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SIMPLEX, A CODE FOR THE SOLUTION OF LINEAR PROGRAMMING PROBLEMS

Hoyt Walker

Laurence S. Hall

June 1975

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LAWRENCE LIVERMORE LABORATORY

University of California, Livermore, California, 94550

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SIMPLEX, A CODE FOR THE SOLUTION OF LINEAR PROGRAMMING PROBLEMS

Abstract

A set of procedures is described for solution of a general linear programming problem that seeks to maximize the linear functional $W = \sum_{j=1}^n c_j x_j$ for coordinates $x_j \geq 0$, subject to m restrictions of the form $\sum_{j=1}^n a_{ij} x_j \leq b_i$ and ℓ restrictions of the form $\sum_{j=1}^n a_{ij} x_j = b_i$. An LRLTRAN computer code, which performs the maximization, has been developed to follow these procedures and is also described. Illustration of the use of the simplex procedure is given.

1. The Linear Programming Problem

A general linear programming problem is one that seeks to maximize the linear functional

$$W = \sum_{j=1}^n c_j x_j \quad (1)$$

for coordinates $x_j \geq 0$ subject to the set of m inequality constraints

$$\sum_{j=1}^n a_{ij} x_j \leq b_i, \quad i = 1, 2, \dots, m \quad (2)$$

and ℓ equality constraints

$$\sum_{j=1}^n a_{ij} x_j = b_i, \quad i = m+1, m+2, \dots, m+\ell. \quad (3)$$

Because the set of points satisfying the constraints is convex (that is, because, in the n -dimensional hyperspace of the vector $\{x_j\}$, every point that lies on a straight line between any two points satisfying the constraints itself satisfies the constraints), the extrema of W will be obtained at one of the extreme points (vertices) of the constraint set.¹ If we introduce the set of slack variables

$$u_i = b_i - \sum_{j=1}^n a_{ij}x_j, \quad (4)$$

then the constraints take the form $x_j \geq 0$, $u_i \geq 0$ [$u_i = 0$, for $(m+1) \leq i \leq (m+l)$], and a vertex corresponds to a point at which n of the variables $\{u_i, x_j\}$ vanish.

If $\{\xi_j\}$, $1 \leq j \leq n$, is a subset of the variables $\{u_i, x_j\}$, all of which vanish when W is maximal, then clearly we may use Eq. (4) to rewrite W , Eq. (1), obtaining

$$W = \sum_{j=1}^n \gamma_j \xi_j + \delta, \quad (5)$$

where $W_{\max} = \delta$ and $\gamma_j \leq 0$, $1 \leq j \leq n$. Likewise $\{\eta_i\}$, $1 \leq i \leq m+l$, the balance of the set of $n+m+l$ variables $\{u_i, x_j\}$, is given in terms of the ξ_j by equations of the form

$$-\eta_i = \sum_{j=1}^n \alpha_{ij} \xi_j + \beta_i, \quad (6)$$

where $\beta_i \leq 0$, $1 \leq i \leq m+l$. The transformation of Eqs. (1) and (4) into Eqs. (5) and (6) with non-positive β_i and γ_j is accomplished by the simplex algorithm. We summarize the procedures of the simplex algorithm in a convenient form by a discussion close to that of G. Owen.¹

Equations (5) and (6) correspond in self-explanatory fashion to the simplex tableau:

	ξ_1	ξ_2	...	ξ_{n-l}	ξ_{n-l+1}	...	ξ_n	1
$-\eta_1$	α_{11}	α_{12}	...	$\alpha_{1,n-l}$	$\alpha_{1,n-l+1}$	...	$\alpha_{1,n}$	β_1
$-\eta_2$	α_{21}	α_{22}	...	$\alpha_{2,n-l}$	$\alpha_{2,n-l+1}$	...	$\alpha_{2,n}$	β_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$-\eta_m$	$\alpha_{m,1}$	$\alpha_{m,2}$	...	$\alpha_{m,n-l}$	$\alpha_{m,n-l+1}$	...	$\alpha_{m,n}$	β_m
$-\eta_{m+1}$	$\alpha_{m+1,1}$	$\alpha_{m+1,2}$	...	$\alpha_{m+1,n-l}$	$\alpha_{m+1,n-l+1}$	...	$\alpha_{m+1,n}$	β_{m+1}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$-\eta_{m+l}$	$\alpha_{m+l,1}$	$\alpha_{m+l,2}$	...	$\alpha_{m+l,n-l}$	$\alpha_{m+l,n-l+1}$	...	$\alpha_{m+l,n}$	β_{m+l}
W	γ_1	γ_2	...	γ_{n-l}	γ_{n-l+1}	...	γ_n	δ

The simplex algorithm systematically chooses pairs of variables (η_i, ξ_j) whose roles are to be exchanged, and then performs the exchanges, one at a time, until the set of coefficients $\{\beta_i, \gamma_j\}$ (within the freedom allowed by the l equality constraints) are all non-positive. The position $(i,j) = (I,J)$ within the simplex tableau that represents the column and row variables to be exchanged is called the pivot. The basic transformation, or pivot step, corresponding to the elimination of ξ_j in terms of η_i , results in the following changes in the simplex tableau:

$$\left. \begin{aligned}
 \alpha_{IJ} &\rightarrow 1/\alpha_{IJ} \\
 \alpha_{iJ} &\rightarrow -\alpha_{iJ}/\alpha_{IJ} \quad , \quad i \neq I \\
 \alpha_{Ij} &\rightarrow \alpha_{Ij}/\alpha_{IJ} \quad , \quad j \neq J \\
 \alpha_{ij} &\rightarrow \alpha_{ij} - (\alpha_{iJ}\alpha_{Ij}/\alpha_{IJ}) \quad , \quad i \neq I, j \neq J, \\
 \xi_J &\leftrightarrow \eta_I
 \end{aligned} \right\} \quad (7)$$

In the above, we read $\alpha_{i,n+1} = \beta_i$, $\alpha_{m+i+1,j} = \gamma_j$, and $\alpha_{m+i+1,n+1} = \delta$.

2. The Simplex Procedures

Beginning with the starting tableau, one first satisfies all equality constraints by pivoting into positions at the head of a column all those slack variables which are known to vanish. Further pivoting involving these slack variables is prohibited. This operation is accomplished by the initialization procedure, described below, which also interchanges the column ordering so that these slack variables which vanish a priori occupy columns $n - i < j < n + 1$. This produces the initialized tableau, upon which further pivoting is permitted only for $1 \leq i \leq m + i$, $1 \leq j \leq n - i$.

After obtaining the initialized tableau, the algorithm sets about finding a basic feasible point, namely a set of variables $\{\xi_j\}$ such that if each ξ_j is taken to vanish, $1 \leq j \leq n - i$, then all constraints $\eta_i \geq 0$ are also satisfied. Obviously, this requires that $\beta_i \leq 0$ for $1 \leq i \leq m + i$.

Procedure I, outlined below, serves to find some one basic feasible point. Then, via Procedure II, the algorithm moves from one basic feasible point to another along a path of nondecreasing W until the maximum feasible point, $\gamma_j \leq 0$ for $1 \leq j \leq n - i$, is reached. The procedures can be shown to converge in a finite number of steps in the absence of degeneracy (no β_i vanishes). Generally, where arbitrary choices are made at indicated points in the procedures, we specify that these are to be made strictly randomly. This avoids any possibility of looping that may otherwise occur in the presence of degeneracy. Other sequencing may, however, be more useful in large linear programming problems.²

INITIALIZATION PROCEDURE

If $\ell = 0$, go on to Procedure I. If $\ell > n$, the system is over-constrained, and no solution exists (in the absence of degeneracy). Otherwise, perform the initialization procedure sequentially for successively larger k , $0 \leq k \leq \ell - 1$, where $I = m + \ell - k$. If $\beta_I > 0$, change the sign of each entry in the I th row: $\alpha_{Ij} \rightarrow -\alpha_{Ij}$, $\beta_I \rightarrow -\beta_I$. For $\beta_I \leq 0$, go on to Step (1).

Step (1)

Find the algebraically greatest $\alpha_{Ij} = \alpha_{I\mu}$, keeping track of ties. If $\alpha_{I\mu} \leq 0$, the restrictions are inconsistent and cannot be satisfied if $\beta_I < 0$. If $\beta_I = 0$, the restriction requires that $\xi_j = 0$ for all $\alpha_{Ij} \neq 0$. If $\alpha_{I\mu} > 0$, go on to Step (2).

Step (2)

Among the set of $\alpha_{Ij} = \alpha_{I\mu}$, choose $\mu = J$ for the algebraically greatest γ_μ . If there are ties, any will do. Perform the pivot step, and then make the column exchange $\alpha_{IJ} \leftrightarrow \alpha_{i,n-k}$, $1 \leq i \leq m + \ell + 1$. When $k = \ell$, go on to Procedure I, unless $\ell = n$. If $\ell = n$, the equality constraints completely specify the problem (in the absence of degeneracy), and no freedom to satisfy the inequality constraints exists.

PROCEDURE I

If $(\beta_i \leq 0, 1 \leq i \leq m + \ell)$, go on to Procedure II. Otherwise, select the greatest $i = \lambda$ for which $\beta_\lambda > 0$. Choose, strictly randomly, any negative coefficient $\alpha_{\lambda j}$, $1 \leq j \leq n - \ell$, determining thereby the pivot column $j = J$.

(If there is no $\alpha_{\lambda j} < 0$, the constraints are inconsistent and no solution exists.) Compute $\beta_{\lambda}/\alpha_{\lambda j}$ and β_i/α_{ij} for all $\alpha_{ij} > 0$, $\lambda < i \leq m + \ell$. Determine the algebraically greatest (numerically least) of these negative ratios, selecting randomly amongst ties. This determines the pivot row $i = I$. Perform the pivot step, Eq. (7), and repeat Procedure I as necessary until all $\beta_i \leq 0$.

PROCEDURE II

If ($\gamma_j \leq 0$, $1 \leq j \leq n - \ell$), the problem solution is $\xi_j = 0$ for $1 \leq j \leq n$, and $W = \delta$. Otherwise, go on to Step (1).

