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by

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ANALYTIC TRANSPORT THEORY SOLUTIONS FOR STEADY SOURCES IN PURE ABSORBERS

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ABSTRACT

The monoenergetic steady-state integral neutron transport equation is solved analytically for several simple source configurations for pure absorbers. Specifically, the scalar neutron flux, $\phi(r)$, is derived for configurations in cartesian geometry representing the point, infinite line, infinite plane, finite line, infinite ribbon, finite ribbon, infinite tube, and finite box sources.

Some of these configurations (point, infinite line, and infinite ribbon sources) were investigated previously by Case, deHoffmann and Placzek¹ and are reviewed here for completeness. The others are more complex and represent new analytic solutions against which multidimensional numerical transport calculational codes may be tested.

I. INTRODUCTION

The monoenergetic steady-state integral neutron transport equation for the scalar flux, $\phi(\vec{r})$, is given by¹

$$\phi(\vec{r}) = \int d\vec{r}' \frac{e^{-\sigma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|^2} [(q(\vec{r}') + \sigma c\phi(\vec{r}'))] \quad (1)$$

where σ is the transport cross section and c is the number of secondary neutrons per collision. We assume that the source function, $q(\vec{r}')$, is given, that σ is constant, and $c = 0$ for pure absorbers. Then, for a number of simple geometric source configurations we may derive exact analytic solutions to Eq. (1).

Specifically we investigate source configurations representing the point, infinite line, infinite plane, finite line, infinite ribbon, finite ribbon, infinite box and finite box sources in cartesian geometries. Some of these source configurations have been investigated previously by Case, de Hoffman, and Placzek¹ (the point, infinite line, and infinite ribbon (or plane) sources), and are reviewed here for completeness. The others, however,

are more complex and represent benchmark solutions using recent new developments in another context.² These new solutions should prove useful in testing multi-dimensional numerical transport calculational codes when used in the flux-source calculational mode.

II. SOURCE DEFINITIONS

The mathematical representations of the various source configurations are summarized in Table I. We use the standard definition for the Dirac Delta function,¹ $\delta(x)$, and define the function $H(x, x_0)$ by

$$H(x, x_0) = \theta(x + x_0) - \theta(x - x_0) \quad (2)$$

where

$$\theta(w) \equiv \begin{cases} 1 & \text{if } w > 0 \\ 0 & \text{if } w \leq 0 \end{cases} \quad (3)$$

In other words, $H(x, x_0)$ is a unit square pulse from $x = x_0$ to $x = -x_0$ of width $2x_0$ centered about $x=0$, see Fig. 1.

Of the 20 source configurations exhibited in Table I and Figs. 2-7 for cartesian geometries, only 8 are unique using the symmetry considerations of rotation-reflection invariance.³

TABLE I
SOURCE CONFIGURATIONS: CARTESIAN GEOMETRIES

Representation	$q(\vec{r}')/q_0$
1. Point	$\delta(x') \delta(y') \delta(z')$
2. Infinite Line	(a) $\delta(x'-x) \delta(y') \delta(z')$ (b) $\delta(x') \delta(y'-y) \delta(z')$ (c) $\delta(x') \delta(y') \delta(z'-z)$
3. Infinite Plane	(a) $\delta(x'-x) \delta(y'-y) \delta(z'-z)$ (b) $\delta(x'-x) \delta(y'-y) \delta(z')$ (c) $\delta(x'-x) \delta(y') \delta(z'-z)$
4. Finite Line	(a) $\delta(x') \delta(y') H(x', x_0)$ (b) $\delta(x') H(y', y_0) \delta(x')$ (c) $H(x', x_0) \delta(y') \delta(x')$
5. Infinite Ribbon	(a) $\delta(x'-x) H(y', y_0) \delta(x')$ (b) $H(x'-x_0) \delta(y'-y) \delta(x')$ (c) $\delta(x', x_0) \delta(y') H(x', x_0)$
6. Finite Ribbon	(a) $\delta(x') H(y', y_0) H(x', x_0)$ (b) $H(x', x_0) \delta(y') H(x', x_0)$ (c) $H(x', x_0) H(y', y_0) \delta(x')$
7. Infinite Tube	(a) $\delta(x'-x) H(y', y_0) H(x', x_0)$ (b) $H(x', x_0) \delta(y'-y_0) H(x', x_0)$ (c) $H(x', x_0) H(y', y_0) \delta(x'-x_0)$
8. Finite Box	(a) $H(x', x_0) H(y', y_0) H(x', x_0)$

Fig. 1. Heaviside unit square pulse.

