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THEORY OF PLASMA SIMULATION
USING MULTIPOLE-EXPANSION
SCHEME

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Theory of Plasma Simulation Using Multipole-Expansion Scheme

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Proposed Running Head: Multipole-Expansion Simulation Theory

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ABSTRACT

Nonphysical grid effects in plasma simulation using the Multipole-Expansion scheme and Gaussian-shape charge particles are studied analytically in one dimension. General expressions for the linear dispersion relation, fluctuation spectra, energy, and momentum conservations are derived and then compared with those of the Cloud-in-Cell (CIC) scheme. The results indicate that the Dipole-Expansion scheme and its subtracted version have, in general, grid effects comparable to CIC. Grid effects are, however, greatly reduced in schemes keeping higher order moments, such as the Quadrupole-Expansion scheme.

1. INTRODUCTION

Recently, Kruer, Dawson, and Rosen [1] have proposed the Multipole-Expansion (MPE) scheme for electrostatic plasma simulations with finite-size (extended) particles. According to this scheme, the moments of the charge density and the force are expanded about the nearest grid point. The above authors have used this scheme up to the dipole moment (Dipole-Expansion, DPE, scheme) in several one-dimensional problems, while Okuda and Dawson [2] applied it to study the plasma transport in multi-dimensions. The results found using DPE are in general quite satisfactory. For example, the total energy is better conserved than in the Nearest-Grid-Point (NGP) scheme [3].

To determine the reliability of producing desirable physics which is the most important property of plasma simulations, one has to examine the nonphysical properties associated with this scheme; notably, the grid effects [6,7]. Although, there is computational evidence that DPE also suffers from the usual grid instabilities with growth rate comparable to that of CIC or PIC [8], no systematic theory exists.

The purpose of the present work is to study the nonphysical grid effects in the MPE scheme and to compare them with the corresponding theory of Langdon for CIC and PIC schemes [7,9]. This is quite useful not only for providing better understandings of the MPE scheme, but also for the comparison among various simulation schemes.

In the following sections, grid effects in the MPE scheme are investigated theoretically. For simplicity, we assume the simulation plasma to be

infinite, uniform and one dimensional. There is no static magnetic field. Ions are immobile and form a neutralizing background. Since we are interested in the spatial grid effects, we assume the time to be continuous; i.e., we take the $\Delta t \rightarrow 0$ limit. Extensions to multi-dimensional, multi-component, periodic, finite Δt or magnetized models are straightforward. In Section 2 a brief description of the algorithm is given. Assuming the simulation plasma is collisionless, linear dispersion relation correct to all moments of expansion is then derived and analyzed in Section 3. Comparisons are made between CIC, DPE, and the Quadrupole-Expansion (QPE) schemes. Section 4 contains derivations and discussions of the fluctuation spectra. Energy and momentum conservations are examined in Section 5. In Section 6, we analyze the Subtracted-Dipole-Expansion scheme, which is faster in field calculations than DPE. Final conclusions and discussions are given in Section 7.

2. ALGORITHM

The Multipole Expansion scheme works in the following way:

1. At each time step t , the moments of a charge particle with respect to its nearest grid point are first calculated. Let the i th particle be in the j th cell; i.e., $|x_i - x_j| \leq \Delta x/2$. Here x_i and $x_j = j \Delta x$ are the positions of the i th particle and the j th grid point, respectively. Δx is the grid spacing. The zeroth moment (monopole) is then given by

$$\rho_0(j, t) = \sum_{i \in j} 1, \quad (1)$$

the first moment (dipole) is

$$\rho_1(j, t) = \sum_{i \in j} [x_i(t) - x_j] , \quad (2)$$

and the ℓ th moment is

$$\rho_\ell(j, t) = \sum_{i \in j} [x_i(t) - x_j]^\ell . \quad (3)$$

2. As the second step, discrete Fourier transforms of $\rho_{g\ell}(x, t)$,

$\rho_{g\ell}(k, t)$, are obtained. Here, $\rho_{g\ell}(x, t) = \sum_j \rho_\ell(j, t) \delta(x - x_j)$ is a function of the grid quantities ρ_ℓ . Since the charge density $\rho(x, t)$ is given by

$$\rho(x, t) = q \sum_{\ell=0}^{\infty} \int dx' S(x - x') \frac{(-1)^\ell}{\ell!} \frac{d^\ell}{dx^\ell} \rho_{g\ell}(x', t) , \quad (4)$$

its Fourier transform then becomes

$$\rho(k, t) = q S(k) \sum_{\ell=0}^{\infty} \frac{(-ik)^\ell}{\ell!} \rho_{g\ell}(k, t) , \quad (5)$$

where

$$S(k) = \exp[-(ka)^2 / 2] \quad (6)$$

is the shape factor of a Gaussian cloud with the "width" $2a$.

