325
APR 7 1964

MASTER

ORNL-3546(Suppl. 1)
UC-38 – Engineering and Equipment
TID-4500 (26th ed.)

PRESTRESSING A TWO-LAYER PRESSURE VESSEL

BY CONTROLLED YIELDING OF THE

INNER LAYER

R. W. Schneider

OAK RIDGE NATIONAL LABORATORY

operated by

UNION CARBIDE CORPORATION

for the

U.S. ATOMIC ENERGY COMMISSION

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Printed in USA. Price: \$1.00 Available from the
Office of Technical Services
U. S. Department of Commerce
Washington 25, D. C.

— LEGAL NOTICE —

This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.

ORNL-3546 (Suppl. 1)

Contract No. W-7405-eng-26

INSPECTION ENGINEERING DEPARTMENT

DIRECTOR'S DIVISION

PRESTRESSING A TWO-LAYER PRESSURE VESSEL BY
CONTROLLED YIELDING OF THE INNER LAYER

R. W. Schneider

APRIL 1964

OAK RIDGE NATIONAL LABORATORY

Oak Ridge, Tennessee

operated by

UNION CARBIDE CORPORATION

for the

U. S. ATOMIC ENERGY COMMISSION

**THIS PAGE
WAS INTENTIONALLY
LEFT BLANK**

CONTENTS

	Page
ABSTRACT	1
INTRODUCTION	1
NOMENCLATURE	4
DESIGN - INFINITELY RIGID JACKET	5
Loading Path During Prestressing	5
Maximum Initial Clearance Between Cylinders	6
Minimum Pressure P_1 Required to Produce Contact	7
Stage of Zero Plastic Flow	10
Plastic Flow Resumes	13
Instability of Inner Shell	15
DESIGN - ELASTIC JACKET	25
General	25
Relationship Between $d\bar{\epsilon}$ and $d\epsilon_c$	25
Loading Path After Inner Shell Contacts Jacket	29
ILLUSTRATIVE PROBLEM - ELASTIC JACKET	33
DISCUSSION AND CONCLUSIONS	39

PRESTRESSING A TWO-LAYER PRESSURE VESSEL BY CONTROLLED YIELDING OF THE INNER LAYER

by R. W. Schneider

ABSTRACT

This paper presents a method of designing a two-layer pressure vessel wherein contact between the layers is produced by controlled yielding of the inner vessel by internal pressure. The amount of prestress depends upon the dimensions of the vessel, the properties of the material of construction and the prestressing pressure. The method takes into account the actual stress-strain curve of the material and satisfies the rules of plastic flow with work hardening.

INTRODUCTION

Large pressure vessels are sometimes fabricated by expanding a closed-end cylindrical shell into an open-end cylindrical jacket. The jacket and inner shell then act as a composite structure to carry the circumferential stress whereas the total axial stress is carried by the inner shell. This method of construction permits the use of two different materials of construction. The greatest benefits are probably derived, however, from the use of thinner material with improved metallurgical properties and the ability to fabricate when rolling capacity is exceeded in single plate construction. Figure 1 illustrates a two-layer vessel before and after the prestressing operation. Contact between the two

shells is produced by plastically deforming the inner shell by means of internal pressure. If the proper pressure is applied the two shells will remain in contact after the pressure is removed, a favorable state of prestress will exist, and the vessel will behave elastically thereafter in service.

A conventional prestressed, multilayer pressure vessel is designed using elastic theory only and a favorable system of residual stresses is produced by controlled tightening of the individual layers. U. S. Patent 2,337,247, dated December 21, 1943 covers the manufacture of multilayer vessels wherein a state of prestress is developed by overpressuring an assembly consisting of a number of concentric, loose-fitting, cylindrical shells.

When prestress is accomplished by plastically expanding the inner shell, in the case of two-layer construction, the design must include a plastic analysis as well as an elastic analysis. The problem is further complicated when a radial clearance exists initially between the two layers since the plasticity analysis must now include the effects of varying stress ratios during the deformation process. Including the elastic behavior of the jacket in the analysis increases the complexity of the problem, however, the elastic "spring-back" of the jacket is essential for producing a state of prestress. For simplicity, the study of the two-layer vessel which follows is divided into two categories, namely, (1) inner vessel enclosed by an infinitely rigid jacket and (2) inner vessel enclosed by an elastic jacket of finite thickness.

E. H. Lee¹ has developed a geometrical and geometrical-analytical method for studying problems involving plastic flow with work hardening. It assumes isotropic work hardening in which the Mises yield criterion applies at all stages in the deformation process and that the dimensions of the yield surface depend on a parameter which increases with continued plastic flow. The method is limited to those cases in which the principal stress directions remain fixed in the material; from symmetry, the vessels shown in Figs. 1 and 2 comply with this condition.

The geometrical-analytical method of Lee will be used to study the vessels in Figs. 1 and 2 and the paper should be referred to for details concerning the theory of its development. The analysis of the design problem which follows is based upon the following assumptions:

1. Ductile materials of construction are used.
2. The design temperature is below the creep range.
3. The plastic properties of the material in the inner vessel are defined by $\bar{\sigma} = K(\bar{\epsilon})^n$.
4. The jacket and inner shell fulfill the requirements of a thin-walled cylinder.
5. The frictional forces at the contact surface may be neglected.
6. The effective stress and effective strain, respectively, are expressed by

¹Lee, E. H., "A Geometrical Construction for Plastic Flow with Work Hardening," Proceedings of the First Midwestern Conference on Solid Mechanics, April 1953, Urbana, Illinois.

$$\bar{\sigma} = \left[1/2 (\bar{\sigma}_c - \bar{\sigma}_a)^2 + \bar{\sigma}_a - \bar{\sigma}_r + \bar{\sigma}_r - \bar{\sigma}_c \right]^{\frac{1}{2}}$$

$$\bar{\epsilon} = \sqrt{2}/3 \left[\bar{\epsilon}_c - \bar{\epsilon}_a + \bar{\epsilon}_a - \bar{\epsilon}_r + \bar{\epsilon}_r - \bar{\epsilon}_c \right]^{\frac{1}{2}}$$

In addition to the above, the analysis is limited to the jacket and to the cylindrical portion of the inner shell. The heads are considered to transmit the axial pressure load to the shell but the discontinuity effects at the junction of the head and shell are disregarded. The same design method may be used for materials with properties which do not conform with the third assumption by deriving the equation of the actual stress-strain curve using any of the usual methods of curve fitting or by working from the actual stress-strain curve.