Step (1)

Determine all $i = \lambda$, $1 \leq i \leq m + \ell$, for which $\beta_{\lambda} = 0$. If there are none, go on to Step (2). Otherwise determine all $j = \mu$, $1 \leq j \leq n - \ell$, for which $\gamma_{\mu} > 0$ and $\alpha_{\lambda\mu} > 0$, for those values of λ just determined. If there are none, go on to Step (2). Otherwise, choose the pivot column $j = J$ randomly from among $j = \mu$, and the pivot row $i = I$ randomly from among those rows $i = \lambda$ for which $\alpha_{\lambda j} > 0$. Perform the pivot step, Eq. (7), and repeat Procedure II as necessary until all $\gamma_j \leq 0$.

Step (2)

Choose, strictly randomly, any $\gamma_j > 0$, thereby determining the pivot column $j = J$. Compute β_i/α_{ij} for all $\beta_i \neq 0$, $\alpha_{ij} > 0$, $1 \leq i \leq m + \ell$. (If $\alpha_{ij} \leq 0$ for $1 \leq i \leq m + \ell$, ξ_j can be increased indefinitely without violating the constraints $\eta_i \geq 0$, and W is unbounded.) Select the algebraically greatest (numerically least) from among the computed negative ratios β_i/α_{ij} , choosing

randomly amongst ties. This determines the pivot row $i = 1$. Perform the pivot step, Eq. (7), and repeat Procedure II as necessary until all $y_j \leq 0$. (Since Procedure II preserves the signs of the β_i , Procedure I is not to be used again. Furthermore, since the pivot step replaces $\hat{z}$ by $\hat{z} - \beta_i y_j / a_{ij}$, and since $\hat{z}_i \leq 0$, $y_j > 0$, $a_{ij} > 0$, the value $W = \hat{z}$ of the basic feasible point cannot decrease.)

3. User-Level Instructions for SIMPLEX

COMMUNICATION

Calling Sequence

Subroutine SIMPLEX is used by adding a CALL SIMPLEX card to the user's program in the appropriate place.

Input

Three parameters and the tableau must be set by the user. NKWN is the number of unknowns in the system of equality and inequality constraints. MNEQ is the number of inequality constraints, and LEQ is the number of equality constraints. These correspond to n , m , and k in the theoretical discussion. These values are defined in PARAMETER statements and must be set in the source file [by TRIX and JED (local editors), etc.] before compilation.

The initial tableau should be set by the user's own input or initialization routine. TABLEAU is a two-dimensional, column-ordered array, dimensioned (MNEQ+LEQ+1, NKWN+1), which stores the tableau at all times during the routine. It is located in common block ARRASML. The last column of TABLEAU

should contain the slack constants, b_i , Eqs. (2) and (3). The last row should hold the c_i , the coefficients of the linear function to be maximized, $\sum_{i=1}^n c_i x_i$, Eq. (1). After the algorithm has completed, the value of the game, $W = \delta$, is the last element of the last row. The rest of the entries are the coefficients, a_{ij} , of the inequality constraints, $\sum_{j=1}^n a_{ij} x_j \leq b_i$, and the equality constraints, $\sum_{j=1}^n a_{ij} x_j = b_i$. The equality constraints must be placed after the inequality constraints in the tableau. For details of the tableau, see Owen.¹

There are also some variables the user may set to make use of some options, but these may be ignored if the defaults produce the required results.

- NCYCSTP is the number of times subroutines PRCDR1 or PRCDR2 will execute their main loops without creating an exit condition. It is set as a default to 200.
- NMBS, NMBSB are the names of the basic and non-basic variables to be printed on output in an A6 format field. As defaults, the basic variables are called "ETA", the non-basic "X".
- ZEROFAC is a fraction which, when multiplied by the largest absolute value of an entry in the initial tableau, gives a relative zero test. It is set to 1.E-10.
- IPRNTFLG determines the amount of I/O desired from the routine. (0 - no output, 1 - some output after the solution is found, 2 - the tableau is printed every pivot.) The default is 0.
- LUN is the logical unit number for printed output. Default is 10.

Output

All output values returned to the calling program are the global variables in common blocks CNSTMP and ARRASMP. The solution of the program is in array X and the value of the game is in W. NEKR should always be checked

upon return for errors. A zero value for NERR indicates a normal return. An error condition will always produce output on logical unit number LUN, which must be assigned to a disk file.

During a normal execution of the routine, when IPRNTFLG is zero, no output will be printed. If IPRNTFLG is set to one, the solution, $X(I)$, and the value of the game will be printed when a solution is found. Also, the initially basic and non-basic variables with zero values will be listed, and the variables with non-zero values will be listed along with their values. When IPRNTFLG is set to two, the tableau is printed every pivot cycle along with some other information. With this option, the code is I/O bound; it runs much more slowly.

Error Returns

After a call to SIMPLEX, a non-zero value for NERR indicates an error return.

- NERR = 1: The constraints are inconsistent. There is a row that ends with a positive entry and has only non-negative entries.
- NERR = 2: The problem is unbounded. There is a column with a positive last entry, and all the remaining entries are non-positive.
- NERR = 3: This flag means that the program has fallen through loop SLI in subroutine STEP1 without finding any elements available for choice as a pivot. This should not happen normally because the tests previous to this loop should assure that an appropriate element exists.
- NERR = 4: Subroutine PRCDRL has pivoted NCYCSTP times without making the last column negative, except for the last entry.

- NERR = 5: Subroutine PRCDR2 has pivoted the tableau NCYCSTP times without making the last row negative, except for the entry.
- NERR = 6: The system is over-constrained. The slack variables for equality constraints must be equal to zero; therefore, the slack variables heading the row of an equality constraint must be pivoted to the head of a column and kept there. Since there are fewer columns than equality constraints, i.e., $l > n$, this cannot be done.
- NERR = 7: The problem is specified by equality constraints. Since $l = n$, there are the same number of columns as equality constraints. Thus, once the equality constraints are pivoted up, no more manipulation can be done on the tableau.
- NERR = 8: No element of an equality constraint was positive when the corresponding β was negative. Thus, the restrictions are inconsistent and cannot be satisfied.
- NERR = 9: Same as 8, except β was zero. Hence, the restrictions can be satisfied only if $\xi_j = 0$ for all non-vanishing α_{ij} .
- NERR = 10: Subroutine PIVOT was called to pivot on a zero tableau element.
- NERR = 11: The error routine was called improperly; i.e., the error number was not between 1 and 10, inclusive.

Common Blocks

- CNSTSMF: This stores all global scalar variables (non-parameters) for SIMPLEX. For additional information, see cliché SMPCLCH (in Appendix A).
- ARRASMP: This stores all global arrays for SIMPLEX. See cliché SMPCLCH (in Appendix A).

Flow Diagram

A flow diagram for the SIMPLEX routine is given in Fig. 1.

RESTRICTIONS

This routine becomes very slow if a large tableau is frequently output. If IPRNTFLG is set to two, the program becomes I/O bound. This can be helped by increasing the output buffer size (to 10004₈, or 20004₈), but this option should be used only on small problems unless necessary. Possible error returns can be avoided if the tableau to be solved has more unknowns than equality constraints and if trivial constraints (all the coefficients are zero) are eliminated.

The only definite restriction on the size of the tableau is that it must fit into SCM on the CDC 7600 along with the code for SIMPLEX and the user's code. If this becomes a restriction, the code could be set up so that the tableau is placed in CDC 7600 large-core memory. Some reprogramming would be necessary, and slower execution would be expected.

EXTERNAL NAMES

- Subroutines supplied: SIMPLEX, SETVAL, INITIALIZE, PKEQJ, PRCDR1, PRCDR2, OUTSOLN, PIVOT, RANDOM, STEP1, STEP2, OUTX, ERRORPR, PRINTTAB.
- Libraries required: ORDERLIB(I/O, RNFL).
- Common blocks: CNSTSMP, ARRASMP.

Fig. 1. Subroutine flow in SIMPLEX. Most of the subroutine calls are controlled by tests, so that the actual flow of the program will depend upon the initial and current states.

SPECIAL INSTRUCTIONS

Care should be taken in combining user's global declarations with the routine's global declarations, which are in cliché SMPCLCH, to avoid overlapping common blocks.

In some situations, ZEROFAC may have to be changed in order to provide correct differentiation between small but significant values and numbers which should be zero but which due to roundoff are very small numbers.

Equality constraints can be handled in a number of ways by the simplex algorithm. Equality constraints can either be placed in the tableau after inequality constraints, or each equality can be replaced by two inequalities; i.e., $a + b = c$ is equivalent to $a + b \leq c$, $-a - b \leq -c$. Depending on the particular problem being solved, the representation of the equality constraints may make a significant difference in the program performance. For the neutral beam injection test problem discussed in Section 4 of this report, the solution was found in 50 pivots, with the equality constraint being placed after the inequality constraints. (In this case, $n = 40$ gave a tableau dimensioned 42 by 41.) When the equality constraint was replaced by two inequalities placed after the normal inequality constraints, the tableau (now 43 by 41) was solved in 56 pivots. By placing the two inequalities before the normal inequality constraints, the result was found in 25 pivots.

The best method of representing the equality constraints will vary from one type of problem to another, and in some cases there will be no significant difference between methods. However, a little experimentation may determine a superior means of representing the equality constraints that will cause significant time saving in the solution of large problems.

AVAILABILITY

- Machines: CDC 7600
- Size: Subroutine SIMPLEX contains 357 LRLTRAN source lines. The program that tests SIMPLEX contains 207 source lines.
- Timing and optimization: Each cycle (pivot element selection and pivot) takes about 0.02 second when no I/O is done in the cycle. If the tableau for a given problem is to be printed often, the output should probably be optimized. This could involve just setting the ORDERLIB output buffer size to a larger value, e.g., $2004_8 + 20004_8$. If more speed is required, subroutine PRINTTAB will probably have to be reworked. If the computational portions of the routine need more speed, BEGINMAP, a local timing routine,³ indicates that 20-25% of the CPU time is spent in loop NXTI in subroutine PIVOT when SIMPLEX is run creating some output (IPRNTFLG=1). Efforts to improve speed should probably begin here.
- Accuracy: For the test case in Section 4 with $N = 40$, the error in the numerical value of the game, compared to the analytic result, was 0.232%. With $N = 100$ over the same interval ($2-1/2$ times as fine a mesh), the error was 0.0596%.

PORTABILITY

This program is written in LRLTRAN, not in FORTRAN. There are a number of differences between the SIMPLEX code as written and FORTRAN. These easily remedied differences are:

- SIMPLEX uses alphabetic rather than numeric statement labels in all situations except format statements.