3. The third step is to obtain the Fourier transforms of the moments of force;

$$F_0(k, t) = q S(k) E(k, t) \quad (7)$$

$$= -i4\pi q S(k) \rho(k, t) / k , \quad (7a)$$

and

$$F_\ell(k, t) = (ik)^\ell F_0(k, t) . \quad (8)$$

4. Inverse discrete Fourier transform $F_\ell(k, t)$ to $F_\ell(j, t)$.

5. The last step is to calculate the force on the i th particle in the j th cell which is given by

$$F(x_i, t) = \sum_{\substack{i \in j \\ \ell=0}}^{\infty} \frac{[x_i(t) - x_j]^\ell}{\ell!} F_\ell(j, t). \quad (9)$$

6. With $F(x_i, t)$ calculated, the i th particle is pushed forward to the next time step $t + \Delta t$ using the usual time-centered leap-frog particle pushing scheme.

3. DISPERSION RELATION

As the first step to analyze the property of the MPE scheme, the dispersion relation is considered here. In order to obtain the linear dispersion relation, we analyze the particle dynamics using the algorithm described in the preceding section. Let us define the NGP weighting function, $\tilde{w}(x)$, as

$$\tilde{w}(x) = \begin{cases} 1/\Delta x, & |x| \leq \Delta x/2 \\ 0, & \text{otherwise.} \end{cases} \quad (10)$$

Then, the ℓ th moment of the particle density is given by

$$\rho_\ell(j, t) = \Delta x \int_{-\infty}^{\infty} dx' (x' - x_j)^\ell \tilde{w}(|x_j - x'|) n(x', t), \quad (11)$$

where

$$n(x, t) = \sum_{i=1}^N \delta[x - x_i(t)] \quad (12)$$

is the microscopic particle density and N is the total number of particles.

Note for an infinite system, we require that as N and L (system length) become infinite, the average uniform density $n_0 = N/L$ stays finite. Now for

$\rho_{gl}(x, t)$ defined as

$$\rho_{gl}(x, t) = \sum_{j=-\infty}^{\infty} \rho_l(j, t) \delta(x - x_j), \quad (13)$$

its (spatial) Fourier transform is

$$\rho_{gl}(k, t) = \sum_{p=-\infty}^{\infty} (-i)^l w^{(l)}(k_p) n(k_p, t), \quad (14)$$

where

$$w(k) = \sin(1/2 k \Delta x) / (1/2 k \Delta x), \quad (15)$$

$w^{(l)}(k) = d^l w(k) / dk^l$, $k_p = k - pk_g$ and $k_g = 2\pi / \Delta x$. Poisson's sum formula is used to derive Eq. (14). Substituting Eq. (14) into Eq. (5), we have

$$\rho(k, t) = qS(k) \sum_p I(k, k_p) n(k_p, t), \quad (16)$$

and

$$I(y, z) = \sum_{l=0}^{\infty} \frac{(-y)^l}{l!} w^{(l)}(z). \quad (17)$$

It is understood that the sum of p runs from $-\infty$ to ∞ .

For the grid force functions, we define

$$F_{gl}(x, t) = \sum_{j=-\infty}^{\infty} F_l(j, t) \delta(x - x_j). \quad (18)$$

From Eq. (9), the force felt by a particle at x is then given by

$$F(x, t) = \Delta x \sum_{\ell=0}^{\infty} \int dx' \frac{(x-x')^{\ell}}{\ell!} \tilde{w}(|x-x'|) F_{g\ell}(x', t). \quad (19)$$

Fourier transform of $F(x, t)$ is

$$F(k, t) = \Delta x \sum_{\ell=0}^{\infty} \frac{(i)^{\ell}}{\ell!} w^{(\ell)}(k) F_{g\ell}(k, t). \quad (20)$$

Again, using Poisson's sum formula and Eq. (8), $F_{g\ell}(k, t)$ can be shown to be

$$F_{g\ell}(k, t) = \frac{1}{\Delta x} \sum_p (ik_p)^{\ell} F_0(k_p, t). \quad (21)$$

Equation (20) then becomes

$$F(k, t) = \sum_p I(k_p, k) F_0(k_p, t). \quad (22)$$

Now, in order to obtain the linear dispersion relation, we assume that the simulation plasma is collisionless (Vlasov) and that time is continuous.