Fig. 2. Point source.

Fig. 3. Finite line sources.

Fig. 4. Infinite ribbon sources.

Fig. 5. Finite ribbon sources.

Fig. 6. Infinite tube sources.

III. INTEGRAL SOLUTIONS

We now solve the integral equation, Eq. (1), for the case of pure absorbers ($c = 0$), constant cross section, σ , and the source configurations given explicitly in Table I.

A. Point Source.¹ From Table I and Fig. 2

$$q(\vec{r}') = q_0 \delta(x') \delta(y') \delta(z') \quad (4)$$

Substituting Eq. (4) into (1) for $c = 0$ we immediately obtain

$$\phi(\vec{r}) = q_0 \frac{e^{-\sigma(x^2+y^2+z^2)^{1/2}}}{4\pi(x^2+y^2+z^2)} = \frac{q_0}{4\pi} \frac{e^{-\sigma r}}{r^2} \quad (5)$$

where $r^2 = x^2 + y^2 + z^2$. This is, of course, a well-known result.¹

B. Infinite Line Source.¹ From Table I and Fig. 3

$$q(\vec{r}') = q_0 \delta(z'-z) \delta(y') \delta(x'). \quad (6)$$

Substituting Eq. (6) into (1) we obtain

$$\phi(\vec{r}) = q_0 \iiint_{-\infty}^{\infty} dx' dy' dz' \exp[-\sigma\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{1/2}] \frac{\delta(z'-z) \delta(y') \delta(x')}{4\pi[(x-x')^2 + (y-y')^2 + (z-z')^2]} \quad (7)$$

Fig. 7. Finite box source.

or

$$\phi(\vec{r}) = \frac{q_0}{4\pi} \frac{e^{-\sigma(x^2+y^2)^{1/2}}}{(x^2+y^2)} = \frac{q_0}{4\pi} \frac{e^{-\sigma R}}{R^2} \quad (8)$$

where $R^2 = x^2 + y^2$.

C. Infinite Plane.¹ From Table I and Fig. 4, the source is given by

$$q(\vec{r}') = q_0 \delta(z') \delta(y'-y) \delta(x'-x) \quad (9)$$

which, when substituted into Eq. (1), gives

$$\phi(\vec{r}) = \frac{q_0}{4\pi} \frac{e^{-\sigma r}}{z^2} \quad (10)$$

The results given in Eqs. (5), (8), and (10) were previously obtained by Case, deHoffman, and Placzek.¹ The remaining cases are somewhat more involved.

D. Finite Line. From Table I and Fig. 3, the source is given by

$$q(\vec{r}') = q_0 \delta(z') \delta(y') H(x', x_0) \quad (11)$$

Substituting Eq. (11) into Eq. (1) we obtain

$$\begin{aligned}\phi(\vec{r}) &= \frac{q_0}{4\pi} \iiint_{-\infty}^{\infty} dx' dy' dz' \\ &\times \frac{\exp(-\sigma[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{1/2})}{(x'-x)^2 + (y'-y)^2 + (z'-z)^2} \\ &= \frac{q_0}{4\pi} \int_{-x_0}^{x_0} dx' \\ &\times \frac{\exp(-\sigma[(x'-x)^2 + y^2 + z^2]^{1/2})}{(x'-x)^2 + y^2 + z^2}\end{aligned}\quad (12)$$

upon taking into account the properties of the delta function.