- In SIMPLEX, the sign of DO loop increments is not always positive (thus allowing the loop to be swept "downward"). Also, the expressions for the DO loop limits are sometimes more complicated than FORTRAN allows.
- Array and dimension indexes in SIMPLEX are sometimes more complicated expressions than FORTRAN allows.
- Variable and subroutine names containing more than six characters have been used.
- Hollerith information is listed within double quotes ("....") in SIMPLEX rather than the standard NH....
- Parentheses rather than slashes are used in the DATA declaration.
- The variable format, using *, used in subroutine PRINTAB, is peculiar to Lawrence Livermore Laboratory.
- The cliché definition makes use of the LRLTRAN macro facility to accomplish global declarations with one statement. The statement, USE SMPCLCH, causes the compiler to insert all the LRLTRAN statements between CLICHE SMPCLCH and ENDCliche into the code at that point.
- The PARAMETER statement, unavailable in FORTRAN, defines a constant value which is not allowed to be changed (the compiler creates a literal).
- The routine RNFL, used in subroutine RANDOM, is part of the LRLTRAN library. It will need to be replaced by a local random generator. RANDOM may have to be reprogrammed to produce the required results with a different random number generator.

For variable definitions, details of the code, etc., see the listing in Appendix A.

4. Examples

We have outlined procedures and a numerical code for solution of the linear programming problem to find the maximum W satisfying Eqs. (1) through (3). This code has been used by Hall, et al.,⁴ to find the optimum distribution of injectors for the problem of neutral beam injection of a plasma in magnetic mirror traps. In this problem, one seeks to maximize the pressure of the contained plasma for a given injected current, and in a certain discrete representation

$$\left. \begin{aligned} W &= \sum_{j=1}^n x_j \ln v_j, \\ \sum_{j=1}^n x_j v_{ij} &\leq 1, \\ \sum_{j=1}^n x_j &= 1/T(R, a, 1), \end{aligned} \right\} \quad (8)$$

where R and a are constants, $1 < a < R$, and

$$v_j = 1/2 + (R - 1)(j - 1)/n, \quad j = 1, 2, \dots, n,$$

$$T(R, \rho, v) = \frac{3}{8} (R\rho)^{1/2} \{ (R + \rho) \tanh^{-1} [(\rho - v)^{1/2} (R - v)^{-1/2}] \\ - (R - v)^{1/2} (\rho - v)^{1/2} \},$$

$$v_{ij} = \begin{cases} T(R, v_j, 1) - T(R, v_j, v_i), & i < j, \\ T(R, v_j, 1) & , \quad i \geq j. \end{cases}$$

When $R = 3$ and $a = 2$, we found the algorithm converged in 24 steps of the simplex procedure when $n = 40$, and the last restriction, Eq. (8), was taken with $=$ replaced by $\leq$. Use of the equality or the equivalent double inequality increased the number of steps as discussed on page 13. The analytic solution

to this problem was also found by Hall, et al., and we include as Appendix B the set of graphs showing the convergence of the simplex algorithm to the analytic solution. We also solved this problem for $n = 10$ and $n = 100$, and Fig. 2 compares the final results for these cases.

Several other test problems, taken from basic linear programming texts, have been solved successfully. An example is Orchard-Hays widget problem.⁵ In this problem, the profits of a widget manufacturer are related to inventory and labor costs and give rise to the profit function

$$W = 5.40x_1 + 7.30x_2 + 12.96x_3 - 6x_4 - 9x_5 - 800,$$

subject to the constraints

$$0.50x_1 + 0.60x_3 - x_4 \leq 80$$

$$0.25x_1 + 0.50x_2 + 0.60x_3 - x_5 \leq 40$$

$$-x_1 + x_2 \leq 0$$

$$100 \leq x_1 \leq 150$$

$$x_4 \leq 20$$

$$x_5 \leq 10.$$

Fig. 2. Comparison of the final results of the SIMPLEX solution of the neutral injection problem for $n = 10$, $n = 40$, and $n = 100$. The solid curve is the analytic solution (on an arbitrary scale), and the bar diagrams give the numerical approximations.

The range of x_1 is equivalent to two inequalities, $x_1 \leq 150$, $-x_1 \leq -100$, and has been expressed in the SIMPLEX tableau, T1.

	x_1	x_2	x_3	x_4	x_5	1
$-\eta_1$	0.500	0.000	0.600	-1.000	0.000	-80.000
$-\eta_2$	0.250	0.500	0.600	0.000	-1.000	-40.000
$-\eta_3$	-1.000	1.000	0.000	0.000	0.000	0.000
T1: $-\eta_4$	1.000	0.000	0.000	0.000	0.000	-150.000
$-\eta_5$	-1.000	0.000	0.000	0.000	0.000	100.000
$-\eta_6$	0.000	0.000	0.000	1.000	0.000	-20.000
$-\eta_7$	0.000	0.000	0.000	0.000	1.000	-10.000
W	5.400	7.300	12.960	-6.000	-9.000	-800.000

Solution of this problem by SIMPLEX produced the final tableau, T2.

	η_5	x_2	η_2	x_4	η_7	1
$-\eta_1$	0.250	-0.500	-1.000	-1.000	-1.000	-5.000
$-x_3$	0.417	0.833	1.667	0.000	1.667	-41.667
$-\eta_3$	-1.000	1.000	0.000	0.000	0.000	-100.000
T2: $-\eta_4$	1.000	0.000	0.000	0.000	0.000	-50.000
$-x_1$	-1.000	0.000	0.000	0.000	0.000	-100.000
$-\eta_6$	0.000	0.000	0.000	1.000	0.000	-20.000
$-x_5$	0.000	0.000	0.000	0.000	1.000	-10.000
W	0.000	-3.500	-21.600	-6.000	-12.600	190.000

The value of the game is $W = 190$, the result found by Orchard-Hays. The detailed result differs from Orchard-Hays, however, who found (upon conversion to our notation) the final tableau, T3.

	η_1	x_2	η_2	x_4	η_7	1
$-\eta_5$	4.000	-2.000	-4.000	-4.000	-4.000	-20.000
$-x_3$	-1.667	1.667	3.333	1.667	3.333	-33.333
$-\eta_3$	4.000	-1.000	-4.000	-4.000	-4.000	-120.000
T3: $-\eta_4$	-4.000	2.000	4.000	4.000	4.000	-30.000
$-x_1$	4.000	-2.000	-4.000	-4.000	-4.000	-120.000
$-\eta_6$	0.000	0.000	0.000	1.000	0.000	-20.000
$-x_5$	0.000	0.000	0.000	0.000	1.000	-10.000
W	0.000	-3.500	-21.600	-6.000	-12.600	190.000

Thus, we have an example of a problem with multiple solutions. The coefficient of η_5 in the expression for W from tableau T2 is zero, so that any pivot in the first column (allowable only if $\alpha_{ij} \neq 0$) will leave the value δ unchanged. Just as in Procedure II, therefore, we may always pivot in the row for which $\alpha_{ij} > 0$ and β_i/α_{ij} is maximal (least negative) to get another feasible solution. Since $J = 1$, this pivot point is $(I,J) = (1,1)$, and performance of the pivot step produces the equal-value Orchard-Hays tableau, T3.

Degeneracies occur whenever a zero value of β_i or γ_j arises. The procedures have been set up to avoid interminable looping when zeros of β_i exist, and these were discussed above. If all solutions $W = W_{\max}$ are required, it is a simple matter to add the additional programming involving pivoting either in the rows with $\beta_i = 0$ or in the columns with $\gamma_j = 0$ in order to produce this set.

References

1. Guillermo Owen, Game Theory (W. B. Saunders Co., Philadelphia, 1968).
2. William Orchard-Hays, Advanced Linear Programming Computing Techniques (McGraw-Hill, New York, 1968).
3. L. J. Sloan, BEGINMAP: A Program that Analyses Computer Programs, Lawrence Livermore Laboratory Rept. UCID-30091 (1974).
4. L. S. Hall, E. B. Hooper, Jr., and W. A. Newcomb, Optimal Injection of Beams in a Mirror Well, Lawrence Livermore Laboratory Rept. UCRL-75561 (1975).
5. William Orchard-Hays, loc. cit., Sec. 3.5.

Appendix A

**Code Listing of SIMPLEX (including routines used
for the neutral injection problem).**

```

1      PROGRAM SIMPLEX(INPUT,SMPSRLT=2004B,TAPE:0=SMPSRLT)
2      CALL CHANGE(IGH+DROPL)
3 C*
4 C SETUP IS A ROUTINE WHICH CALLS OTHER ROUTINES
5 C THESE ROUTINES INPUT ANY DATA, SET UP THE TABLEAU, COMPUTE ANY
6 C PROGRAM CONSTANTS,ETC.
7      CALL SETUP
8 C*
9 C SIMPLEX MANIPULATES THE TABLEAU TO COMPUTE THE MAXIMUM
10 C VALUE OF THE GAME. W.
11      CALL SIMPLEX
12 C*
13 C OUTSMP PRINTS ANY ADDITIONAL INFORMATION THAT
14 C IS DESIRABLE TO HAVE.
15      CALL OUTSMP
16      CALL EXIT(1)
17      END

18 C
19 C
20 C***** THIS SECTION OF ROUTINES EXECUTES THE SIMPLEX ALGORITHM.
21 C
22 C
23      SUBROUTINE SIMPLEX
24 C* ** ** ** **
25 C**
26 C**      SUBROUTINE SIMPLEX COMPUTES THE SOLUTION OF A **
27 C**      LINEAR PROGRAMMING PROBLEM VIA THE SIMPLEX ALGORITHM. **
28 C**      SEE GAME THEORY ,G. OWEN,CH. 3. **
29 C**
30 C**      PROGRAMMED BY H. WALKER **
31 C**
32 C**      TO USE THIS ROUTINE, FIRST SET THE **
33 C**      PARAMETERS IN CLICHE SMPCLCH (SEE BELOW) **
34 C**      TO THE VALUES APPROPRIATE FOR THE PROBLEM TO **
35 C**      BE SOLVED. THEN SET THE TABLEAU TO BE SOLVED BY AN **
36 C**      INPUT OR INITIALIZATION ROUTINE. IT SHOULD **
37 C**      BE FORMATTED AS IN GAME THEORY . THIS ROUTINE SHOULD **
38 C**      BE COMPILED UNDER CHAT WITH ORDERLIB. **
39 C**      THE SUBROUTINE SIMPLEX CAN THEN BE CALLED. **
40 C**      IF OUTPUT IS DESIRED SOME OPTIONS ARE AVAILABLE **
41 C**      (SEE IPRINTFLG, IN SMPCLCH), OR SOME **
42 C**      OTHER ROUTINES CAN BE WRITTEN BY THE USER. **
43 C**
44 C**      DUE TO THE USE OF COMMON, CLICHE SMPCLCH SHOULD EITHER **
45 C**      BE THE FIRST SET OF DECLARATIONS FOR COMMON, **
46 C**      OR OTHER DECLARATIONS WILL HAVE TO BE **
47 C**      PLACED BEFORE ALL USE SMPCLCH LINES IN SIMPLEX. **
48 C**
49 C**      ON ERROR RETURN FROM ANY ROUTINE, SIMPLEX **
50 C**      CALLS ERRORPR, THEN RETURNS TO THE CALLING **
51 C**      ROUTINE WITH THE ERROR SENTINEL, NERR, SET. **
52 C**
53 C**      SIMPLEX IS CALLED BY: USER'S ROUTINE **
54 C**      CAN CALL SETVAL,INITIALIZE,PRCDRI, **
55 C**      PRCDR2,OUTSOLN,ERRORPR **
56 C**
57 C** ** ** **
58 C*
59      CLICHE SMPCLCH
60 C* ** ** **
61 C**      VARIABLE DEFINITIONS FOR SMPCLCH **
62 C**      ***** **
63 C**
64 C** *PARAMETERS* **
65 C**      NKNW - NUMBER OF UNKNOWN IN SYSTEM OF EQUALITIES **
66 C**      AND INEQUALITIES **
67 C**      MNEQ - NUMBER OF INEQUALITY CONSTRAINTS **
68 C**      IEQ - NUMBER OF EQUALITY CONSTRAINTS **
69 C**      IGRW - LAST ROW OF TABLEAU. INITIALLY STORES **