Then the force and density perturbation are related through the Vlasov equation.

In the linear regime we obtain for the perturbed density $n_1(x, t)$

$$n_1(k, \omega) = -i (n_0/m) \psi(k, \omega) F(k, \omega), \quad (23)$$

where

$$\psi(k, \omega) = \int \frac{df_0/dv}{\omega - kv} dv. \quad (24)$$

$f_0(v)$ is the equilibrium velocity distribution function, and $\text{Im } \omega \geq 0$ for the Laplace transform in time. From Eqs. (7), (16), (22), and (23) and note

$n(k, \omega) = 2\pi n_0 \delta(k) + n_1(k, \omega)$, we find for $k \neq 0$,

$$4\pi\rho(k) = -i\omega_p^2 S(k) \sum_p \sum_r S(k_r) I(k, k_p) I(k_r, k_p) \psi(k_p, \omega) E(k_r, \omega) \quad (25)$$

ω_p is the plasma frequency. Since generally $a \sim O(\Delta x)$, hence, for $|k\Delta x| < \pi$ (i.e., $|k| < k_g/2$) $|S(k)| < 1$ for Gaussian shape particles except when $r=0$. That is, the dominant contribution in the r -sum comes from the $r=0$ term. Equation (25) then becomes approximately

$$4\pi\rho(k) \simeq -i\omega_p^2 S^2(k) \sum_p I^2(k, k_p) \psi(k_p, \omega) E(k, \omega) \quad (26)$$

Substituting Eq. (25) into Poisson's equation

$$ikE(k, \omega) = 4\pi\rho(k, \omega) \quad (27)$$

we obtain the following approximate linear dispersion relation for $|k\Delta x| < \pi$

$$\epsilon(k, \omega) = 1 + \omega_p^2 [S^2(k)/k] \sum_p I^2(k, k_p) \psi(k_p, \omega), \quad \text{Im } \omega \geq 0. \quad (28)$$

(The exact dispersion relation is derived in the Appendix A.) The dispersion relation of the standard charge-sharing scheme is [7]

$$\tilde{\epsilon}(k, \omega) = 1 + (\omega_p^2/\kappa) \sum_p |w_p^2(k_p)|^\alpha \psi(k_p, \omega), \quad \text{Im } \omega \geq 0. \quad (29)$$

Here $\kappa = \tan(1/2 k\Delta x)(\Delta x/2)^{-1}$, $\alpha=1$ for NGP and $\alpha=2$ for CIC or PIC with cloud size Δx . The κ factor comes from finite differencing the Poisson's equation. By comparing the forms of ϵ and $\tilde{\epsilon}$, one can easily see that MPE scheme has the usual grid instabilities; e.g., in a stationary Maxwellian with $\lambda_D/\Delta x \sim 0.1$, as the charge-sharing schemes [7,8].

To compare the grid effects in the various schemes, we use three functions, $A(k)$, $D(k)$, and $B(k)$, defined as

$$\left. \begin{array}{l} A(k) \\ D(k) \end{array} \right\} = \frac{\text{Coefficient of } \psi(k_{\pm 1}, \omega)}{\text{Coefficient of } \psi(k, \omega)},$$

and

$$B(k) = \frac{\text{Coefficient of } \psi(k, \omega)}{\text{Coefficient of the gridless result}}.$$

$A(k)$ and $D(k)$, therefore, indicate the relative importance of the grid aliases ($p \neq 0$ terms). We choose the $p = 1$ and $p = -1$ aliases for $A(k)$ and $D(k)$ because they are generally the most serious ones.