If we define

$$\begin{aligned}a^2 &= y^2 + z^2, \\ w^2 &= (x'-x)^2 + a^2 \\ w_{\pm}^2 &= (\pm x_0 - x)^2 + a^2 > a^2 > 0,\end{aligned}\quad (13)$$

then

$$dx' = \frac{w dw}{(w^2 - a^2)^{1/2}}. \quad (14)$$

$$E_n(z) \equiv \int_1^{\infty} \frac{dt e^{-zt}}{t^n}, \quad n = 0, 1, 2, \dots, \quad (18)$$

$\operatorname{Re}(z) > 0$

then the expression for the flux, Eq. (15), becomes

$$\begin{aligned}\frac{4\pi}{q_0} \phi(\vec{r}) &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} \left[\left(\frac{-a^2}{w_-^2} \right)^n \frac{1}{w_-} \right. \\ &\times E_{2n+2}(\sigma w_-) - \left. \left(\frac{-a^2}{w_+^2} \right)^n \frac{1}{w_+} E_{2n+2}(\sigma w_+) \right]\end{aligned}\quad (19)$$

where $a^2 = y^2 + z^2$ and $w_{\pm}^2 = a^2 + (\pm x_0 - x)^2$.

It is of interest to examine the limit of Eq. (19) for $\sigma = 0$, which is the idealized case of a source in vacuum. Using the recursion relation on the exponential integral, (2)

$$\frac{E_{2n+2}(\sigma w)}{w} = \frac{1}{2n+1} \left[\frac{e^{-\sigma w}}{w} - \sigma E_{2n+1}(\sigma w) \right] \quad (20)$$

Eq. (19) takes the form

$$\begin{aligned}\frac{4\pi}{q_0} \phi(\vec{r}) &= \frac{1}{\sqrt{\pi}} \frac{e^{-\sigma w_-}}{w_-} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n! (2n+1)} \left(\frac{-a^2}{w_-^2} \right)^n \frac{e^{-\sigma w_+}}{w_+} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n! (2n+1)} \left(\frac{-a^2}{w_+^2} \right)^n \\ &\times \left[\left(\frac{-a^2}{w_-^2} \right)^n E_{2n+1}(\sigma w_-) - \left(\frac{-a^2}{w_+^2} \right)^n E_{2n+1}(\sigma w_+) \right].\end{aligned}\quad (21)$$

Using this transformation, Eq. (12) becomes

$$\phi(\vec{r}) = \frac{q_0}{4\pi} \int_{w_-}^{w_+} \frac{dw}{(1 - a^2/w^2)^{1/2}} \frac{e^{-\sigma w}}{w^2}. \quad (15)$$

If we use the identity (4)

$$\frac{1}{\sqrt{1 - a^2/w^2}} \equiv \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} (-a^2/w^2)^n, \quad (16)$$

where

$$\Gamma(n+1/2) \equiv \sqrt{\pi} (2n-1)!!/2^n, \quad (17)$$

and the definition of the exponential integral (5),

$E_n(z)$,

Using the definition of $\Gamma(n+1/2)$, Eq. (17), and

$$2n! = 2^n n! (2n-1)!! \quad (22)$$

we may write the coefficient in Eq. (21) as

$$\frac{\Gamma(n+1/2)}{n! (2n+1)} = \frac{2n! \sqrt{\pi}}{2^{2n} (n!)^2 (2n+1)}. \quad (23)$$

Finally using the series representation⁵

$$\begin{aligned}\operatorname{arc} \sin(x) &= xF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right) \\ &= \sum_{k=0}^{\infty} \frac{2k!}{2^{2k}} \frac{x^{2k+1}}{(k!)^2 (2k+1)}, \quad (24) \\ x^2 &< 1\end{aligned}$$

and taking the $\sigma \rightarrow 0$ limit in Eq. (21) we find

$$\frac{4\pi}{q_0} \phi(\vec{r}) = -\frac{1}{a} \sin^{-1} \left(\frac{a}{w_-} \right) + \frac{1}{a} \sin^{-1} \left(\frac{a}{w_+} \right) \quad (25)$$

where $a^2 = x^2 + y^2$ and $w_{\pm}^2 = a^2 + (\pm x_0 - x)^2$.