```

```

70 C**      THE COEFFICIENTS OF THE LINEAR FUNCTIONAL      **
71 C**      BEING MAXIMIZED                                **
72 C**                                                     **
73 C**      PARAMETER (NKNW=40)                             **
74 C**      PARAMETER (MNEQ=21)                             **
75 C**      PARAMETER (LEQ=0)                               **
76 C**      PARAMETER (IGMRW=MNEQ+LEQ+1)                   **
77 C**                                                     **
78 C** *SCALARS (IN COMMON BLOCK CNSTMP)*                  **
79 C**      IDENTFI - NAME OF RUN, TO BE READ IN AS DATA AND **
80 C**      PROVIDED ON OUTPUT (A10) TO IDENTIFY RUN.      **
81 C**      IPIV - ROW OF PIVOT ELEMENT                     **
82 C**      IPRINTFLG - FLAG CONTROLLING THE AMOUNT OF PRINTED OUTPUT **
83 C**      0=NO OUTPUT, 1= SOME OUTPUT, 2= A LOT OF OUTPUT **
84 C**      DEFAULT IS 0.                                    **
85 C**      JPIV - COLUMN OF PIVOT ELEMENT                   **
86 C**      LUN - LOGICAL UNIT NUMBER FOR PRINTED OUTPUT.  **
87 C**      DEFAULT=10.                                      **
88 C**      NAMEARA - 10 CHARACTER (A10) IDENTIFIER WHICH LABELS **
89 C**      THE TABLEAU WHEN IT IS OUTPUT VIA PRINTGRID   **
90 C**      NCYC1 - NUMBER OF TIMES PROCRI HAS BEEN EXECUTED **
91 C**      NCYC2 - NUMBER OF TIMES PROC2 HAS BEEN EXECUTED **
92 C**      NCYCSTP - AFTER THIS MANY CYCLES OF EITHER PROCRI OR PROC2 **
93 C**      MULT EXECUTION WITH A MESSAGE. DEFAULT IS 200.  **
94 C**      NERR - ERROR SENTINEL 1=0-NORMAL RETURN, >0=ERROR RETURN) **
95 C**      NMBS0 - NAME OF INITIALLY BASIC VARIABLES (AS FIELD) **
96 C**      DEFAULT IS ETA.                                  **
97 C**      NMNBSC - NAME OF INITIALLY NON-BASIC VARIABLES.  **
98 C**      DEFAULT IS X.                                    **
99 C**      ZEROFAC - FRACTION OF LARGEST INITIAL TABLEAU VALUE TO BE **
100 C**      USED AS TEST FOR ZERO. DEFAULT IS 1.E-10.      **
101 C**      ZEROTOLR - VALUE TO BE USED AS TEST FOR ZERO.  **
102 C**                                                     **
103 C**      COMMON /CNSTMP IDENTFI,IPIV,IPRNTFLG,JPIV,LUN,  **
104 C**      NAMEARA,NCYC1,NCYC2,NCYCSTP,NERR,NMBS0,NMNBSC,  **
105 C**      ZEROFAC,ZEROTOLR                                **
106 C**                                                     **
107 C** *ARRAYS (IN COMMON BLOCK ARRASHMP)*                  **
108 C**      LAMBDA - HOLDS INDEXES OF ELEMENTS OF A COLUMN  **
109 C**      TO BE PICKED FROM RANDOMLY.                     **
110 C**      LOCUS1 - SAME AS LAMBDA                          **
111 C**      LOCUSJ - HOLDS INDEXES OF ELEMENTS OF A ROW TO BE **
112 C**      PICKED FROM RANDOMLY.                           **
113 C**      LSTORDER - STORES THE ORDER OF THE BASIC AND   **
114 C**      NON-BASIC VARIABLES.                            **
115 C**      MU - SAME AS LOCUSJ                              **
116 C**      TABLEAU - 2-DIM ARRAY WHICH HOLDS THE TABLEAU TO **
117 C**      BE OPERATED ON. THE SLACK CONSTANT'S ARE IN     **
118 C**      THE LAST COLUMN OF THE ARRAY, WHICH IS         **
119 C**      EQUIVALENCED TO 1-DIM ARRAY CNSTRNT             **
120 C**      THE VALUE OF THE GAME IS STORED IN              **
121 C**      TABLEAU(MNEQ+LEQ+1,NKNW+1), AND THIS ELEMENT  **
122 C**      IS EQUIVALENCED TO W.                            **
123 C**      X - STORES FINAL VALUES OF THE                 **
124 C**      INITIALLY NON-BASIC VARIABLES.                  **
125 C**                                                     **
126 C**      COMMON /ARRASHMP LAMBDA(MNEQ+LEQ+1,LOCUS1)(MNEQ+LEQ+1,  **
127 C**      LOCUSJ(NKNW+1,LSTORDER)(MNEQ+LEQ+1,NKNW+2),MU(NKNW+1,  **
128 C**      TABLEAU(IGMRW+1,NKNW+1),X,NKNW+1)              **
129 C**                                                     **
130 C**      CNSTRNT - HOLDS THE SLACK VARIABLES, AND         **
131 C**      IS PART OF THE TABLEAU                         **
132 C**      W - THE VALUE OF THE GAME, ALSO PART OF THE TABLEAU. **
133 C**                                                     **
134 C**      DIMENSION CNSTRNT(MNEQ+LEQ)                     **
135 C**      EQUIVALENCE (TABLEAU((IGMRW)*NKNW+1),CNSTRNT)  **
136 C**      (TABLEAU((IGMRW)*NKNW+1),W)                     **
137 C**                                                     **
138 C** *DEFAULT VALUES ARE SET IN THIS DATA STATEMENT*    **
139 C**      ( CAN BE CHANGED BY AN INPUT OR INITIALIZE ROUTINE) **
140 C**                                                     **
141 C**      DATA (NCYCSTP=200, (IPRNTFLG=0), (LUN=10),     **
142 C**      (NMBS0= 'ETA'), (NMNBSC= 'X'), (ZEROFAC=1.E-10)  **
143 C**                                                     **
144 C** *PERFORM ELEMENTARY OPTIMIZATION FOR 7600*           **
145 C**      OPTIMIZE

```

```

146 C**
147 C** *****
148 C**      END OF CLICHE SMPCLCH
149 C** .. .. .. .. ..
150      ENDCliche
151      USE SMPCLCH
152 C*
153 C DO SOME INTERNAL SETTING OF VALUES
154      CALL SETVAL
155 C*
156 C PIVOT EQUALITY CONSTRAINTS TO CORRECT PLACE IN TABLEAU
157      CALL INITIALIZE
158      IF (NERR .NE. 0) GO TO ERPRC
159 C*
160 C FIND BASIC FEASIBLE POINT
161      CALL PRCDRI
162      IF (NERR .NE. 0) GO TO ERPRC
163 C*
164 C FIND MAXIMUM FEASIBLE POINT, THE SOLUTION
165      CALL PRCDR2
166      IF (NERR .NE. 0) GO TO ERPRC
167 C*
168 C PRINT SOLUTION
169      CALL OUTSOLN
170      RETURN
171 ERPRC CALL ERRORPR
172      RETURN
173      END

174 C
175 C
176 C*****
177 C
178 C
179      SUBROUTINE SETVAL
180 C** .. .. .. .. ..
181 C**
182 C**      SETVAL DOES SOME INITIALIZATION WHICH IS
183 C**      INTERNAL TO SIMPLEX.
184 C**
185 C**      ALL VALUES ARE PASSED IN COMMON BLOCKS
186 C**      ARRASHP AND CONSTSHP. (SEE SMPCLCH)
187 C**
188 C**      SETVAL IS CALLED BY: SIMPLEX
189 C**      CAN CALL: (NONE)
190 C**
191 C** *LOCAL VARIABLES*
192 C**      I,J      - RUNNING VARIABLES FOR LOOPS
193 C**      TABLMX   - LARGEST ABSOLUTE VALUE OF ELEMENTS OF TABLEAU
194 C**      TESTED IN LOOP BGI
195 C**
196 C** .. .. .. .. ..
197 C
198 C*
199 C SMPCLCH DEFINED IN ROUTINE SIMPLEX
200      USE SMPCLCH
201 C*
202 C SET CYCLE COUNTERS TO ZERO
203      NCYC1=0
204      NCYC2=0
205 C*
206 C INITIALIZE LSTORDER TO 1,2,3,...
207      DO 10 I=1,MNEG+LEG+NKNH+2
208          LSTORDER(I)=1
209 10 CONTINUE
210 C*
211 C FIND LARGEST ABSOLUTE VALUE IN INITIAL TABLEAU.
212 C USE THIS VALUE, TABLMX TO DETERMINE A TOLERANCE TO ZERO
213 C THAT IS APPROPRIATE TO THE PROBLEM. THIS IS DONE.
214 C BY TAKING A SMALL FRACTION OF THIS LARGEST VALUE.

