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CROSS SECTIONS

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Study of Magnetohydrodynamic Modes in Tokamak Configurations With Noncircular Cross Sections *

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ABSTRACT

Stability criteria for localized instabilities for both ideal and resistive models are derived and evaluated for general axisymmetric configurations. A numerical scheme for determining the spectrum in these configurations is developed. Application is made to the special case of an elliptical cylindrical plasma column, with finite pressure and length, carrying a uniform axial current.

Present interest in tokamaks with noncircular cross sections motivates study of the magnetohydrodynamic stability properties of general configurations. In addition, the possibility of wave heating makes desirable the determination of the stable spectrum, together with the structure of the associated eigenfunctions. With this motivation, we evaluate analytic expressions to gain an understanding of localized instabilities and resistive modes and then study the Lagrangian associated with linearized perturbations about the equilibrium to determine the spectrum and its associated eigenfunctions.

We first discuss the equilibrium. Since current interest is concentrated on tokamak devices, it is reasonable to restrict consideration to axially symmetric systems. Then we can decompose the magnetic field into poloidal and toroidal components:

$$\vec{B} = B_0 [f(\psi)\vec{\nabla}\phi \times \vec{\nabla}\psi + Rg(\psi)\vec{\nabla}\phi] , \quad (1)$$

with the azimuthal angle ϕ the ignorable coordinate and ψ an arbitrary surface label. The constant R is the major radius. The equilibrium is determined by solving the usual equation,

$$X \frac{\partial}{\partial X} \frac{1}{X} \frac{\partial \Psi}{\partial X} + \frac{\partial^2 \Psi}{\partial Z^2} = -XJ_\phi = -X^2 \frac{dp}{d\psi} - R^2 B_0^2 g \frac{dg}{d\psi} , \quad (2)$$

for a reasonable prescription of the pressure p and toroidal field $B_0 g$, given the total toroidal current inside the plasma. Here $\Psi = 2\pi B_0 \int f d\psi$

is the poloidal flux inside a magnetic surface, X is the distance from the major axis, and Z is the distance along this axis. The boundary conditions are fixed by specifying the current in external coils. Thus no conducting boundary is required. Using an efficient numerical program, we have solved Eq. (2) for many possible configurations in the projected Princeton PDX divertor experiment. A typical case is shown in Fig. 1. Note that two poloidal divertors are located on the outside of the plasma so that one could expect to find poor stability properties. Minor modification of the currents in the external coils drastically changes the shapes of the surfaces so that the separatrices can lie on the inside of the plasma.

We next consider stability criteria with respect to localized modes. The criterion is well known for the ideal model [1,2]. Resistive mode criteria have been derived for special one-dimensional configurations [3,4] as well as for general toroidal equilibria in the low-pressure limit [5]. We give here the results of extending this work to systems with finite plasma pressure. Details of the work are forthcoming [6]. We introduce the parameters

$$D \equiv \frac{\langle B^2 / |\vec{\nabla}V|^2 \rangle}{\Lambda^2} \left[J' \phi'' - I' \psi'' + \Lambda \frac{\langle \sigma B^2 \rangle}{\langle B^2 \rangle} + p'^2 \left\langle \frac{1}{B^2} \right\rangle + \left\langle \frac{\sigma^2 B^2}{|\vec{\nabla}V|^2} \right\rangle - \frac{\langle \sigma B^2 / |\vec{\nabla}V|^2 \rangle^2}{\langle B^2 / |\vec{\nabla}V|^2 \rangle} \right], \quad (3)$$

$$H \equiv \frac{\langle B^2 / |\vec{\nabla}V|^2 \rangle}{\Lambda} \left[\frac{\langle \sigma B^2 / |\vec{\nabla}V|^2 \rangle}{\langle B^2 / |\vec{\nabla}V|^2 \rangle} - \frac{\langle \sigma B^2 \rangle}{\langle B^2 \rangle} \right], \quad (4)$$