NOMENCLATURE

P = pressure in inner shell, psig

P_c = interfacial contact pressure between shell and jacket, psig

t = instantaneous thickness at some pressure P, in.

R = instantaneous radius at some pressure P', in.

$\bar{\sigma}$ = effective stress, psi

$\bar{\epsilon}$ = effective strain, in/in

$\sigma_c, \sigma_a, \sigma_r$ = true principal stresses, circumferential, axial, and radial respectively, psi.

$\epsilon_c, \epsilon_a, \epsilon_r$ = natural principal strains, in/in

S_u^{*} = conventional or engineering ultimate tensile strength of the material in the inner shell, psi

n = strain hardening exponent of the material in the inner shell

K = a constant in the equation of stress-strain curve $\bar{\sigma} = K(\bar{\epsilon})^n$

δ = radial deformation, in.

s = the distance of the strain point from the origin, at some pressure P , measured along the trajectory of the strain point, in.

E = modulus of elasticity, psi

Many of the symbols are used with subnumerals; they are defined in the text where they first appear. The subnumerals denote a specific condition at a specific time during the prestressing operation and the subnumeral "0" denotes an initial state before the application of pressure. Arabic subnumerals are used for states of stress and Roman subnumerals for states of strain; the corresponding numbers refer to the same instant of time in the loading cycle. All symbols refer to the inner cylindrical shell unless used with a subscript "j" to denote the jacket.

DESIGN - INFINITELY RIGID JACKET

Loading Path During Prestressing

The discussion which follows assumes that the jacket is infinitely rigid as shown in Fig. 2. The loading path of the inner shell for the entire prestressing operation is illustrated in Fig. 3; however, the path is continued to where instability of the inner shell occurs. The important features of the construction method are:

1. The true principal stress and natural principal strain axes are inclined 120° to each other on one plane.
2. The point representing the relative magnitude of the three principal stress components is called the stress point. The length of the stress vector from the origin "0" to the stress point is equal to $\bar{\sigma}$ and is thus the radius of the current yield circle.
3. The point representing the relative magnitude of the three principal strain components is called the strain point. The length "s" of the trajectory of the strain point is equal to $3\bar{\epsilon}/2$.
4. During plastic flow the tangent to the strain trajectory must be parallel to the associated stress vector. Stated differently, if the stress point moves a distance $d\bar{\sigma}$ from the origin, the strain point will move a distance $3/2 d\bar{\epsilon}$ parallel to the stress vector. If the loading cycle is changed so that plastic flow ceases, the stress point moves on or within the current yield circle and the strain point remains stationary.

Maximum Initial Clearance Between Cylinders

Cooper² has considered the instability condition of a closed-end unrestrained, thin-walled pressure vessel where the material follows the equation $\bar{\sigma} = K(\bar{\epsilon})^n$. The maximum pressure is given by

$$P_u = \frac{2 t_o S_u^* (3)^{-(n+1)/2}}{R_o} \quad (1)$$

²Cooper, W. E., "The Significance of the Tensile Test to Pressure Vessel Design," The Welding Journal, 36, (1) Research Suppl., 49-s-54-s, January 1957.

and the radial deformation at the instability pressure P_u is

$$\delta_u = R_o (e^{n/2} - 1) \quad (2)$$

Obviously, then, the initial radial clearance δ_o between the two layers should not exceed δ_u or the inner vessel will fail by instability before contacting the outer shell.

Minimum Pressure P_1 Required to Produce Contact

Figure 4 shows the loading path from the start of pressurization of the inner shell until the shell just contacts the jacket at pressure P_1 .

During the initial stage of prestress the inner shell yields as a simple, closed-end cylinder under internal pressure. Since the ratio of the circumferential stress to the axial stress is two, the stress point moves along the line 0-1. Point 1 represents the stress point at the instant of contact; the length of the stress vector 0-1 is the effective stress $\bar{\sigma}_1$ corresponding with P_1 .

The length of a closed-end cylinder does not change during plastic flow under internal pressure, therefore, ϵ_a equals zero. To satisfy the incompressibility condition, the radial component of strain is equal in magnitude, but of opposite sign, to the circumferential component of strain. Accordingly, the strain point moves along line 0-I and the length of the strain point trajectory equals $3\bar{\epsilon}_1/2$.

From the geometry of the construction the effective stress and effective strain, respectively, are

$$\bar{\sigma}_1 = \frac{\sqrt{3}}{2} \sigma_{c1} \quad (3)$$

$$\bar{\epsilon}_I = \frac{2}{\sqrt{3}} \epsilon_{cI} \quad (4)$$

During yielding the true principal natural strains are

$$\epsilon_{aI} = \ln \frac{L_1}{L_0} = 0 \quad (5a)$$

$$\epsilon_{cI} = \ln \frac{R_1}{R_0} \quad (5b)$$

$$\epsilon_{rI} = \ln \frac{t_1}{t_0} \quad (5c)$$

and the principal stresses are

$$\sigma_{a1} = \frac{P_1 R_1}{2 t_1} \quad (6a)$$

$$\sigma_{c1} = \frac{P_1 R_1}{t_1} \quad (6b)$$

$$\sigma_{r1} = \begin{array}{l} -P_1 \text{ at inner surface of shell} \\ 0 \text{ at outer surface of shell} \end{array} \quad (6c)$$

where R_1 and t_1 are instantaneous values of the radius and thickness at pressure P_1 .

