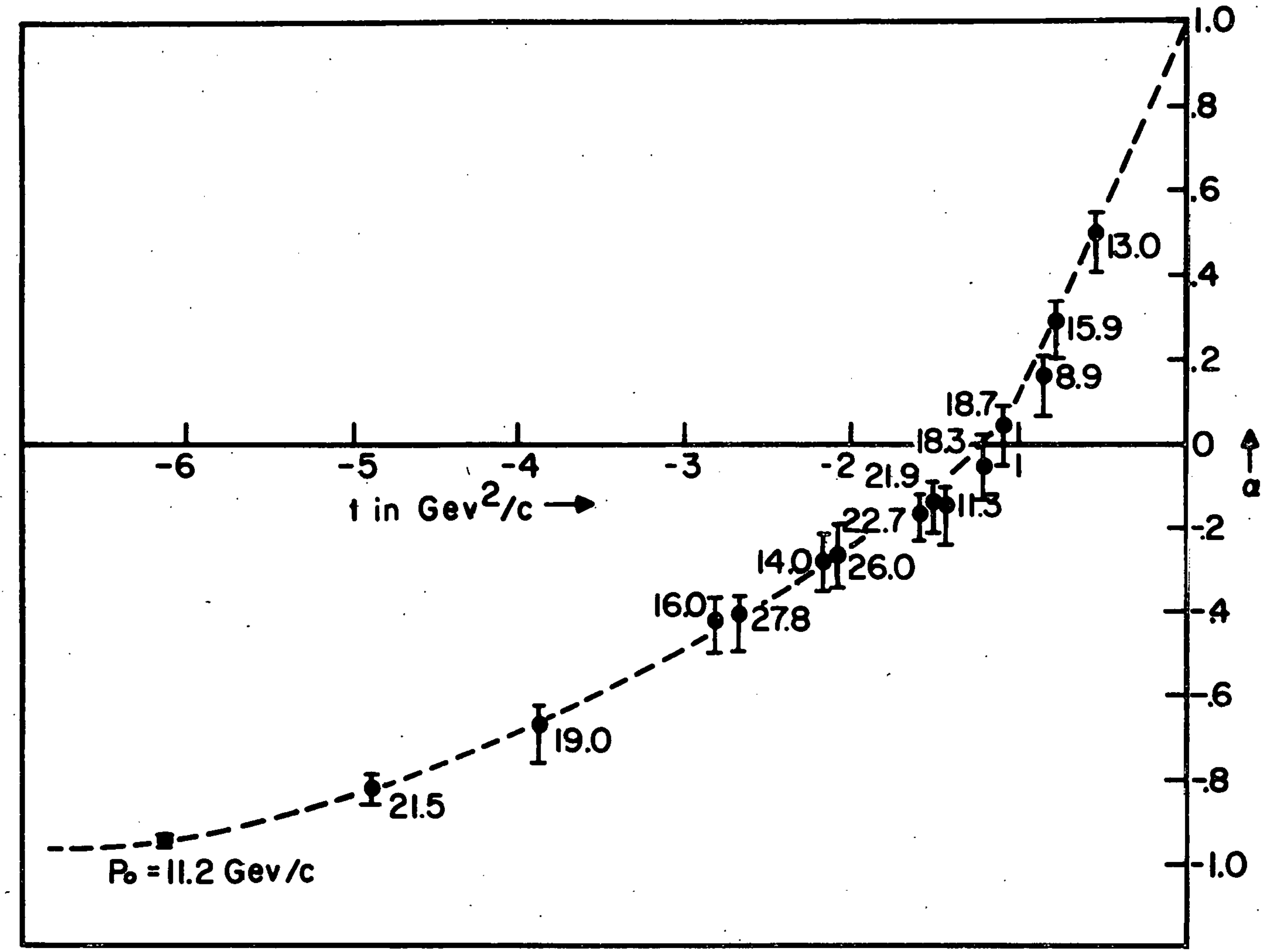

1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
α
-6
-5
-4
-3
-2
-1
t in Gev²/c
13.0
15.9
8.9
18.7
18.3
21.9
11.3
22.7
14.0
26.0
16.0
27.8
19.0
21.5
P₀ = 11.2 Gev/c