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THE TWO-PION EXCHANGE CONTRIBUTION TO THE HIGHER PARTIAL WAVES OF NUCLEON-NUCLEON SCATTERING

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Peter Cziffra

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Contents

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ABSTRACT

By means of the Mandelstam representation, expressions are obtained for the two-pion-exchange contributions to the higher partial waves of nucleon-nucleon scattering. A set of ten invariant amplitudes is selected, of which each member obeys the Mandelstam representation. Dispersion relations are written for the amplitudes in which the discontinuities are absorptive parts for nucleon-antinucleon scattering. By means of the unitarity condition the absorptive parts are expressed as a partial-wave expansion in terms of the $\pi\pi \rightarrow n\bar{n}$ partial-wave amplitudes of Frazer and Fulco, except for the contributions of the pole in the pion-nucleon system which are treated exactly in order to ensure better convergence of the partial-wave expansion. Finally, the nucleon-nucleon transition amplitudes in the angular momentum representation are expressed in terms of the invariant amplitudes.

I. INTRODUCTION

Application of meson theory to the two-nucleon interaction has to date, been fraught with great difficulty.[1,2] The perturbation method of quantum field theory which worked so well in quantum electrodynamics is stymied in meson theory by the large magnitude of the pion-nucleon coupling constant, so that the convergence of an expansion of the scattering amplitudes in powers of this constant is extremely slow $\frac{1}{m}$ if the series converges at all. Recently a new approach has entered the picture, that of the dispersion relation of spectral representation, the most powerful variety of which is the two-dimensional dispersion relations first proposed by Mandelstam.[3] The validity of these dispersion relations, unfortunately, has only been proved to sixth order in perturbation theory,[3,4,5] and a rigorous proof based on the general principles of quantum field theory is not in sight. Nevertheless, the Mandelstam representation is plausible, and we shall assume it to be correct for the purposes of this paper. Indeed, the most convincing proof of its correctness would be if it led to results that agree with experiment. A recent article by G. F. Chew reviews the philosophy and practice of dispersion relations, both one- and two-dimensional, with copious references.[6]

Dispersion theory is concerned with the study of the singularities of the scattering amplitude. These singularities occur for unphysical as well as physical values of the variables that describe the scattering amplitude, and are associated with the possible real (i.e., with momenta on the mass shell) intermediate states into which the scattering amplitude can be expanded (see Ref. 6 for details). In making approximations, the main assumption is that the closer a singularity is to the physical region the more important its contribution to the scattering amplitude will be. This assumption is necessary, since the close singularities are usually the only ones tractable by present methods, and it is also a reasonable one. For instance, if the residues of two poles are of the same order of magnitude then, obviously, the pole closer to the physical region will make the larger contribution to the amplitude. Even if the more distant pole has a larger residue, the change in the amplitude as a function of the variable in which the poles occur will be induced to a much larger extent by the nearer pole. In many cases the amplitude is normalized at some point (i.e., a subtracted dispersion relation is used) by means of information, usually experimental, not contained in the dispersion relations; it is then the change in the function that is of interest. The same reasoning applies to the branch-cuts.

In the nucleon-nucleon problem the closest singularities are the two one-pion exchange poles, whose use has already borne considerable fruit. First there is the proposal of Chew,[7] as carried out by Cziffra and Moravcsik,[8] for the determination of the pion-nucleon coupling constant directly from n-p angular distributions. There is the modified phase shift analysis,[9, 10] first proposed by Moravcsik, in which the higher angular momentum states are given directly by the pole term while the lower ones are treated phenomenologically. There is the calculation of the Asymptotic D-wave function of the deuteron by Wong,[11] and the modifications of the effective range formula for nucleon-nucleon scattering of Cini, Fubini and Stanghellini, and of Noyes and Wong;[12] the latter works, however, involve more of the Mandelstam representation than just the poles.

After the poles, the closest singularity is the branch cut due to the two-pion intermediate state; it is with this that the present work is concerned. Let p' and p, be respectively the final and initial four-momenta of one of the nucleons, and $t = -(p'-p)^2$ be the invariant momentum transfer (we use the metric such that $p^2 = \underset{\sim}{p}^2 - p_0^{\,2}$). In nucleon-nucleon scattering the physical region has $t \leq 0$, the pole occurs at $t=\mu^2$, where μ is the pion mass, the two-pion branch cut starts at $t=(2\mu)^2$, and the contribution of the next heaviest intermediate state, viz. the three-pion state, starts at $t=(3\mu)^2$. Thus the three-pion singularity is not much further from the physical region than the two-pion singularity. There are, however, two main reasons for ignoring singularities other than the poles and the two-pion cut. Firstly, at present we do not know how to treat the more distant singularities, especially those involving intermediate states of more than two particles. Secondly, it is hoped that the pion-pion resonance recently conjectured in order to describe nucleon electromagnetic structure,[13] will serve to increase the contribution of the two-pion state to the nucleon-nucleon scattering amplitude. In the phase-shift analysis of proton-proton scattering at 310 Mev[9, 10] the two one-meson exchange poles were found quite capable of determining the higher phase shifts (from L = 4 on up). The present calculation should be able to predict some of the higher phase shifts for which the one-pion exchange poles are inadequate. That the lighter intermediate states should determine the higher angular momentum states is very plausible on elementary grounds, since the lighter the intermediate state the longer the range of the force to which it gives rise.

Briefly, our method is as follows: The nn amplitude can be expressed in terms of a set of ten invariant functions, which we shall call "Mandelstam functions" because they are assumed to obey the Mandelstam representation. These functions also describe $n\bar{n}$ scattering and can be related by means of the substitution law to the $n\bar{n}$ amplitude. The unitarity condition for the $n\bar{n}$ scattering amplitude can be written symbolically: $2\mathrm{Im}\left\langle n\bar{n} \,\middle|\, n\bar{n} \right\rangle = \sum_i \left\langle n\bar{n} \,\middle|\, i \right\rangle \left\langle n\bar{n} \,\middle|\, i \right\rangle^*$, where the sum is to be taken over all permissible real intermediate states. The intermediate state with the lowest mass is the one-pion state which gives rise to the one-pion exchange pole. The next least massive state is the two-pion state which gives rise to the two-pion branch cuts in the Mandelstam functions. For the reasons given above, states heavier than the two-pion will be ignored. It should be noted that since the 2π intermediate state starts at an energy less than the lowest possible energy for a physical state, we are using the unitarity condition in an unphysical region; this has recently been justified by Mandelstam.[14] The functions $\left\langle n\bar{n} \,\middle|\, 2\pi \right\rangle$ have been studied by Frazer and Fulco[15] (hereafter referred to as FF) on the basis of the Mandelstam representation. These functions can be evaluated by use of available pion-nucleon scattering data if the pion-pion phase shifts are known. The latter are now being calculated by Chew and Mandelstam,[16] again utilizing the Mandelstam representation. From the imaginary part of the $n\bar{n}$ amplitude as given by the unitarity condition, we can determine the absorptive part of the Mandelstam functions, and by a dispersion relation get the complete function, which in turn will give us the nn amplitude.

Frazer and Fulco's calculation gives $\langle n\bar{n} \mid 2\pi \rangle$ partial wave amplitudes; consequently, the $n\bar{n}$ absorptive part will be given as a partial wave (i.e., Legendre polynomial) expansion. Except for very low nucleon-nucleon energies, however, the absorptive part becomes singular for values of t just above $4\mu^2$, the latter being the lower limit of the dispersion relation. Consequently, the expansion fails to converge over a large part of the region of integration of the dispersion relation. The first singularity in the absorptive part is due to the existence of the one-nucleon pole of the pion-nucleon interaction (hereafter called the πn-pole). This pole leads to the "box-diagram" in the nucleon-nucleon system, which corresponds to the fourth-order two-pion exchange Feynman diagram of perturbation theory. Fortunately, as Mandelstam has shown,[4,5] the contribution of the box diagram to the absorptive part can be evaluated exactly (cf. Section VIII), so that only the remainder of the absorptive part need be given as a Legendre polynomial expansion. It can be shown that mathematically this expansion converges for values of t even greater than the three-pion exchange threshold, although there, of course, it soon ceases to give a reasonable approximation to the actual nn amplitude. It is hoped that the convergence is rapid enough for S and P wave two-pion intermediate states to suffice for the determination of the higher phase shifts of $n\bar{n}$ scattering.

The portion of the absorptive part that is analytically continued by a partial wave expansion has its singularities neglected. This means that the imaginary part of the nucleon-nucleon partial wave amplitudes will come exclusively from the box diagram, and that the imaginary part due to the other contributions must be small for our method to be feasible. In general, this will occur only for partial waves of sufficiently high order and consequently small magnitude.

II. THE FRAZER-FULCO FUNCTIONS

The πn-pole gives rise to an anomalously large S-wave contribution to pion-nucleon scattering; a contribution presumably suppressed by higher-order terms.[17] The corresponding terms should also be suppressed in the nn amplitude. Frazer and Fulco's calculation, however, does not appear to contain a mechanism which will bring this suppression about, the restriction that the $\pi\pi \rightarrow n\bar{n}$ amplitude have the phase of $\pi\pi$ scattering probably not being sufficient. A phenomenological means of avoiding this difficulty is based on the observation by Chew[18] that the annihilation amplitudes of FF at zero incoming energy were very simply related to pion-nucleon scattering amplitudes at zero momentum transfer. In fact, the variable t of FF is the total energy for the annihilation process and the momentum transfer for pion-nucleon scattering. Thus, by using experimental pion-nucleon data in forward scattering dispersion relations, the $\pi\pi \rightarrow n\bar{n}$ amplitudes at zero total energy can be calculated and a subtraction made in FF's integral equations to normalize the functions. This has been done by D. Y. Wong,[19] who finds that at zero energy the S-wave FF function so calculated is very much less than the value due to the πn-pole. It will be remembered that in FF the left-hand cut is determined from the πn pole plus what is frequently called the "rescattering correction", which consists of a partial wave expansion as a function of the pion-nucleon scattering angle, and uses experimental pion-nucleon phase shifts. According to FF, this expansion should converge up to $t = -26\mu^2$. Wong[19] has compared the correct P-wave annihilation amplitude at $t = 0$, as determined from Chew's suggestion, with the one determined from FF's integral using the rough estimate of the pion-pion phase shift obtained by FF from the nucleon electromagnetic structure. He found that in order to get agreement between the two values, it was necessary to extend the partial wave expansion far beyond $t=-26\mu^2$. This indicates that the left-hand cut of the FF functions cannot be determined from pion-nucleon scattering merely by a partial wave expansion. However, the method permitting us to calculate the FF functions at $t = 0$ also permits us to determine the derivative at the same point.

By normalizing both the functions and their derivatives, it appears possible to determine what we shall call a "modified FF function"; which should be reliable; of course, the pion-pion phase shifts are still needed.[19]

III. THE INVARIANT AMPLITUDES

The S-matrix for a nucleon-nucleon scattering process may be written:

$$\left\langle p', r', \alpha'; q', s', \beta', \left| S \right| p, r, \alpha; q, s, \beta \right\rangle =$$

$$\delta_{21} + \frac{i}{4\pi^2} \delta^{(4)}(p' + q' - p - q)\left(\frac{m^4}{p'_o q'_o p_o q_o}\right)^{1/2}$$

$$\overline{U}_{r'\alpha'}^{(1)}(p')\,\overline{U}_{s'\beta'}^{(2)}(q')\,M(p', q', p, q)\,U_{r\alpha}^{(1)}(p)\,U_{s\beta}^{(2)}(q)$$

(III-1)

Here p', q' and p, q, are the four-momenta of the two final and initial particles respectively; r', s' and r, s their final and initial spins or helicities and α', β' and α, β their final and initial i-spins. The Dirac-spinors $U_{r\alpha}(p)$ are eight component entities in the product space of the i-spin and Dirac-spinor spaces; they may be written more explicitly as $U_{r\alpha}(p) = u_r(p)\,\chi_\alpha$ where:

$$\chi_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ for the proton}$$

$$\chi_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for the neutron}$$

and $u_r(p)$ is a four-component Dirac-spinor such that $(i\gamma \cdot p + m)u_r(p)=0$. The matrix $M(p', q', p, q)$ is a 64-by-64 matrix in the product space of the two initial and two final particles. The definition of the S-matrix used here corresponds to that of Jauch and Rohrlich.[20]

According to the substitution rule, the matrix $M(p', q', p, q)$ describes nucleon-antinucleon and antinucleon–antinucleon scattering as well as nucleon-nucleon scattering. This rule is implicit in the structure of perturbation theory (Ref. 20, Sec. 8-5) and also follows from the reduction formulae of Lehmann, Symanzik, and Zimmermann,[21,22] as will be shown in Section V. For the scattering of a nucleon of four-momentum q and an antinucleon of momentum q' into a nucleon of momentum p' and antinucleon of momentum p the rule gives:

$$\left\langle p', \bar{p} \left| S-1 \right| q, \bar{q}' \right\rangle = \pm \frac{1}{4\pi^2} \delta^4 (p' - q' + p - q) \left(\frac{m^4}{p'_o p_o q'_o q_o} \right)^{1/2}$$

$$\bar{U}^{(1)}(p') \bar{V}^{(2)}(q') M(p', -q', -p, q) V^{(1)}(p) U^{(2)}(q). \qquad \text{(III-2)}$$

The bars over p and q' on the left merely indicate that p and q' are momenta of antinucleons. The spin and i-spin indices have been suppressed, and the δ_{21} of (III-1) has been absorbed into the matrix element on the left. The overall sign of the left-hand side is not obvious and will be determined in Section V. The Dirac-spinors $V_{r\,a}(p)$ are also eight-component entities and may be decomposed into $V_{r\,a}(p) = v_r(p)\, \chi_a$, where $v_r(p)$ is a four-component, negative energy Dirac-spinor such that $(-i\gamma \cdot p + m)\, v_r(p) = 0$, and the χ_a are:

$\chi_{\bar{p}} = \chi_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for the proton and antiproton

$\chi_{\bar{n}} = \chi_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the neutron and antineutron.

Thus the i-spin spinor for an outgoing antinucleon stands on the right just its Dirac-spinor does. The use of these i-spin spinors will be further discussed in Appendix A.

We shall assume that the $n\bar{n}$ interaction is charge independent, in which case the S-matrix must be invariant under rotations in i-spin space. Since only two invariants may be formed from the i-spin matrices in the product space of the two particles, the matrix M may be split into two parts:

$$M = M^- + \underset{\sim}{\tau}^{(1)} \cdot \underset{\sim}{\tau}^{(2)} M^+ . \qquad \text{(III-3)}$$

Throughout this paper our convention will be that the superscript "1" refers to particles with momenta p' and p whereas "2" refers to those with momenta q' and q even though, as will be seen below, two particles with, for example, momenta p' and p may both be in the initial state.

Lorentz invariance ensures that the $M^{\pm}$ can be split up further:

$$M^{-}(p', q' p, q) = \sum_n A^n(s, t, \bar{t})\, X^n \tag{III-4}$$

$$M^{+}(p', q' p, q) = \sum_n B^n(s, t, \bar{t})\, X^n$$

where the X^n are 16-by-16 matrices which may be functions of the four-momenta and the $A^n(s,t,\bar{t})$ and $B^n(s,t,\bar{t})$ are arbitrary functions of the invariant scalars s, t, and $\bar{t}$ only. For the momentum definitions of (III-1) and the process shown in Fig. 1, the latter can be written:

$$s = -(p+q)^2 = -(p'+q')^2$$

$$t = -(p'-p)^2 = -(q'-q)^2 \tag{III-5}$$

$$\bar{t} = -(p'-q)^2 = -(q'-p).$$

In the barycentric system, with z_1 the cosine of the scattering angle and p_1 the modulus of barycentric three-momentum, these variables become:

$$s = 4(p_1^2 + m^2) = 4E_1^2$$

$$t = 2p_1^2(1 - z_1) \tag{III-6}$$

$$\bar{t} = -2p_1^2(1 + z_1)$$

where m is the nucleon mass. Comparing (III-1) and (III-2) we see that in the transition from the process described by the first equation

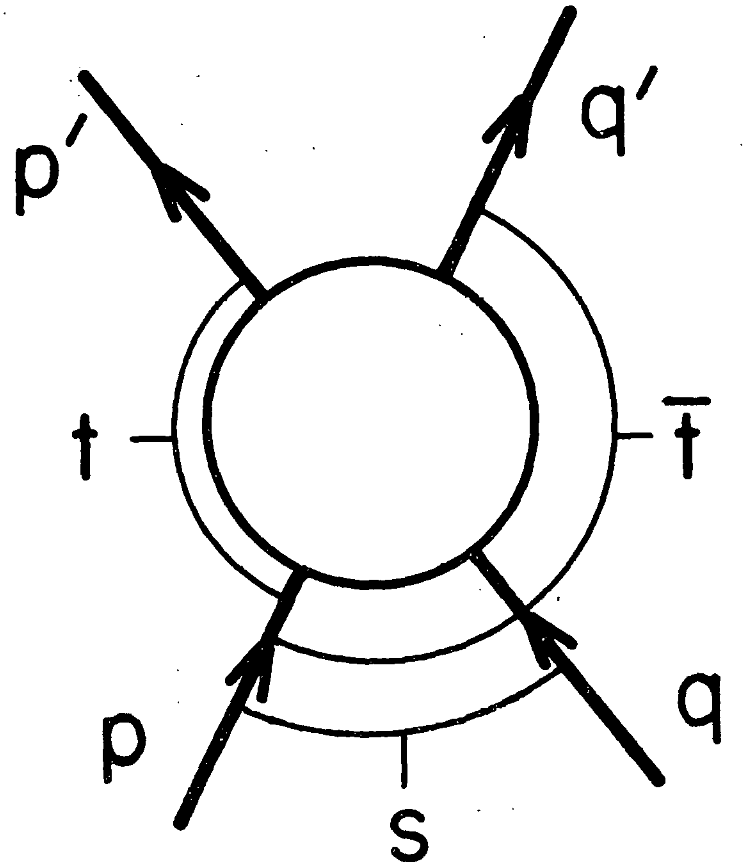

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Fig. 1. Nucleon-nucleon scattering: channel 1. The time direction is upward.

in Fig. 2, the matrix $M(p', q', p, q)$ goes to $M(p', -q', -p, q)$. Consequently the scalars become for the second process:

$$s = -(q-p)^2 = -2p_2^2(1+z_2)$$

$$t = -(p'+p)^2 = 4(p_2^2 + m^2) \qquad \text{(III-7)}$$

$$\bar{t} = -(p'-q)^2 = -2p_2(1-z_2)$$

where p_2 is the modulus of the barycentric three-momentum and z_2 is the barycentric scattering angle taken, as will always be the case for $n\bar{n}$ processes, between the two nucleons. It will be noted that in (III-6) s gave the total energy of the system, whereas in (III-7) the total energy was given by t. We shall accordingly call the process in which s was the total energy "channel 1", and that in which t was the total energy "channel 2". In addition there is the channel, shown in Fig. 3, in which $\bar{t}$ gives the total energy; this will be called "channel 3". According to the substitution rule, all three channels are described by the same matrix M.