The effective strain at the instant contact is made with the rigid jacket is found by substituting Eq. (5b) into Eq. (4) and then by setting R_1 equal to $R_0 + \delta_0$, or

$$\bar{\epsilon}_I = \frac{2}{\sqrt{3}} \ln \left(\frac{R_0 + \delta_0}{R_0} \right) \quad (7)$$

Combining Eqs. (6b) and (3) gives the corresponding effective stress

$$\bar{\sigma}_1 = \frac{\sqrt{3} P_1 (R_o + \delta_o)}{2 t_1} \quad (8)$$

where P_1 is the pressure required to just produce contact and t_1 is the new thickness of the shell.

During straining $\epsilon_r = -\epsilon_c$. If Eqs. (5b) and (5c) are equated, it is found that at contact the shell thickness is no longer t_o but a lesser value t_1 , where

$$t_1 = \frac{R_o t_o}{R_o + \delta_o} \quad (9)$$

and Eq. (8) then becomes

$$\bar{\sigma}_1 = \frac{\sqrt{3} P_1 (R_o + \delta_o)^2}{2 R_o t_o} \quad (10)$$

Since the stress-strain curve of the material is expressed by

$\bar{\sigma} = K(\bar{\epsilon})^n$, Eqs. (7) and (10) are related by

$$\frac{\sqrt{3} P_1 (R_o + \delta_o)^2}{2 R_o t_o} = K \left[\frac{2}{\sqrt{3}} \ln \left(\frac{R_o + \delta_o}{R_o} \right) \right]^n \quad (11)$$

and the pressure P_1 required to just produce contact is

$$P_1 = \frac{2 R_o t_o}{\sqrt{3} (R_o + \delta_o)^2} K \left[\frac{2}{\sqrt{3}} \ln \left(\frac{R_o + \delta_o}{R_o} \right) \right]^n \quad (12)$$

The three principal stresses in the shell at instant of contact are found from Eqs. (6), (9) and (12) to be

$$\sigma_{a1} = \frac{P_1(R_o + \delta_o)^2}{2 R_o t_o} \quad (13a)$$

$$\sigma_{c1} = \frac{P_1(R_o + \delta_o)^2}{R_o t_o} \quad (13b)$$

$$\sigma_{r1} \approx 0 \quad (13c)$$

Stage of Zero Plastic Flow

After contact is made, the jacket prevents further circumferential straining of the inner shell, thus, $d\epsilon_{cI} \approx 0$. When plastic flow resumes, $d\epsilon_r$ must equal $-d\epsilon_a$ in order to maintain the condition of incompressibility and the strain point must move parallel to the associated stress vector. The strain trajectory, where $d\epsilon_r = -d\epsilon_a$, is shown in Fig. 3 as line I - IV - V. In order for the associated stress vector to be tangent to the strain trajectory, plastic flow must cease until the stress point moves from 1 to 3 on or within the yield circle of radius $\bar{\sigma}_1$. While this occurs the strain point remains stationary and the single strain point I, II and III corresponds to stress points 1, 2 and 3, respectively.

It is necessary to see if it is possible for the stress point to move from 1 to 3 within the circle of radius $\bar{\sigma}_1$. Assume that it is possible! The expression for the effective stress as a function of the three principal stresses is given by

$$\bar{\sigma} = \left[1/2 \left(\overline{\sigma_c - \sigma_a}^2 + \overline{\sigma_a - \sigma_r}^2 + \overline{\sigma_r - \sigma_c}^2 \right) \right]^{\frac{1}{2}} \quad (14)$$

The three principal stresses in the inner shell at the instant of contact at pressure P_1 are given by Eq. (6). The effective stress $\bar{\sigma}_1$, corresponding to pressure P_1 , is found by substituting the values from Eq. (6) into Eq. (14) or

$$\bar{\sigma}_1 = \sqrt{\frac{3}{4}} \frac{P_1 R_1}{t_1} \quad (15)$$

At some pressure P greater than P_1 , but less than P_3 , the three principal stresses are

$$\sigma_a = \frac{P R_1}{2 t_1} \quad (16a)$$

$$\sigma_c = \frac{(P - P_c) R_1}{t_1} \quad (16b)$$

$$\sigma_r \approx 0 \quad (16c)$$

where P_c is the interfacial contact pressure. The effective stress at pressure P , found by substituting Eq. (16) into Eq. (14), cannot exceed $\bar{\sigma}_1$ or the stress point would be moving outside the circle of radius $\bar{\sigma}_1$, which is contrary to the assumption, therefore

$$\left[\frac{3 P^2 R_1^2}{4 t_1^2} - \frac{1.5 P P_c R_1^2}{t_1^2} + \frac{P_c^2 R_1^2}{t_1^2} \right]^{\frac{1}{2}} \leq \bar{\sigma}_1 \quad (17)$$

The value of $\bar{\sigma}_1$, from Eq. (15) is substituted into Eq. (17) to yield

$$4 P_c^2 - 6 P P_c + 3(P^2 - P_1^2) \leq 0 \quad (18)$$

If the stress point moves inside the circle of radius $\bar{\sigma}_1$, there are no plastic strains, hence, the change in the circumferential strain, referred

to the condition at P_1 , becomes

$$\Delta(\epsilon_c)_{\text{elastic}} = \frac{1}{2E} [2(\Delta\sigma_c) - (\Delta\sigma_a)] = 0 \quad (19)$$

where

$$\Delta\sigma_c = \frac{(P - P_c) R_1}{t_1} - \frac{P_1 R_1}{t_1} \quad (20a)$$

$$\Delta\sigma_a = \frac{P R_1}{2t_1} - \frac{P_1 R_1}{2t_1} \quad (20b)$$

If Eq. (20) is substituted into Eq. (19), the interfacial contact pressure is found to be

$$P_c = \frac{3}{4} (P - P_1) \quad (21)$$

and the substitution of Eq. (21) into Eq. (18) yields the inequality

$$P^2 \leq P_1^2 \quad (22)$$

Since P was assumed to be greater than P_1 , Eq. (22) is obviously not valid and thus the stress point cannot move within the circle of radius $\bar{\sigma}_1$.