where $\Lambda \equiv \phi' \psi'' - \psi' \phi''$, $\sigma \equiv \vec{J} \cdot \vec{B} / B^2$, ϕ , ψ , I , and J are toroidal and poloidal field and current fluxes, primes denote derivatives with respect to the volume V , and brackets denote field line averages. The system is stable with respect to localized interchanges on the ideal model if

$$D_I \equiv D + H - \frac{1}{4} < 0. \quad (5)$$

If the system is stable on the ideal model, resistive effects may be important. The resistive problem is reduced to one of solving the dispersion relation

$$\Delta = \frac{(2V_s/X_o)^{1-2H} \pi \Gamma(1/4) \Gamma^2(1 - H/4) \Gamma(3/4 - H/2)}{(1 - 2H) \Gamma(1 - H/2) [\cos(H\pi/2) \Gamma(1/4 + H/4) \Gamma(1 - H)]^2} \times Q^{(2H+5)/4} \left[1 - \frac{\Gamma(3/4) \Gamma^2(1/2 - H/4) \Gamma(1/4 - H/2) D_R}{\Gamma(1/4) \Gamma^2(1 - H/4) \Gamma(3/4 - H/2) 4Q^{3/2}} \right], \quad (6)$$

with Δ a generalization of the jump in the logarithmic derivative of the perturbed field across the resistive layer, X_o/V_s the ratio of a resistive scale length to a magnetohydrodynamic one which varies as $\eta^{1/3}$, Q a scaled growth rate with a similar dependence on η , and

$$D_R \equiv D + H^2. \quad (7)$$

Equation (6) is analyzed using Nyquist techniques. If D_R is positive, the

resistive interchange is recovered, with a growth rate that scales as $\eta^{1/3}$. Thus the resistive analogue to Eq. (5) is

$$D_R < 0 \quad (8)$$

for stability. If $D_R \approx 0$, the tearing mode is recovered, with a growth rate that depends on Δ and scales as $\eta(3-2H)/(5+2H)$. Negative values of D_R provide strong stabilization of the tearing mode. If the resistivity is not too small, overstable modes can be found.

Values of D_I , D_R , and q for the configuration of Fig. 1 are shown in Fig. 2. The value of H is quite small in this case. The destabilizing tendency arises from q being less than unity near the axis, as in the simplified version of the Mercier criterion, and from the distortion induced by divertors placed outside the plasma. According to the ideal criterion, this destabilization is overridden by the stabilizing effect of shear, except for surfaces so close to the magnetic axis that they cannot be calculated accurately and thus have been omitted from Fig. 2. On the other hand, shear is ineffective in stabilizing the dangerous resistive modes. Hence, $D_R > 0$ throughout the plasma.

Although much can be learned from the evaluation of equilibrium quantities and localized criteria, it is necessary to undertake a more extensive investigation in order to determine the general magnetohydrodynamic stability properties and to obtain the spectrum. In formulating such a computational program, it is essential to anticipate the physical results to be expected and to choose the representation so that the numerical approximations do not introduce serious distortions. Since current can flow freely along magnetic field lines and fluid cannot cross them, it is clear that plasma behavior is quite anisotropic. Mathematically, the equations are higher order in derivatives within the magnetic surfaces than in derivatives across them. For this reason, we must choose a natural coordinate system based on the magnetic field lines.

The symmetry of the configurations makes it very reasonable to use ϕ as one of the coordinates. As has just been noted, it is necessary to choose ψ , or some function of it, as the second one. We choose the third coordinate θ so that the magnetic field lines are straight; i.e.,

$$\vec{B} \cdot \vec{\nabla} = \alpha(\psi, \theta) (\partial/\partial\theta + \beta(\psi) \partial/\partial\phi), \quad (9)$$

since this particular operator enters the calculation in many places. With this it is convenient to set the Jacobian

$$J \equiv (\vec{\nabla}\psi \times \vec{\nabla}\theta \cdot \vec{\nabla}\phi)^{-1} = (X^2/4\pi^2) \int_p d\tau/X^2, \quad (10)$$

so that $\psi = 1$ on the plasma-vacuum interface. Since this interface is a coordinate surface, it is easy to use Green's function techniques to express the extremized contribution of the Lagrangian from the vacuum region in terms of the distortion of the plasma surface.