As long as the incoming and outgoing particles are on the mass shell, the variables s, t, and $\bar{t}$ are not independent, being related by equation:

$$s + t + \bar{t} = 4m^2. \qquad \text{(III-8)}$$

In addition to requiring charge independence and Lorentz invariance, we shall assume that our interaction is invariant under charge conjugation, parity, and time reversal. There is at present no reason to believe that any of these invariance principles are violated in strong coupling physics. The matrices X^n must accordingly be chosen so that the interaction will be invariant under all these transformations. The procedure for finding a complete set of X^n is as follows:

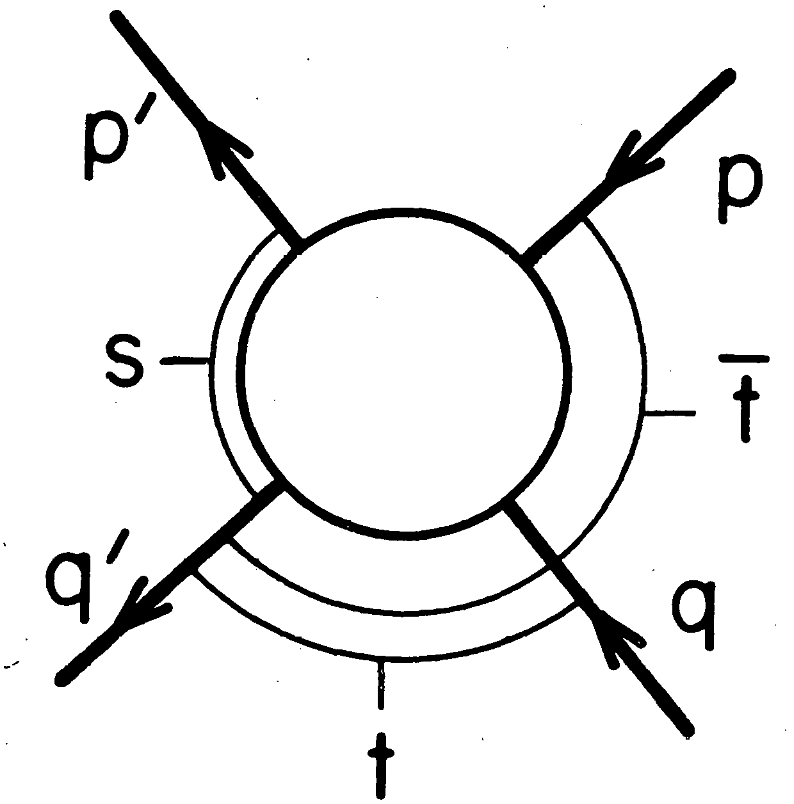

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Fig. 2. Nucleon-antinucleon scattering: channel 2.

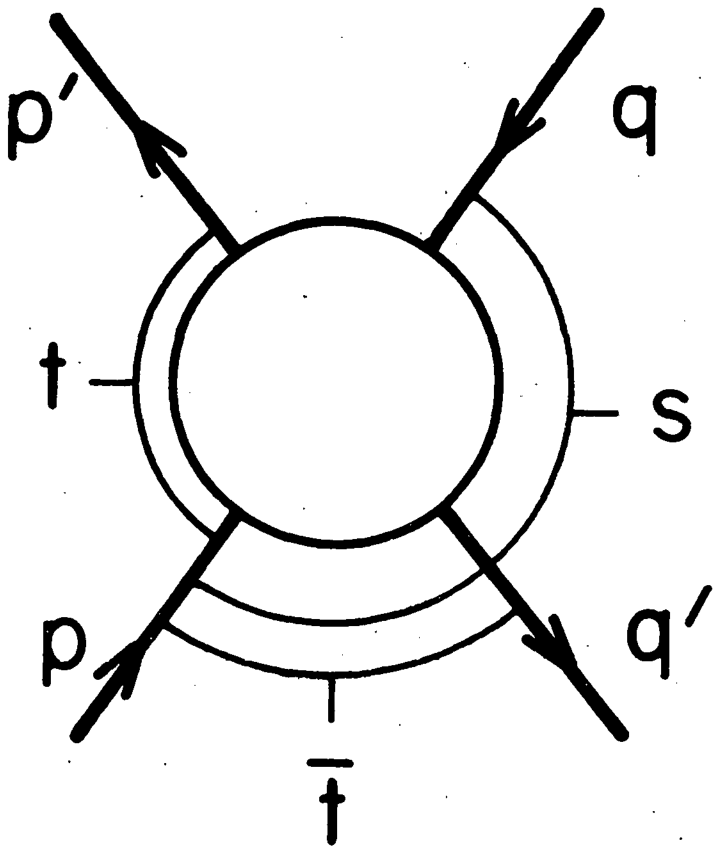

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Fig. 3. Nucleon-antinucleon scattering: channel 3.

In the composite space of the two particles construct all scalars (with respect to Lorentz transformations) which can be constructed out of the three independent momentum vectors and the γ matrices. Eliminate matrices which can be reduced to another matrix because the spinors obey the Dirac-equation, and also eliminate those which do not lead to invariance under time reversal, charge conjugation, and parity. This, it turns out eliminates all but the following eight forms:

$$i\gamma^{(1)} \cdot (q'+q) + i\gamma^{(2)} \cdot (p' + p);\ i\gamma^{(1)} \cdot (q' + q)\ i\gamma^{(2)} \cdot (p' + p);$$

$$\left[\gamma_5^{(1)}\, i\gamma^{(1)} \cdot (q' + q)\right]\left[\gamma_5^{(2)}\, i\gamma^{(2)} \cdot (p' + p)\right];$$

$$1^{(1)}\, 1^{(2)};\quad \gamma_5^{(1)}\, \gamma_5^{(2)};$$

$$\gamma^{(1)} \cdot \gamma^{(2)};\ i\gamma_5^{(1)}\, \gamma^{(1)} \cdot i\gamma_5^{(2)}\, \gamma^{(2)};\ \frac{1}{2} \sum_{\mu,\nu=0,1,2,3} \sigma^{(1)}{}_{\mu\nu}\, \sigma_{\mu\nu}^{(2)};$$

(III-9)

where our representation is such that

$$\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix},\ j = 1, 2, 3;\ \beta = i\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma_5 = \gamma_1\, \gamma_2\, \gamma_3\, \beta = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \sigma_{\mu\nu} = \frac{1}{2i}\,(\gamma_\mu\, \gamma_\nu - \gamma_\nu\, \gamma_\mu).$$

Not all eight of these can be independent in the subspace in which the incoming and outgoing particles are positive energy nucleons. By using an explicit representation of the Dirac-spinors, e.g.

$$u_r(p) = \frac{-i\gamma \cdot p + m}{\left[2m\,(p_0+m)\right]^{1/2}} \begin{pmatrix} \chi_r \\ 0 \end{pmatrix}$$

where χ_r is a two-component Pauli-spinor, we find that the eight forms in (III-9) reduce to five forms, namely those of Wolfenstein and Ashkin,[23] which are frequently misnamed the "non-relativistic forms." Since there are only five independent matrices in the nn channel, the scattering must be completely describable by only five arbitrary complex functions. Thus only five of the eight matrices of (III-9) are independent, and any five linearly independent ones should be sufficient to determine the scattering amplitude. In earlier works, e.g. Goldberger, Nambu and Oehme[24], the first five were chosen; they are, however, less satisfactory than the last five for two reasons. Firstly, the latter give rise to simpler crossing relations whereas in the former the crossing relations are complicated by having the X^n be explicit functions of the four-momenta. Secondly, and more importantly, the explicit momentum dependence of the first five matrices forces the A^n and B^n to have so-called "kinematical singularities" — singularities not associated with any intermediate states, but due entirely to extraneous momentum factors. It has been shown by Grisaru and Wong[25] that the last five matrices, which are in fact the Fermi β-decay matrices, do not develop any extraneous singularities; we shall briefly describe their arguments.

For convenience let us discuss only M^-, the arguments for M^+ being practically identical. Thus

$$M^- = \sum_n X^n A^n(s, t, \bar{t}) \tag{III-10}$$

where the sum runs over $n = S, P, V, A$, and T, the letters standing for scalar, pseudoscalar, vector, axial vector and tensor, respectively; and the X^n are defined by

$$X^S = 1^{(1)}\, 1^{(1)};\; X^P = \gamma_5^{(1)}\, \gamma_5^{(2)} \;;\; X^V = \gamma^{(1)} \,.\, \gamma^{(2)} ;$$

$$X^A = i\gamma_5^{(1)}\, \gamma^{(1)} \,.\, i\gamma_5^{(2)}\, \gamma^{(2)} \;;\; X^T = \frac{1}{2} \sum_{\mu\nu = 0,1,2,3} \sigma^{(1)}_{\mu\nu}\, \sigma^{(2)}_{\mu\nu} \,. \tag{III-11}$$

The denominators occurring in a perturbation-theoretic expression for M^- would be no different from those occurring in a spinless, scalar theory having the same spectrum as the present theory. Consequently, it seems very reasonable to assume that each element of the matrix

$$\mathcal{M}^- = L^{(1)}(p') \, L^{(2)}(q') \, M^- \, L^{(1)}(p) \, L^{(2)}(q) \qquad \text{(III-12)}$$

with

$$L(p) = \frac{(-i\gamma \cdot p + m)}{2m}$$

is an analytic function of the four four-vector variables p', q', p and q, except in those regions in which the amplitude of the spinless scalar theory would not be analytic. It then follows that for any n the function $\mathrm{Tr}\,(\mathcal{M}^- X^n)$ would also be an analytic function of the four-vector variables with the same region of analyticity as that of $\mathcal{M}^-$. This with the fact that the trace is invariant under the orthochronous Lorentz group implies, according to the theorem of Hall and Wightman,[26] that the trace is an analytic function of the invariant scalars, except, of course, in the region mentioned above. From (III-10) and (III-12) we get

$$\mathrm{Tr}\,(\mathcal{M}^- X^n) = \sum_m \alpha_{nm}(s, t, \bar{t}) \, A^m(s, t, \bar{t})$$

where

$$\alpha_{nm}(s, t, \bar{t}) = \mathrm{Tr}\left[L^{(1)}(p') \, L^{(2)}(q') \, X^m \, L^{(1)}(p) \, L^{(2)}(q) \, X^n \right].$$

The $\alpha_{nm}(s, t, \bar{t})$ are obviously analytic functions of s, t, and $\bar{t}$, but they may vanish for some value of the invariant scalars, forcing $A^n(s, t, \bar{t})$ to have a pole at that point, unless the traces on the left happened to vanish there too. Solving the above set of equations for A^m we obtain:

$$A^m(s, t, \bar{t}) = \sum_n \frac{\beta_{mn}(s, t, \bar{t})}{\Delta} \, T_r(\mathcal{M}^- X^n)$$

where the β_{mn} are analytic functions and $\Delta = \det \| \alpha_{mn} \|$. This determinant, according to Grisaru and Wong,[25] is given by $\Delta = c(s, t, \bar{t})^3$ where c is a constant. Thus if our invariant amplitudes A^n and B^n have any extraneous singularities they can only be the poles that could occur when one of the invariant scalars vanishes. In Section VI we will derive a definite relation between A^n, B^n, and $n\bar{n}$ transition amplitudes for helicity states in channel 2. In this channel s is the momentum transfer and $\bar{t}$ the crossed-momentum transfer, and it can be made plausible that both A^n and B^n remain finite as s or $\bar{t}$ goes to zero. We shall return to this point in Section VI. The same procedure involving either channels 1 or 2 would establish that A^n and B^n remain finite as t goes to zero. We may thus conclude that the only singularities occurring in the invariant amplitudes $A^n(s, t, \bar{t})$ and $B^n(s, t, \bar{t})$ are those which also occur in the scalar, spinless theory, provided we choose the set of X^n given in (III-11).

IV. THE MANDELSTAM REPRESENTATION

At this point we make the crucial assumption that the analyticity properties of the amplitude of our spinless scalar theory are such that the amplitude has a Mandelstam representation. Beyond what has already been said in the introduction (Section I), we shall not attempt to justify this assumption here. The arguments of the previous section then show that if we choose the X^n of (III-11), i.e., n = S, T, V, A, P, both the $A^n(s,t,\overline{t})$ and $B^n(s,t,\overline{t})$ will also have a Mandelstam representation. Consequently, $A^n(s, t, \overline{t})$ may be written:

$$A^n(s,t,\overline{t}) = \text{poles} + \frac{1}{\pi}\int_{(3\mu)^2}^{\infty} dt' \frac{a_2{}^n(t')}{t' - t} + \frac{1}{\pi}\int_{(3\mu)^2}^{\infty} d\overline{t}' \frac{a_3{}^n(\overline{t}')}{\overline{t}' - \overline{t}}$$

$$+ \frac{1}{\pi^2}\int_{(2m)^2}^{\infty} ds' \int_{(2\mu)^2}^{\infty} dt' \frac{a_{12}{}^n(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{(2m)^2}^{\infty} ds' \int_{(2\mu)^2}^{\infty} \frac{a_{13}{}^n(s',\overline{t})}{(s'-s)(\overline{t}'-\overline{t})}$$

$$+ \frac{1}{\pi^2}\int_{(2\mu)^2}^{\infty} dt' \int_{(2\mu)^2}^{\infty} d\overline{t}' \frac{a_{23}{}^n(t',\overline{t}')}{(t'-t)(\overline{t}'-\overline{t})} . \tag{IV-1}$$

with a similar expression for $B^n(s,t,\overline{t})$. The poles have been adequately discussed in Ref. 9 and will be ignored hereafter. The next two terms of (IV-1) are frequently called "subtraction terms," and correspond to diagrams of the type shown in Figs. 4a and 4b. Figure 4b involves a two-nucleon intermediate state, and Fig. 4a shows the three-pion intermediate state that is the lightest intermediate state which can occur in a subtraction term. Since we are not including anything more massive than two-pion states, we can ignore all the subtraction terms. The weight functions $a_{12}{}^n$, $a_{13}{}^n$ and $a_{23}{}^n$ are everywhere real, and each vanishes outside a region bounded by a curve whose asymptotes are:

$s = (2m)^2$ and $t = (2\mu)^2$ for $a_{12}{}^n(s,t)$, $s = (2m)^2$ and $\overline{t} = (2\mu)^2$ for a_{13} $(s, \overline{t})$, and finally, $t = (2\mu)^2$ and $\overline{t} = (2\mu)^2$ for $a_{23}(t, \overline{t})$.

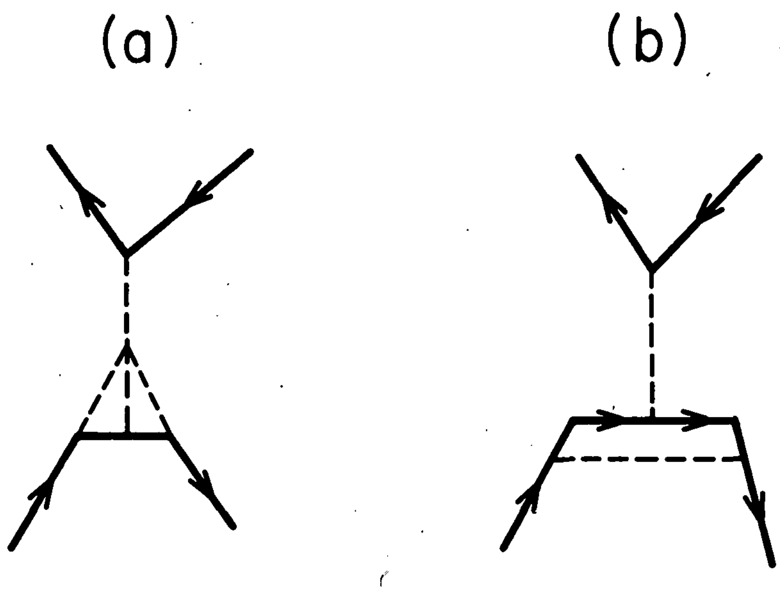

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Fig. 4a and 4b. Typical "subtraction terms."

Mandelstam[4, 5] has calculated these boundary curves using the spinless scalar theory which, according to our earlier discussion, must give the same results as the pseudoscalar spin-one-half theory. Indeed, we shall find the boundary curves as a by-product of our calculation of the effect of the π-n pole in Section VIII, and they will turn out to be those predicted by Mandelstam. Spin, it appears, is not an essential complication, it merely complicates the algebra. We give below the curves obtained in Ref. 5. The weight function $a_{12}(s, t)$ is non-zero inside the parabolic boundary curve C_{12}, (cf. Fig. 5), which is:

$$(s - 4m^2)(t - 4\mu^2) = 4\mu^4. \qquad \text{(IV-2a)}$$

The curve C_{13} for a_{13} $(s,\bar{t})$ is the same as C_{12} except that $t \rightarrow \bar{t}$. The curve C_{23} is the boundary of the union of the areas bounded by the two parabolas:

$$\begin{aligned} (t - 4m^2)(\bar{t} - 4\mu^2) &= 4\mu^4 \\ (t - 4\mu^2)(\bar{t} - 4m^2) &= 4\mu^4 . \end{aligned} \qquad \text{(IV-2b)}$$

The denominator of the last term in (IV-1) may be split into partial fractions:

$$\frac{1}{(t' - t)(\bar{t}' - \bar{t})} = \frac{1}{(t' + \bar{t}' + s - 4m^2)} \left(\frac{1}{t' - t} + \frac{1}{\bar{t}' - \bar{t}} \right),$$

where we have used the relation: $s + t + \bar{t} = 4m^2$. With the aid of the above result, (IV-1) may be rewritten as a one-dimensional dispersion relation:

$$A^n(s, t, \bar{t}) = \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \, \frac{A_2^{\,n}(s, t')}{t' - t} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} d\bar{t}' \, \frac{A_3^{\,n}(s, \bar{t}')}{\bar{t}' - \bar{t}}, \qquad \text{(IV-3)}$$

where we have set:

$$A_2^{\,n}(s, t') = \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \, \frac{a_{12}^{\,n}(s', t')}{s' - s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} d\bar{t}' \, \frac{a_{23}^{\,n}(t', \bar{t}')}{t' + \bar{t}' + s - 4m^2}$$

$$A_3^n(s, \bar{t}') = \frac{1}{\pi} \int_{4m^2}^{\infty} ds' \frac{a_{13}^n(s', \bar{t}')}{s' - s} + \frac{1}{\pi} \int_{4\mu^2}^{\infty} dt' \frac{a_{23}^n(t', \bar{t}')}{t' + \bar{t}' + s - 4m^2} \tag{IV-4}$$

The expressions for $B^n(s, t, \bar{t})$ are precisely parallel.