It can be shown that in the case of an elastic, strain-hardening material, the radius of the yield circle expands slightly and the stress point moves between points 1 and 3 on the expanding circle. For engineering design purposes, the stress point can be assumed to move from 1 to 3 along the circle of constant radius $\bar{\sigma}_1$, as shown in Fig. 3.

Since the thickness of the inner shell is constant between stress points 1 and 3, the axial stress is proportional to the pressure in the

vessel. Thus, when the internal pressure P_2 satisfies the condition $P_3 \geq P_2 \geq P_1$, the axial stress is given by

$$\sigma_{a2} = \frac{P_2 (R_o + \delta_o)^2}{2 R_o t_o} \quad (23)$$

According to Fig. 3, the circumferential stress at pressure P_2 is represented by the length of the line 0-2'. The circumferential stress at pressure P_1 is given by the line 0-1' and by Eq.(13b), thus

$$\sigma_{c2} = \frac{P_1 (R_o + \delta_o)^2 (0-2')}{R_o t_o (0-1')} \quad (24)$$

Equation (24) may also be written in terms of the internal pressure P_2 and the interfacial contact pressure P_{c2} , whereby

$$\sigma_{c2} = \frac{(P_2 - P_{c2}) (R_o + \delta_o)^2}{R_o t_o} \quad (25)$$

thus, when Eqs.(24) and (25) are equated, the interfacial contact pressure is found to be

$$P_{c2} = P_2 - \frac{(0-2')}{(0-1')} P_1 \quad (26)$$

Plastic Flow Resumes

After stress point 3 is reached, plastic flow resumes. The stress point moves along line 3-4 and the stress vector 0-3-4 is parallel to the associated strain trajectory I-IV to fulfill a condition of plastic flow.

Since no yielding occurs between stress points 1 and 3, the thickness of the shell remains unchanged, therefore, $t_3 = t_1$ and from Eq.(9)

$$t_3 = t_1 = \frac{R_o t_o}{R_o + \delta_o} \quad (27)$$

From the geometry of construction, Fig. 3, the axial stress corresponding to stress point 3 is double the axial stress of stress point 1. The circumferential stress corresponding to stress point 3 is one-half the circumferential stress of stress point 1. Since the axial stress varies directly with the prestressing pressure in the inner shell during the stage of zero plastic flow, the pressure P_3 , corresponding to stress point 3, is double the pressure which was required to initially produce contact between the layers, or $P_3 = 2 P_1$. The three principal stresses represented by stress point 3 are readily obtained from Eq.(13), or

$$\sigma_{a3} = \frac{P_1 (R_o + \delta_o)^2}{R_o t_o} = \frac{P_3 (R_o + \delta_o)^2}{2 R_o t_o} \quad (28a)$$

$$\sigma_{c3} = \frac{P_1 (R_o + \delta_o)^2}{2 R_o t_o} = \frac{P_3 (R_o + \delta_o)^2}{4 R_o t_o} \quad (28b)$$

$$\sigma_{r3} \approx 0 \quad (28c)$$

An interfacial contact pressure (P_{c3}) exists since the inner shell is in contact with the rigid jacket. The circumferential stress, which is produced by the differential pressure $P_3 - P_{c3}$ across the inner shell, must equal σ_{c3} of Eq.(28b), or

$$\frac{P_3 (R_o + \delta_o)^2}{4 R_o t_o} = \frac{(P_3 - P_{c3})(R_o + \delta_o)^2}{R_o t_o} \quad (29)$$

Solving Eq.(29) yields the interfacial contact pressure while the inner shell is under pressure P_3 , or

$$P_{c3} = \frac{3}{4} P_3 \quad (30)$$

The shell behaves elastically as the prestressing pressure P_3 is reduced to zero. A simple elastic analysis shows that the radius of the shell will tend to decrease by the amount δ_R , or

$$\delta_R = \frac{(R_o + \delta_o)}{E} \left[\sigma_{c3} - \nu \sigma_{a3} \right] \quad (31)$$

Since σ_{c3} is greater than $\nu \sigma_{a3}$, according to Eq.(28), the inner shell actually shrinks away from the jacket by the amount δ_R and the inner shell is left in a neutral or non-prestressed condition.

As plastic flow continues beyond stress point 3, the ratio of the axial stress to the circumferential stress remains constant. Accordingly, it is impossible to prestress the inner shell by controlled yielding under internal pressure when the shell is contained within a rigid, non-yielding jacket. If a jacket with elastic properties is substituted for the rigid enclosure, prestressing is possible due to the elastic recovery of the jacket upon release of the pressure.

Instability of Inner Shell

Stress point 5 and the corresponding strain point V represent the values at the maximum or instability pressure. The length of the strain trajectory "s" from 0 to V along O-I-IV-V is equal to $3\bar{\epsilon}_V/2$. From the

geometry of the construction, Fig. 3, the length of the strain trajectory increment between I and V is $\sqrt{3}\epsilon_a$ where ϵ_a is the axial strain between I and V. The total length of the strain trajectory from the origin is then

$$s_V = \frac{3}{2} \bar{\epsilon}_I + \sqrt{3} \epsilon_a \quad (32)$$

but s_V also equals

$$s_V = \frac{3}{2} \bar{\epsilon}_V \quad (33)$$

therefore

$$\bar{\epsilon}_V = \bar{\epsilon}_I + \frac{2}{\sqrt{3}} \epsilon_a \quad (34)$$