```



```

285 C*
286 C SWITCH VARIABLE POSITIONS
287 C LSTORDER(NKWN+1,LOCUSJ)
288 C LSTORDER(NKWN+1,LOCUSJ+1)
289 C LSTORDER(L+1,LSTMP)
290 PWSH CONTINUE
291 C*
292 C IF LEO = NKWN, THEN THE EQUALITY CONSTRAINTS COMPLETE.
293 C SPECIFY THE PROBLEM IN THE ABSENCE OF DEGENERACY
294 C ERROR RETURN, THE TABLEAU WILL BE PRINTED BY FPRDPR
295 C IF LEO = NKWN RETURN
296 CERR=1
297 RETURN
298 END

299 C
300 C
301 C****
302 C
303 C
304 C SUBROUTINE PREG, *
305 C* * * * *
306 C**
307 C** PREG, ATTEMPTS TO FIND AN APPROPRIATE
308 C** PIVOT ELEMENT IN A ROW WHICH IS AN EQUALITY
309 C** CONSTRAINT. THE ROUTINE, IN CHOOSING A ELEMENT,
310 C** ATTEMPTS TO MINIMIZE THE NUMBER OF PIVOTS REQUIRED
311 C** TO FIND THE MAXIMUM FEASIBLE POINT
312 C**
313 C** PREG IS CALLED BY: INTRA
314 C** CAN CALL: RANDOM
315 C**
316 C** ALL VALUES ARE PASSED IN COMMON BLOCKS: ARITHM
317 C** AND CONSTMP. SEE SMPLOC.
318 C**
319 C** LOCAL VARIABLES
320 C** GAMMA - LARGEST TABLEAU ENTRY IN THE LAST ROW AND
321 C** ALSO IN A COLUMN LISTED IN LOCUSJ
322 C** L - NUMBER OF ELEMENTS AVAILABLE TO BE PICKED
323 C** FROM RANDOMLY OUT OF LOCUSJ, L=1 ONLY WHEN
324 C** TIES OCCUR IN THE VALUE OF ROWMAX
325 C** MU - NUMBER OF ELEMENTS AVAILABLE TO BE PICKED
326 C** FROM RANDOMLY OUT OF MU, MU=1 ONLY WHEN
327 C** TIES OCCUR IN THE VALUE OF GAMMA
328 C** ROWMAX - LARGEST VALUE IN ROW IPIV
329 C**
330 C** * * * * *
331 C*
332 C SMPLOC DEFINED IN ROUTINE SIMPLEX
333 C USE SMPLOC
334 C*
335 C IF CONSTRAINTS PIVOT, CHANGE THE SIGN OF EACH ENTRY IN
336 C IN ROW IPIV
337 C IF CONSTRAINT PIVOT LE -ZERO THEN GO TO STOP
338 C DO SWSON L+1,NKWN+1
339 C TABLEAU(IPIV,L+1)=TABLEAU(IPIV,J)
340 SWSON CONTINUE
341 C*
342 C FIND THE ALGEBRAICALLY GREATEST TABLEAU(IPIV,J)
343 C KEEP TRACK OF TIES IN LOCUSJ
344 C STOP L=1
345 C LOCUSJ=L+1
346 C ROWMAX=TABLEAU(IPIV,L)
347 C DO SLLRG L=2,NKWN+1
348 C IF TABLEAU(IPIV,J) LT ROWMAX GO TO SLLRG
349 C IF TABLEAU(IPIV,J) GT ROWMAX GO TO LRGST
350 C L=L+1
351 C LOCUSJ=L+1
352 C GO TO SLLRG
353 C STOP L=1

```



```

+24      L=0
+25 C*
+26 C CHOOSE STRICTLY RANDOMLY, ANY NEGATIVE TABLEAU(LAMBDA, I)
+27 C :LAMBDA=LEQ IF THERE ARE NONE, THE CONSTRAINTS ARE
+28 C INCONSISTENT AND THERE IS NO SOLUTION, ERROR RETURN
+29 C LOCUS, MODS THE INDEXES ELIGIBLE FOR CHOICE.
+30 C
+31 C THE CHOICE OF TABLEAU(LAMBDA, J) DETERMINES JPIV, THE PIVOT COLUMN.
+32 DO SELECT JPIV, MIN=LEQ
+33 IF TABLEAU(LAMBDA, J) GE -ZEROTOLRIGO TO SELECT
+34 IF
+35 LOCUS=LPIV
+36 SELECT CONTINUE
+37 IF L=LEQ DINNER=1
+38 IF NERR NE 0 RETURN
+39 CALL RANDOM(LOCUS, L, JPIV)
+40 IF
+41 C COMPUTE CNSTRNT(LAMBDA+TABLEAU(LAMBDA, J),
+42 C AND CNSTRNT(I):TABLEAU(I, J) FOR ALL TABLEAU(I, J) > 0, J=JPIV.
+43 C LAMBDA=LAMBDA+LEQ
+44 C
+45 C DETERMINE THE RATIO CLOSEST TO ZERO OF THESE NEGATIVE
+46 C RATIOS, SELECT RANDOMLY AMONG 1:25 THIS DETERMINES THE PIVOT
+47 C ROW, LPIV, KEEP INDEXES OF ELIGIBLE ELEMENTS IN LOCUS1.
+48 C BIGNAT=CNSTRNT(LAMBDA+TABLEAU(LAMBDA, JPIV)
+49 C L=1
+50 LOCUS1=L:LAMBDA
+51 DO SELECT LAMBDA=L+MIN=LEQ
+52 IF TABLEAU(L, JPIV) LE -ZEROTOLRIGO TO SELECT
+53 IF CNSTRNT(I):TABLEAU(I, JPIV)-BIGNAT) LT -ZEROTOLR
+54 GO TO SELECT
+55 IF CNSTRNT(I):TABLEAU(I, JPIV)-BIGNAT) GT -ZEROTOLR
+56 GO TO LOST
+57 L=L+1
+58 LOCUS1=L+1
+59 GO TO SELECT
+60 LOST L=1
+61 LOCUS1=L+1
+62 BIGNAT=CNSTRNT(I):TABLEAU(I, JPIV)
+63 SELECT CONTINUE
+64 CALL RANDOM(LOCUS1, L, IPIV)
+65 C*
+66 C PERFORM THE PIVOT STEP AND REPEAT PROCEDURE 1. (IF PRCDR1 HAS NOT
+67 C COMPLETED MORE THAN A SPECIFIED NUMBER OF CYCLES)
+68 NCYC=NCYC+1
+69 CALL PIVOT
+70 IF NERR NE 0 RETURN
+71 IF NCYC GT NCYCSTP NERR=4
+72 IF NERR NE 0 RETURN
+73 GO TO AGN
+74 END

```

```

+75 C
+76 C
+77 C*****
+78 C
+79 C
+80 SUBROUTINE PRCDR2
+81 C** * * * *
+82 C**
+83 C** PRCDR2 ATTEMPTS TO FIND A MAXIMUM FEASIBLE POINT, I.E. **
+84 C** IT MANIPULATES THE TABLEAU UNTIL ALL OF THE **
+85 C** LAST ROW OF THE TABLEAU, EXCEPT THE LAST ELEMENT, IS **
+86 C** NON-POSITIVE. **
+87 C**
+88 C** PRCDR2 IS CALLED BY: SIMPLEX **
+89 C** CAN CALL: PIVOT, STEP1 **
+90 C**
+91 C** ALL VALUES ARE PASSED IN COMMON BLOCKS ARRASHP **
+92 C** CNSTSHP (SEE SIMPLCH) **
+93 C**

```



```

494 C** *LOCAL VARIABLES* **
495 C**      - RUNNING VARIABLE FOR LOOP **
496 C**      - AFTER COMPLETING LOOP SEL, IL IS THE **
497 C**      NUMBER OF POSITIVE ELEMENTS IN THE LAST ROW **
498 C**      OF THE TABLEAU EXCEPT THE LAST ONE IN. **
499 C** **
500 C** * * * * * **
501 C**
502 C SMPLOCN DEFINED IN ROUTINE SIMPLEX
503 C USE SMPLOCN
504 C*
505 C IF TABLEAU(MRWL) ARE ALL 0, IT IS UNKNOWN, THEN
506 C A SOLUTION HAS BEEN FOUND. RETURN TO OUTPUT RESULTS
507 C STORE INDEXES OF POSITIVE ELEMENTS IN LOCUS.
508 C ACN JUL=0
509 C DO SEL=JUL,UNKNOWN
510 C IF TABLEAU(MRWL) LE ZERO OR GO TO SEL
511 C JUL=JUL+1
512 C LOCUS(JUL)=0
513 C SEL CONTINUE
514 C IF JUL EQ. 0 RETURN
515 C*
516 C STEP1 FINDS A PIVOT UNDER CERTAIN CONDITIONS, AND IF THOSE
517 C CONDITIONS ARE NOT MET THEN IT CALLS STEP2 WHICH USES
518 C A DIFFERENT METHOD TO FIND A PIVOT ELEMENT
519 C CALL STEP1(JUL)
520 C IF INERR NE 0 RETURN
521 C*
522 C PERFORM PIVOT ON ELEMENT RETURNED INDEXES IN (PIV,PIV)
523 C IN CNSTSM1 AND IS1 THE NUMBER OF CYCLES
524 C NCYC2=NCYC2+1
525 C CALL PIVOT
526 C IF INERR NE 0 RETURN
527 C IF NCYC2 GT NCYCSTP INERR=5
528 C IF INERR NE 0 RETURN
529 C GO TO 40
530 C END

531 C
532 C
533 C*****
534 C
535 C
536 C SUBROUTINE STEP1(JUL)
537 C** * * * * **
538 C**
539 C** STEP1 FINDS A PIVOT ELEMENT WHEN SOME OF THE **
540 C** CONSTRAINTS ARE ZERO AND IF CERTAIN ELEMENTS OF THE **
541 C** TABLEAU SATISFY PARTICULAR CONDITIONS WHICH CAUSE **
542 C** THESE ZERO CONSTRAINTS TO BE SIGNIFICANT. **
543 C**
544 C** JL IS THE NUMBER OF POSITIVE VALUES IN THE **
545 C** LAST ROW OF TABLEAU (INDEXES ARE IN LOCUS). IT **
546 C** IS PASSED FROM PROCOR2, THE CALLING ROUTINE. ALL **
547 C** OTHER VARIABLES ARE PASSED IN ARRAYSMP OR CNSTSM1 **
548 C** **
549 C**
550 C** STEP1 IS CALLED BY: PROCOR2
551 C** CAN CALL: RANDOM,STEP2
552 C**
553 C** *LOCAL VARIABLES* **
554 C**      - I,J - RUNNING VARIABLES FOR LOOPS **
555 C**      - IL - AFTER LOOP PSON, IL IS THE NUMBER OF ZERO **
556 C**      CONSTRAINTS **
557 C**      - ML - AFTER LOOP SEL, ML IS THE NUMBER OF COLUMNS WITH A **
558 C**      POSITIVE LAST ELEMENT AND A POSITIVE ENTRY IN A ROW **
559 C**      WITH A ZERO CONSTRAINT. **
560 C**      - L - AFTER LOOP SLL, L IS THE NUMBER OF POSITIVE **
561 C**      ELEMENTS IN THE PIVOT COLUMN IN ROWS **
562 C**      WITH ZERO CONSTRAINTS. **