The range of the invariant scalars for an actual physical $n\bar{n}$ scattering process in channel 2 is such that $t \geq 4m^2$ and $s, \bar{t} \leq 0$. In this range none of the denominators in (IV-4) can vanish, so that both A_2^n and A_3^n are real; only the first denominator of (IV-3) can vanish. Therefore,

$$\text{Im } A^n(s, t, \bar{t}) = A_2^n(s, t); \quad t \geq 4m^2; \quad s, \bar{t} \leq 0. \tag{IV-5}$$

Thus, once $\text{Im } A^n(s, t, \bar{t})$ is known in the physical region for channel 2, $A_2(s, t)$ can be determined everywhere by analytic continuation.

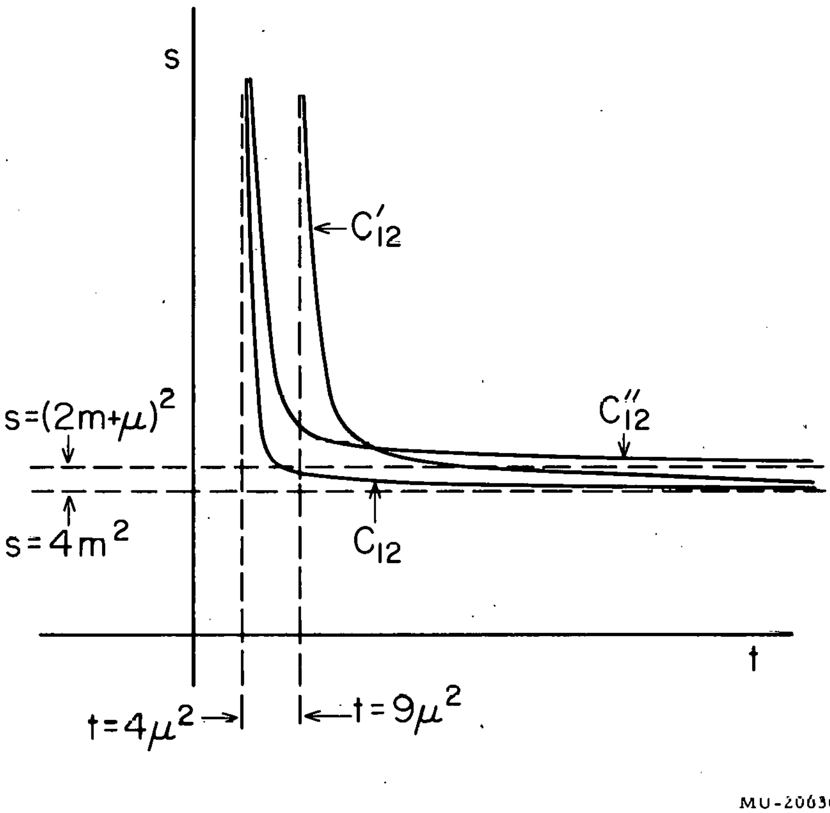

Fig. 5. Boundaries of the functions $a_{12}{}^{n}(s,t)$ and $b_{12}{}^{n}(s,t)$.

V. SUBSTITUTION RULE AND CROSSING

In this section we shall use the reduction formulae to determine the sign of the right-hand side of (III-2), and the relation between the $A_2^{\,n}(s, t)$ and $A_3^{\,n}(s, \bar{t})$ of (IV-3).

If $a^\dagger(p)$ is the creation operator for a nucleon in an asymptotic state of momentum p, we define a two-nucleon asymptotic state by $|p, q\rangle = a^\dagger(p)\, a^\dagger(q)\, |0\rangle$, and the conjugate state by $\langle p, q| = \langle 0|\, a(q)\, a(p)$, where $|0\rangle$ is the vacuum.

For convenience we will let

$$M'(p', q', p, q) = \delta^{(4)}(p' + q' - p - q)\, M(p', q', p, q)$$

where M is the 64-by-64 matrix of (III-1). Suppressing spin and i-spin indices, we may rewrite the latter equation:

$$\langle p', q' \,|\, S-1 \,|\, p, q\rangle = \frac{1}{4\pi^2}\left(\frac{m^4}{p'_0\, q'_0\, p_0\, q_0}\right)^{1/2}$$

$$\bar{U}_i(p')\, \bar{U}_j(q')\, M'_{(ij)(k\ell)}(p', q', p, q)\, U_k(p)\, U_\ell(q)\,. \qquad \text{(V-1)}$$

The indices i, j, k, and ℓ, refer to the rows and columns of the matrices and not necessarily to spin or i-spin quantum numbers.

The reduction formulae of Lehmann, Symanzik and Zimmerman[21] have been extended to spinors by Schweber,[22] whose formalism gives us:

$$\langle p', q' | S-1 | p, q\rangle = \frac{(i)^4}{(2\pi)^6}\left(\frac{m^4}{p'_0\, q'_0\, p_0 q_0}\right)^{1/2} \bar{U}_i(p')\, \bar{U}_j(q')$$

$$\int d^4x' \int d^4y' \int d^4x \int d^4y\; e^{-i(p'\cdot x' + q'\cdot y' - p\cdot x - q\cdot y)}$$

$$\langle 0 \,|\, T\left\{\Omega_j(y')\, \Omega_i(x')\, \bar{\Omega}_k(x)\, \bar{\Omega}_k(x)\, \bar{\Omega}_\ell(y)\right\} |\, 0\rangle\; U_k(p)\, U_\ell(q). \qquad \text{(V-2)}$$

The functions $\Omega_j(x)$ is the jth component of the eight-component entity that is the source function for the nucleon system, i. e. $(\gamma^{\mu}\partial_{\mu} + m)\, \psi(x) = \Omega(x)$, $\psi(x)$ being the field variable for the nucleon field. For $\overline{\Omega}(x)$ we have the relation: $\overline{\Omega}(x) = \overline{\psi}(x)(\partial_{\mu}\gamma^{\mu} + m)$. Regarding the time-ordered product inside the vacuum-expectation value, the only property which concerns us is that its factors anticommute. Comparing (V-1) and (V-2) we see that:

$$M'_{(ij)(k\ell)}(p', q', p, q) = \frac{(i)^3}{(2\pi)^4} \int d^4x' \int d^4y' \int d^4x \int d^4y\; e^{-i(p'.x' + q'.y' - p.x - q.y)}$$

$$\left\langle 0 \left| T\left\{ \Omega_j(y')\Omega_i(x')\, \overline{\Omega}_k(x)\, \overline{\Omega}_\ell(y) \right\} \right| 0 \right\rangle . \tag{V-3}$$

For a $n\overline{n}$ scattering in channel 2 we have:

$$\left\langle p', \overline{p} \left| S-1 \right| q, \overline{q}' \right\rangle = \frac{(i)^4}{(2\pi)^6} \sqrt{\frac{m^4}{p'_0 q'_0 p_0 q_0}}\; \overline{U}_i(p')\, \overline{V}_j(q')$$

$$\int d^4x' \int d^4y' \int d^4x \int d^4y\; e^{-i(p'.x' + p.x - q'.y' - q.y)}$$

$$\left\langle 0 \left| T\left\{ \overline{\Omega}_k(x)\, \Omega_i(x')\, \overline{\Omega}_\ell(y)\, \Omega_j(y') \right\} \right| 0 \right\rangle V_k(p)\, U_\ell(q) \tag{V-4}$$

Since the components of the source functions anticommute we find that:

$$T\left\{ \overline{\Omega}_k(x)\, \Omega_i(x')\, \overline{\Omega}_\ell(y)\, \Omega_j(y') \right\} =$$

$$+ \quad T\left\{ \Omega_j(y')\, \Omega_i(x')\, \overline{\Omega}_k(x)\, \overline{\Omega}_\ell(y) \right\} . \tag{V-5}$$

Combining the last three equations we observe that the positive sign is the correct one in (III-2), that is:

$$\left\langle p', \overline{p} \left| S-1 \right| q, \overline{q}' \right\rangle = \frac{+i}{4\pi^2}\; \delta^{(4)}(p' + p - q' - q) \left(\frac{m^4}{p'_0 q'_0 p_0 q_0} \right)^{1/2}$$

$$\overline{U}^{(1)}(p')\, \overline{V}^{(2)}(q')\, M(p', -q', -p, q)\, V^{(1)}(p)\, U^{(2)}(q). \tag{V-6}$$

The analogous equation for $n\bar{n}$ scattering in channel 3, i.e., for $\left\langle p', \bar{q} \,|\, S-1 \,|\, p, \bar{q}' \right\rangle$, has a minus sign in front of the right-hand member.

Let us now turn to the relation between $A_2^{\,n}(s,t)$ and $A_3^{\,n}(s,\bar{t})$. If in (V-2) the particle designated by p' is interchanged with that designated by q', the reduction formula becomes:

$$\left\langle q', p' \,|\, S-1 \,|\, p, q \right\rangle = \frac{i^4}{(2\pi)^6} \sqrt{\frac{m^4}{p'_0 q'_0 p_0 q_0}} \; \bar{U}_i(p')\, \bar{U}_j(q')$$

$$\int d^4x' \int d^4y' \int d^4x \int d^4y \; e^{-i(p'\cdot x' + q'\cdot y' - p\cdot x - q\cdot y)}$$

$$\left\langle 0 \,|\, T\left\{ \Omega_i(x')\, \Omega_j(y')\, \bar{\Omega}_k(x)\, \bar{\Omega}_\ell(y) \right. \,|\, 0 \right\rangle \; U_k(p)\, U_\ell(q). \qquad \text{(V-7)}$$

The time-ordered product in (V-7) differs from the one in (V-3) only in the order of the first two factors; consequently

$$\left\langle q', p' \,\middle|\, S-1 \,\middle|\, p, q \right\rangle = -\frac{i}{4\pi^2} \left(\frac{m^4}{p'_0\, q'_0\, p_0\, q_0} \right)^{1/2} \bar{U}_i(p')\; \bar{U}_j(q')$$

$$M'_{(ij)(k\ell)}(p', q'; p, q)\; U_k(p)\, U_\ell(q).$$

Hence,

$$\left\langle q', p' \,\middle|\, S-1 \,\middle|\, p, q \right\rangle = -\left\langle p', q' \,\middle|\, S-1 \,\middle|\, p, q \right\rangle, \qquad \text{(V-8)}$$

where the right-hand term is given by (III-1). On the other hand, merely by interchanging labels in the final state of (III-1) we get:

$$\left\langle q', s', \beta', p', r', \alpha' \,\middle|\, S-1 \,\middle|\, p, r, \alpha; q, s, \beta \right\rangle =$$

$$\frac{i}{4\pi^2}\; \delta^{(4)}(p' + q' - p - q) \left(\frac{m^4}{p'_0\, q'_0\, p_0\, q_0} \right)^{1/2}$$

$$\bar{U}^{(1)}_{s'\beta'}(q')\, U^{(2)}_{r'\alpha'}(p')\, M(q', p', p, q)\, U^{(1)}_{r\alpha}(p)\, U^{(2)}_{s\beta}(q).$$

(V-9)

In terms of the invariant scalars the interchange $q' \leftrightarrow p'$ implies that $t \leftrightarrow \bar{t}$. Comparing (V-8) and (V-9), and making use of the expansion of M as given by (III-3) and (III-4) we obtain:

$$- \sum_n X^n \left[\delta_{\alpha'\alpha}\, \delta_{\beta'\beta}\, A^n(s,t,\bar{t}) + \underset{\sim}{\tau}_{\alpha'\alpha} \cdot \underset{\sim}{\tau}_{\beta'\beta}\, B^n(s,t,\bar{t}) \right]$$

$$= \sum_m \bar{X}^m \left[\delta_{\alpha'\beta}\, \delta_{\beta'\alpha}\, A^m(s,\bar{t}\, t) + \underset{\sim}{\tau}_{\beta'\alpha} \cdot \underset{\sim}{\tau}_{\alpha'\beta}\, B^m(s,\bar{t},t) \right], \qquad \text{(V-10)}$$

where $\bar{X}^m$ is defined by:

$$\bar{u}_{r'}^{(1)}(p')\, \bar{u}_{s'}^{(2)}(q')\, \bar{X}^m u_r^{(2)}(p)\, u_s^{(2)}(q) =$$

$$\bar{u}_{s'}^{(1)}(q')\, \bar{u}_{r'}^{(2)}(p')\, X^m u_r^{(1)}(p)\, u_s^{(2)}(q).$$

It can easily be shown that:

$$\left.\begin{aligned} \delta_{\alpha'\beta}\,\delta_{\beta'\alpha} &= \frac{1}{2}\left(\delta_{\alpha'\alpha}\,\delta_{\beta'\beta} + \underset{\sim}{\tau}_{\alpha'\alpha} \cdot \underset{\sim}{\tau}_{\beta'\beta} \right) \\ \underset{\sim}{\tau}_{\alpha'\beta} \cdot \underset{\sim}{\tau}_{\beta'\alpha} &= \frac{1}{2}\left(3\,\delta_{\alpha'\alpha}\,\delta_{\beta'\beta} - \underset{\sim}{\tau}_{\alpha'\alpha} \cdot \underset{\sim}{\tau}_{\beta'\beta} \right) \end{aligned}\right\} \qquad \text{(V-11)}$$

and $\bar{X}^m$ is related to X^n by the well-known "reshuffle theorem" of Fierz,[27] which gives the relation

$$\bar{X}^m = \sum_n Z_{mn} X^n, \qquad \text{(V-12)}$$

with

$$\left\| Z_{mn} \right\| = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}, \qquad \text{(V-13)}$$

where the order of the rows and columns is: S, V, T, A, P. Substituting (V-11) and (V-12) into (V-10) and comparing the coefficients of the i-spin and β-decay matrices, we obtain:

$$A^n(s,t,\bar{t}) = -\frac{1}{2}\sum_m Z_{mn}\left[A^m(s,\bar{t},t) + 3B^m(s,\bar{t},t)\right]$$
$$B^n(s,t,\bar{t}) = -\frac{1}{2}\sum_m Z_{mn}\left[A^m(s,\bar{t},t) - B^m(s,\bar{t},t)\right] \qquad \text{(V-14)}$$

Equation (IV-3) states that:

$$A^m(s,\bar{t},t) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} d\bar{t}'\,\frac{A_2{}^m(s,\bar{t}')}{\bar{t}'-\bar{t}} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\,\frac{A_3{}^n(s,t')}{t'-t} \qquad \text{(V-15)}$$

with a similar expression for $B^m(s,\bar{t},t)$. Upon substituting (IV-3) and (V-15) into (V-14) and equating the integrands over $\bar{t}'$ we get, finally:

$$\left.\begin{aligned} A_3{}^n(s,\bar{t}) &= -\frac{1}{2}\sum_m Z_{mn}\left[A_2^m(s,\bar{t}) + 3B_2{}^m(s,\bar{t})\right] \\ B_3{}^n(s,\bar{t}) &= -\frac{1}{2}\sum_m Z_{mn}\left[A_2{}^m(s,\bar{t}) - B_2{}^m(s,\bar{t})\right]. \end{aligned}\right\} \qquad \text{(V-16)}$$

In the nn channel (channel 1), according to Eq. (III-6), $t = 2p_1{}^2\,(1-z_1)$ and $\bar{t} = -2p_1{}^2(1+z_1)$. For convenience, let us define:

$$\left.\begin{aligned} F^n(p_1{}^2, z_1) &= \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\,\frac{A_2{}^n(s,t')}{t'-t} \\ G^n(p_1{}^2, z_1) &= \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt'\,\frac{B_2{}^n(s,t')}{t'-t} \end{aligned}\right\} \qquad \text{(V-17)}$$

Then, Eqs. (V-16) and (IV-3) imply that:

$$A^n(s,t,\bar{t}) = F^n(p_1{}^2, z_1) - \frac{1}{2}\sum_m Z_{mn}\left[F^m(p_1{}^2, -z_1) + 3G^m(p_1{}^2, -z_1)\right]$$
$$B^n(s,t,\bar{t}) = G^n(p_1{}^2, z_1) - \frac{1}{2}\sum_m Z_{mn}\left[F^m(p_1{}^2, -z_1) - G^m(p_1{}^2, -z_1)\right]$$
$$\text{(V-18)}$$

For the phase shift analysis it may be desirable to directly determine coefficients $A_\ell^n(s)$ and $B_\ell^n(s)$ of the expansions:

$$\left.\begin{aligned} A^n(s,\ t,\ \overline{t}) &= \sum_{\ell=0}^{\infty} (2\ell+1)\ A_\ell^n(s)\ B_\ell(z_1) \\ B^n(s,\ t,\ \overline{t}) &= \sum_{\ell=0}^{\infty} (2\ell+1)\ B_\ell^n(s) P_\ell(z_1). \end{aligned}\right\} \tag{V-19}$$

Now, according to Heine,[28, 29]

$$\frac{1}{t'-t} = \frac{1}{2p_1^2\left[\frac{t'}{2p_1^2} + 1 - z_1\right]} = \sum_{\ell=0}^{\infty} (2\ell+1)\frac{1}{2p_1^2}\ Q_\ell\left(\frac{t'}{2p_1^2}+1\right) P_\ell\ (z_1). \tag{V-20}$$

where Q_ℓ is a Legendre function of the second kind, so that by virtue of (V-17) and (V-18) and the relation $P_\ell(z_1') = (-1)^\ell P_\ell(z_1)$, we may write:

$$\left.\begin{aligned} A_\ell^n(s) &= F_\ell^n(p_1^2) - \frac{1}{2}(-1)^\ell \sum_m Z_{mn}\left[F_\ell^m(p_1^2) + 3\ G_\ell^m(p_1^2)\right] \\ B_\ell^n(s) &= G_\ell^n(p_1^2) - \frac{1}{2}(-1)^\ell \sum_m Z_{mn}\left[F_\ell^m(p_1^2) - G_\ell^m(p_1^2)\right] \end{aligned}\right\} \tag{V-21}$$

where:

$$\left.\begin{aligned} F_\ell^n(p_1^2) &= \frac{1}{2\pi p_1^2} \int_{4\mu^2}^{\infty} dt'\ Q_\ell\left(\frac{t'}{2p_1^2} + 1\right) A_2(s,\ t') \\ G_\ell^n(p_1^2) &= \frac{1}{2\pi p_1^2} \int_{4\mu^2}^{\infty} dt'\ Q_\ell\left(\frac{t'}{2p_1^2} + 1\right) B_2(s,\ t') \end{aligned}\right\} . \tag{V-22}$$

The Q_ℓ are fairly simple functions which can easily be calculated.: they are tabulated in Ref. 29.