Incidentally, the last term of Eq. (21) can be shown to vanish in the $c = 0$ limit by using the recursion relation ($n > 1$) for exponential integrals,⁵

$$E_n(z) = \theta(n-1) \frac{e^{-z}}{(n-1)!} \sum_{k=1}^{\infty} (n-1-k)! (-z)^{k-1} + \frac{(-z)^{n-1}}{(n-1)!} E_1(z), \quad (26)$$

the series representation for $E_1(z)$

$$E_1(z) = -\gamma - \ln z - \sum_{p=1}^{\infty} \frac{(-z)^p}{p \cdot p!} \quad (27)$$

where $\gamma = 0.577215\dots$ = Euler's constant, and the result that $\lim_{x \rightarrow 0} x \ln x = 0$.

E. Infinite Ribbon. From Table I and Fig. 4, the source is given by

$$q(\vec{r}) = q_0 \delta(z'-z) H(y', y_0) \delta(x'). \quad (28)$$

Substituting Eq. (28) into Eq. (1) we obtain

$$\phi(r) = \frac{q_0}{4\pi} \int_{-z_0}^{y_0} dy' \frac{e^{-\sigma[(y'-y)^2 + x^2]^{1/2}}}{(y'-y)^2 + x^2}. \quad (29)$$

If we define, analogous to the finite line problem,

$$\begin{aligned} a^2 &= x^2 \\ w^2 &= (y'-y)^2 + x^2 \\ w_{\pm}^2 &= (\pm y_0 - y)^2 + x^2 > a^2 > c \end{aligned} \quad (30)$$

then

$$\phi(\vec{r}) = \frac{q_0}{4\pi} \int_{w_-}^{w_+} \frac{dw}{\sqrt{1-a^2/w^2}} \frac{e^{-\sigma w}}{w^2} \quad (31)$$

which, analogous to Eq. (19), has the solution

$$\begin{aligned} \frac{4\pi}{q_0} \phi(r) &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} \left[\left(\frac{-a^2}{w_-^2} \right)^n \frac{1}{w_-} E_{2n+2}(\sigma w_-) \right. \\ &\quad \left. - \left(\frac{-a^2}{w_+^2} \right)^n \frac{1}{w_+} E_{2n+2}(\sigma w_+) \right] \end{aligned} \quad (32)$$

where $a^2 = x^2$ and $w_{\pm}^2 = a^2 + (\pm y_0 - y)^2$.

The vacuum limit, $\sigma = 0$, corresponding to Eq. (32) is

$$\frac{4\pi}{q_0} \phi(\vec{r}) = -\frac{1}{a} \sin^{-1} \frac{a}{w_-} + \frac{1}{a} \sin^{-1} \frac{a}{w_+}. \quad (33)$$

F. Finite Ribbon. From Table I and Fig. 5, the source is given by

$$q(\vec{r}) = q_0 \delta(z') H(y', y_0) H(x', x_0). \quad (34)$$

Substituting Eq. (34) into Eq. (1) we obtain

$$\begin{aligned} \frac{4\pi}{q_0} \phi(\vec{r}) &= \int_{-y_0}^{y_0} dy' \int_{-x_0}^{x_0} dx' \\ &\times \frac{\exp(-\sigma(z^2 + (y'-y)^2 + (x'-x)^2)^{1/2})}{z^2 + (y'-y)^2 + (x'-x)^2} \end{aligned} \quad (35)$$

The first integral of Eq. (35) is, of course, related in form to the result obtained for finite line and infinite ribbon.

If we define

$$\begin{aligned} a^2 &= z^2 + (x'-x)^2 \\ w^2 &= z^2 + (y'-y)^2 + (x'-x)^2 \\ w_{\pm}^2 &= z^2 + (\pm y_0 - y)^2 + (x'-x)^2 \end{aligned} \quad (36)$$

then

$$wdw = (x'-x) dx' + (y'-y) dy' \quad (37)$$

and

$$\frac{\partial y'}{\partial w} = \frac{1}{\sqrt{1-a^2/w^2}} \quad (38)$$

so that

$$\frac{4\pi}{q_0} \phi(\vec{r}) = \int_{-x_0}^{x_0} dx' \int_{w_-}^{w_+} \frac{dw}{\sqrt{1-a^2/w^2}} \frac{e^{-\sigma w}}{w^2} \quad (39)$$