```

```

563 C**
564 C**
565 C**
566 C SYMBOLIC DEFINED IN ANY ONE SIMPLEX
567 C USE SYMBOLIC
568 C*
569 C DETERMINE ALL LAMBDA+LAMBDA+PNEQLEQ.
570 C FOR WHICH CONSTRAINT LAMBDA+LAMBDA+PNEQLEQ. THERE ARE NONE SKIP TO STEP 2
571 C DO
572 C DO PSCN+LAMBDA+PNEQLEQ
573 C IF ABS(CONSTANT) GT ZERO THEN GO TO PSCN
574 C ELSE
575 C LAMBDA+LAMBDA+PNEQLEQ
576 PSCN+LAMBDA+PNEQLEQ
577 C IF LAMBDA+LAMBDA+PNEQLEQ
578 C CALL STEP2 JL
579 C RETURN
580 C*
581 C OTHERWISE DETERMINE ALL LAMBDA+LAMBDA+PNEQLEQ.
582 C FOR WHICH TABLEAU LAMBDA+LAMBDA+PNEQLEQ.
583 C TABLEAU LAMBDA+LAMBDA+PNEQLEQ. FOR THOSE VALUES OF LAMBDA JUST
584 C DETERMINED IF THERE ARE NONE SKIP TO STEP 2
585 C DO
586 C DO SELM+LAMBDA+PNEQLEQ
587 C DO SELM+LAMBDA+PNEQLEQ
588 C IF TABLEAU LAMBDA+LAMBDA+PNEQLEQ. THEN GO TO ZERO THEN GO TO ENDM
589 C ELSE
590 C CONTINUE
591 C DO TO SELM
592 C IF LAMBDA+LAMBDA+PNEQLEQ
593 C SELM+LAMBDA+PNEQLEQ
594 C IF LAMBDA+LAMBDA+PNEQLEQ
595 C CALL STEP2 JL
596 C RETURN
597 C*
598 C IF THERE ARE, CHOOSE A PIVOT COLUMN JPIV.
599 C RANDOMLY FORM AMONG JAMB. AND THE
600 C PIVOT ROW IPIV. RANDOMLY FROM AMONG THOSE ROWS
601 C LAMBDA FOR WHICH TABLEAU LAMBDA+LAMBDA+PNEQLEQ. HOLDS
602 C THE ELEMENTS TO BE CHOSEN FROM
603 C FOSM CALL RANDOM(M,L,JPIV)
604 C DO SELM+LAMBDA+PNEQLEQ
605 C IF TABLEAU LAMBDA+LAMBDA+PNEQLEQ. THEN GO TO SELM
606 C ELSE
607 C LOCUS=LAMBDA+LAMBDA+PNEQLEQ
608 C CONTINUE
609 C IF LAMBDA+LAMBDA+PNEQLEQ
610 C IF LAMBDA+LAMBDA+PNEQLEQ
611 C CALL RANDOM(LOCUS,L,IPIV)
612 C RETURN
613 C END

```

```

614 C
615 C
616 C****
617 C
618 C
619 C SUBROUTINE STEP2(JL)
620 C**
621 C**
622 C** STEP2 FINDS A PIVOT ELEMENT WHEN STEP1 DOESNT APPLY.
623 C**
624 C** JL IS THE NUMBER OF POSITIVE VALUES IN THE
625 C** LAST ROW OF TABLEAU (INDEXES ARE IN LOCUSJ). ALL
626 C** OTHER VARIABLES ARE PASSED IN ARRAYSMP OR CNSTGMP
627 C**
628 C** STEP2 IS CALL BY: STEP1
629 C** CAN CALL: RANDOM
630 C**
631 C** *LOCAL VARIABLES*

```

```

641 **          + NO NO VARIABLE FOR LOOP **
642 **          NUMBER OF NEGATIVE RATIOS. % LOOP SELECT **
643 **          + NO NO VARIABLE FOR LOOP **
644 **          NUMBER OF NEGATIVE RATIOS **
645 **          + NO NO VARIABLE FOR LOOP **
646 **          NUMBER OF NEGATIVE RATIOS **
647 **          + NO NO VARIABLE FOR LOOP **
648 **          NUMBER OF NEGATIVE RATIOS **
649 **          + NO NO VARIABLE FOR LOOP **
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691 **          + NO NO VARIABLE FOR LOOP **
692 **          NUMBER OF NEGATIVE RATIOS **
693 **          + NO NO VARIABLE FOR LOOP **
694 **          NUMBER OF NEGATIVE RATIOS **
695 **          + NO NO VARIABLE FOR LOOP **
696 **          NUMBER OF NEGATIVE RATIOS **
697 **          + NO NO VARIABLE FOR LOOP **
698 **          NUMBER OF NEGATIVE RATIOS **
699 **          + NO NO VARIABLE FOR LOOP **
700 **          NUMBER OF NEGATIVE RATIOS **
701 **          + NO NO VARIABLE FOR LOOP **
702 **          NUMBER OF NEGATIVE RATIOS **

```

```

703 C*
704 C* COLUONE DEFINED IN ROUTINE SIMPLEX
705 C* USE SIMPLEX
706 C*
707 C* DETERMINE NEW PIVOT ELEMENT
708 C* (TABLEAUPIVLOCPIV) = 0.0
709 C* (TABLEAUPIVLOCPIV) = 0.0
710 C* (TABLEAUPIVLOCPIV) = 0.0
711 C*
712 C* PIVOT ARRAY (NOT INCLUDING PIVOT ROW AND PIVOT COLUMN)
713 C* DO NATL = 1, NROW
714 C* IF (U EQ UPIV) GO TO NATL
715 C* DO NATL = 1, NROW
716 C* IF (U EQ UPIV) GO TO NATL
717 C* (TABLEAUPIVLOCPIV) = (TABLEAUPIVLOCPIV) * (TABLEAUPIVLOCPIV)
718 C* (TABLEAUPIVLOCPIV) = (TABLEAUPIVLOCPIV) * (TABLEAUPIVLOCPIV)
719 NATL CONTINUE
720 NATL CONTINUE
721 C*
722 C* CALCULATE PIVOT ROW
723 C* DO PTRW = 1, NROW
724 C* IF (U EQ UPIV) GO TO PTRW
725 C* (TABLEAUPIVLOCPIV) = (TABLEAUPIVLOCPIV) * (TABLEAUPIVLOCPIV)
726 PTRW CONTINUE
727 C*
728 C* CALCULATE PIVOT COLUMN
729 C* DO PTRC = 1, NCOL
730 C* IF (U EQ UPIV) GO TO PTRC
731 C* (TABLEAUPIVLOCPIV) = (TABLEAUPIVLOCPIV) * (TABLEAUPIVLOCPIV)
732 PTRC CONTINUE
733 C*
734 C* SWITCH BASIC AND NON-BASIC VARIABLE POSITIONS
735 C* (MPI) = (STORDERINKNW) * (MPI)
736 C* (STORDERINKNW) = (MPI) * (STORDERINKNW)
737 C* (STORDERINKNW) = (MPI) * (STORDERINKNW)
738 C* (MPI) = (STORDERINKNW) * (MPI)
739 C* THIS IS SPECIAL OUTPUT FOR THIS TEST CASE
740 C* CALL OUTSIG
741 C* CALL PLOTSIGOF
742 C* (MPI) = (STORDERINKNW) * (MPI)
743 C* (MPI) = (STORDERINKNW) * (MPI)
744 C* RETURN
745 C* END

```

```

746 C
747 C
748 C****
749 C
750 C
751 C SUBROUTINE RANDOMLOCUS, NUMCH, KPIV
752 C**
753 C**
754 C** THIS ROUTINE PICKS A RANDOM ELEMENT OUT OF ARRAY LOCUS
755 C** THIS ELEMENT IS RETURNED VIA KPIV TO BE USED AS A PIVOT
756 C** ELEMENT. NUMCH CONTAINS THE NUMBER OF ELEMENTS IN
757 C** LOCUS.
758 C**
759 C** RANDOM IS CALLED BY: PRORD; STEP1, AND STEP2
760 C** CAN CALL: RNFL
761 C**
762 C** RNFL IS AN ORDERLIB ROUTINE WHICH RETURNS PSEUDO-
763 C** RANDOM NUMBERS IN (0,1). INT IS THE CHAT MACHINE
764 C** FUNCTION TRUNCATION INTEGER
765 C**
766 C** *LOCAL VARIABLES*
767 C** (TMP) - TEMPORARY VARIABLE STORES WHICH ELEMENT
768 C** OF LOCUS TO USE AS KPIV
769 C** Q - DUMMY VARIABLE FOR RNFL
770 C** RNDNUM - STORES THE RANDOM REAL NUMBER RETURNED
771 C** BY RNFL. IT IS ALSO USED AS A TEMPORARY
772 C** IN FINDING THE INTEGER INDEX TO BE USED