VI. THE $n\bar{n}$ T-MATRIX AND ITS RELATION TO THE MANDELSTAM AMPLITUDES

A function with a more convenient unitarity condition than that of the S-matrix is the T-matrix as defined by Moller.[30] If γ_2 and γ_1 refer to miscellaneous quantum numbers in the final and initial states, respectively, we have in the barycentric system:

$$\left\langle k'_2, k_2; \gamma_2 \left| S - 1 \right| k'_1, k_1; \gamma_1 \right\rangle =$$
$$i\,\delta^{(4)}(k'_2 + k_2 - k'_1 - k_1) \left(\frac{\omega'_2 + \omega_2}{|\underset{\sim}{k}_2|\,\omega'_2\omega_2} \right)^{1/2} \left(\frac{\omega'_1 + \omega_1}{|\underset{\sim}{k}_1|\,\omega'_1\omega_1} \right)^{1/2}$$
$$\left\langle \theta_2\, \phi_2; \gamma_2 \left| T \right| \theta_1\, \phi_1; \gamma_1 \right\rangle , \qquad \text{(VI-1)}$$

where $\theta_2\,\phi_2$ and $\theta_1\,\phi_1$ are the barycentric scattering angles of the particles designated by k'_2 and k'_1 respectively, and ω is the barycentric energy. If we restrict ourselves to two-particle intermediate states, the unitarity condition for our S-matrix is

$$\sum_{\gamma_1} \int d^3k'_i\, d^3k_i \left\langle k'_2, k_2; \gamma_2 \left| S \right| k'_i, k_i; \gamma_i \right\rangle \left\langle k'_1, k_1; \gamma_1 \left| S \right| k'_i, k_i; \gamma_i \right\rangle^*$$
$$= \delta_{\gamma_2\gamma_1}\, \delta^{(3)}(\underset{\sim}{k}'_2 - \underset{\sim}{k}'_1)\, \delta^{(3)}(\underset{\sim}{k}_2 - \underset{\sim}{k}_1) . \qquad \text{(VI-2)}$$

On substituting (VI-1) into (VI-2) and carrying out the integrations over the intermediate momenta we get:

$$2\,\mathrm{Im} \left\langle \theta_2 0;\; \gamma_2 \left| T \right| \theta_1 0;\; \gamma_1 \right\rangle =$$

$$\sum_{\gamma_i} \int_{-1}^{1} d(\cos\theta_i) \int_0^{2\pi} d\phi_i \left\langle \theta_2 0; \gamma_2 \left| T \right| \theta_i \phi_i; \gamma_i \right\rangle \left\langle \theta 0;\; \gamma_1 \left| T \right| \theta_i\, \phi_i;\; \gamma_i \right\rangle^* ,$$

$$\text{(VI-3)}$$

where we have assumed that the T-matrix is symmetric, i.e.

$$\left\langle \theta_2 0;\ \gamma_2 \middle| T^{\dagger} \middle| \theta_1 0;\gamma_1 \right\rangle = \left\langle \theta_2 0;\gamma_2 \middle| T \middle| \theta_1 0;\gamma_1 \right\rangle^{*},$$ which is true in our case only for $\phi_2 = \phi_1 = 0$

For orientation purposes it may be noted that the T-matrix used here is related to the differential cross-section for distinguishable particles by:

$$\sigma_{21}(\theta,\phi) = \left| \frac{2\pi}{|\underset{\sim}{k}_1|} \left\langle \theta\phi, \gamma_2 \quad T \quad 00;\gamma_1 \right\rangle \right|^2 .$$

In the barycentric systematic the S-matrix for channel 2 is related to the T-matrix by:

$$\left\langle p', r', a'; \bar{p}, r, a, \middle| S - 1 \middle| q, s, \beta; \bar{q}'', s', \beta \right\rangle$$

$$= i\delta^{(4)}(p' + p - q - q')\frac{2}{p_2 E_2} \left\langle \theta'\phi'; r', r; a' a \middle| \bar{T} \middle| \theta\phi; s, s'; \beta, \beta' \right\rangle \tag{VI-4}$$

The angles give the direction of the nucleons (not antinucleons); the indices r and s refer to helicity states as defined by Jacob and Wick,[31] rather than to the more usual z-component of spin states. In (VI-4) the nucleon helicity and i-spin indices are always written before those of the anti-nucleon, and their somewhat unusual assignment stems from the desire to keep the same set of indices for the same nucleon line, no matter how the latter may be twisted in going from one channel to another under the substitution rule. Finally, the bar over the T indicates that we are referring to the T-matrix for channel 2.

From (V-6) and (VI-4) we get:

$$\left\langle \theta'\phi'; r', r; a', a \middle| \bar{T} \middle| 0; s, s'; \beta, \beta' \right\rangle =$$

$$\frac{m^2 p_2}{8\pi^2 E_2} \bar{U}_{r'a'}^{(1)}(p') \bar{V}_{s'\beta'}^{(2)}(q') M(p', -q', -p, q) V_{ra}^{(1)}(p) U_{s\beta}^{(2)}(q) \tag{VI-5}$$

according to (III-3) and Appendix A

$$M = 2(M^- \bar{P}_0 + M^+ \bar{P}_1), \tag{VI-6}$$

where $\bar{P}_I$ is the projection operator for a state with total i-spin I in channel 2. Thus the T-matrix for a state with definite i-spin is:

$$\left\langle \theta'\phi'; r', r \left| \bar{T}^I \right| 0; s, s' \right\rangle =$$

$$\frac{m^2 p_2}{4\pi^2 E_2} \bar{u}_{r'}^{(1)}(p') \bar{v}_{s'}^{(2)}(q') M^{\pm}(p', -q'-p, q) v_r^{(1)}(p) u_s^{(2)}(q) \tag{VI-7}$$

where we must choose M^+ for $I = 1$ and M^- for $I = 0$.

For a particular i-spin the matrix M can be expressed in terms of five arbitrary functions, which implies that only five of the sixteen possible combinations of initial and final helicity states can be independent. This can also be shown by applying time reversal and parity invariance, and charge independence, directly to the helicity-state amplitudes, using the rule given by Jacob and Wick.[31] It will be found that the following five matrix elements are independent; they will be designated by the numbers 1 through 5:

$$\begin{array}{ccccccc} \mu & = & 1 & 2 & 3 & 4 & 5 \\ (r'r)(ss') & = & (++)(++) & (++)(+-) & (++)(--) & (+-)(+-) & (+-)(-+) . \end{array} \tag{VI-5}$$

For simplicity we write:

$$\left\langle \theta_2 0; r', r \left| \bar{T}^I \right| 00; s, s' \right\rangle = \bar{T}_\mu^I (z_2)$$

where the matrix element is between the five basic states of (VI-8). Using (III-4) and (VI-4) we can express the T-matrix elements in terms of the Mandelstam functions:

$$\overline{T}_{\mu}^{\;0}(z_2) = \frac{m^2 p_2}{4\pi^2 E_2} \sum_n a_{\mu n} A^n(s, t, \overline{t}) \qquad \text{(VI-9)}$$

$$\overline{T}_{\mu}^{\;1}(z_2) = \frac{m^2 p_2}{4\pi^2 E_2} \sum_n a_{\mu n} B^n(s, t, \overline{t})$$

where the sum runs over $n = S, T, V, A, P$, and where,

$$a_{\mu n} = \overline{u}_r^{(1)}(p') \, \overline{v}_s^{(2)}(q') \, X^n \, v_r^{(1)}(p) \, u_s^{(2)}(q), \qquad \text{(VI-10)}$$

The μ refers to the five basic amplitudes of (VI-8). In Table I the functions $a_{\mu n}$ are given; they have been calculated using the explicit representation of the helicity-spinors described in Appendix B. In the table, p and E are the barycentric three-momentum and energy respectively, $\Delta^2 = 2p^2(1-z)$, $\zeta^2 = 2p^2(1+z)$, where z is the barycentric scattering angle. In terms of the invariant scalars we would have in channel 2 the relations: $\Delta^2 = -\overline{t}$, $\zeta^2 = -s$, and $4E^2 = 4(p^2 + m^2) = t$.

Equation (VI-9) may be solved for the Mandelstam functions:

$$A^n(s,t,\overline{t}) = \frac{4\pi^2 E_2}{m^2 p_2} \sum_{\mu=1}^{5} b_{n\mu} \overline{T}_{\mu}^{\;0}(z_2) \qquad \text{(VI-11)}$$

$$B^n(s,t,\overline{t}) = \frac{4\pi^2 E_2}{m^2 p_2} \sum_{\mu=1}^{5} b_{n\mu} \overline{T}_{\mu}^{\;1}(z_2)$$

where $b_{n\mu} = (a^{-1})_{n\mu}$. The matrix $\left\|b_{n\mu}\right\|$ is given in Table II.

In the physical region for channel 2 the functions $b_{n\mu}$ are real; from (IV-5) we have, therefore, in this region:

$$A_2^n(s,t) = \frac{4\pi^2 E_2}{m^2 p_2} \sum_{\mu-1}^{5} b_{n\mu} \, \mathrm{Im}_2 \, \overline{T}^{\,0}_{\mu(z_2)} \qquad \text{(VI-12)}$$

$$B_2^{\;n}(s,t) = \frac{4\pi^2 E_2}{m^2 p_2} \sum_{\mu-1}^{5} b_{n\mu} \, \mathrm{Im}_2 \, \overline{T}^{\,1}_{\mu(z_2)}$$

Table I

The Matrix $||a_{\mu n}||$ of Equation (VI-10)

μ $(r'r)(ss')$ \ n	S	V	T	A	P
1 $(++)(++)$	p^2/m^2	$\frac{\Delta^2-\zeta^2}{4p^2}$	$\frac{\Delta^2-\zeta^2}{4p^2}$	-1	$-E^2/m^2$
2 $(++)(+-)$	0	$\frac{\Delta\zeta E}{2mp^2}$	$\frac{\Delta\zeta E}{2mp^2}$	0	0
3 $(++)(--)$	p^2/m^2	$\frac{\Delta^2-\zeta^2}{4p^2}$	$\frac{(2E^2-m^2)(\Delta^2-\zeta^2)}{4p^2m^2}$	1	E^2/m^2
4 $(+-)(+-)$	0	$\frac{-E^2\zeta^2}{2m^2p^2}$	$-\zeta^2/2p^2$	$\zeta^2/2m^2$	0
5 $(+-)(-+)$	0	$\frac{-\Delta^2E^2}{2p^2m^2}$	$-\Delta^2/2p^2$	$-\Delta^2/2m^2$	0

Table II

The Matrix $\|b_{n\mu}\|$ Being the Inverse of $\|a_{\mu n}\|$

μ → / $(r'r)(ss')$ / n ↓	1 $(++)(++)$	2 $(++)(+-)$	3 $(++)(--)$	4 $(+-)(+-)$	5 $(+-)(-+)$
S	$m^2/2p^2$	$\frac{m(\zeta^2-\Delta^2)(E^2+m^2)}{2\Delta\zeta E p^2}$	$m^2/2p^2$	$\frac{-m^2(2p^2-\zeta^2)}{2p^2\zeta^2}$	$\frac{m^2(2p^2-\Delta^2)}{2p^2\Delta^2}$
V	0	$\frac{-2m^3}{\Delta\zeta E}$	0	$-m^2/\zeta^2$	$-m^2/\Delta^2$
T	0	$2mE/\Delta\zeta$	0	m^2/ζ^2	m^2/Δ^2
A	0	0	0	m^2/ζ^2	$-m^2/\Delta^2$
P	$-m^2/2E^2$	$\frac{m(\zeta^2-\Delta^2)}{2\Delta\zeta E}$	$m^2/2E^2$	$\frac{-m^2(2E-\zeta^2)}{2E^2\zeta^2}$	$\frac{m^2(2E^2-\Delta^2)}{2E^2\Delta^2}$

where the subscript 2 after the "Im" indicates that we are referring to the imaginary part in the physical region for channel 2.

Finally, we fulfil our promise of Section III and indicate that the A^n and B^n remain finite as s or $\overline{t}$ goes to zero. We shall specifically discuss A^n, the arguments for B^n being identical. According to Jacob and Wick,[31] the amplitude $\overline{T}^0_\mu(z)$ may be written as a partial wave expansion of the form:

$$\left\langle \theta 0;\ r', r \left| \overline{T}^0 \right| 00;\ s, s' \right\rangle = \sum_{J=0}^{\infty} \left(\frac{2J+1}{4\pi} \right) e^{-i(\lambda' - \lambda)\phi}\, d_{\lambda\lambda'}{}^J(\theta)\ \overline{T}_\mu{}^\theta(J) \tag{VI-13}$$

where λ' and λ are the differences between the nucleon and antinucleon helicities for the final and initial states respectively. The functions $d_{\lambda\lambda'}{}^J(\theta)$ are given in Ref. 31 and

$$\overline{T}_\mu^{(I)}(J) = \left\langle r',\ r \left| T(J, I) \right| s,\ s' \right\rangle .$$

When $\overline{t}$ is zero, so is Δ^2, and if in addition $p_2 \neq 0$ then θ is zero, too. From Table II we see that as $\Delta \to 0$, $b_{n2} \sim 1/\Delta$ and $b_{n5} \sim 1/\Delta^2$ for all n, while the remaining $b_{n\mu}$ stay finite. For $\mu = 2$ and $\mu = 5$ the d-functions in (VI-13) are $d_{01}{}^J(\theta)$, respectively; and from Appendix A of Ref. 31 we obtain:

$$d_{01}{}^J(\theta) = \frac{\sin\theta\, P'_J(\cos\theta)}{\sqrt{J(J+1)}}$$

$$d_{-11}{}^J(\theta) = (-1)^{J+1}\, d_{11}{}^J(\pi - \theta) =$$

$$\frac{(1-\cos\theta)}{J(J+1)} \left[P'_J(\cos\theta) - (1+\cos\theta)\, P''_J(\cos\theta) \right].$$

Thus, as $\theta \to 0$ both $\Delta^{-1} d_{01}{}^J(\theta)$ and $\Delta^{-2} d_{-11}{}^J(\theta)$ remain finite. Hence, by virtue of (VI-11), it is reasonable to assume that $A^n(s, t, \overline{t})$ remains finite as $\overline{t} \to 0$. This is not by any means a conclusive proof, since there is no guarantee that the series $(1-z_2)^{-1/2}\,\overline{T}^{\,0}{}_2\,(z_2)$ and $(1-z_2)^{-1}\,\overline{T}^{\,0}{}_5\,(z_2)$ remain finite as z_2 approaches one. A similar argument could be used to discuss the point $s = 0$ (i.e., $\zeta^2 = 0$).

VII. THE POLYNOMIAL EXPANSION

We now turn to the problem of determining the functions $A_2{}^n(s, t)$ and $B_2{}^n(s, t)$. According to (VI-12), these functions can be given in terms of $\mathrm{Im}_2\, \overline{T}_\mu{}^I(z_2)$, which in turn can be determined by means of the unitarity condition (VI-3). For the reasons indicated in Section I we shall assume that only the two-pion intermediate state contributions to the unitarity condition need be considered, and except for the box diagram which will be calculated exactly, that, Λ, S and P wave two-pion states will be sufficient to determine the higher angular momentum states of $n\bar{n}$ scattering. Thus, by using unitarity we can get $\mathrm{Im}_2\, T_\mu{}^I(z_2)$ in the physical region for channel 2, i.e., $s \leqslant 0$ and $t \geqslant 4m^2$, in terms of the FF functions, and then (VI-12) will give $A_2{}^n(s, t)$ and $B_2{}^n(s, t)$ and $B_2{}^n(s, t)$ in this same region. Since their analyticity properties are known from (IV-4), the absorptive parts may be analytically continued into the region $s \geqslant 4m^2$ and $t \geqslant 4\mu^2$, in which they are required for the dispersion relations (V-17). It should be emphasized that whereas $\mathrm{Im}_2\, T_\mu{}^I(z_2)$ and $b_{n\mu}$ of (VI-12) may individually be singular at many points in the unphysical region for channel 2, when combined according to (VI-12) the result must have the analyticity properties indicated by (IV-4), if the Mandelstam functions actually obey the Mandelstam representation.

In the two-pion approximation, then, the unitarity condition (VI-3) may be written:

$$\mathrm{Im}_2\, \overline{T}_\mu{}^I(z_2) = 1/2 \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi \left\langle \theta_2 0; \lambda' \left| T^I \right| \theta\phi \right\rangle \left\langle 00; \lambda \left| T^I \right| \theta\phi \right\rangle^* \tag{VII-1}$$

where λ and λ' are the nucleon minus the antinucleon helicities for the initial and final states respectively, and

$\left\langle \theta' \phi'; \lambda \left| T^I \right| \theta\phi \right\rangle$ is a T-matrix element for the process $\pi\pi \to n\bar{n}$.

Our T-matrix element is related to the $\mathcal{F}$ of FF by:

$$\left\langle \theta'\phi'; 0 \left| T^I \right| \theta\phi \right\rangle = 1/\sqrt{2}\ \frac{(p_2\ k)^{1/2}}{2\pi}\ \mathcal{F}_{++}^{\ I}$$

$$\left\langle \theta'\phi'; +1 \left| T^I \right| 00 \right\rangle = - \left\langle \theta' 0; -1 \left| T^I \right| 00 \right\rangle = 1/\sqrt{2}\ \frac{(p_2\ k)^{1/2}}{2\pi}\ \mathcal{F}_{+-}^{\ I}\ , \tag{VII-2}$$

Here p_2 and k are the barycentric nucleon and pion momenta respectively, i.e., $p_2 = \left[(1/4)\ t - m^2\right]^{1/2}$ $k = \left[(1/4)\ t - \mu^2\right]^{1/2}$, and the $\mathcal{F}_{\underline{++}}^{\ I}$ are the functions of FF Eqs. (3.9) and (3.10), except that the i-spin eigenamplitudes A^0 and A^1 of FF Eq. (2.8) are used in FF Eqs. (3.3) and (3.4). Finally, the S-matrix of FF has been multiplied by $2^{-1/2}$ to take into account the indistinguishability of the initial pions when in a state of definite i-spin.