$B(k)$ indicates the degree that the simulation code can reproduce the gridless results. For CIC

$$\left. \begin{array}{l} A_C(k) \\ D_C(k) \end{array} \right\} = [w^4(k_{\pm 1})/w^4(k)] \quad (30)$$

$$B_C(k) = w^2(k)/(k/\kappa); \quad (31)$$

and for DPE and QPE

$$\left. \begin{array}{l} A_M(k) \\ D_M(k) \end{array} \right\} = [I_M^2(k, k_{\pm 1})] [I_M^2(k, k)]^{-1}, \quad (32)$$

$$B_M(k) = I_M^2(k, k), \quad M = D \text{ or } Q \quad (32)$$

where $\ell_D = 1$, $\ell_Q = 2$ and

$$I_M(y, z) = \sum_{\ell=0}^{\ell_M} \frac{(-y)^\ell}{\ell!} w^{(\ell)}(z). \quad (34)$$

Figures 1, 2, and 3 show numerical calculations of $A(k)$'s, $D(k)$'s, and $B(k)$'s corresponding to CIC, DPE, and QPE for $k\Delta x$ from 0.2 to 3.0. Since the short-wavelength ($|k\Delta x| > 1$) modes are generally

suppressed in simulations either by the gaussian shape form factor or by the additional k -space smoothing as practiced often in CIC scheme, we, therefore, concentrate primarily on the long-wavelength ($|k\Delta x| < 1$) regime which is the region of physical interest. Figures 1 and 2 show that for $|k\Delta x| < 1$ alias effects in QPE are much smaller than those in CIC and DPE, $A_Q \sim D_Q \sim 0(k/k_g)^6$, $A_C \sim D_C \sim A_D \sim D_D \sim 0(k/k_g)^4$. The latter two are comparable, consistent with simulation results [8]. Figure 3 shows that for $|k\Delta x| < 1$ $B_D \approx B_Q \approx B_C \approx 1$, indicating the modification due to the grids is quite small. The fact that B_C diverges from 1 for $|k\Delta x| \gtrsim 1$, however, is mainly due to the κ factor arising from finite differencing the Poisson's equation.

The above results, thus, indicate that for the physically important long-wavelength regime, grid effects in CIC and DPE are comparable, but much larger than those in QPE.

4. FLUCTUATION SPECTRUM

Let us consider a homogeneous, stationary, and stable (physically as well as nonphysically) ensemble of simulation plasmas. Using Eq. (15), and the well-known result of density fluctuation in a plasma, the spectrum of charge density fluctuation for MPE can be readily obtained as

$$(\rho^2)_{k,\omega} = \frac{2\pi n_0 q^2 S^2(k)}{|\epsilon(k,\omega)|^2} \sum_p I^2(k, k_p) \int dv f_0(v) \delta(\omega - kv). \quad (35)$$

The electric field fluctuation spectrum then is

$$(E^2)_{k, \omega} = (4\pi/k)^2 (\rho^2)_{k, \omega} . \quad (36)$$

As a comparison, the corresponding expression for CIC with cell-size clouds is [9]

$$(\tilde{E}^2)_{k, \omega} = (4\pi/k)^2 \frac{2\pi n_0 q^2}{|\tilde{\epsilon}(k, \omega)|^2} \sum_p w^4(k_p) \int dv f_0(v) \delta(\omega - kv) . \quad (37)$$

Thus both spectra yield essentially the gridless spectra for $|k \Delta x| < 1$.

We can also find the force fluctuation spectrum from Eqs. (7), (28), and (36); i.e.,

$$(F^2)_{k, \omega} = q^2 \sum_p I^2(k_p, k) S^2(k_p) (E^2)_{k_p, \omega} . \quad (38)$$

Using $(F^2)_{k, \omega}$, a corresponding Balescu-Lenard kinetic equation can be constructed following the method of Langdon for the charge-sharing scheme [9].

This is straightforward and we will not go into it here.