The derivative of Eq. (34) is

$$d\epsilon_a = \frac{\sqrt{3}}{2} d\bar{\epsilon}_V \quad (35)$$

The relationship between the effective stress and the axial stress at stress point 5 is found from the geometry of Fig. 3 to be

$$\bar{\sigma}_5 = \frac{\sqrt{3}}{2} \sigma_{a5} \quad (36)$$

and the derivative becomes

$$d\sigma_{a5} = \frac{2}{\sqrt{3}} d\bar{\sigma}_5 \quad (37)$$

The axial stress corresponding to stress point 5 is

$$\sigma_{a5} = \frac{P_5 (R_0 + \delta_0)}{2 t_5} \quad (38)$$

or

$$P_5 = \frac{2 t_5 \sigma_{a5}}{R_0 + \delta_0} \quad (39)$$

The maximum pressure is found by taking the derivative of Eq.(39) and setting dP equal to zero. The expression becomes

$$d\sigma_{a5} + \sigma_{a5} \frac{dt_5}{t_5} = 0 \quad (40)$$

whereby

$$\frac{d\sigma_{a5}}{\sigma_{a5}} = - \frac{dt_5}{t_5} \quad (41)$$

but since $\epsilon_r = -\epsilon_a$ between strain points I (or II and III) and V, Eq.(41) becomes

$$\frac{d\sigma_{a5}}{\sigma_{a5}} = d\epsilon_a \quad (42)$$

or

$$\frac{d\sigma_{a5}}{d\epsilon_a} = \sigma_{a5} \quad (43)$$

Substituting Eqs.(35), (36) and (37) into Eq.(43) yields

$$\frac{d\bar{\sigma}_5}{d\bar{\epsilon}_V} = \frac{\sqrt{3}}{2} \bar{\sigma}_5 \quad (44)$$

The effective stress and effective strain are related by the expression $\bar{\sigma}_5 = K(\bar{\epsilon}_V)^n$. After the occurrence of necking in a tensile specimen at $\bar{\epsilon} = n$, the true stress-strain curve is usually extended as a straight line tangent to the exponential curve. For simplicity, the equations which

follow are based on the exponential curve even for effective strains greater than "n". The derivative of the stress-strain equation is

$$\frac{d\bar{\sigma}_5}{d\bar{\epsilon}_V} = K n (\bar{\epsilon}_V)^{n-1} \quad (45)$$

therefore, Eq.(44) may be set equal to Eq.(45) or

$$K n (\bar{\epsilon}_V)^{n-1} = \frac{\sqrt{3}}{2} \bar{\sigma}_5 = \frac{\sqrt{3}}{2} K (\bar{\epsilon}_V)^n \quad (46)$$

From Eq.(46) the effective strain at instability becomes

$$\bar{\epsilon}_V = \frac{2}{\sqrt{3}} n \quad (47)$$

If the equivalent value of $\bar{\epsilon}_V$ from Eq.(34) is substituted into Eq.(47), the value of the axial strain at instability is determined,

$$\epsilon_a = n - \frac{\sqrt{3}}{2} \bar{\epsilon}_I \quad (48)$$

but since $\epsilon_r = -\epsilon_a$ between strain points I (or II and III) and V, Eq.(48) may be rewritten

$$\ln \frac{t_3}{t_5} = \ln \frac{t_1}{t_5} = n - \frac{\sqrt{3}}{2} \bar{\epsilon}_I \quad (49)$$

Solving Eq.(49) gives the thickness of the shell at instability, which is

$$t_5 = \frac{t_3}{e^{(n - \sqrt{3} \bar{\epsilon}_I / 2)}} \quad (50)$$

Eqs.(36) and (47), substituted into the equation of the material stress-strain curve yields

$$\frac{\sqrt{3}}{2} \sigma_{a5} = K \left(\frac{2n}{\sqrt{3}} \right)^n \quad (51)$$

which is the same as

$$\frac{\sqrt{3} P_5 (R_0 + \delta_0)}{4 t_5} = K \left(\frac{2}{\sqrt{3}} \right)^n (n)^n \quad (52)$$

or the maximum prestressing pressure P_5 , which is the pressure which produces instability of the inner shell, is

$$P_5 = \frac{4 t_5 K(n)^n}{\sqrt{3} (R_0 + \delta_0)} \left(\frac{2}{\sqrt{3}} \right)^n \quad (53)$$

Substituting Eq. (50) into Eq. (53) and $e^n S_u^*$ for $K(n)^n$ yields

$$P_5 = \frac{4 t_3 e^n S_u^* (2/\sqrt{3})^n}{\sqrt{3} (R_0 + \delta_0) (e)^{(n - \sqrt{3} \bar{\epsilon}_I/2)}} \quad (54)$$

which when simplified becomes

$$P_5 = \frac{4 t_3 S_u^* (2/\sqrt{3})^n (e)^{(\sqrt{3} \bar{\epsilon}_I/2)}}{\sqrt{3} (R_0 + \delta_0)} \quad (55)$$

Eq. (7) may be rearranged to give

$$e^{(\sqrt{3} \bar{\epsilon}_I/2)} = \frac{R_0 + \delta_0}{R_0} \quad (56)$$

and this may be used to further simplify Eq. (55), whereby

$$P_5 = \frac{4 t_3 S_u^* (2/\sqrt{3})^n}{\sqrt{3} R_0} \quad (57)$$

Finally, if the value of t_3 from Eq. (27) is substituted into Eq. (57), the

instability pressure of the inner shell is found in terms of the original dimensions of the vessel. Accordingly, the instability pressure becomes

$$P_u = P_5 = \frac{4}{\sqrt{3}} \left(\frac{2}{\sqrt{3}} \right)^n \left(\frac{t_o S_u^*}{R_o + \delta_o} \right) \quad (58)$$

The term $R_o + \delta_o$ is a constant and equal to the inside radius of the rigid jacket. Thus, the ultimate pressure of the inner cylinder is independent of the initial clearance between the two shells provided the inner shell does not fail by instability before contacting the jacket.