Since the $\pi\pi \to n\bar{n}$ amplitudes are obtained as partial waves of definite helicity, we make a partial wave expansion according to the method of Jacob and Wick:[31]

$$\left\langle \theta'\phi'; \lambda \left| T^I \right| \theta\phi \right\rangle =$$

$$\sum_{J,\,M} \left(\frac{2J+1}{4\pi}\right) e^{i(M-\lambda)\phi'}\ e^{-iM\theta} \left\langle \lambda \left| T^I (J, M) \right| \ \right\rangle d_{M\lambda}^{\ J}(\theta')\ d_{M0}^{\ J}(\theta), \tag{VII-3}$$

where the d-functions are those of Ref. 31, M is the z-component of the total angular momentum, and the blank in the ket on the right-hand side refers to the lack of helicity of the two-pion state. Substituting (VII-3) into the unitarity condition (VII-1), and making use of the relations:

$$\int_0^{2\pi} e^{i(M-M')\phi}\ d\phi = 2\pi\ \delta_{MM'}$$

$$\int_0^{\pi} \sin\theta\ d\theta\ d_{M0}^{\ J'}(\theta)\ d_{M0}^{\ J}(\theta) = \frac{2\ \delta_{JJ'}}{2J+1}$$

$d_{M\lambda}^{\ J}(0) = \delta_{M\lambda}$, we get:

$$\mathrm{Im}_2 \overline{T}_\mu^{\ I}(z_2) = 1/2 \sum_J \left(\frac{2J+1}{4\pi}\right) d_{\lambda\lambda'}^{\ J}(\theta_2) \left\langle \lambda' \middle| T^I(J, \lambda) \middle| \right\rangle \left\langle \lambda \middle| T^I(J, \lambda) \middle| \right\rangle \tag{VII-4}$$

The functions $\left\langle \lambda \middle| T^I(J, M) \middle| \right\rangle$ are actually independent of M owing to rotational invariance; they are related to the $f_\pm^{\ J}(t)$ of FF by:

$$\left\langle 0 \middle| T^I(J) \middle| \right\rangle = 1/\sqrt{2} \sqrt{\frac{k}{p_2}} \frac{(p_2 k)^J}{E_2} f_{+I}^{\ J}(t)$$

$$\left\langle +1 \middle| T^I(J) \middle| \right\rangle = \left\langle -1 \middle| T^I(J) \middle| \right\rangle = 1/\sqrt{2} \sqrt{\frac{k}{p_2}} (p_2 k)^J f_{-I}^{\ J}(t), \tag{VII-5}$$

in which the $f_{\pm I}^{\ J}(t)$ are those of FF, except, again, that the i-spin eigenamplitudes of FF Eq. (2.8) must be used. For example, Eq. (3.16) of FF would read:

$$f_{-I}^{\ J} = 1/8\pi \frac{\sqrt{J(J+1)}}{2J+1} \frac{1}{(pq)^{J-1}} (B_{J-1}^{\ I} - B_{J+1}^{\ I})$$

with $B_J^{\ 0} = \sqrt{6}\ B_J^{(+)}$, $B_J^{\ 1} = 2B_J^{(-)}$. Actually, the $f_\pm^{\ J}(t)$ should be the modified FF functions discussed in Section II of the present work.

In order to calculate the absorptive parts, we substitute the $\mathrm{Im}_2 \overline{T}_\mu^{\ I}(z_2)$ of (VII-4) into (VI-12) and express z_2, k, E_2 and p_2 in terms of s and t. The result is a Legendre function expansion in terms of

$$z_2 = -\left(\frac{s}{2p^2} + 1\right) = -\left(\frac{2s + t - 4m^2}{t - 4m^2}\right).$$

In the region of interest for the dispersion relation we have $s > 4m^2$ and $\infty \geq t \geq 4\mu^2$, so that $|z_2| > |$ for all of the range of integration in (IV-3), and the expansion may diverge. According to Neumann's theorem,[32] a Legendre function expansion in $z = \cos\theta$ converges inside an ellipse in the complex z-plane that has foci at $+1$ and -1, and passes through the nearest singularity. In the present case z_2 is always real, and it is easily seen that the expansion for $A_2^{\ n}(s, t)$ will converge except for points at which $A_2^{\ n}(s, t)$ is singular. From (IV-4) we note that for $s \geq 4m^2$, $A_2(s,t)$ has a singularity in the region in which $a_{12}^{\ n}(s, t)$ is non-zero; this, according to (IV-2a), will occur when: $t \geq 4\mu^2 + 4\mu^4(s-4m^2)^{-1}$, which would give us a very short range of integration before the expansion begins to diverge. The curve C_{12} is, according to Mandelstam,[4, 5] the boundary of the contribution of the π-n pole alone to the spectral functions $a_{12}^{n}(s,t)$ and $b_{12}^{n}(s,t)$, other contributions not entering until we reach the curves C'_{12} and C''_{12} of Fig. 5. Since the π-n pole contributions to the $\pi\pi \to n\bar{n}$ amplitude are easily determined, we can by means of (VII-1) and (VI-12) calculate its effect on the absorptive parts, without recourse to a partial wave expansion; this is done in Section VIII. Each absorptive part in the two-pion approximation will consequently consist of three terms:

$$A_2^{n}(s,\ t) = A'^{\,n}_{2}(s,\ t) + A''^{\,n}_{2}(s,t) - \tilde{A}_2^{n}(s,\ t). \qquad \text{(VII-6)}$$

with a similar expression for $B_2^{n}(s,\ t)$. In the above equation $A'^{\,n}_{2}(s,t)$ is the partial wave expansion in terms of the modified **FF** functions, $A''^{\,n}_{2}(s,\ t)$ gives the π-n pole term (i.e. the box-diagram) in unexpanded form, and $\tilde{A}_2^{n}(s,t)$ is a partial wave expansion of the pole term, involving the same number of partial waves as $A'^{\,n}_{2}(s,t)$. The function $\tilde{A}_2^{n}(s,t)$ is required because the **FF** functions already contain the π-n pole contributions in partial wave form which must be subtracted out by means of $\tilde{A}_2^{n}(s,t)$.

The curve C'_{12} in Fig. 5 is the boundary of the three-pion contribution and can be calculated using Mandelstam's method[4, 5] by considering an intermediate state involving a pion and a particle of twice the pion mass.[33] The result is:

$$t - 4\mu^2 = \mu^2 + \frac{8\mu^2}{s-4m^2} + 4\mu^2 \left[\left(1 + \frac{\mu^2}{s-4m^2}\right)\left(1 + \frac{4\mu^2}{s-4m^2}\right)\right]^{1/2},$$

with the asymptotes: $t = 9\mu^2$, $s = 4m^2$. The curve C''_{12} is the boundary of the higher order two-pion exchange contributions, and has asymptotes $t = (2\mu)^2$ and $s = (2m + \mu)^2$. It can be determined by merely replacing one of the nucleons in the calculation of C_{12} by a particle of the mass of a nucleon plus a pion, the result is:

$$t - 4\mu^2 = \frac{4\mu^2\left[2\mu(m+\mu)\ s + (m^2-\mu^2)(2m + \mu)^2\right]}{\left[(s - \mu^2)\quad s - (2m + \mu)^2\right]}.$$

For s less than the asymptote of C''_{12}, i.e., for a nucleon kinetic energy in the laboratory system $T_L \leqslant 287$ Mev, $(s-4m^2=4p_1^2=2m\ T_L)$, the series for $A_2^n(s,t)$, in the two-pion approximation, will converge for all values of t; however, once t crosses C'_{12} the two-pion approximation soon loses its validity, although it will not do so immediately since the three-pion contributions will, in all probability, be initially small. For $T_L \approx 700$ Mev, C'_{12} and C''_{12} intersect at $t \approx 9.2\mu^2$; thus, once the π-n pole term has been subtracted out, the partial wave expansion will converge up to $t = 9.2$ for $T_L \leqslant 700$ Mev. For values of $T_L > 700$ Mev the expansion will converge for values of t given by C''_{12}.

The foregoing remarks illustrate a general property of scattering amplitudes that was first pointed out by Mandelstam.[4,5] Consider a scattering amplitude in the approximation that only the lowest mass two-particle intermediate state is included. If the interaction is such that no box-type diagram exists, i.e., there is no three-particle vertex like the pion-nucleon vertex, then in the lowest approximation the actual values of the two-dimensional spectral functions may be ignored, and only the boundary curves are needed. This, for example, is the state of affairs in the pion-pion problem. If, however, there is a three-particle vertex such that a box-type diagram exists, the value of the spectral function due to the box-diagram must be known in closed, i.e., not partial wave, form. In principle, as we shall see in the next section, this is always possible.

Finally, we write $A'^{\,n}_2(s,t)$ and $B'^{\,n}_2(s,t)$ in terms of the modified FF functions. From Appendix A of Ref. 31, we get for the functions: $d_{\lambda\lambda'}^{J}(\theta)$, with $z = \cos\theta$

$$d_{00}^{D}(\theta) = 1, \quad d_{\lambda\lambda'}^{0}(\theta) = 0, \quad \lambda \text{ and } \lambda' \neq 0$$

$$d'_{00}(\theta) = z, \quad d'_{10}(\theta) = -1/\sqrt{2}\sqrt{1-z^2}, \quad d'_{11}(\theta) = 1/2\,(1+z)$$

$$d'_{-11}(\theta) = 1/2\,(1-z). \tag{VII-7}$$

Using (VII-4) with J = 0 and 1 only, together with (VII-5), (VII-7) and

(VI-12), and recalling from FF that $f_{\pm 0}^{\,2J+1}(t) = f_{\pm 1}^{\,2J}(t) = 0$, we get:

$$A'^{\,S}_{2}(s,t) = \frac{4\pi}{(t-4m^2)^2}\sqrt{\frac{t-4\mu^2}{t}}\,\left| f_{+0}^{\,0}(t) \right|^2$$

$$A'^{\,V}_{2}(s,t) = A'^{\,T}_{2}(s,t) = A'^{\,A}_{2}(s,t) = A'_{2}(s,t) = 0$$

$$B'^{\,S}_{2}(s,t) = -\frac{3\pi}{32t^{1/2}}\,\frac{(t-4\mu^2)^{3/2}}{(4m^2-t)^2}\,(2s+t-4m^2\left[8\left| f_{+1}^{\,1}(t)\right|^2 \right.$$

$$\left. -\sqrt{\frac{2}{m}}\,(t+4m^2)\, f_{+1}^{\,1}(t)\, f_{-1}^{\,1*}(t) + t\left| f_{-1}^{\,1}(t)\right|^2 \right]$$

$$B'^{\,V}_{2}(s,t) = -\frac{3\pi}{32}\,\frac{(t-4\mu^2)^{3/2}}{(4m^2-t)t^{1/2}}\left[4\sqrt{2}m\; f_{+1}^{\,1}(t)\; f_{-1}^{\,1*}(t) \right.$$

$$\left. - t\left| f_{-1}^{\,1}(t)\right|^2 \right]$$

$$B'^{\,T}_{2}(s,t) = \frac{3\pi}{32}\,\frac{(t-4\mu^2)^{3/2}t^{1/2}}{4m^2-t}\left[\frac{\sqrt{2}}{m} f_{+1}^{\,1}(t)\, f_{-1}^{\,1*}(t) \right.$$

$$\left. - \left| f_{-1}^{\,1}(t)\right|^2 \right]$$

$$B'^{\,A}_{2}(s,t) = 0$$

$$B'^{\,P}_{2}(s,t) = -\frac{3\pi}{32}\,\frac{(t-4\mu^2)^{3/2}}{(4m^2-t)\,t^{1/2}}\,(2s+t-4m^2)\left[\frac{\sqrt{2}}{m}\; f_{+1}^{\,1}(t)\, f_{-1}^{\,1}(t) \right.$$

$$\left. - \left| f_{-1}^{\,1}(t)\right|^2 \right] . \qquad \text{(VII-8)}$$

The expressions for $\tilde{A}_2^{\,n}(s,t)$ and $\tilde{B}_2^{\,n}(s,t)$ may be obtained from (VII-8) by merely substituting the functions $g_{\pm I}^{\,J}(t)$ for the $f_{\pm I}^{\,J}(t)$, where the former are the π-n pole term partial waves given in (VIII-2).

VIII. THE π-n POLE CONTRIBUTIONS

In this section we calculate the contribution to $A_2^n(s, t)$ and $B_2^n(s, t)$ due to the π-n pole, or box-diagram, both in terms of partial waves and as a closed expression.

The functions corresponding to the $f_{\pm I}^J(t)$, but containing only the π-n pole term will be denoted by $g_{\pm I}^J(t)$. They are easily obtained from FF. From Eq. (4.1) of FF we get for the pole term in the barycentric system of the process $\pi\pi \to n\vec{n}$:

$$A_P^{\pm} = 0$$

$$B_P^{(\pm)}(z) = 4\pi g^2 \left(\frac{1}{2E^2 - \mu^2 - 2pkz} \mp \frac{1}{2E^2 - \mu^2 + 2pkz} \right) \qquad \text{(VIII-1)}$$

where $A^{\pm}$ and $B^{\pm}$ are the invariant functions of FF with the subscript-P standing for pole, g is the renormalized unrationalized pion-nucleon coupling constant ($g^2 \approx 14.4$), p and k are respectively the nucleon and pion momentum, z is the cosine of the barycentric scattering angle, and $E^2 = p^2 + m^2 = k^2 + \mu^2$. Making use of FF Eqs. (3.17), (3.15), and (2.8), as well as the expansion used in our (V-20) we easily obtain:

$$g_{+0}^J(t) = \frac{\sqrt{6}\, g^2 m}{(pk)^J} \left[\frac{\gamma}{2p}\, Q_J\left(\frac{\gamma}{2p}\right) - \delta_{J0} \right], \quad J \text{ even}$$

$$= 0, \qquad J \text{ odd.}$$

$$g_{-0}^J(t) = \frac{\sqrt{6}\, g^2 \sqrt{J(J+1)}}{(2J+1)(pk)^J} \left[Q_{J-1}\left(\frac{\gamma}{2p}\right) - Q_{J+1}\left(\frac{\gamma}{2p}\right) \right], \quad J \text{ even}$$

$$= 0 \qquad J \text{ odd}$$

$$g_{+1}^J(t) = \frac{2g^2 m}{(pk)^J}\, \frac{\gamma}{2p}\, Q_J\left(\frac{\gamma}{2p}\right), \qquad J \text{ odd}$$

$$= 0 \qquad J \text{ even}$$

$$g_{-1}{}^J(t) = \frac{2g^2\sqrt{J(J+1)}}{(2J+1)(pk)^J}\left[Q_{J-1}\left(\frac{\gamma}{2p}\right) - Q_{J+1}\left(\frac{\gamma}{2p}\right)\right], \qquad J \text{ odd}$$

$$= 0, \qquad J \text{ even.} \qquad \text{(VIII-2)}$$

The Q_J are the Legendre functions of the second kind used in (V-20); and since t, which is the total energy in our channel 2, is also the total energy for the process $\pi\pi \to n\bar{n}$, we have:

$$t = 4E^2 = 4(k^2 + \mu^2) = 4(p^2 + m^2)$$

and

$$\gamma = \frac{2E^2 - \mu^2}{k} = \frac{t - 2\mu^2}{\sqrt{t - 4\mu^2}} .$$

To get the $\tilde{A}_2{}^n(s, t)$ and $\tilde{B}_2{}^n(s, t)$ we need merely to replace $f_{\pm I}{}^J(t)$ by $g_{\pm I}{}^J(t)$ in (VII-8).