5. ENERGY AND MOMENTUM CONSERVATIONS

Since the check on energy and momentum conservations is always an assurance for good physics and is being widely used, we examine it here for the MPE scheme. The time rate of change in the field energy is given by

$$\begin{aligned} \frac{d}{dt} (F.E.) &= \frac{d}{dt} \frac{1}{16\pi^2} \int_{-k_g/2}^{k_g/2} dk |E(k,t)|^2 \\ &= -\frac{1}{2\pi} \sum_p \int_{-k_g/2}^{k_g/2} dk S(k) I(k, k_p) E(k,t) J(-k_p, t) k_p/k. \end{aligned} \quad (39)$$

Here, $J(k,t)$ is the current density. In deriving Eq. (39), we have used the Poisson's equation, Eq. (16) and the continuity equation. Now, the time rate of change in the kinetic energy is given by

$$\frac{d}{dt} K.E. = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \frac{F(k,t)}{q} J(-k,t), \quad (40)$$

which becomes with Eqs.(7) and (22)

$$\frac{d}{dt} K.E. = \frac{1}{2\pi} \sum_p \int_{-k_g/2}^{k_g/2} dk S(k) I(k, k_p) E(k,t) J(-k_p, t). \quad (41)$$

Combining Eqs. (39) and (41), we have

$$\frac{d}{dt} (T.E.) = \frac{1}{2\pi} \sum_{p \neq 0} \int_{-k_g/2}^{k_g/2} dk S(k) I(k, k_p) E(k,t) J(-k_p, t) (1 - k_p/k). \quad (41)$$

Note $k_p = k$ for $p=0$. Thus, energy is not conserved due to the presence of the grid aliases ($p \neq 0$ terms) as in the CIC scheme.

As to the momentum, it is conserved (not so obviously) for the MPE as well as charge-sharing schemes. The proof is given in Appendix B.

6. SUBTRACTED-DIPOLE-EXPANSION SCHEME

In order to reduce the number of Fourier and Inverse Fourier transforms required in the standard DPE scheme (four times in one dimension), Kruer et al., [1] proposed an alternative algorithm, the Subtracted-Dipole-Expansion (SDPE) scheme, which requires only two transforms in one dimension; i.e., same as in NGP. SDPE differs from DPE in that the derivatives are replaced by the difference between the neighboring grid points. That is, in charge density calculation we have at each grid point an effective grid charge

$$\rho_e(j, t) = \rho_0(j, t) - [\rho_1(j+1, t) - \rho_1(j-1, t)] / 2\Delta x. \quad (42)$$

The corresponding $\rho(k, t)$ then is given by

$$\rho_{SD}(k, t) = qS(k) \rho_{ge}(k, t), \quad (43)$$

where $\rho_{ge}(x, t) = \sum_j \rho_e(j, t) \delta(x - x_j)$. As in DPE, we obtain $F_0(k, t)$ and, hence, $F_0(j, t)$ from $\rho_{SD}(k, t)$ using Eq. (7). The force felt by the i th particle in the j th cell, however, is modified to

$$F_{i \in j}(x_i) = F_0(j, t) + (x_i - x_j) [F_0(j+1, t) - F_0(j-1, t)] / 2\Delta x. \quad (44)$$

Since this scheme is faster than DPE in field calculations and appears to give satisfactory results [1], it is worthwhile to examine SDPE in more details. In the follosings, we derive the corresponding dispersion relation and discuss its grid properties with respect to other schemes. The analyses are similar to those in Section 3.

From Eq. (42), $\rho_{ge}(k, t)$ can be shown to be

$$\rho_{ge}(k, t) = \rho_{g0}(k, t) - ikw(2k) \rho_{g1}(k, t). \quad (45)$$

Using Eq. (14) for ρ_{g0} and ρ_{g1} in Eq. (45), Eq. (43) reduces to

$$\rho_{SD}(k, t) = qS(k) \sum_p I_{SD}(k, k_p) n(k_p, t), \quad (46)$$

where

$$I_{SD}(y, z) = w(z) - y w(2y) w^{(1)}(z). \quad (47)$$

Similarly, we find for the force

$$F(k, t) = q \sum_p I_{SD}(k_p, k) S(k_p) E(k_p, t). \quad (48)$$

Combining Eqs. (23), (46) and (48), we obtain

$$4\pi\rho_{SD}(k, \omega) = -i\omega_p^2 S(k) \sum_p \sum_r S(k_r) I_{SD}(k, k_p) I_{SD}(k_r, k_p) \cdot \psi(k_p, \omega) E(k_r, \omega), \quad (49)$$

which, again, for $|k \Delta x| < \pi$, becomes approximately

$$4\pi\rho_{SD}(k, \omega) \simeq -i\omega_p^2 S^2(k) \sum_p I_{SD}^2(k, k_p) \psi(k_p, \omega) E(k, \omega). \quad (50)$$