or

$$\begin{aligned} \frac{4\pi}{q_0} \phi(\vec{r}) &= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} \int_{-x_0}^{x_0} dx' \\ &\times \left[\left(\frac{-a^2}{w_-^2} \right)^n \frac{1}{w_-} E_{2n+2}(\sigma w_-) \right. \\ &\quad \left. - \left(\frac{-a^2}{w_+^2} \right)^n \frac{1}{w_+} E_{2n+2}(\sigma w_+) \right] \end{aligned} \quad (40)$$

where a^2 and $w_{\pm}^2$ are defined by Eq. (36).

Equation (40) can be rewritten as

$$\frac{4\pi}{q_0} \phi(\vec{r}) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} (I_n^- - I_n^+) \quad (41)$$

where we have defined

$$I_n^{\pm} = (-)^n \int_{w_{\pm}(-x_0)}^{w_{\pm}(x_0)} dw_{\pm} \left(\frac{a^2 + w_{\pm}^2 - b_{\pm}^2}{w_{\pm}^2} \right)^n \quad (42)$$

$$\times \frac{1}{\sqrt{1 - b_{\pm}^2/w_{\pm}^2}} \frac{E_{2n+2}(0w_{\pm})}{w_{\pm}}$$

using the definitions

$$\begin{aligned} a^2 &= z^2 \\ b_{\pm}^2 &= z^2 + (z y_0 - y)^2 \\ w_{\pm}^2 &= (x' - x)^2 + b_{\pm}^2 > b_{\pm}^2 > 0 \end{aligned} \quad (43)$$

and the result

$$\frac{\partial w_{\pm}}{\partial w_{\pm}} = \frac{1}{\sqrt{1 - b_{\pm}^2/w_{\pm}^2}} \quad (44)$$

If we further define

$$c_{\pm}^2 = a^2 - b_{\pm}^2 = - (z y_0 - y)^2 \quad (45)$$

and use the expansion given by Eq. (16)

$$\frac{1}{\sqrt{1 - b_{\pm}^2/w_{\pm}^2}} = \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{\Gamma(m+1/2)}{m!} \left(-b_{\pm}^2/w_{\pm}^2 \right)^m \quad (46)$$

as well as the binomial theorem⁵

$$(1 - c_{\pm}^2/w_{\pm}^2)^n = \sum_{p=0}^n \binom{n}{p} (-c_{\pm}^2/w_{\pm}^2)^p \quad (47)$$

then Eq. (42) becomes

$$\begin{aligned} I_n^{\pm} &= \frac{(-)^n}{\sqrt{\pi}} \sum_{m=0}^{\infty} \sum_{p=0}^n \frac{\Gamma(m+1/2)}{m!} \binom{n}{p} \\ &\times (c_{\pm}^2)^p (-b_{\pm}^2)^m K_{n,p}^{\pm} \end{aligned} \quad (48)$$

where

$$K_{n,p}^{\pm} \equiv \int_{w_{\pm}(-x_0)}^{w_{\pm}(x_0)} dw_{\pm} \frac{E_{2n+2}(0w_{\pm})}{w_{\pm}^{2m+2p+1}} \quad (49)$$

The integral in Eq. (49) can be rewritten as

$$K_{n,p}^{\pm} = \frac{1}{w_{\pm}^{2m+2p} (-x_0)} \int_1^{\infty} dt \frac{E_{2n+2}(0w_{\pm}(-x_0)t)}{t^{2m+2p+1}} \quad (50)$$

$$- \frac{1}{w_{\pm}^{2m+2p} (x_0)} \int_1^{\infty} dt \frac{E_{2n+2}(0w_{\pm}(x_0)t)}{t^{2m+2p+1}}$$

or, defining⁽²⁾ the integral

$$D_{n,p}^{(1)}(\lambda) \equiv \int_1^{\infty} dx x^n E_p(\lambda x) \quad (51)$$