We now turn to the problem of calculating the unexpanded π-n pole terms $A_2{}''^n(s, t)$ and $B_2{}''^n(s, t)$. From the equation in **FF**, and our (VII-2), the T-matrix for the pole term in the process $\pi\pi \to n\bar{n}$ can be written:

$$\left\langle \theta' \phi' ; \lambda \left| T_P{}^{(\pm)} \right| \theta\phi \right\rangle = \left(\frac{pk}{2}\right)^{1/2} \frac{1}{(4\pi)^2 E} B_P{}^{(\pm)}(Z)\, h_\lambda(\theta'\phi', \theta\phi),$$

here:

$$h_\lambda(\theta'\phi, \theta\phi) = \bar{U}_{r'}(p')\, i\, \underset{\sim}{\gamma} \cdot \underset{\sim}{k}_1\, V_r(p), \qquad \lambda = r' - r, \qquad \text{(VIII-3)}$$

$\underset{\sim}{k}_1$ is the barycentric three-momentum of one of the incoming pions — it makes no difference which one, since overall signs are irrelevant for our purposes — $B_P{}^{\pm}(Z)$ is given by (VIII-1) in which:

$$Z = \frac{\underset{\sim}{p}' \cdot \underset{\sim}{k}_1}{pk} = z'z + y'y \cos(\theta' - \phi),$$

where

$$z' = \cos\theta', \quad y' = \sin\theta', \text{ etc.}$$

The functions h_λ, when evaluated in terms of the helicity spinors given in Appendix B, are

$$h_+(\theta'\phi', \theta\phi) = -\frac{Ek}{m}\left[y\, e^{i\phi} - \frac{y'\, e^{-i\phi'}(Z+z)}{1+z'}\right]$$

$$h_-(\theta'\phi', \theta\phi) = \frac{Ek}{m}\left[y\, e^{i\phi} - \frac{y'\, e^{i\phi'}(Z+z)}{1+z'}\right]$$

$$h_0(\theta'\phi', \theta\phi) = kZ. \qquad \text{(VIII-4)}$$

The unitarity condition (VII-1) now tells us that:

$$\mathrm{Im}_2\, \overline{T}_{P\mu}^{(\pm)}(z_2) = \frac{pk}{4(4\pi)^4 E^2}\int_{-1}^{1} dz \int_0^{2\pi} d\phi\, h_\lambda'(\theta_2 0, \theta\phi)\, h_\lambda^*(00, \theta\phi)$$

$$B_P^{(\pm)}(Z)\, B_P^{(\pm)}(z). \qquad \text{(VIII-5)}$$

here

$$Z - z_2 z + y_2 y \cos\phi.$$

In (VIII-4) the subscript "P" shows that the pole contribution is meant, and the "μ" refers to the five basic helicity states of (VI-8). Equation (2.8) of FF implies that:

$$\mathrm{Im}_2\, \overline{T}_{P\mu}^{(0)}(z_2) = 6\, \mathrm{Im}_2\, \overline{T}_{P\mu}^{(+)}(z_2)$$

$$\mathrm{Im}_2\, \overline{T}_{P\mu}^{1}(z_2) = 4\, \mathrm{Im}_2\, \overline{T}_{P\mu}^{(-)}(z_2).$$

The integrals (VIII-5) can be performed, but since they are messy we shall not burden the reader with the intermediate details, but merely with the results. In Appendix C we shall indicate how the integrations may be done. The result is:

$$\mathrm{Im}\ \overline{T}_{P1}{}^{0}(z) = \mathrm{Im}\ \overline{T}_{P3}{}^{0}(z)$$

$$= \frac{3g^4 m^2 k}{8\pi E^2 p}\left[\frac{1}{2}(W_1 + W_2) - \frac{\gamma}{2p}\ \ell n\left(\frac{\gamma + 2p}{\gamma - 2p}\right) + 1\right]$$

$$\mathrm{Im}\ \overline{T}_{P2}{}^{0}(z) = -\frac{3g^4 mk}{16\pi E\gamma p}\left\{(1 - z)\, W_1\right.$$

$$\left. - (1 + z)\, W_2\right] + \frac{\gamma_z}{2p}\ \ell n\left(\frac{\gamma + 2p}{\gamma - 2p}\right)\Bigg\}$$

$$\mathrm{Im}\ \overline{T}_{P4}{}^{0}(z) = \frac{3g^4 k}{16\pi p(1+z)}\left\{\left[\frac{4p^2}{\gamma^2}(1 - z) - (3 - z)\right] W_1\right.$$

$$\left. - \left(\frac{4p^2}{\gamma^2} - 1\right)(1 + z)\, W_2 \quad \frac{\gamma}{p}\ \ell n\left(\frac{\gamma + 2p}{\gamma - 2p}\right) - 2(1 + z)\right\}$$

$$\mathrm{Im}\ \overline{T}_{P5}{}^{0}(z) = -\frac{-3g^4 k}{16\pi\, p\, (1 - z)}\left\{\left(1 - \frac{4p^2}{\gamma^2}\right)(1 - z)\, W_1\right.$$

$$- \left[(3 + z) - \frac{4p^2}{\gamma^2}(1 - z)\right] W_2 + \frac{\gamma}{p}\ \ell n\left(\frac{\gamma + 2p}{\gamma - 2p}\right)$$

$$\left. - 2(1 - z)\right\}$$

$$\mathrm{Im}\ \overline{T}_{P1}{}^{1}(z) = \mathrm{Im}\ \overline{T}_{P3}{}^{1}(z) = \frac{g^4 m^2 k}{8\pi E^2 p}(W_1 - W_2)$$

$$\mathrm{Im}\ \overline{T}_{P2}{}^{1}(z) = \frac{-g^4 mk}{8\pi p E\gamma}\left[(1 - z)\, W_1 + (1 + z)\, W_2 - \frac{\gamma}{2p}\ \ell n\left(\frac{\gamma + 2p}{\gamma - 2p}\right)\right]$$

$$\mathrm{Im}\ \overline{T}_{P4}{}^{1}(z) = \frac{-g^4 k}{8\pi p(1 + z)}\left\{\left[3 - z - \frac{4p^2}{\gamma^2}(1 + z)\right] W_1\right.$$

$$+ \left(1 - \frac{4p^2}{\gamma^2}\right)(1+z)\, W_2 - \frac{\gamma}{p}\, \ell n \left(\frac{\gamma + 2p}{\gamma - 2p}\right)\Bigg\}$$

$$\mathrm{Im}\, \overline{T}_{P5}{}^{1}(z) = - \frac{g^4 k}{8\pi p\,(1-z)} \left\{\left(1 - \frac{\gamma^2}{4p^2}\right)(1-z)\, W_1\right.$$

$$\left. + \left[3 + z - \frac{4p^2}{\gamma^2}(1-z)\right] W_2 - \frac{\gamma}{p}\, \ell n \left(\frac{\gamma + 2p}{\gamma - 2p}\right)\right\} . \qquad \text{(VIII-6)}$$

Here, γ has the same meaning as in (VIII-2), y is $\sin\theta$, and

$$W_1 = \frac{\gamma^2}{4p^2(1-z)\, x_1}\, \ell n \left(\frac{x_1 + 1}{x_1 - 1}\right)$$

$$W_2 = \frac{\gamma^2}{4p^2(1+z)\, x_1}\, \ell n \left(\frac{x_2 + 1}{x_2 - 1}\right)$$

$$x_1 = \left[\frac{\gamma^2 - 4p^2(1+z)}{4p^2(1-z)}\right]^{1/2}, \qquad x_2 = \left[\frac{\gamma^2 - 4p^2(1-z)}{4p^2(1+z)}\right]^{1/2} .$$

From (VI-12) and (VIII-6), together with the relations $s = -2p^2(1+z)$, $t = 4(p^2+m^2) = 4E^2$, we can get the π-n pole contributions to the absorptive parts in channel 2; these turn out to be:

$$\frac{A_2{}''^{S}(s,t)}{3\eta} = \frac{2}{s(s+t-4m^2)}\left[\frac{4m^2}{t} - \frac{(2s+t-4m^2)^2}{s(s+t-4m^2)}\right]$$

$$- \left\{\frac{4}{t} - \frac{(4m^2-t)^2}{st(s+t-4m^2)} + (4m^2-t)\left[\frac{1}{s^2} + \frac{1}{(s+t-4m^2)^2}\right]\right\} \frac{W_0}{(4m^2-t)}$$

$$+ \frac{1}{s}\left[\frac{2}{t} + \frac{2}{s} + \frac{1}{(s+t-4m^2)} + \frac{2s+t-4m^2}{\gamma^2(s+t-4m^2)}\right] W_1$$

$$- \frac{1}{(s+t-4m^2)}\left[\frac{2}{t} - \frac{2}{s+t-4m^2} - \frac{1}{s} + \frac{2s+t-4m^2}{s\gamma^2}\right] W_2$$

$$\frac{A_2^{\prime\prime V}(s,t)}{3\eta} = -\frac{2(2s+t-4m^2)}{s(s+t-4m^2)}\left[\frac{1}{t} - \frac{(4m^2-t)}{s(s+t-4m^2)}\right]$$

$$-\frac{(2s+t-4m^2)}{st(s+t-4m^2)}\left[4m^2 - \frac{t(4m^2-t)^2}{s(s+t-4m^2)}\right]\frac{W_0}{(4m^2-t)}$$

$$+\frac{1}{s}\left[\frac{2}{s}+\frac{2}{t} - \frac{1}{s+t-4m^2} - \frac{(4m^2-t)}{\gamma^2(s+t-4m^2)}\right] W_1$$

$$+\frac{1}{(s+t-4m^2)}\left[\frac{2}{t} - \frac{2}{s+t-4m^2}+\frac{1}{s}+\frac{(4m^2-t}{\gamma^2 s}\right] W_2$$

$$\frac{A_2^{\prime\prime T}(s,t)}{3\eta} = -\frac{2(2s+t-4m^2)(4m^2-t)}{s^2(s+t-4m^2)^2} + \frac{(t+2s-4m^2)}{s(s+t-4m^2)}$$

$$\times\left[1 - \frac{(4m^2-t)^2}{s(s+t-4m^2)}\right]\frac{W_0}{(4m^2-t)} - \frac{1}{s}\left[\frac{2}{s} - \frac{1}{s+t-4m^2} - \frac{(4m^2-t)}{\gamma^2(s+t-4m^2)}\right] W_1$$

$$+\frac{1}{(s+t-4m^2)}\left[\frac{2}{s+t-4m^2} - \frac{1}{s} - \frac{(4m^2-t)}{\gamma^2 s}\right] W_2$$

$$\frac{A_2^{\prime\prime A}(s,t)}{3\eta} = -\frac{2}{s(s+t-4m^2)}\left[1 - \frac{(2s+t-4m^2)^2}{s(s+t-4m^2)}\right] + \left[\frac{1}{s^2} + \frac{1}{(s+t-4m^2)^2}\right] W_0$$

$$-\frac{1}{s}\left[\frac{2}{s} + \frac{1}{s+t-4m^2} + \frac{(2s+t-4m^2)}{\gamma^2(s+t-4m^2)}\right] W_1$$

$$-\frac{1}{(t+s-4m^2)}\left[\frac{2}{s+t-4m^2} + \frac{1}{s} - \frac{(2s+t-4m^2)}{\gamma^2 s}\right] W_2$$

$$\frac{A''^{P}_{2}(s,t)}{3\eta} = \frac{2}{s(t+s-4m^2)}\left[\frac{4m^2}{t} - \frac{(2s+t-4m^2)^2}{s(s+t-4m^2)}\right]$$

$$-\left\{\frac{4}{t} - \frac{(4m^2-t)^2}{st(s+t-4m^2)} + (4m^2-t)^2\left[\frac{1}{s^2} + \frac{}{(s+t-4m^2)}\right]\right\}\frac{W_0}{(4m^2-t)}$$

$$+\frac{1}{s}\left[\frac{2}{s} + \frac{2}{t} + \frac{2}{s+t-4m^2} + \frac{(4m^2-t)}{\gamma^2(s+t-4m^2)}\right] W_1$$

$$-\frac{1}{(s+t-4m^2)}\left[\frac{2}{t} - \frac{2}{s+t-4m^2} - \frac{1}{s} + \frac{2s+t-4m^2}{\gamma^2 s}\right] W_2$$

$$\frac{B_2''^{S}(s,t)}{2\eta} = -\frac{(2s+t-4m^2)}{t\,s^2(s+t-4m^2)^2}\left[t^2 + (t+s)(s-4m^2)\right](W_0+2)$$

$$+\frac{1}{s}\left[\frac{2}{s} + \frac{2}{t} + \frac{1}{t+s-4m^2} + \frac{(2s+t-4m^2)}{\gamma^2(s+t-4m^2)}\right] W_1$$

$$+\frac{1}{(s+t-4m^2)}\left[\frac{2}{t} - \frac{2}{t+s-4m^2} = \frac{1}{s} + \frac{(2s+t-4m^2)}{\gamma^2 s}\right] W_2$$

$$\frac{B''^{V}(s,t)}{2\eta} = \frac{-1}{s(s+t-4m^2)}\left[2 - \frac{4m^2}{t} + \frac{(4m^2-t)^2}{s(s+t-4m^2)}\right](W_0+2)$$

$$+\frac{1}{s}\left[\frac{2}{s} + \frac{2}{t} - \frac{-1}{s+t-4m^2} - \frac{(4m^2-t)}{\gamma^2(s+t-4m^2)}\right] W_1$$

$$-\frac{1}{(s+t-4m^2)}\left[\frac{2}{t} - \frac{2}{t+s-4m^2} + \frac{1}{s} + \frac{(4m^2-t)}{\gamma^2 s}\right] W_2$$

$$\frac{B_2{}^{''T}(s,\ t)}{2\eta} = \frac{1}{s^2(s+t-4m^2)^2}\left[(4m^2-t)^2+s^2+(4m^2-t)\,s\right](W_0+2)$$

$$-\frac{1}{s}\left[\frac{2}{s}-\frac{1}{s+t-4m^2}-\frac{(4m^2-t)}{\gamma^2(s+t-4m^2)}\right]W_1$$

$$-\frac{1}{(s+t-4m^2)}\left[\frac{2}{s+t-4m^2}-\frac{1}{s}-\frac{(4m^2-t)}{\gamma^2\ s}\right]W_2$$

$$\frac{B_2{}^{''A}(s,\ t)}{2\eta} = -\frac{(4m^2-t)(t+2s-4m^2)}{s^2(s+t-4m^2)^2}\ (W_0+2)$$

$$-\frac{1}{s}\left[\frac{2}{s}+\frac{1}{s+t-4m^2}+\frac{(2s+t-4m^2)}{\gamma^2(s+t-4m^2)}\right]W_1$$

$$+\frac{1}{s+t-4m^2}\left[\frac{2}{s+t-4m^2}+\frac{1}{s}-\frac{(2s+t-4m^2)}{\gamma^2\ s}\right]W_2$$

$$\frac{B_2{}^{''P}(s,\ t)}{2\eta} = -\frac{(2s+t-4m^2)}{ts^2(s+t-4m^2)^2}\left[t^2+(t+s)(s-4m^2)\right](W_0+2)$$

$$+\frac{1}{s}\left[\frac{2}{s}+\frac{2}{t}+\frac{1}{s+t-4m^2}+\frac{(2s+t-4m^2)}{\gamma^2(s+t-4m^2)}\right]W_1$$

$$+\frac{1}{(s+t-4m^2)}\left[\frac{2}{t}-\frac{2}{s+t-4m^2}-\frac{1}{s}+\frac{(2s+t-4m^2)}{\gamma^2\ s}\right]W_2,$$

(VIII-7)

where

$$\eta = \pi g^4\ Ek = \frac{\pi\ g^4\left[t(t-4\mu^2)\right]^{1/2}}{4}$$

$$W_0 = \frac{\gamma}{2p}\, \ell n \left(\frac{\gamma + 2p}{\gamma - 2p}\right) - 2 = \frac{\gamma}{(t - 4m^2)^{1/2}}\, \ell n \left[\frac{\gamma(t - 4m^2)^{-1/2} + 1}{\gamma(t - 4m^2)^{-1/2} - 1}\right] - 2.$$

In terms of the scalar invariants of channel 2 the functions W_1 and W_2 are:

$$W_1 = \frac{\gamma^2}{2(s + t - 4m^2)\, x_1}\, \ell n \left(\frac{x_1 + 1}{x_1 - 1}\right)$$

$$W_2 = \frac{\gamma^2}{2s\, x_2}\, \ell n \left(\frac{x_2 + 1}{x_2 - 1}\right)$$

$$x_1 = \left[\frac{4\mu^4 + (t - 4\mu^2)(s + t)}{(t = 4\mu^2)(t + s - 4m^2)}\right]^{1/2}$$

$$x_2 = \left[\frac{4\mu^4 - (t - 4\mu^2)(s - 4m^2)}{-s(t - 4\mu^2)}\right]^{1/2}$$

$$\gamma = \frac{t - 2\mu^2}{(t - 4\mu^2)^{1/2}}$$

The above logarithmic functions are defined on those Riemann sheets which make $A_2''^{n}(s, t)$ and $B_2''^{n}(s, t)$ real in the physical region for channel 2. For the dispersion relation, however, they must be analytically continued into the region $s > 4m^2$, $t > 4\mu^2$ and care must be taken to remain on that branch of the logarithm which gives a real absorptive part for $s < 0$, $t > 4m^2$.

The function W_0 is actually;

$$W_0 = 2Q_1\left(\frac{\gamma}{(t - 4m^2)^{1/2}}\right)$$

the Q_1 being a Legendre function of the second kind, and both it and its argument are real for $t > 4m^2$. For $4\mu^2 < t < 4m^2$, W_0 becomes:

$$W_0 = \frac{\gamma}{i(4m^2 - t)^{1/2}} \ell n \left(\frac{\gamma(4m^2 - t)^{-1/2} + i}{\gamma(4m^2 - t)^{-1/2} - i} \right) - 2$$

$$= \frac{2\gamma}{(4m^2 - t)^{1/2}} \tan^{-1}\left[\frac{(4m^2 - t)^{1/2}}{\gamma} \right] - 2, \quad 4\mu^2 < t < 4m^2.$$

Note that there is no discontinuity at $t = 4m^2$, and that

$$(t - 4m^2)^{-1} W_0 \rightarrow (3\gamma^2)^{-1} \quad \text{as} \quad t \rightarrow 4m^2.$$

The function W_1 is no problem since it has no singularities in the region of interest and x_1 is real throughout.

In the function W_2, x_2 becomes pure imaginary for:

$$s > 0, \quad 0 < (t - 4\mu^2) < \frac{4\mu^4}{s - 4m^2},$$

so that W_2 must be written:

$$W_2 = \frac{-\gamma^2}{2si|x_2|} \ell n \left(\frac{i|x_2| + 1}{i|x_2| - 1} \right)$$

$$= \frac{\gamma^2}{s|x_2|} \tan^{-1} \frac{1}{|x_2|}, \quad 0 \leqslant \tan^{-1} \frac{1}{|x_2|} \leqslant \frac{\pi}{2}.$$

W_2 becomes singular when

$$4\mu^4 - (t - 4\mu^2)(s - 4m^2) = 0,$$

i.e., at the curve C_{12} of (IV-2a), and is complex inside the region bounded by C_{12},

$$W_2 = -\frac{\gamma^2}{2s|x_2|} \left(\ell n \left| \frac{1 + |x_2|}{1 - |x_2|} \right| \right) + \frac{\gamma^2 \pi i}{2s|x_2|}, \quad t - 4\mu^2 > \frac{4\mu^4}{s - 4m^2}, \quad s > 4m^2.$$

The Mandelstam functions, as calculated here, will be real except for an imaginary part coming from the imaginary part of W_2. The nn phase shifts for which our calculation has validity will be small, so that the imaginary parts of the amplitudes will be negligible. Moreover, the imaginary part of the amplitude can easily be determined once the real part is known, We will consequently ignore the imaginary part of W_2.

To summarize, for $s > 4m^2$, $t > 4\mu^2$ the three functions W_0, W_1, and W_2 are:

$$W_0 = \frac{\gamma}{(t - 4m^2)^{1/2}} \ln \left[\frac{\gamma(t - 4m^2)^{-1/2} + 1}{\gamma(t - 4m^2)^{-1/2} - 1}\right] - 2, \quad t > 4m^2$$

$$= \frac{2\gamma}{(4m^2 - t)^{1/2}} \tan^{-1} \left[\frac{(4m^2 - t)^{1/2}}{\gamma}\right] - 2, \quad 4\mu^2 < t < 4m^2$$

$$W_1 = \frac{\gamma^2}{2(s + t - 4m^2)\, x_1} \ln\left(\frac{x_1 + 1}{x_1 - 1}\right), \quad t > 4\mu^2$$

$$x_1 = \left[\frac{4\mu^4 + (t - 4\mu^2)(s + t)}{(t - 4\mu^2)(t + s - 4m^2)}\right]^{1/2}$$

$$W_2 = \frac{\gamma^2}{s|x_2|} \tan^{-1}\left(\frac{1}{\sqrt{|x_2|}}\right), \quad 0 < t - 4\mu^2 < \frac{4\mu^4}{s - 4m^2}$$

$$0 < \tan^{-1} \frac{1}{|x_2|} < \frac{\pi}{2}$$

$$= - \frac{\gamma^2}{2s\,|x_2|} \ln\left(\frac{1 + |x_2|}{1 - |x_2|}\right), \quad t - 4\mu^2 > \frac{4\mu^4}{s - 4m^2}$$

$$x_2 = \left[\frac{4\mu^4 - (t - 4\mu^2)(s - 4m^2)}{- s\,(t - 4\mu^2)}\right]^{1/2}$$

$$\gamma = \frac{t - 2\mu^2}{(t - 4\mu^2)^{1/2}} \qquad \text{(VIII-8)}$$

IX. ANGULAR MOMENTUM DECOMPOSITION OF nn AMPLITUDES

In this section we relate the amplitudes $A_\ell^{\,n}(s)$ and $B_\ell^{\,n}(s)$ of (V-19) to nucleon-nucleon phase shifts. Since all previous phase-shift calculations have been done in terms of z-component of spin rather than helicity states, we will use the former throughout this section.