We next consider the decomposition of the displacement vector in the plasma region. Keeping in mind the fact that the Lagrangian is diagonalized by the normal modes of the physical system, we should choose components of the perturbations that represent the polarizations of the various modes, at least in the low-pressure, long-wavelength limit that is reasonable for present-day devices. This is also likely to be the most troublesome regime since the range between the highest and lowest eigenvalues is most exaggerated here, spanning many orders of magnitude. The lowest

frequency, or acoustic, modes consist of flow along the magnetic field lines. The next branch, that of shear Alfvén waves, has a divergence-free motion perpendicular to the field. The fast magnetosonic mode is primarily due to the perpendicular compressibility. Since the frequencies of these different branches can be widely separated, it is important to select the representation so that evaluation of the spectrum does not rely on cancellation of large terms for its accurate calculation. The sound waves are well treated by the projection

$$\vec{\xi} = (J\xi_\psi/R^2 B_{0g}) \vec{\nabla}\theta \times \vec{B} + i(J\xi_s/R^2 B_{0g}) \vec{B} \times \vec{\nabla}\psi + i(\tau/B_0) \vec{B}, \quad (11)$$

while, at least in the long-wavelength limit, the shear Alfvén and fast compressional branches are decoupled by the transformation

$$\xi_\psi = \delta - 2\pi i \partial\zeta/\partial\theta, \quad (12)$$

$$\xi_s = 2\pi \partial\zeta/\partial\psi. \quad (13)$$

We pose the problem of determining the spectrum as one of finding estimates of the eigenvalues ω^2 and eigenfunctions $\vec{\xi}$ that make the Lagrangian

$$L = \omega^2 K(\vec{\xi}^*, \vec{\xi}) - \delta W(\vec{\xi}^*, \vec{\xi}) \quad (14)$$

stationary with respect to variations of $\vec{\xi}^*$. Here $\text{Re}[\vec{\xi}(\vec{r}) \exp(-i\omega t)]$ is the displacement of a fluid element from its equilibrium position $\vec{r}$, and $\omega^2 K$ and δW are the kinetic and potential energy functionals: [7]

$$2K = \int_p d\tau \rho |\vec{\xi}|^2, \quad (15)$$

$$2\delta W = \int_p d\tau \left[|\vec{Q} - \vec{B} \frac{\vec{\xi} \cdot \vec{\nabla} p}{B^2}|^2 + \frac{\vec{J} \cdot \vec{B}}{B^2} \vec{B} \times \vec{\xi}^* \cdot \vec{Q} - 2\vec{\xi} \cdot \vec{\nabla} p \vec{\xi}^* \cdot \vec{\kappa} + \gamma p |\vec{\nabla} \cdot \vec{\xi}|^2 \right] + \int_v d\tau |\vec{\nabla} \times \vec{A}|^2, \quad (16)$$

with ρ the plasma density, $\vec{Q} \equiv \vec{\nabla} \times (\vec{\xi} \times \vec{B})$ the perturbed magnetic field, $\vec{\kappa} \equiv (\vec{B}/B) \cdot \vec{\nabla} (\vec{B}/B)$ the local magnetic field line curvature, γ the ratio of specific heats, and $\vec{A}$ the perturbed vector potential in the vacuum. The normal component of the perturbed magnetic field must be continuous at the plasma-vacuum interface and vanish at the vacuum wall.