```

```

783 C**
784 C**
785 C**
786 C**
787 C**
788 C**
789 C**
790 C**
791 C**
792 C**
793 C**
794 C**
795 C**
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818 C**
819 C**
820 C**
821 C**
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823 C**
824 C**
825 C**
826 C**
827 C**
828 C**
829 C**
830 C**
831 C**
832 C**
833 C**
834 C**
835 C**
836 C**
837 C**
838 C**
839 C**
840 C**
841 C**

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790 C
 791 C
 792 C
 793 C
 794 C
 795 C
 796 C
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 799 C
 800 C
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 830 C
 831 C
 832 C
 833 C
 834 C
 835 C
 836 C
 837 C
 838 C
 839 C
 840 C
 841 C

SUBROUTINE OUTSOLN

 OUTPUT LISTS THE VARIABLES PSI WHICH ARE EQUAL TO
 ZERO, THEN LISTS THE VARIABLES ETA AND THEIR
 VALUES, -CONSTANT().
 ARRAY LSTORDER HOLDS THE ORDER OF THE ORIGINAL
 VARIABLES. THE VALUE OF THE GAME IS ALSO PRINTED
 ALL VALUES ARE PASSED IN COMMON BLOCKS ARRASHP
 AND CNSTNMP. (SEE SMPCLCH)
 OUTSOLN IS CALLED BY: SIMPLEX
 CAN CALL: OUTX
 LOCAL VARIABLES
 I,J - RUNNING VARIABLES FOR LOOPS
 LOCAL ARRAYS
 IPLACE - THIS ARRAY HOLDS THE INDEX OF THE ORIGINAL
 POSITION OF THE VARIABLE
 NMVAR - STORES WHETHER THE ORIGINAL VARIABLE HAS
 A BASIC OR A NON-BASIC VARIABLE
 VALVAR - STORES THE VALUES OF THE NON-ZERO VARIABLES

 C SMPCLCH DEFINED IN ROUTINE SIMPLEX
 C USE SMPCLCH
 C DIMENSION IPLACE(MNEQ+LEQ+NKN+1),NMVAR(MNEQ+LEQ+NKN+1),
 C VALVAR(MNEQ+LEQ+NKN+1)
 C
 C PRINT SOLUTION VECTOR X
 C CALL OUTX
 C
 C LOOP THROUGH LSTORDER TO FIND EACH INITIAL VARIABLES
 C FINAL POSITION.
 C DO OROLP I=1,MNEQ+LEQ+NKN+1
 C IF(LSTORDER(I).LT.(NKN+1))GO TO EX
 C IF(LSTORDER(I).EQ.(NKN+1))GO TO ONE
 C NMVAR(I)=NMBS
 C IPLACE(I)=LSTORDER(I)-(NKN+1)
 C GO TO SETVL
 C ONE
 C NMVAR(I)=ONE
 C IPLACE(I)=NKN+1
 C GO TO SETVL
 C EX
 C NMVAR(I)=NMBS
 C IPLACE(I)=LSTORDER(I)

```

842 SETVL (I,1) (1, NKWN+1) VALVAR (1+CNSTRT) (NKWN+1)
843 ORDLP CONTINUE
844 WRITE (LUN,10)
845 10 FORMAT (1X, 'THESE VARIABLES ARE EQUAL TO ZERO: ')
846 WRITE (LUN,20) (NVAR(I),1,PLACE(I),1,NKWN+1)
847 20 FORMAT (B16.6, ' ',13, ' ',13, ' ')
848 WRITE (LUN,30)
849 30 FORMAT (1X, 'THE VALUES OF THE REMAINING VARIABLES ARE: ')
850 WRITE (LUN,40) (NVAR(I),1,PLACE(I),1,VALVAR(I),1,NKWN+1) (10PW)
851 40 FORMAT (B16.6, ' ',13, ' ',13, ' ',15, B16.6)
852 RETURN
853 ENDO

```

```

854 C
855 C
856 C*****
857 C
858 C
859 C
860 C SUBROUTINE OUTX
861 C** * * * *
862 C** OUTSOLN PRINTS THE VALUES OF X(I), THE INITIALLY **
863 C** NON-BASIC VARIABLES. THESE X MAXIMIZE **
864 C** THE SUM OF THE C(I)*X(I), WHERE C(I) ARE THE INITIAL **
865 C** TABLEAU(10MRH,1). **
866 C**
867 C** ALL VARIABLES ARE PASSED IN COMMON BLOCKS ARRASHP **
868 C** AND CNSTMP. (SEE SMPOLCH) **
869 C**
870 C** OUTSOLN CAN BE CALLED FROM WHERE EVER IT **
871 C** IS APPROPRIATE. **
872 C**
873 C** OUTX IS CALLED BY: OUTSOLN **
874 C** CAN CALL: NONE **
875 C**
876 C** *LOCAL VARIABLES* **
877 C** I,J = RUNNING VARIABLES FOR LOOPS **
878 C** **
879 C** * * * * *
880 C**
881 C SMPOLCH DEFINED IN ROUTINE SIMPLEX
882 USE SMPOLCH
883 C*
884 C AN X VARIABLE IS INDICATED BY LSTORDER(I) < NKWN.
885 C THE FIRST NKWN+1 INDEXES OF LSTORDER HAVE ZERO VALUES
886 DO RSLT (1),NKWD+1,NKWN+1
887 IF(LSTORDER(I)) .GT. NKWN GO TO RSLT
888 IF(I .LE. NKWN+1) GO TO ZERO
889 X(LSTORDER(I)) = CNSTRT(1-(NKWN+1))
890 GO TO RSLT
891 ZERO X(LSTORDER(I)) = 0.
892 RSLT CONTINUE
893 WRITE (LUN,10) (NVCY1,NCYCP,NMNBSC
894 10 FORMAT(11X, 'NVCY1= ',15,6,11X, 'NCYCP= ',15,6,11X, 'NMNBSC= ',15,6,11X, ' ')
895 WRITE (LUN,20) (X(I),1,NKWN+1)
896 20 FORMAT(15I7,B16.6)
897 RETURN
898 ENDO

```

```

899 C
900 C
901 C*****
902 C
903 C
904 C SUBROUTINE ERRORPR

```



```

975      JMOB=100(NKNW+1,B)
976 C*
977 C OUTPUT TABLEAU HEADINGS
978      WRITE (LUN,10) NAMECARA
979      10 FORMAT(1,'****',A10,'****')
980      WRITE (LUN,20) (PIV,JPIV,NCYC1,NCYC2
981      20 FORMAT(//,1PIV=,15.5X,JPIV=,15.//, NCYC1=,15.4X, NCYC2=,15)
982 C*
983 C PRINT B COLUMNS OF THE TABLEAU PER PAGE
984      DO OUTD K=0,JOB-1
985          WRITE (LUN,30) (B*K)+1,B*(K+1)
986          30 FORMAT(1,J=,15,TO,15)
987          WRITE (LUN,40) ((TABLEAU(I,J),J=(B*K)+1,B*(K+1)),
988              1=1,IGMRH)
989          40 FORMAT(//,51,1,BE14.5)
990      OUTD CONTINUE
991 C*
992 C PRINT REMAINING COLUMNS OF THE TABLEAU
993      IF JMOB.EQ.0 GO TO OUTLS
994          WRITE (LUN,50) (JOB*B)+1,NKNW+1
995          WRITE (LUN,50) (JMOB,(TABLEAU(I,J),J=(B*JOB)+1,NKNW+1),
996              1=1,IGMRH)
997          50 FORMAT(//,51,1,1E14.5)
998 C*
999 C LIST LSTORDER
1000      OUTLS WRITE (LUN,60) (LSTORDER(I),I=1,MNEQ+LEQ+NKNW+2)
1001      60 FORMAT(251,2015)
1002      RETURN
1003      END

```

```

1004 C
1005 C
1006 C***** ALL ROUTINES AFTER THIS POINT ARE FOR THE TEST CASE
1007 C
1008 C
1009 C SUBROUTINE SETUP
1010 C USE SMPCLCH
1011 C*
1012 C THIS ROUTINE CALLS ROUTINES WHICH SET UP THE TEST CASE
1013 C AND DO SOME I/O
1014 C CLICHE TSTCLCH
1015 C PARAMETER (N=NKNW)
1016 C PARAMETER (PI=3.14159265358979)
1017 C PARAMETER (ONTHRD=.13)
1018 C PARAMETER (THREDB=.8)
1019 C COMMON/CONST1(AE,AM,ARE,ARM,ARMION,ARMIONEN,ARMA,
1020 C CHIRAI,ENDARM,ONQZPA,ONQA,
1021 C ONQARMA,ONOR,PA,PHIRAI,PLTRANGE,SQTARQA,
1022 C TANHINX,UPSLNIAI)
1023 C COMMON/ARRAY1(RHOINI,ENUINI,SIGMAINI,SIGMAAINI,
1024 C ENUOFSTI2*NI,SIGOFSTI2*NI)
1025 C ENDCLICHE
1026 C USE TSTCLCH
1027 C CALL READNSET
1028 C CALL SETTAB
1029 C CALL SOLNANI
1030 C CALL PLOTSIG
1031 C CALL LISTCONST
1032 C RETURN
1033 C END

```

```

1034 C
1035 C
1036 C*****
1037 C

```



```

1030 C
1039 SUBROUTINE READNSET
1040 USE SMPCLCH
1041 USE TSTCLCH
1042 C*
1043 C THIS ROUTINE READS TEST PARAMETERS AND
1044 C COMPUTES CONSTANTS.
1045 READ (2,10) IDENFFI
1046 10 FORMAT(A10)
1047 READ (2,20) ARE,AEE,PLTRANGE
1048 20 FORMAT(5E16.6)
1049 IF (ARE .LT. AEE) GO TO ERRR
1050 IF (ARE .LT. 1.) GO TO ERRR
1051 AMI=AEE-1.
1052 ARMI=ARE-1.
1053 ONOA=1./AEE
1054 ONOR=1./ARE
1055 ARMA=ARE-AEE
1056 ONDARMA=1./ARMA
1057 ARMION=ARMI/N
1058 ARMIOZN=.5*ARMION
1059 ENOARM=1./ARMION
1060 SQTAROA=SQRT(ARE*ONOA)
1061 TANMINX=TANMINV*SQRT(AMI*ARMI)
1062 CHIRAI=.5*ONOA*SQTAROA*(SQRT(ARMI*AMI)-ARMA*TANMINX)
1063 PHIRAI=.2*LOG1(SQRT(AEE*ARMI*ONDARMA)+SQRT(ARE*AMI*ONDARMA))
1064 Z=SQTAROA*TANMINX
1065 UPSLRAI=UPSILON/AEE
1066 PA=IONOA*ONOR*(IONOA*ON*HRD*ONOR)*(PHIRAI+C-IPAI-AEE*ONOR)
1067 CHIRAI=UPSLRAI
1068 ONORPA=.5*PA
1069 C*
1070 C SET UP ORDERLIB PLCTING
1071 CALL KEEP80(1,ORDXL,NPLTGG)
1072 CALL CRTID(2,NH,1,1)
1073 CALL FRAME
1074 CALL SETCH(15,10,0,0,3)
1075 WRITE (10,30) IDENFFI
1076 30 FORMAT(' *** A10, *** ')
1077 CALL FRAME
1078 RETURN
1079 ERRR WRITE(LUN,40)
1080 40 FORMAT(' *BAD INPUT
1081 CALL EXIT(1)
1082 END