The T-matrix in channel 1 is (cf VI-5):

$$\left\langle \theta'\phi';\, r',s';\, \alpha',\, \beta' \middle| T \middle| 00;\, r,\, s;\, \alpha,\, \beta \right\rangle$$

$$= \frac{m^2 p}{8\pi^2 E}\, \bar{U}_{r'\alpha'}^{(1)}(p')\, \bar{U}_{s'\beta'}^{(2)}(q')\, M(p',\, q',\, p,\, q)\, U_{r\alpha}^{(1)}(p)\, U_{s\beta}^{(2)}(q), \qquad \text{(IX-1)}$$

where the indices r', s', r, s now refer to z-components of spin rather than to helicities. The i-spin projection operators in channel 1 are:

$$P_0 = \frac{1}{4}(1 - \underset{\sim}{\tau}^{(1)} \cdot \underset{\sim}{\tau}^{(2)})$$

$$P_1 = \frac{1}{4}(3 + \underset{\sim}{\tau}^{(1)} \cdot \underset{\sim}{\tau}^{(2)})$$

so that:

$$M = M^- + \underset{\sim}{\tau}^{(1)} \cdot \underset{\sim}{\tau}^{(2)}\, M^+$$

$$= (M^- - 3M^+)\, P_0 + (M^- + M^+)\, P_1 \qquad \text{(IX-2)}$$

The T-matrix for a scattering in a state of definite i-spin and definite initial and final total spin is:

$$\left\langle \theta\,\phi,\, \sigma' \middle| T^I \middle| 00,\, \sigma \right\rangle = \frac{m^2 p}{(4\pi)^2} \sum_n c_{\sigma'\sigma}\, {}^n D^{In}(s,t,\bar{t})\, e^{-i(\sigma' - \sigma)\phi} \qquad \text{(IX-3)}$$

where σ and σ' are 1, 0, and -1 for the triplet spin state, and $\sigma = S$ will designate the singlet state. The functions $c_{\sigma'\sigma}{}^n$ are the proper combinations of the terms:

$$\bar{u}_{r'}^{(1)}(p')\,\bar{u}_{s'}^{(2)}(q')\,X^n\,u_r^{(1)}(p)\,u_s^{(2)}(q)$$

to give the final and initial total-spin states designated by σ' and σ; they are tabulated in Table III, and have been calculated using the explicit Dirac-spinor representation of Appendix B. The $c_{\sigma'\sigma}{}^n$ of Table III are not all independent, since according to Wolfenstein and Ashkin,[23] time reversal invariance gives rise to the relations:

$$\sqrt{2}\; z\,(c_{10}{}^n + c_{01}{}^n) = y\,(c_{11}{}^n - c_{00}{}^n - c_{1-1}{}^n).$$

From (III-4) and (IX-2) the functions $D^{In}(s, t, \bar{t})$ are:

$$\left.\begin{aligned} D^{0n} &= A^n - 3B^n \\ D^{In} &= A^n + B^n \end{aligned}\right\} \qquad \text{(IX-4)}$$

Finally, the right-hand side of (IX-3) has been multiplied by 1/2 because nucleons when in states of definite i-spin and definite ordinary spin are indistinguishable particles, whereas the matrix M is calculated as though they were distinguishable.

The D^{In} can be expanded in terms of Legendre polynomials:

$$D^{In} = \sum_{\ell=0}^{\infty} (2\ell + 1)\, D_\ell{}^{In}\, P_\ell(z) \qquad \text{(IX-5)}$$

where the $D_\ell{}^{In}$ are related to the $A_\ell{}^n$ and $B_\ell{}^n$ of (V-19) and (V-21) by

$$\left.\begin{aligned} D_\ell{}^{0n} &= A_\ell{}^n - 3\,B_\ell{}^n \\ D_\ell{}^{ln} &= A_\ell{}^n + B_\ell{}^n \end{aligned}\right\}. \qquad \text{(IX-6)}$$

Table III

The Functions $c_{\sigma'\sigma}^{\,n}$ of Equation (IX-3)

The Table actually contains $2m^2\, c_{\sigma'\sigma}^{\,n}$. $z = \cos\theta$, $y = \sin\theta$.

n	S	V	T	A	P
$c_{ss}^{\,n}$	$p^2(1-z) + 2m^2$	$4p^2 + 2m^2$	$-2(p^2 z + 3E^2)$	$-2(2E^2 + m^2)$	$p^2(1 - z)$
$c_{00}^{\,n}$	$(E-m)^2 z^2 - p^2 z + 2Em$	$-2m(E-m)\, z^2 + 4p^2 z + 2Em$	$2[E(E-m)z^2 + 3p^2 z + Em]$	$-2[(E-m)mz^2\ 2p^2 z - Em]$	$-p^2 z\,(1-z)$
$c_{10}^{\,n} = -c_{-10}^{\,n}$	$\frac{(E-m)y}{2}[E+m - (E-m)z]$	$\sqrt{2}\,y(E-m)[mz - 2(E+m)]$	$-\sqrt{2}(E-m)y[Ez + 3(E+m)]$	$\sqrt{2}\,(E-m)y[mz + (E+m)]$	$\frac{p^2 y\,(1-z)}{\sqrt{2}}$
$c_{01}^{\,n}$, $-c_{0-1}^{\,n}$	$-\frac{(E-m)y}{2}[E+m - (E-m)z\,]$	$\sqrt{2}\,(E-m)y(Ez + E + m)$	$-\sqrt{2}\,(E-m)myz$	$\sqrt{2}\,(E-m)y(Ez + E + m)$	$\frac{p^2 y\,(1-z)}{\sqrt{2}}$
$c_{11}^{\,n}$, $c_{-1-1}^{\,n}$	$\frac{1}{2}[(E-m)^2 z^2 - 2p^2 z + (E+m)^2$	$E(E-m)z^2 + 2p^2 z + E\,(E+m)$	$-m[(E-m)z^2 = (E+m)]$	$E(E-m)z^2 + 2p^2 z + E\,(E+m)$	$-\frac{1}{2}p^2\,(1 - z)^2$
$c_{1-1}^{\,n} = c_{-11}^{\,n}$	$\frac{1}{2}(E-m)^2\,(1-z^2)$	$E(E-m)(1-z^2)$	$-m(E-m)(1-z^2)$	$E(E-m)(1-z^2)$	$-\frac{1}{2}\,p^2\,(1-z^2)$

The purpose of this section is to relate the D_ℓ^{In} which are obtained from the dispersion relations to T-matrix elements in the angular momentum representation: $\left\langle J, M, L', S \middle| T^I \middle| J, M, L, S \right\rangle$, where L' and L are the final and initial orbital angular momenta, S the total spin and $M = J_z$. Note that J, M, S and I are all conserved and that the matrix element must be independent of M owing to rotational invariance. To see that S is conserved we observe that an exchange of particles in the initial state vector produces a factor:

$$(-1)^{L + (S+1) + (I+1)} = (-1)^{L+S+I}$$

which must be negative by the Pauli principle; that is, L + S + I must be odd. Now I is conserved, and parity conservation requires that $L' - L$ be even, so that if L + S + I is to be odd for the final as well as for the initial state, $S' - S$ must be even. For the scattering of two spin-1/2 particles $S_f - S_i = 0$ or 1, hence in this case $S_f - S_i = 0$, and S is conserved. Consequently, we may write the T-matrix elements that are non-zero as follows:

For the spin-singlet:

$$T_J^{\,I} = \left\langle J, M, J, 0 \middle| T^I \middle| J, M, J, 0 \right\rangle . \tag{IX-7a}$$

For the spin-triplet, we have matrix elements $T_{LJ}^{\,I}$ when the initial and final L values are the same; specifically these elements are:

$$T_{JJ}^{\,I} = \left\langle J, M, J, 1 \middle| T^I \middle| J, M, J, 1 \right\rangle$$

$$T_{J\pm 1\,J} = \left\langle J, M, J \pm 1, 1 \middle| T^I \middle| J, M, J, \pm 1, 1 \right\rangle ,$$

and when $L' - L = \pm 2$: (IX-7b)

$$T^{IJ} = \left\langle J, M, J+1, 1 \middle| T^I \middle| J, M, J-1, 1 \right\rangle$$

$$= \left\langle J, M, J-1, 1 \middle| T^I \middle| J, M, J+1, 1 \right\rangle$$

where the latter equality comes from the symmetry of the T-matrix. The above expressions are related to the a_{LJ} of Stapp et.al.[34] as follows:

$$a_L = iT_L, \quad a_{LJ} = iT_{LJ}, \quad a^J = -iT^J.$$

The T-matrix can be written:

$$\left\langle \theta\phi, \sigma' \middle| T^I \middle| \theta_0\phi_0, \sigma \right\rangle = \sum_{L'L} \left\langle \theta\phi \middle| L'L'_z \right\rangle \left\langle L'L'_z\sigma' \middle| T^I \middle| LL_z\sigma \right\rangle \left\langle LL_z \middle| \theta_0\phi_0 \right\rangle$$

where:

$$\left\langle \theta\phi \middle| LL_z \right\rangle = Y_{LL_z}(\theta\phi),$$

$Y_{LL_z}(\theta\phi)$ being a spherical harmonic, defined as in Appendix A of Blatt and Weisskopf.[35] Since

$$Y_{LL_z}(00) = \sqrt{\frac{2L+1}{4\pi}}\ \delta_{L_z 0},$$

we obtain

$$\left\langle \theta\phi, \sigma' \middle| T^I \middle| 00, \sigma \right\rangle = \sum_{L', L} Y_{L'L'_z}(\theta\phi) \left\langle L'L'_z\,\sigma' \middle| T^I \middle| L0\sigma \right\rangle \sqrt{\frac{2L+1}{4\pi}} \tag{IX-8}$$

where now $L'_z = \sigma - \sigma'$ by conservation of the z-component of total angular momentum.

The $c_{\sigma'\sigma}^{\ n}$ of (IX-3) are functions of $\sin\theta$ and $\cos\theta$, and by using the recursion relations for Legendre polynomials, (IX-3) can be rewritten in the form:

$$\left\langle \theta\phi, \sigma' \middle| T^I \middle| 00, \sigma \right\rangle = \sum_{L'} Y_{L'L'_z}(\theta\phi)\, K_{\sigma'\sigma}^{\ IL'} \tag{IX-9}$$

where the $K_{\sigma'\sigma}^{IL'}$ are known functions of the D_ℓ^I, and of E and p, but have no angular dependence. Comparing (IX-8) and (IX-9) we see that:

$$K_{\sigma'\sigma}^{IL'} = \sum_L \left\langle L', L'_z = \sigma - \sigma', \sigma' \middle| T^I \middle| L0\sigma \right\rangle \sqrt{\frac{2L+1}{4\pi}} . \qquad \text{(IX-10)}$$

In the spin-singlet case $L' = I = J$, and we immediately get the relation:

$$T_J^I = \sqrt{\frac{4\pi}{2J+1}} \quad K_{ss}^{IJ} \qquad \text{(IX-11)}$$

In the spin-triplet case matters are not as simple, since we may have $J = L'$ or $L' + 1$ and $L = L'$ or $L' + 2$, and it is necessary to project out the T-matrix element referring to the various values of J and L. For this purpose we calculate from the $K_{\sigma'\sigma}^{IL'}$ the functions:

$$H_{L'J}^{I\sigma} = \sum_{\sigma' = 0 \pm 1} \left\langle J\,\sigma, L', 1 \middle| L', L'_z = \sigma - \sigma', \sigma \right\rangle K_{\sigma'\sigma}^{IL'} , \qquad \text{(IX-12)}$$

where $\left\langle J, \sigma, L', 1 \middle| L' L'_z = \sigma - \sigma', \sigma' \right\rangle$ is a Clebsch-Gordon coefficient that relates the $LL_z\sigma$ representation (recall that σ stands for $S = 1$, $S_z = \sigma$) to the J M L S representation. From conservation of angular momentum:

$$M = L'_z + \sigma' = L_z + \sigma ,$$

but $L_z = 0$, so that $M = \sigma$ and $L'_z = \sigma - \sigma'$. Upon combining (IX-10) and (IX-12), and making use of the orthogonality properties of Clebsch-Gordon coefficients, we get:

$$H_{L'J}^{I\sigma} = \sum_{L = L', L' \pm 2} \left\langle J\,\sigma, L', 1 \middle| T^I \middle| L, 0, \sigma \right\rangle \sqrt{\frac{2L+1}{4\pi}} . \qquad \text{(IX-13)}$$

Now:

$$\left. L0\sigma \right\rangle = \sum_{J1} \left| J', \sigma, L, 1 \right\rangle \left\langle J', \sigma, L, 1 \middle| L, 0, \sigma \right\rangle$$

and since the interaction conserves J only, the $J' = J$ term will contribute in (IX-13) thus:

$$H_{L'J}^{I\sigma} = \sum_{L=L', L'+2} \left\langle J, \sigma, L', 1 \middle| T^I \middle| J, \sigma, L, 1 \right\rangle \left\langle J, \sigma, L, 1 \right. \sqrt{\frac{2L+1}{4\pi}} . \tag{IX-14}$$

For $L' = J$ only one term occurs in Eq. (IX-14), namely the one in which L = J; thus we have

$$H_{JJ}^{I\sigma} = T_{JJ}^{I} \left\langle J, \sigma, J, 1 \middle| J, 0, \sigma \right\rangle \sqrt{\frac{2J+1}{4\pi}} .$$

Using this equation for $\sigma = +1$ or -1 (for $\sigma = 0$ both sides vanish identically) we can express the T_{JJ}^{I} in terms of the $H_{JJ}^{I\sigma}$. When $L' = J + 1$, for example, there will be two terms in (IX-14) corresponding to $L = J + 1$ and $L = J - 1$, hence

$$H_{J+1J}^{I\sigma} = T_{J+1J}^{I} \left\langle J, \sigma, J+1, 1 \middle| J+1, 0, \sigma \right\rangle \sqrt{\frac{2J+3}{4\pi}}$$
$$+ T^{IJ} \left\langle J, \sigma, J-1, 1 \middle| J-1, 0, \sigma \right\rangle \sqrt{\frac{2J-1}{4\pi}} . \tag{IX-15}$$

Equation (IX-15) is actually a set of three equations, one for each permissible value of σ, and each equation involves the same two quantities T_{J+1J}^{I} and T^{IJ}. Any two of these relations are independent and can be used to solve for $T_{J+1,J}^{I}$ and T^{IJ} in terms of the $H_{J+1J}^{I\sigma}$. Similarly, $T_{J-1,J}^{I}$ and T^{IJ} can be obtained in terms of $H_{J-1J}^{I\sigma}$. Since the $H_{L'J}^{I\sigma}$ are known functions of the D_{ℓ}^{I} by virtue of (IX-12), the T-matrix elements have been expressed in terms of the Mandelstam amplitudes.

In (IX-16) below we give give the result of this procedure; the functions designated by $\hat{T}$ are the T-matrix elements due to the one-pion exchange pole, which we have ignored up to now, and which have been given previously.[9] The functions c_j^n ($j = 0, 1, 2, 3, 4$; $n =$ S, V, T, A, P) are given in Table IV. For the spin-singlet:

$$T_J^I = \hat{T}_J^I + \frac{p}{8\pi E(2J+1)} \sum_n \left\{ (2J+1)\, c_0^n\, D_J^{In} + p^2 (c_2^n - c_1^n) \left[J D_{J-1}^{In} + (J+1)\, D_{J+1}^{In} \right] \right\} .$$

For the spin-triplet:

$$T_{JJ}^I = \hat{T}_{JJ}^I + \frac{p}{8\pi E(2J+1)} \sum_n \left\{ \left[(E+m)^2\, c_1^n + p^2\, c_2^n \right] (2J+1)\, D_J^{In} + p^2\, c_3^n \left[J\, D_{J+1}^{In} + (J+1)\, D_{J-1}^{In} \right] \right\}$$

$$T_{J+1J}^I = \hat{T}_{J+1J}^I + \frac{p}{8\pi E(2J+1)^2} \sum_n \left\{ \left[2J^2 (E+m)^2\, c_1^n - 2Em\, c_1^n + (2J+1) \left[(E+m)^2\, c_1^n - p^2\, c_2^n) \right] D_{J+1}^{In} + 2J(J+1)(E-m)^2\, c_1^n\, D_{J-1}^{In} + (2J+1) \left[2(J+1)\, c_4^n - c_3^n \right] \times p^2\, D_J^{In} \right\}$$

$$T_{J-1J}^I = \hat{T}_{J-1J}^I + \frac{p}{8\pi E(2J+1)^2} \sum_n \left\{ 2J(J+1)(E-m)^2\, c_1^n\, D_{J+1}^{In} + \left[2(J+1)^2 (E+m)^2\, c_1^n - 2Em\, c_1^n - (2J+1)\ (E+m)^2\, c_1^n - p^2 c_2^n) \right] D_{J-1}^{In} + (2J+1)(2J\, c_1^n + c_3^n)\, p^2\, D_J^{In} \right\}$$

Table IV

The Function C_j^n of Equation (IX-16)

j \ n	S	V	T	A	P
0	$p^2 + 2m^2$	$2(2p^2 + m^2)$	$-6E^2$	$-2(2p^2 + 3m^2$	p^2
1	1	1	1	1	0
2	0	1	-1	1	-1
3	-1	2	0	2	1
4	-1	3	3	-1	0

$$T^{IJ} = \hat{T}^{IJ} + \frac{p\sqrt{J(J+1)}}{8\pi E(2J+1)^2} \sum_n \left[(2J+1)\, p^2\, c_2^{\,n} - (E-m)^2\, c_1^{\,n} \right] D_{J+1}^{\,In}$$

$$+ \left[(2J+1)\, p^2\, c_2^{\,n} + (E-m)^2\, c_1^{\,n} \right] D_{J-1}^{\,In} + (2J+1) 2p^2 (c_2^{\,n} - c_4^{\,n}) D_J^{\,In} \Big\} . \qquad \text{(IX-16)}$$

Note that because of the Pauli principle L' and L are odd for $I = S$ and even for $I \neq S$, where S is the total spin.