We adopt the Galerkin method for making Eq. (14) stationary. Approximating $\vec{\xi}$ by a linear superposition of M linearly independent expansion functions $\vec{\phi}_m^{(M)}$,

$$\vec{\xi}^{(M)} = \sum_{m=1}^M a_m^{(M)} \vec{\phi}_m^{(M)}, \quad (17)$$

and varying Eq. (14) with respect to $a_m^{(M)*}$, we obtain the matrix eigenvalue problem

$$\sum_{m=1}^M \left[\omega_m^{(M)2} \langle \vec{\phi}_{m'}^{(M)*} | K | \vec{\phi}_m^{(M)} \rangle - \langle \vec{\phi}_{m'}^{(M)*} | \delta W | \vec{\phi}_m^{(M)} \rangle \right] a_m^{(M)} = 0 \quad (18)$$

for $\omega_m^{(M)2}$ and the expansion coefficients $a_m^{(M)}$.

It is convenient, in choosing the $\vec{\phi}_m^{(M)}$'s, to use Fourier series in θ and ϕ ,

$$\vec{\xi}(\vec{r}) = \sum_{\ell, n} \vec{\xi}_{\ell, n}(\psi) \exp i(\ell\theta - n\phi) . \quad (19)$$

The different terms in n decouple, so we can treat them separately. A careful study [8] led us to adopt a finite-element expansion in ψ ; i.e.,

$$\begin{aligned} \phi_{\ell, m}^{(M)} = MT_{\ell}(\psi) [& (\psi - \psi_{m-1})H(\psi - \psi_{m-1})H(\psi_m - \psi) \\ & - (\psi - \psi_{m+1})H(\psi - \psi_m)H(\psi_{m+1} - \psi)] \end{aligned} \quad (20)$$

to represent δ_{ℓ} and ζ_{ℓ} , and

$$\phi_{\ell, m}^{(M)} = H(\psi - \psi_{m-1})H(\psi_m - \psi) \quad (21)$$

for the τ_{ℓ} . The Heaviside functions $H(p) \equiv (p + |p|)/2p$ make the ϕ 's vanish except between the nodal points ψ_{m-1} and ψ_{m+1} with $\psi_m \equiv m/M$; $T_{\ell}(\psi)$ is chosen to be unity for even ℓ and $\psi^{1/2}$ for odd ℓ because of analytic considerations.

Analytic determination of spectral properties is quite difficult. However, some understanding of the nature and number of continuous and discrete spectra can be obtained. This usefully complements the numerical program; this is especially important since those modes belonging to continuous spectra must be represented carefully near singular surfaces. We find only two classes of continuous spectra, associated with the acoustic and shear Alfvén modes, in agreement with results for a diffuse linear pinch [9]. These lead to the accumulation points of instability growth rates associated with the Mercier criterion and are the starting place for evaluating growth rates of resistive modes associated with Eq. (6).

As an application of the computational program, we adopt a simple system for which we can determine the equilibrium and coordinate system analytically. We consider an elliptical cylindrical plasma column, embedded in an external axial magnetic field B_0 , and carrying a uniform axial current J_z . To do this, we let $x = X - R$, $y = Z$, and $z = R\phi$, and introduce a periodicity length $L = 2\pi R$. Then $f(\psi) = b^2 a^2 L J_z / 4\pi B_0 (b^2 + a^2)$, so that, from $\nabla^2 \psi = L J_z / 2\pi B_0 f$, we obtain

$$\psi = x^2/a^2 + y^2/b^2 . \quad (22)$$

Note that $2\pi f B_0 J_z / L = -p' - B_0^2 g g'$ and we can adjust g to maintain a constant J_z . We use the coordinate system (ψ, θ, z) with

$$x = a\psi^{1/2} \cos \theta, \quad y = b\psi^{1/2} \sin \theta . \quad (23)$$

This model is an extension of previous studies [8, 10] to include effects of finite pressure and finite wave length. In addition, it provides a bridge between studies of configurations where exact analytic solutions can be found to verify the accuracy of the code and of realistic models with only numerical equilibrium solutions.