1083 C
1084 C*
1085 C****
1086 C
1087 C
1088 SUBROUTINE SETTAB
1089 USE SMPCLCH
1090 USE TSTCLCH
1091 C*
1092 C COMPUTE PROBLEM COORDINATES AT HALF INTERVALS
1093 DO RHOX J=1,N
1094 RHO(J)=IFLOAT(J)-.5)*ARMION+1.
1095 RHOX CONTINUE
1096 DO ENUX J=1,N
1097 ENUX(J)=IFLOAT(J)-.5)*ARMION+1.
1098 ENUX CONTINUE
1099 C*
1100 C COMPUTE DOUBLED UP COORDINATES FOR BAR PLOTS
1101 ENUOFST(1)=1.
1102 ENUOFST(2*N)=ARE
1103 DO STNU J=2,2*N-1,2
1104 ENUOFST(J)=FLOAT(J)*ARMIOZN+1.
1105 ENUOFST(J+1)=ENUOFST(J)
1106 STNU CONTINUE
1107 C*

```

```

1108 C SET UP TABLEAU
1109 DO TBLJ J=1,N
1110   TABLEAU(1,J)=1.
1111   TABLEAU(2,J)=1.
1112   TABLEAU(10MRH,J)=LOG(ENUTJ)
1113   DO TBLI I=3,N+2
1114     IF(RHO(I-2).LE. ENU(J))GO TO SMP
1115     TABLEAU(I,J)=UPSILON(RHO(I-2),1.1-UPSILON(RHO(I-2),ENU(J))
1116     GO TO TBLI
1117 SMP   TABLEAU(I,J)=UPSILON(RHO(I-2),1.1)
1118 TBLI CONTINUE
1119 TBLJ CONTINUE
1120 C*
1121 C ASSIGN INITIAL SLACK VARIABLES
1122 DO KNTRT I=3,N+2
1123   CNSTRNT(I)=1.
1124 KNTRT CONTINUE
1125 CNSTRNT(1)=1./UPSILNR41
1126 CNSTRNT(2)=CNSTRNT(1)
1127 W=0.
1128 RETURN
1129 END

```

```

1130 C
1131 C
1132 C*****
1133 C
1134 C
1135   FUNCTION UPSILON(RO,ENDD)
1136 C*
1137 C FUNCTION USED IN INITIALIZING THE TABLEAU AND
1138 C IN COMPUTING THE ANALYTIC RESULT.
1139 USE SMPCLCH
1140 USE TSTCLCH
1141 RTIO=SQRT(1-RO-ENDD) (ARE=ENDD)
1142 PRDD=SQRT(1-RO-ENDD)**(ARE+ENDD)
1143 UPSILON=THREOB*SQRT(ARE*RO)**(ARE+RO)**TANHINV.RTIO) PRDD)
1144 RETURN
1145 END

```

```

1146 C
1147 C
1148 C*****
1149 C
1150 C
1151   FUNCTION TANHINV(ARGMNT)
1152 C*
1153 C THIS FUNCTION RETURNS THE INVERSE HYPERBOLIC TANGENT.
1154 USE SMPCLCH
1155 C*
1156 C IF THE ARGUMENT SQUARED IS NOT LESS THAN ONE.
1157 C THEN AN ERROR RESULTS.
1158 IF (ARGMNT**2)GE .1. GO TO BDRG
1159 RTIO=(1.+ARGMNT)/(1.-ARGMNT)
1160 TANHINV=.5*LOG(RTIO)
1161 RETURN
1162 BDRG WRITE (LUN,10) ARGMNT
1163   DO FORMAT( '*** BAD TANHINV ARGUMENT = ',E15.6)
1164   CALL EXIT(1)
1165 RETURN
1166 END

```

```

1167 C
1168 C
1169 C*****
1170 C
1171 C
1172 SUBROUTINE SOLNANL
1173 USE SMPCLCH
1174 USE TSTCLCH
1175 C*
1176 C COMPUTES ANALYTIC SOLUTION TO TEST PROBLEM.
1177 DO NXTJ J=1,N
1178 IF (ENU(J) .LE. AEE/100 TO ZERO
1179 ARNU=ARE-ENU(J)
1180 ENUMA=ENU(J)-AEE
1181 ONONU=1./ENU(J)
1182 SA=((2./PI)*PA*SQRT(AEE*ENUMA), (ENUMA*ENUMA))**.5, *ONTRD*
1183 ONONU*ONONU*ONONU*(ENU(J)*ARMA+.5*ARNU*ENUMA)-.5*ARMA*
1184 SQRTARD*(J*USLNR1))**.5*((ENUMA*SQRT(ARMA*AM1))-ARNU*AM1*
1185 ARMA*ENUMA))-TANH(X1)*(CHIRAL*USLNR1)*AEE*
1186 ONONU*ONONU*(ENU(J)*ARMA*2.*ARNU*ENUMA),
1187 SIGMA(IJ)=ONQ2PA*SQRT(ONDR*ARNU)*SA
1188 GO TO NXTJ
1189 ZERO SIGMA(IJ)=0.
1190 NXTJ CONTINUE
1191 RETURN
1192 END

```

```

1193 C
1194 C
1195 C*****
1196 C
1197 C
1198 SUBROUTINE OUTSIGA
1199 C*
1200 C THIS ROUTINE PRINTS THE ANALYTIC SOLUTION.
1201 USE SMPCLCH
1202 USE TSTCLCH
1203 WRITE (LUN,10)
1204 10 FORMAT(///, SIGMA(IJ)= )
1205 WRITE (LUN,20)(SIGMA(IJ),J=1,N)
1206 20 FORMAT(0E15.6),
1207 RETURN
1208 END

```

```

1209 C
1210 C
1211 C*****
1212 C
1213 C
1214 SUBROUTINE PLOTSIG
1215 USE SMPCLCH
1216 USE TSTCLCH
1217 C*
1218 C PLOTS ANALYTIC SOLUTION
1219 SIGMAX=SIGMA(I)
1220 DO LSIG J=1,N
1221 IF (SIGMA(IJ) .GT. SIGMAX) SIGMAX=SIGMA(IJ)
1222 LSIG CONTINUE
1223 SIGMAX=1.1*SIGMAX
1224 SIGNIN=PLTRANGE*SIGMAX
1225 CALL MAPS(ENU(1),ENU(N),SIGNIN,SIGMAX)
1226 CALL SETCH(2.,15.,0.0,1.1)
1227 WRITE (100,10)
1228 10 FORMAT( SIGMA(ENU) )
1229 CALL SETCH(20.,2.,0.0,1.0)

```



```

1290 C
1291 C
1292 C*****
1293 C
1294 C
1295     SUBROUTINE OUTSIG
1296     USE SMPCLCH
1297     USE TSTCLCH
1298 C*
1299 C THIS ROUTINE PRINTS THE NUMERICAL RESULT
1300     DO RSLT ! = 1, MNEQ*LEG*NGWNI
1301         IF (LSTORDER()) .GT. NGWNI GO TO RSLT
1302         IF (1 .LE. NGWNI) GO TO ZERO
1303         SIGMA(LSTORDER()) = -ENDARM1*CONSTANT(1-1/NGWNI)
1304         GO TO RSLT
1305     ZERO SIGMA(LSTORDER()) = 0.
1306     RSLT CONTINUE
1307     WRITE (LUN,10) NCYC1,NCYC2,W,IPIV,JPIV
1308     10 FORMAT(///, NCYC1=,15.3X, NCYC2=,15.3X, W=,E15.6,
1309             , 3X, IPIV=,15.3X, JPIV=,15.3X, SIGMA(J)= )
1310     WRITE (LUN,20) (SIGMA(J), J=1,NGWNI)
1311     20 FORMAT(15('1.5E11.5'))
1312     RETURN
1313     END

1314 C
1315 C
1316 C*
1317 C
1318 C
1319     SUBROUTINE PLOTSIGOF
1320     USE SMPCLCH
1321     USE TSTCLCH
1322 C*
1323 C PLOTSIGOF CREATES A BAR PLOT OF SIGMA
1324 C OVER A SMOOTH PLOT OF THE ANALYTIC RESULT.
1325     DO STSG J=1,2*N,2
1326         SIGOFST(J)=SIGMA(J+1)/2
1327         SIGOFST(J+1)=SIGOFST(J)
1328     STSG CONTINUE
1329 C*
1330 C DETERMINE MAXIMUM VALUE TO BE PLOTTED.
1331     SIGMAX=SIGMAA(1)
1332     DO LSIG J=1,N
1333         IF (SIGMAA(J) .GT. SIGMAX) SIGMAX=SIGMAA(J)
1334         IF (SIGOFST(J) .GT. SIGMAX) SIGMAX=SIGOFST(J)
1335         IF (SIGOFST(J+N) .GT. SIGMAX) SIGMAX=SIGOFST(J+N)
1336     LSIG CONTINUE
1337     SIGMIN=1.1*SIGMAX
1338     SIGMIN=PLTRANGE*SIGMAX
1339 C*
1340 C CREATE GRID AND LABELS.
1341     CALL MAPSIENU(1,ENU,N1,SIGMIN,SIGMAX)
1342     CALL SETCH(2,,16,,0.0,1)
1343     WRITE (100,10)
1344     10 FORMAT( SIGOFST(ENU): N:SIGOFST , A:SIGMAA 0)
1345     CALL SETCH(20,,2,,0.0,1.00)
1346     WRITE (100,20) NCYC1,NCYC2,W
1347     20 FORMAT( ENJ ,5X, NCYC1=,15.3X, NCYC2=,15.3X, W=,E15.6)
1348 C*
1349 C PLOT CURVES.
1350     CALL TRACEC(11H,ENU,SIGMAA,N)
1351     CALL TRACEC(11H,ENUOFST,SIGOFST,2*N)
1352     CALL FRAME
1353     RETURN
1354     END

```

Appendix B

**The Sequence of Iterations Showing the Convergence
of the Simplex Algorithm for $n = 40$ in the
Neutral Injection Problem. (The solid curve is the
analytic result. The vertical scale is arbitrary and is
set by the computer. Relative values may be determined
by the height of the analytic curve.)**