UNCLASSIFIED
ORNL-DWG 63-4648

Fig. 1 A Two-Layer Pressure Vessel

Fig. 2 A Cylindrical Vessel with an Infinitely Rigid Jacket

UNCLASSIFIED
ORNL-DWG 63-4651

Fig. 4 The Loading Path from the Start of Pressurization until the Two Layers Make Contact

DESIGN-ELASTIC JACKET

General

In the preceding analysis it was shown that a state of prestress cannot be induced when an infinitely rigid jacket is used. It was also shown that the stress point follows three distinct loading paths before the inner shell fails by instability, specifically, (1) from the start of pressurization until the inner shell contacts the jacket at pressure P_1 , (2) a zone of zero plastic flow between P_1 and $2P_1$ and (3) a continuation of plastic flow commencing at pressure $2P_1$.

When an infinitely rigid jacket is used the change in circumferential strain ($d\epsilon_c$) is zero after the inner shell makes contact with the jacket. When an elastic jacket is considered, $d\epsilon_c$ is no longer zero, the zone of zero plastic flow disappears, and the inner shell continues to undergo plastic flow but along a different path.

The discussion which follows pertains to the type of vessel illustrated in Fig. 1, that is, a closed-end cylindrical shell enclosed by an elastic jacket.

Relationship Between $d\bar{\epsilon}$ and $d\epsilon_c$

After the inner layer contacts the jacket, the plastic behavior of the inner shell is established by the elastic behavior of the jacket under the influence of the interfacial contact pressure. Conversely, the interfacial contact pressure is governed by the flexibility of the jacket.

During plastic flow the tangent to the strain trajectory must be

parallel to the associated stress vector. When the stress point moves a distance $d\bar{\sigma}$ from the origin, the strain point moves a distance $3/2 d\bar{\epsilon}$ parallel to the stress vector. Since the change in circumferential strain of the inner shell must equal the change in circumferential strain of the jacket ($d\epsilon_c = d\epsilon_{cj}$)*, it is necessary to consider the relationship between $d\epsilon_c$ and $d\bar{\epsilon}$.

Figure 5 shows the strain trajectory plus an incremental length orientated at β^0 from the horizontal axis. Since $d\epsilon_c + d\epsilon_r + d\epsilon_a = 0$, it can be shown that

$$d\bar{\epsilon} = \frac{2 d\epsilon_c}{\sqrt{3} \cos \beta - \sin \beta} \quad (59)$$

or

$$d\bar{\epsilon} = X d\epsilon_c \quad (60)$$

where

$$X = \frac{2}{\sqrt{3} \cos \beta - \sin \beta} \quad (61)$$

The function X is given in Table 1, in increments of 2^0 , for values of β between 0^0 and 60^0 .

*In the analyses which follow, Arabic subnumerals have been substituted for Roman subnumerals in the strain and effective strain terms. The change was made to shorten terms carrying both subscripts and subnumerals.

UNCLASSIFIED
ORNL-DWG 64-299

Fig. 5 Relationship Between $d\bar{\epsilon}$ and $d\epsilon_c$

Table 1 X as a Function of the Angle β

β	X	β	X
0°	1.155	32°	2.130
2°	1.179	34°	2.281
4°	1.206	36°	2.459
6°	1.236	38°	2.670
8°	1.269	40°	2.924
10°	1.305	42°	3.236
12°	1.346	44°	3.628
14°	1.390	46°	4.134
16°	1.440	48°	4.810
18°	1.495	50°	5.759
20°	1.556	52°	7.186
22°	1.624	54°	9.567
24°	1.701	56°	14.337
26°	1.788	58°	28.653
28°	1.887	60°	—
30°	2.000		

Loading Path After Inner Shell Contacts Jacket

Figure 6 shows an incremental change in the strain point and stress point after the inner shell makes contact with the jacket. Obviously, the motion of the stress point and strain point, before contact is established, is independent of the jacket.

When the pressure is increased from P_3 to P_4 , the stress point moves from point 3 to point 4 along an arc of increasing radius such that $\bar{\sigma}_4 > \bar{\sigma}_3$. The effective stress at pressure P_4 is

$$\bar{\sigma}_4 = \bar{\sigma}_3 + d\bar{\sigma}_3 \quad (62)$$

but since the stress-strain curve of the material is given by $\bar{\sigma} = K(\bar{\epsilon})^n$, Eq. (62) becomes

$$\bar{\sigma}_4 = \bar{\sigma}_3 + Kn(\bar{\epsilon}_3)^{n-1} d\bar{\epsilon}_3 \quad (63)$$

Equation (60) may be substituted into Eq. (63) to yield

$$\bar{\sigma}_4 = \bar{\sigma}_3 + Kn(\bar{\epsilon}_3)^{n-1} X d\epsilon_{c3} \quad (64)$$

The change in circumferential strain of the jacket ($d\epsilon_{cj3}$), when the internal pressure changes from P_3 to P_4 , may be expressed in terms of the interfacial contact pressure, or

$$d\epsilon_{cj3} = \frac{(R_o + \delta_o)}{E_j t_j} (P_{c4} - P_{c3}) \quad (65)$$

A constant value is used for the radius $(R_o + \delta_o)$ since, in practical

situations, prestressing is always accomplished before any significant increase in radius occurs. Since $d\epsilon_{c3} = d\epsilon_{cj3}$, Eq. (65) may be substituted into Eq. (64) to give

$$\bar{\sigma}_4 = \bar{\sigma}_3 + \frac{1}{E_j t_j} \left[K n (\bar{\epsilon}_3)^{n-1} X(R_o + \delta_o) (P_{c4} - P_{c3}) \right] \quad (66)$$

where X is a function of the angle β_3 . Values of X are given in Table 1.