For the sake of completeness we give the one-pion exchange terms explicitly:

For the spin-singlet:

$$\hat{T}_J^{\,I} = \frac{a_I p g^2}{(2J+1)2E} \left[(J+1)\, Q_{J+1} + J\, Q_{J-1} - (2J+1)\, Q_J \right], \quad J \neq 0$$

$$= \frac{a_I p g^2}{2E} (Q_1 - Q_0) \qquad J = 0.$$

For the spin-triplet:

$$\hat{T}_{JJ}^{\,I} = \frac{-a_I p g^2}{(2J+1)\, 2E} \left[J Q_{J+1} + (J+1)\, Q_{J-1} - (2J+1)\, Q_J \right]$$

$$\hat{T}_{J+1\,J}^{\,I} = \frac{-a_I\, p g^2}{(2J+1)\;2E} (Q_{J+1} - Q_J)$$

$$\hat{T}_{J-1\,J}^{\,I} = \frac{-a_I\, p g^2}{(2J+1)\;2E} (Q_J - Q_{J-1})$$

$$\hat{T}^{IJ} = \frac{a_I\, p g^2}{(2J+1)2E} \sqrt{J(J+1)} \left[Q_{J+1} + Q_{J-1} - 2\, Q_J \right] \qquad \text{(IX-17)}$$

where $a_1 = 1$, $a_0 = -3$; and $Q_J\left(1 + \frac{\mu^2}{2p^2}\right)$ is a Legendre function of the second kind. Note that in (IX-17) Q_{-1} must be taken to be identically zero. For the relations between our T-matrix elements and phase shifts, see Ref. 34.

X. CONCLUSION

Because of the lack of reliable values for the FF functions, no numerical results could be included. It is hoped that the calculations of the modified FF functions currently being made by Ball and Wong[36] will soon remedy this lack, and it should then be possible to calculate several of the phase-shifts just below those adequately given by the one-pion exchange pole.

Calculations very similar to, but much more ambitious in scope than the present ones, are being carried out by Goldberger, Grisaru, McDowell, Noyes, and D. Wong.[37] These authors write dispersion relations for partial wave amplitudes in the nucleon-nucleon channel, which will enable them to involve the unitarity condition in that channel, and thus derive a set of coupled integral equations by means of the n/D technique of Chew and Mandelstam.[16] They also include coulomb corrections, as well as phenomenological singularities to represent three-pion and higher mass contributions, and should consequently be able to predict successfully the values of the phase-shifts of much lower angular momentum states than can be done by the method presented here.

In our procedure we have neglected what in the language of partial wave dispersion relations is called the right-hand or unitarity cut (cf. Ref. 16), for example, except for the contributions of the π-n pole; thus the amplitude that we get is an integral over the left-hand cut alone.

Considerations similar to ours have been employed in a recent paper by Amati, Leader, and Vitale,[38] although these authors do not include the complete unexpanded π-n pole term.

APPENDICES

A. The I-Spin Formalism

The i-spin formalism for antinucleons used in this thesis is not new, having been treated by Malenka and Primakoff,[39] yet seems sufficiently rare in actual application to merit some discussion.

The formalism is based on the observation that the field variable $\psi(x)$ for a four-component-spinor field involves both the particle and its anti-particle, so that when the neutron and proton fields are combined into one eight-component-spinor field it is natural to write its field variable:

$$\psi(x) = \begin{pmatrix} \psi_p(x) \\ \psi_n(x) \end{pmatrix} \qquad \text{(A-1)}$$

where $\psi_p(x)$ and $\psi_n(x)$ are the proton and neutron field variables respectively. The adjoint spinor is then

$$\overline{\psi}\,(x) = (\overline{\psi}_p(x),\ \ \overline{\psi}_n(x) \qquad \text{(A-2)}$$

where $\overline{\psi}_p(x) = \psi_p^{\dagger}(x)\,\beta$ etc., and β is given just below (III-9). Expanding $\psi(x)$ in terms of creation and destruction operators:

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3p \sqrt{\frac{m}{E}}\ \sum_{\alpha} \left\{ a_\alpha(p)\, U_\alpha(p)\, e^{ip\cdot x} + b_\alpha^{\dagger}(p)\, V_\alpha(p)\, e^{-ip\cdot x} \right\} \qquad \text{(A-2)}$$

where the index α can be $+$ or $-$ corresponding to $I_3 = +1$ or -1 respectively, destroys a particle and b_α an antiparticle, and all ordinary-spin indices have been suppressed. The U_α and V_α are defined as follows:

$$\begin{aligned} U_\alpha(p) &= u(p)\,\chi_\alpha \\ V_\alpha(p) &= v(p)\,\chi_{\overline{\alpha}} \end{aligned} \qquad \text{(A-3)}$$

with $\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_- = \begin{pmatrix}0\\1\end{pmatrix}$ and $\chi_{\bar{a}} = \chi_{(-a)}$. The u(p) and v(p) are ordinary four-component Dirac-spinors.

It is immediately obvious that with these definitions the matrix $\underset{\sim}{\tau}$ loses all meaning as an i-spin operator for antiparticles; instead, the i-spin operator for both particles and antiparticles is now:

$$\underset{\sim}{T} = \frac{1}{2} : \int \overline{\Psi}\, \beta\, \underset{\sim}{\tau}\, \psi\, d^3x : \tag{A-4}$$

where β is actually $\begin{pmatrix}\beta & 0\\ 0 & \beta\end{pmatrix}$ and the ":" indicates that a normal product is to be formed. The important point here is that the i-spinor of the outgoing antinucleon stands to the right of τ while that of the ingoing antinucleon stands to the left, just as their "spin-spinors" do. Consequently, under the substitution rule the U and V spinors behave precisely as the u and v spinors do.

From (A-4) and the relations:

$$\begin{cases} \bar{u}(p)\, \beta\, u(p) = \bar{v}(p) \beta v(p) = E/m \\ \bar{v}(p)\, \beta\, u(p) = \bar{u}(p)\, \beta\, v(p) = 0 \end{cases}$$

we get:

$$\begin{cases} T_+ = T_1 + iT_2 = \int d^3p \left[a_+^\dagger(p)\, a_-(p) - b_+^\dagger(p)\, b_-(p) \right] \\ T_- = T_1 - iT_2 = \int d^3p \left[a_-^\dagger(p)\, a_+(p) - b^\dagger(p)\, b_+(p) \right] \\ T_3 = \frac{1}{2} \int d^3p \left[a_+^\dagger(p) a_+(p) - a_-^\dagger(p)\, a_-(p) + b_+^\dagger(p)\, b_+(p) - b_-^\dagger(p) b_-(p) \right] \end{cases} \tag{A-5}$$

With these rules, states consisting of a nucleon and an antinucleon no longer combine into states of total i-spin by means of the usual Clebsch-Gordon coefficients. If the state with total i-spin 1 and $I_3 = +1$ is

$$\left| 1, 1 \right\rangle = \left| p\bar{n} \right\rangle$$

where:

$$\left| p\bar{n} \right\rangle = a_+^\dagger \; b_+^\dagger \left| vac \right\rangle ,$$

then:

$$\left| 1,\ 0 \right\rangle = \frac{1}{\sqrt{2}}\ T_- \left| p\,\bar{n} \right\rangle$$

$$= \frac{1}{\sqrt{2}} \left[\left| n\bar{n} \right\rangle - \left| p\bar{p} \right\rangle \right] .$$

Similarly:

$$\left| 1,\ -1 \right\rangle = \frac{1}{\sqrt{2}}\ T_- \left| 1\ 0 \right\rangle$$

$$= - \left| n\,\bar{p} \right\rangle .$$

Since the $I = 0$ state is not coupled to the $I = 1$ states the overall sign of the former is arbitrary. We thus have:

$$\begin{aligned}
\left| 1,\ 1 \right\rangle &= \left| p\,\bar{n} \right\rangle \\
\left| 1,\ 0 \right\rangle &= \frac{1}{\sqrt{2}} \left[\left| n\,\bar{n} \right\rangle - \left| p\,\bar{p} \right\rangle \right] \\
\left| 1,\ -1 \right\rangle &= - \left| n\,p \right\rangle \\
\left| 0,\ 0 \right\rangle &= \frac{1}{\sqrt{2}} \left[\left| n\,\bar{n} \right\rangle + \left| p\,\bar{p} \right\rangle \right] .
\end{aligned} \tag{A-6}$$

In channel 2 a projection operator for a state with total i-spin I may be written:

$$\bar{P}_I = a_I\, 1 + b_I\ \underset{\sim}{\tau}^{(1)} \cdot \underset{\sim}{\tau}^{(2)} . \tag{A-7}$$

In this channel, however, "particle 1" refers to an outgoing nucleon and outgoing antinucleon, whereas "particle 2" refers to an incoming nucleon and antinucleon (cf. (III-2) . Thus, for example, we should have in this channel:

$$\left\langle p\,\bar{n}\left|\underset{\sim}{\tau}^{(1)}\cdot\underset{\sim}{\tau}_2\right|n\,\bar{p}\right\rangle = \chi_p{}^{\dagger}\underset{\sim}{\tau}\chi_{\bar{n}}\cdot\chi_{\bar{p}}{}^{\dagger}\underset{\sim}{\tau}\chi_n$$

where $\chi_p = \chi_+ = \chi_{\bar{p}}$, $\chi_{\bar{n}} = \chi_- = \chi_n$,

and

$$\left\langle p\,\bar{n}\left|1\right|n\bar{p}\right\rangle = \chi_p{}^{\dagger}\chi_{\bar{n}}\;\chi_{\bar{p}}{}^{\dagger}\chi_n\,.$$

If we apply the i-spin states of (A-6) to the operators 1 and $\underset{\sim}{\tau}^{(1)}\cdot\underset{\sim}{\tau}^{(2)}$ with the proviso that the states are in channel 2, we get:

$$\left\langle I', I'_3\left|1\right|I, I_3\right\rangle = 2\,\delta_{I'I}\;\delta_{I'_3 I_3} \qquad I = 0$$

$$= 0 \qquad I = 1$$

$$\left\langle I', I'_3\left|\underset{\sim}{\tau}^{(1)}\cdot\underset{\sim}{\tau}^{(2)}\right|I, I_3\right\rangle = 2\delta_{I'I}\;\delta_{I'_3 I_3} \quad I = 1$$

$$= 0 \qquad I = 0 \qquad \text{(A-8)}$$

From Eqs. (A-7) and (A-8) we find:

$$a_0 = \frac{1}{2},\quad b_0 = 0;\quad a_1 = 0,\quad b_1 = \frac{1}{2}$$

so that

$$\bar{P}_0 = \frac{1}{2}\,1,\quad \bar{P}_1 = \frac{1}{2}\;\underset{\sim}{\tau}^{(1)}\cdot\underset{\sim}{\tau}^{(2)}\,.$$

B. The Dirac-Spinors

This appendix is concerned with the explicit representation of the Dirac-spinors, helicity and z-component of spin, used in this thesis.

The z-component of spin spinors $u_r(p)$ are obtained by applying a Lorentz transformation in the $-\underset{\sim}{p}$ direction to a spinor having the spin r in the rest system.

$$u_r(p) = \sqrt{\frac{2m}{m + p_0}} \; L(p) \begin{pmatrix} \chi_r \\ 0 \end{pmatrix} \tag{B-1}$$

where: $L(p) = \dfrac{-i\gamma \, . \, p + m}{2m}$, as in (III-12)

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The spinor $v_r(p)$ is the charge conjugate of $u_r(p)$

$$v_r(p) = C^* \, u^*_r(p) \tag{B-2}$$

where

$$C = \begin{pmatrix} 0 & i\sigma_y \\ -i\sigma_y & 0 \end{pmatrix}.$$

Since

$$C \gamma_\mu C^{-1} = \gamma_\mu^*$$

we have

$$v_r(p) = \sqrt{\frac{2m}{m + p_0}} \; L(-p) \begin{pmatrix} 0 \\ -i\sigma_y \chi_r \end{pmatrix}.$$

The helicity spinors are obtained by first rotating a spinor in the rest system until its spin direction is the same as, or opposite to, the direction of the momentum $\underset{\sim}{p}$ and then applying a Lorentz transformation in the direction $-\underset{\sim}{p}$. Thus

$$u_r(p) = \sqrt{\frac{2m}{m + p_0}}\; L(p)\; \Sigma(\hat{n}) \begin{pmatrix} \chi_r \\ 0 \end{pmatrix} \tag{B-3}$$

$$\Sigma(\hat{n}) = \frac{1 + \underset{\sim}{\sigma} : \hat{n}\, \sigma_z}{\sqrt{2(1 + \cos\theta)}} ,$$

where $\hat{n}$ is a unit vector in the direction of p, θ is the angle between $\hat{n}$ and an arbitrarily chosen z-axis. The operator $\Sigma(\hat{n})$ is then the rotation operator $R_{\phi, \theta, -\phi}$ of Jacob and Wick[31] for the spinor case. For antiparticle we use (B-2):

$$v_r(p) = \sqrt{\frac{2m}{m+p_0}}\; C^*\, L^*(p)\, \Sigma^*(n) \begin{pmatrix} \chi_r \\ 0 \end{pmatrix}$$

$$= \sqrt{\frac{2m}{m+p_0}}\; L(-p)\, \Sigma \ell n) \begin{pmatrix} 0 \\ -i\,\sigma_y\, \chi_r \end{pmatrix} . \tag{B-4}$$

In Ref. 31 the two particle states are defined so that in the barycentric system the same rotation can be applied to both particles. Consider two particles of momenta p_1 and p_2 and helicities r and s in the barycentric system. Then the spinor for one particle is:

$$u_r(p_1) = \sqrt{\frac{2m}{m + E}}\; L(p_1)\, \Sigma(\hat{n}_1) \begin{pmatrix} \chi_r \\ 0 \end{pmatrix}$$

where n_1 is in the direction of $\underset{\sim}{p}_1$, whereas the spinor for the other particle is:

$$\tilde{u}_s(p_2) = \sqrt{\frac{2m}{m + E}}\; L(p_2)\, \Sigma(\hat{n}_1) \begin{pmatrix} \chi_{-s} \\ 0 \end{pmatrix} ,$$

in accordance with Eqs. (13), (14) and (15) of Ref. 31. In Section VI the antiparticles of channel 2 were always taken to be spinors of this latter type, so that the direction of scattering was defined by the nucleons rather than by the anti-nucleons.

C. Integrals

We sketch here the method by which the integrals of Section VIII may be performed.

The integrals of ϕ are done first. They are most easily performed by making the substitution $\xi = e^{i\phi}$ and then integrating over the unit circle.

The resulting expressions can always be written as a sum of integrals of the following types:

$$I_1 = \int_{-1}^{1} \frac{d_z}{\alpha+\beta_z},$$

$$I_2 = \int_{-1}^{1} \frac{d_z}{x^{1/2}},$$

$$I_3 = \int_{-1}^{1} \frac{d_z}{(\alpha + \beta' z)\, X^{1/2}},$$

where

$$X^{1/2} = \sqrt{\beta^2 z^2 + 2\alpha z z_2 + \alpha^2 - \beta^2(1-z_2^2)}$$

with $(\beta')^2 = \beta^2; \quad \alpha^2 > \beta^2,$

and where it must be remembered that the integrals should be performed when the variables that are not being integrated over are in the physical range for channel 2.

Since I_1 and I_2 are merely special cases of I_3 we shall only show explicitly how the latter may be done.

For this purpose consider the integral:

$$I = \int_{-1}^{1} dz \int_{0}^{2\pi} d\phi \; \frac{1}{(\alpha+\beta' z)(\alpha + \beta Z)}$$

where

$$Z = z_2 z + (1 - z^2)^{1/2} \; (1 - z_2{}^2)^{1/2} \; \cos\phi.$$

The integral over ϕ gives:

$$I = 2\pi \int_{-1}^{1} \frac{dz}{(\alpha+\beta' z) \; X^{1/2}}$$

so that $I = 2\pi \, I_3$.

On the other hand, define three unit vectors $\hat{n}$, $\hat{n}_1$, and $\hat{n}_2$ such that

$$z = \hat{n} \cdot n_1, \quad Z = \hat{n} \cdot \hat{n}_2, \quad z_2 = \hat{n}_1 \cdot \hat{n}_2$$

and the integral becomes:

$$I_3 = \frac{1}{2\pi} \int \frac{d^3\hat{n}}{(\alpha + \beta' \hat{n} \cdot \hat{n}_1)(\alpha+\beta\hat{n} \cdot \hat{n}_2)} \; .$$

By means of the Feynman rules for the combination of denominators[40] we find that

$$I_3 = \frac{1}{2\pi} \int_{0}^{1} dt \int \frac{d^3\hat{n}}{(\alpha+\hat{n} \cdot \underset{\sim}{N})^2}$$

where

$$\underset{\sim}{N} = \beta' \hat{n}_1 \, t + \beta \, \hat{n}_2 \, (1 - t) \; .$$

Let us pick a coordinate system in which $\underset{\sim}{N}$ is along the z-axis, and let $\theta' \phi'$ be the direction angles of $\hat{n}$ with respect to this new axis. Then since $\hat{n} \cdot \underset{\sim}{N}$ is independent of ϕ' we get

$$I_3 = \int_0^1 dt \int_{-1}^{1} \frac{dz'}{(\alpha + N_z')^2}$$

where $z' = \cos\theta'$ and $N = \left| \underset{\sim}{N} \right|$.

The last expression is easily integrated over z to give:

$$I_3 = 2 \int_0^1 \frac{dt}{\alpha^2 - N^2}$$

so that:

$$I_3 = \frac{2}{(\beta^2 - \beta\beta' z_2)^{1/2} (2\alpha^2 - \beta^2 - \beta\beta' z_2)^{1/2}} \ \ell n \left(\frac{x+1}{x-1} \right)$$

$$x = \left(\frac{2\alpha^2 - \beta^2 - \beta\beta' z_2}{\beta^2 - \beta\beta' z_2} \right)^{1/2} .$$

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