Eq. (50) becomes

$$\begin{aligned} K_{n,p}^{\pm} &= \frac{1}{w_{\pm}^{2m+2p} (-x_0)} D_{-(2m+2p+1), 2n+2}^{(1)}(0w_{\pm}(-x_0)) \\ &- \frac{1}{w_{\pm}^{2m+2p} (x_0)} D_{-(2m+2p+1), 2n+2}^{(1)}(0w_{\pm}(x_0)) \end{aligned} \quad (52)$$

Substituting Eq. (52) into (48) and (41) we obtain the result

$$\frac{4\pi}{q_0} \Phi(\vec{r}) = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{n!} \frac{\Gamma(m + \frac{1}{2})}{m!} \binom{n}{p} \left\{ \begin{array}{l} (-b_-^2)^m (-c_-^2)^k \left\{ \frac{1}{v_-^{2m+2p}} (-x_0) D_{-(2m+2p-1), 2n+2}^{(1)} (\sigma v_- (-x_0)) \right\} \\ \frac{1}{v_-^{2m+2p}} (x_0) D_{-(2m+2p+1), 2n+2}^{(1)} (\sigma v_- (x_0)) \right\} \\ -(-b_+^2)^m (-c_+^2)^k \left\{ \frac{1}{v_+^{2m+2p}} (-x_0) D_{-(2m+2p+1), 2n+2}^{(1)} (\sigma v_+ (x_0)) \right\} \\ -\frac{1}{v_+^{2m+2p}} (x_0) D_{-(2m+2p+1), 2n+2}^{(1)} (\sigma v_+ (x_0)) \right\} \end{array} \right\} \quad (53)$$

where

$$\begin{aligned} b_{\pm}^2 &= z^2 + (\pm y_0 - y)^2 \\ c_{\pm}^2 &= -(\pm y_0 - y)^2 \end{aligned} \quad (54)$$

and the integral $D_{n,s}^{(1)}(\lambda)$ is given by²

$$\begin{aligned} D_{n,s}^{(1)}(\lambda) &\equiv \int_1^{\infty} dx x^n E_s(\lambda x) = \left\{ \left[(1-\delta_{s,0}) \frac{(-\lambda)^{s-1}}{(s-1)!} \frac{\theta(n+s)}{n+s} \left\{ \alpha_{n+s}(\lambda) - E_1(\lambda) \right\} \right. \right. \\ &\quad \left. \left. + \delta_{n+s-1,-1} \left\{ \frac{1}{2} \left[\frac{\pi^2}{6} + (\gamma + \ln \lambda)^2 \right] + \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{2^p p!} \right\} \right. \right. \\ &\quad \left. \left. + \theta(-n-s) (1+n-s-1) (E_1(\lambda) - E_{1+n-s-1}(\lambda)) \right] \right\} \\ &\quad + \delta_{s,0} \frac{1}{\lambda} \left\{ \theta(n) \alpha_{n-1}(\lambda) + \theta(1-n) E_{|n-1|}(\lambda) \right\} \quad (55) \\ &\quad + (1-\delta_{s,0}) \frac{\theta(s-1)}{(s-1)!} \sum_{k=1}^{s-1} (s-1-k)! (-\lambda)^{k-1} \left\{ \theta(n+k) \alpha_{n+k-1}(\lambda) \right. \\ &\quad \left. + \theta(1+k-n) E_{|n|+1+k}(\lambda) \right\}. \end{aligned}$$

where

$$\alpha_n(\lambda) \equiv \int_1^{\infty} dx x^n e^{-\lambda x} = \frac{n!}{\lambda^{n+1}} e_n(\lambda),$$

$$e_n(\lambda) = \sum_{k=0}^n \frac{(\lambda)^k}{k!} \quad (56)$$

$$E_n(\lambda) = \int_1^{\infty} dx \frac{e^{-\lambda x}}{x^n}, \quad n = 0, 1, \dots, \operatorname{Re}(\lambda) > 0, \quad (57)$$

$$E_1(\lambda) = -\gamma - \ln \lambda - \sum_{p=1}^{\infty} \frac{(-\lambda)^p}{p p!}, \quad \operatorname{Re}(\lambda) > 0, \quad (58)$$

$$E_{n+1}(\lambda) = \frac{1}{n} (e^{-\lambda} - \lambda E_n(\lambda)), \quad n \geq 1 \quad (59)$$