From the geometry of the construction, Fig. (6)

$$\sigma_{a4} = \bar{\sigma}_4 \left(\sin \phi_4 + \frac{\sqrt{3}}{3} \cos \phi_4 \right) \quad (67)$$

but the axial stress also equals

$$\sigma_{a4} = \frac{P_4 (R_o + \delta_o)}{2 t_1} \quad (68)$$

where t_1 is given by Eq. (9). A constant value is used for the thickness of the inner shell for the same reason that a fixed radius was used.

Equation (68) substituted into Eq. (67) yields

$$\bar{\sigma}_4 = \frac{P_4 (R_o + \delta_o)}{2 t_1 \left(\sin \phi_4 + \frac{\sqrt{3}}{3} \cos \phi_4 \right)} \quad (69)$$

From the geometry of the construction, Fig. (6)

$$\sigma_{c4} = \frac{2 \bar{\sigma}_4 \cos \phi_4}{\sqrt{3}} \quad (70)$$

but the circumferential stress also equals

$$\sigma_{c4} = \frac{(P_4 - P_{c4})(R_o + \delta_o)}{t_1} \quad (71)$$

and combining Eqs. (70) and (71) gives

$$\bar{\sigma}_4 = \frac{\sqrt{3}(R_o + \delta_o)(P_4 - P_{c4})}{2 t_1 \cos \phi_4} \quad (72)$$

Equations (66), (69) and (72) are three equations with three unknowns, thus, their simultaneous solution will yield $\bar{\sigma}_4$, P_4 and P_{c4} corresponding to the arbitrarily selected angle ϕ_4 . The stresses in the inner shell, when the pressure in the inner vessel is P_4 , may be calculated from Eqs. (68) and (71). Knowing the interfacial contact pressure (P_{c4}) permits the calculation of the circumferential stress in the jacket (σ_{cj4}). Finally, the state of prestress in both shells, when the internal pressure is released, is found by a simple elastic analysis based upon compatible deformations of the two layers. An actual problem is solved by repeating the above calculations for increments of the angle ϕ , starting with $\phi = 0$, until the desired state of prestress is achieved.

The above derivation applies as long as the jacket does not yield and the radius and thickness of the inner shell do not change significantly. Significant changes in the radius and thickness of the inner shell, with increasing pressure, can be accounted for by using the instantaneous values of R and t in Eqs. (62) through (72).

ILLUSTRATIVE PROBLEM-ELASTIC JACKET

A vessel of the type shown in Fig. 1 is to be prestressed by controlled yielding of the inner shell under internal pressure. Determine the loading path for prestressing pressures up to about 3600 psi.

Given:

$$R_o = 20.000 \text{ in}$$

$$t_o = 1.000 \text{ in}$$

$$\delta_o = 0.125 \text{ in}$$

$$\bar{\sigma} = 127,200(\bar{\epsilon})^{.25}, \text{ stress-strain curve of inner shell material}$$

$$S_u^* = 70,000 \text{ psi, conventional UTS of inner shell material}$$

$$E = 30 \times 10^6 \text{ psi}$$

$$E_j = 30 \times 10^6 \text{ psi}$$

$$t_j = 0.750 \text{ in}$$

The calculations are performed as follows:

1. Use Eq. (2) to verify that $\delta_o < \delta_\mu$.
2. Use Eq. (12) and calculate the pressure P_1 required to just produce contact. The corresponding values of $\bar{\epsilon}_1$, t_1 , $\bar{\sigma}_1$, σ_{a1} and σ_{c1} are found by solving Eqs. (7), (9), (10), (13a) and (13b) respectively. These values correspond to $\phi_1 = 0$.
3. The angle ϕ will be taken in increments of 4° to $\phi = 24^\circ$. Start with $\phi_2 = 4^\circ$ ($\beta_1 = 2^\circ$). The simultaneous solution of Eqs. (66), (69) and (72) yields $\bar{\sigma}_2$, P_2 and P_{c2} corresponding to $\phi_2 = 4^\circ$. The stresses

in the inner shell are found from Eqs. (68) and (71); $\bar{\epsilon}_2$ is determined directly from the stress-strain curve of the inner shell material

$$\left[\bar{\sigma}_2 = 127,200(\bar{\epsilon}_2)^{.25} \right].$$

Since the interfacial contact pressure P_{c2} is now known, the circumferential stress in the jacket (σ_{cj2}) may be

determined from $\sigma_{cj2} = P_{c2}(R_o + \delta_o)/t_j$.

4. Step 3 is repeated for $\phi_3 = 8^\circ$ and $\beta_2 = 6^\circ$, $\phi_4 = 12^\circ$ and $\beta_3 = 10^\circ$, etc., until $\phi_7 = 24^\circ$ is reached.
5. The residual stresses are found by a simple elastic analysis based upon compatible deformations of the two layers.
6. The pertinent stresses and strains in the inner shell and jacket, while the inner vessel is at pressure P_1 , P_2 , - - -, P_7 , respectively, are given in Table 2.
7. The residual stresses in the inner shell and jacket, produced by a prestressing pressure of P_6 (3399 psi) and P_7 (3629 psi), are given in Table 3.
8. The operating stresses are found by superposition, that is, the membrane stress induced by the operating pressure is added algebraically to the residual stresses produced by the prestressing operation.