Substituting Eq. (50) into the Poisson's equation gives the dispersion relation

$$\epsilon_{SD}(k, \omega) = 1 + \omega_p^2 \frac{S^2(k)}{k} \sum_p I_{SD}^2(k, k_p) \psi(k_p, \omega), \quad \text{Im } \omega \geq 0. \quad (51)$$

Equation (51) indicates that the usual grid instability, again, would occur in

SDPE. To compare the grid effects with those in other schemes, we compute

$A_{SD}(k)$, $D_{SD}(k)$, and $B_{SD}(k)$ over the same range of $k\Delta x$, from 0.2 to 3.0.

The results are plotted in Figs. 1, 2, and 3 respectively. Again, we concentrate on the long-wavelength ($|k\Delta x| < 1$) modes, where $A_{SD} \sim D_{SD} \sim 0(k/k_g)^4$ is small and comparable to A_D, D_D, A_C , and D_C . As to $B_{SD}(k)$, similar to B_C, B_D , and B_Q , $B_{SD} \approx 1$ for $|k\Delta x| < 1$. Thus, one may conclude that for long wavelength ($|k\Delta x| < 1$) grid effects in SDPE are comparable to those in CIC and DPE. In view of its faster field calculations, SDPE, therefore, appears very attractive as an efficient simulation scheme. Although we have only analyzed the Subtracted-Dipole case, the above analyses can be extended to include the higher moments.

7. CONCLUSIONS AND DISCUSSIONS

In the previous sections, using a one-dimensional model, we have theoretically analyzed the Multipole-Expansion simulation scheme proposed by Kruer, et al., [1], and compared it with other schemes. A linear dispersion relation correct for all moments of expansion is derived, which shows the existence of nonphysical grid effects. Comparison of grid effects in CIC, DPE, and QPE indicate that for long wavelength ($|k\Delta x| < 1$), where the physics is most important, the grid effects in CIC and DPE are comparable, but larger than those in QPE. We also derive expressions of fluctuations in the MPE scheme. We demonstrate that, similar to CIC, for MPE the total energy is not conserved due to the alias effects ($p \neq 0$ terms). The MPE scheme, however, conserves the total momentum as in the CIC scheme.

We also examine the Subtracted-Dipole-Expansion scheme, which reduces the number of Fourier transforms required in DPE and, hence, is faster in field calculations. The corresponding dispersion relation is derived and analyzed. For $|k\Delta x| < 1$, grid effects are comparable to CIC and DPE.

The above results, thus, suggest that nonphysical grid effects associated with the MPE and the Subtracted-MPE schemes, such as DPE and SDPE, are in general comparable to those in CIC. Although, we have only treated the one-dimensional cases, similar properties are expected in higher dimensions.

As we have shown, by keeping higher moments in the multipole expansion, grid effects can be greatly reduced in the long wavelength ($|k\Delta x| < 1$) regime, which is generally the most important regime of physical interest. It, thus, appears that the MPE scheme can also be an efficient way to reduce the grid effects and, therefore, in approaching the gridless results.

One of the examples which require the high accuracy of electric field calculations is the two-dimensional guiding center model in a strong magnetic field [10]. The particle motion is approximated as being a massless guiding center in this

model and, therefore, there is no kinetic energy associated with particles. The electric field calculations then determine the accuracy of the evolution of the system and it is so essential to have a high accuracy of computation when determining the irreversible transport processes such as particle diffusion in this guiding center model. Unfortunately, it is known [11] that the standard charge-sharing scheme fails to keep sufficient high accuracy of computation for a long time, while one could expect that the higher-order MPE scheme, such as QPE, saves the situation.

Recently, Chen, et al., [12] propose grid "jiggling" and "interlacing" schemes in order to reduce the grid effects in charge-sharing schemes, such as CIC. They find that "jiggling" produces nonphysical high-frequency ($\sim \Delta t^{-1}$) modes which can be either stable or unstable depending on the velocity distribution. "Interlacing" eliminates certain groups of aliases and has no such ill effects. It, however, requires that the particles be processed at least twice in each time step and, hence, is computationally expensive. Although, it is hard to compare quantitatively the reduction factors in the "interlacing" and MPE scheme, the MPE certainly looks more attractive just from the computational point of view.