We show in Fig. 3 the spectrum for a system with a circular cross section where the different Fourier modes are completely decoupled. Note that the values of ω^2 vary from -0.6 for the unstable kink mode, through values as small as 5×10^{-3} for sound waves, to 7×10^5 for the fast wave. The total number of modes observed is determined by the number of expansion functions. In this case we study even ℓ modes between $\ell = -4$ and $\ell = 4$

with six finite elements to represent behavior in ψ . Even with so few functions the continuum nature of the spectrum is apparent. In general, the different modes retain their identities as the plasma cross section is elongated. The lower branches of the spectrum of Fig. 3 are followed in Fig. 4 as the ellipticity is increased. The acoustic modes are unaffected by ellipticity, but the shear Alfvén frequencies are decreased markedly. When $b/a \geq 4$, an $\ell = 2$ mode goes unstable. This is the result of a strong coupling, primarily to the $\ell = 4$ kink mode, such that it is pulled unstable by the unfavorable curvature of the magnetic field lines.

In Fig. 5 we show ξ_ψ , the component of the displacement normal to the surface, over the cross section for the unstable kink mode with $b/a = 10$. The eigenfunction is nearly a pure $\ell = 2$ motion; ellipticity does not have a significant effect on this mode. On the other hand, as can be seen in Fig. 6, in the mode that emerges from the lowest $\ell = 2$ shear Alfvén wave when it goes unstable, the $\ell \neq 2$ components are large.

It is remarkable that the shapes of the eigenmodes, as well as the spectrum, are represented so well with such a small number of expansion functions. This provides some assurance in the efficacy of the program for studying more realistic configurations.

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Fig. 1. Plot of surfaces of constant Ψ for a possible tokamak experiment with outside divertors. The pressure has been taken to be proportional to $\Psi_{lim} - \Psi$, and the toroidal field, $X(B_\phi - B_\phi^{vac})/R$, has been taken proportional to $(\Psi_{lim} - \Psi)^2$, where Ψ_{lim} is the value of Ψ at the plasma-vacuum boundary adjacent to the separatrix and B_ϕ^{vac} is the applied vacuum field. The constants of proportionality yield $\beta_{0av} = 0.37$ and, near the magnetic axis, $q = 0.8$. The aspect ratio is approximately 4.

Fig. 2. The stability parameters D_I and D_R , and q , as functions of ψ for the configuration of Fig. 1. Here $\psi = 0$ at the magnetic axis and 0.14 at the separatrix. For this configuration $0.005 < H < 0.01$, and H is thus impotent.

Fig. 3. Calculated spectrum for a constant-current, straight, circular pinch with pressure proportional to $(\Psi_{lim} - \Psi)$. For this case $\beta = 0.4$, $q_{lim} = 1.5$, and the ratio of periodicity length to plasma radius is 40π . The unstable $\ell = 2$ kink is represented by a cross. On each branch the upper 5 points represent the fast magnetosonic modes. The six lower points represent the continuum associated with the acoustic modes, and the middle group represent shear Alfvén waves. Stabilized kink modes appear above the Alfvén continuum in the -4 and -2 branches, but are mixed with the continuum in the 0 and 4 branches.

Fig. 4. Lower part of the calculated spectrum as a function of the ellipticity parameter a/b . The left edge of this plot is contained in Fig. 3. The growth rate of the $\ell = 2$ kink mode, represented by the dashed curve, is increased by ellipticity. Only the $\ell = 2$ branch of the shear Alfvén modes (connected by solid lines), is shown. These decrease in frequency, and the lowest mode becomes unstable. The other shear Alfvén modes have a higher frequency. The acoustic continuum modes (identified by their ℓ values on the right-hand side of the figure) are nearly unaffected by ellipticity.

Fig. 5. Plot of the radial perturbation, $\vec{\xi} \cdot \vec{\nabla} \psi$, for the unstable $\ell = 2$ kink mode of Fig. 4 with $a/b = 0.1$. Note that the x scale has been stretched.

Fig. 6. Plot of the radial perturbation, $\vec{\xi} \cdot \vec{\nabla} \psi$, for the second unstable mode of Fig. 4, with $a/b = 0.1$.

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