Table 2. Stresses and Strains in Inner Shell
and Jacket at Prestressing Pressure P

	INNER SHELL							JACKET
ϕ	P psi	P _c psi	$\bar{\sigma}$ psi	$\bar{\epsilon}$ in/in	σ_a psi	σ_c psi	ds in	σ_{cj} psi
$\phi_1 = 0^\circ$	2112	0	37039	0.00719	21385	42770		0
$\phi_2 = 4^\circ$	2384	258	37391	0.00747	24141	43057	0.000417	6930
$\phi_3 = 8^\circ$	2650	519	37746	0.00775	26835	43158	0.000430	13920
$\phi_4 = 12^\circ$	2908	782	38112	0.00806	29447	43057	0.000458	21000
$\phi_5 = 16^\circ$	3158	1048	38499	0.00839	31979	42733	0.000500	28120
$\phi_6 = 20^\circ$	3399	1314	38912	0.00876	34419	42226	0.000547	35300
$\phi_7 = 24^\circ$	3629	1580	39338	0.00915	36749	41497	0.000585	42397

Table 3. Residual Stresses in Inner Shell and Jacket

		INNER SHELL	JACKET
ϕ	PRESTRESSING PRESSURE psi	σ_c psi	σ_{cj} psi
$\phi_6 = 20^\circ$	$P_6 = 3399$	- 1457	+ 1930
$\phi_7 = 24^\circ$	$P_7 = 3629$	- 5120	+ 6790

The loading path for values of ϕ from 0° (ϕ_1) to 24° (ϕ_7) is illustrated in Fig. 7 by a heavy line. An enlarged view of the strain trajectory is given in Fig. 8. If the jacket had been infinitely rigid the stress point would have followed the broken line from stress point 1. On the other hand, a less rigid jacket would have caused the stress points to be displaced further from the broken line. A continuation of the line 0 - 1 represents the motion of the stress point when the inner shell is not enclosed by a jacket.

* ϵ_c = TOTAL CIRCUMFERENTIAL STRAIN OF INNER SHELL
AFTER INNER SHELL CONTACTS JACKET.

Fig. 8 Enlarged View of Strain Trajectory for the Illustrative Problem

DISCUSSION AND CONCLUSIONS

In the preceding analysis it was shown that a state of prestress cannot be produced when the inner shell is expanded against an infinitely rigid jacket. The example illustrates that by properly proportioning the thickness of the inner shell and jacket, the elastic spring-back of the jacket more than compensates for the elastic spring-back of the inner shell, when the pressure is removed, and a system of residual stresses develops.

In the illustrative problem the angle ϕ was increased from 0° to 24° in increments of 4° , however, in many instances the use of larger increments will yield satisfactory results. The sample problem was also solved in a single step using $\phi_2 = 24^\circ$ and $\beta_1 = 12^\circ$. The following values were obtained:

$$P_2 = 3644 \text{ psi}$$

$$P_{c2} = 1586 \text{ psi}$$

$$\sigma_{a2} = 36,900 \text{ psi}$$

$$\sigma_{c2} = 41,680 \text{ psi}$$

$$\sigma_{cj2} = 42,557 \text{ psi}$$

The stresses occur when the inner vessel is at pressure P_2 (3644 psi). The excellent agreement may be observed by comparing these values with the stresses in Table 2 for $\phi_7 = 24^\circ$. Caution must be exercised, however, in using large increments between succeeding values of ϕ

particularly when ϕ exceeds about 30° or when the jacket is significantly more flexible than the inner shell.

In designing a two-layer vessel the dimensions may be estimated quite easily. The inner shell must be thick enough to carry the axial stress produced by the maximum internal operating pressure. Since the jacket and inner shell must carry the circumferential stress, the thickness of the jacket will be approximately the same as the thickness of the inner shell unless different materials of construction are used. Some adjustment in the thickness of the jacket may be necessary depending upon the amount of prestress required.

The prestressing operation can be observed and controlled by the use of strain gages applied to the jacket. Such a procedure would permit making on the spot adjustments in the controlled yielding operation and provide valuable data for the design of similar vessels. It would also show the effects of anisotropy and frictional forces between the two layers, both of which were ignored in the preceding analyses.

ORNL-3546 (Suppl. 1)
UC-38 — Engineering and Equipment
TID-4500 (26th ed.)

INTERNAL DISTRIBUTION

1. Biology Library	57. C. E. Larson
2-4. Central Research Library	58. R. P. Levey, Jr.
5. Reactor Division Library	59. W. D. Manly
6-7. ORNL — Y-12 Technical Library	60. W. L. Marshall
Document Reference Section	61. E. C. Miller
8-27. Laboratory Records Department	62. S. E. Moore
28. Laboratory Records, ORNL R.C.	63. G. Morris
29. D. L. AuBuchon	64. C. E. Muzzall
30. J. R. Avery	65. J. W. Paul
31. J. R. Barkman	66. H. R. Payne
32. R. G. Berggren	67. H. A. Pohto
33. W. D. Box	68. A. S. Quist
34. G. E. Boyd	69. S. A. Rabin
35. G. B. Child	70. M. E. Ramsey
36. C. E. Childress	71. E. G. Richardson, Jr.
37. C. J. Claffey	72. R. C. Robertson
38. G. W. Clark	73. A. F. Rupp
39. W. G. Cobb	74. G. Samuels
40. W. B. Cottrell	75. R. E. Schappel
41. G. A. Cristy	76-79. R. W. Schneider
42. R. G. Domer	80. M. J. Skinner
43. D. A. Douglas	81. C. O. Smith
44. J. C. Ebersole	82. A. Spaller
45. S. T. Ewing	83. I. Spiewak
46. B. R. Fish	84. W. J. Stelzman
47. R. M. Fuller	85. J. A. Swartout
48. C. H. Gabbard	86. J. R. Tallackson
49. W. R. Gall	87. W. C. Ulrich
50. C. W. Gartrell	88. J. T. Venard
51. B. L. Greenstreet	89. A. M. Weinberg
52. K. W. Haff	90. J. R. Weir
53. R. Helton	91. F. J. Witt
54. P. P. Holz	92. F. C. Zapp
55. T.B. Jernigan	93. B. Zimmerman (K-25)
56. J. W. Keisling	94. E. Zurcher

EXTERNAL DISTRIBUTION

- 95. L. H. Jackson, Director, Engineering Division, AEC, Oak Ridge
- 96. Research and Development Division, AEC, ORO
- 97-611. Given distribution as shown in TID-4500 (26th ed.) under
Engineering and Equipment category (75 copies - OTS)