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APPENDIX A

Exact Linear Dispersion Relation. Here we derive the exact dispersion relation for the MPE scheme. From Eqs. (7), (14), (22), and (23), we can obtain the following relation among the grid charge functions, ρ_{gl} ;

$$\rho_{gl}(k, \omega) + \sum_{s=0}^{\infty} \alpha_{ls}(k, \omega) \rho_{gs}(k, \omega) = 0, \quad (A.1)$$

where

$$\alpha_{ls}(k, \omega) = \omega_p^2 \frac{(-i)^{l+s}}{s!} \sum_p \sum_r (k_r)^{s-1} S^2(k_r) I(k_r, k_p) w^{(l)}(k_p) \psi(k_p, \omega). \quad (A.2)$$

In deriving Eq. (A.1) we note that $\rho_{gl}(k_p, \omega) = \rho_{gl}(k, \omega)$. Written in matrix form, Eq. (A.1) becomes

$$\epsilon_{\approx} \cdot \rho_{\approx g} = 0, \quad (A.3)$$

with the elements of the infinite matrix $\epsilon_{\approx}$ being

$$\epsilon_{ls} = \delta_{ls} + \alpha_{ls}; \quad l, s = 0, \dots, \infty. \quad (A.4)$$

A formal exact dispersion relation can then be obtained by setting the determinant of $\epsilon_{\approx}$ to vanish, i.e.,

$$\epsilon = |\epsilon_{\approx}| = 0, \quad \text{Im } \omega \geq 0. \quad (A.5)$$

For example, we have for DPE

$$\begin{aligned}
 \epsilon_D = 1 + \omega_p^2 \sum_{p, r} \frac{S^2(k_r)}{k_r} I_D^2(k_r, k_p) \psi(k_p, \omega) \\
 + \omega_p^4 \sum_{p, p', r, r'} S^2(k_r) S^2(k_{r'}) w(k_p) w^{(1)}(k_{p'}) I_D(k_r, k_p) I_D(k_{r'}, k_{p'}) \\
 \left(\frac{1}{k_r} - \frac{1}{k_{r'}} \right).
 \end{aligned}
 \tag{A.6}$$

If we make the approximation that $|S^2(k_r)| < 1$ for $|k \Delta x| < \pi$ except when $r = 0$, Eq. (A.6) reduces to the approximate dispersion relation for DPE expressed in Eq. (28). Similar analysis can be carried out for schemes with higher moments.

APPENDIX B

Momentum Conservation. Here we prove that the total momentum is conserved in the MPE scheme. As usual, the time rate of change in the total momentum is given by

$$\frac{d}{dt} \text{T.M.} = \frac{1}{2\pi} \cdot \frac{1}{m} \int_{-\infty}^{\infty} dk F(k, t) n(-k, t). \quad (\text{B.1})$$

Using Eqs. (7), (16) and (22), Eq. (B.1) becomes

$$\begin{aligned} \frac{d}{dt} \text{T.M.} = & - \frac{i}{2\pi n_0} \omega_p^2 \int_{-k_g/2}^{k_g/2} dk n(-k_p, t) n(k_r, t) I(k, k_p) I(k, k_r) \\ & S^2(k)/k. \end{aligned} \quad (\text{B.2})$$

It is easy to show that the integrand is an odd function in k .

Hence, we have

$$\frac{d}{dt} \text{T.M.} = 0. \quad (\text{B.3})$$

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LIST OF SYMBOLS

ρ	=	rho
ϵ	=	epsilon
Δ	=	u.c. delta
Σ	=	u.c. sigma
δ	=	l.c. delta
ℓ	=	script L
ψ	=	l.c. psi
ω	=	l.c. omega
α	=	alpha
κ	=	kappa

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Fig. 1. Plot of $A(k)$ for CIC, DPE, SDPE, and QPE schemes.

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Fig. 2. Plot of $D(k)$ for CIC, DPE, SDPE, and QPE schemes.

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Fig. 3. Plot of $B(k)$ for CIC, DPE, SDPE, and QPE schemes.

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