The integrals in Eq. (53) involving the function $D_{n,s}^{(1)}(\lambda)$ given by Eq. (55) have the index constraints $s \geq 2$ and $n < -1$, so that applying Eq. (55) to (53) we have

$$\begin{aligned}
D_{-(2m+2p+1), 2n+2}^{(1)}(\Lambda) = & \frac{(-\Lambda)^{2n+1}}{(2n+1)!} \left\{ \frac{\theta(2n+1-2m-2p)}{2n+1-2m-2p} \left[\alpha_{2n+1-2m-2p}(\Lambda) - E_1(\Lambda) \right] \right. \\
& + \theta(2m+2p-1-2n) (1-2n-2m-2p-1) \left[E_1(\Lambda) - E_{2|n-m-p|}(\Lambda) \right] \Big\} \\
& + \frac{\theta(2n+1)}{(2n+1)!} \sum_{k=1}^{2n+1} (2n+1-k)! (-\Lambda)^{k-1} \\
& \times \left\{ \begin{aligned} & \theta(k-2m-2p-1) \alpha_{k-2-2m-2p}(\Lambda) \\ & + \theta(2+k+2m+2p) E_{1-2m+2p+1+1+k}(\Lambda) \end{aligned} \right\}. \quad (60)
\end{aligned}$$

Therefore, Eqs. (53), (54), (56) - (60) constitute the analytical solution to the finite ribbon source problem.

G. Infinite Tube. From Table I and Fig. 6 the source is given by

$$q(\vec{r}') = q_0 \delta(z'-z) H(y', y_0) H(x', x_0). \quad (61)$$

Substituting Eq. (61) into Eq. (1) we obtain

$$\begin{aligned}
\frac{4\pi}{q_0} \phi(\vec{r}) = & \int_{-x_0}^{x_0} dx' \int_{-y_0}^{y_0} dy' \\
& \times \frac{\exp(-\sigma[(x'-x)^2 + (y'-y)^2]^{1/2})}{(x'-x)^2 + (y'-y)^2} \quad (62)
\end{aligned}$$

which is the finite ribbon problem, Eq. (35), for $z = 0$. Hence the solution to Eq. (62) is given by Eqs. (53), (54), (56) - (60) with $z = 0$.

H. Finite Box. From Table I and Fig. 7 the source is given by

$$q(\vec{r}) = q_0 H(z', z_0) H(y', y_0) H(x', x_0). \quad (63)$$

Substituting Eq. (63) into Eq. (1), we obtain

$$\begin{aligned}
\frac{4\pi}{q_0} \phi(\vec{r}) = & \int_{-x_0}^{x_0} dx' \int_{-y_0}^{y_0} dy' \int_{-z_0}^{z_0} dz' \\
& \times \frac{\exp(-\sigma[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{1/2})}{(x'-x)^2 + (y'-y)^2 + (z'-z)^2} \quad (64)
\end{aligned}$$

Following the analysis of the finite ribbon problem we integrate over x' and y' to obtain

$$\begin{aligned}
\frac{4\pi}{q_0} \phi(\vec{r}) = & \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} \frac{\Gamma(m+1/2)}{m!} \left(\frac{n}{p} \right) \int_{-z_0}^{z_0} dz' \\
& \times \left\{ \begin{aligned} & (-b_-^2)^n (-c_-^2)^p \left[\frac{1}{w_-^{2m+2p}(-x_0)} D_{-(2m+2p+1), 2n+2(\sigma w_-(-x_0))}^{(1)} \right. \\ & \quad \left. - \frac{1}{w_-^{2m+2p}(x_0)} D_{-(2m+2p+1), 2n+2(\sigma w_-(x_0))}^{(1)} \right] \\ & - (-b_+^2)^n (-c_+^2)^p \left[\frac{1}{w_+^{2m+2p}(-x_0)} D_{-(2m+2p+1), 2n+2(\sigma w_+(-x_0))}^{(1)} \right. \\ & \quad \left. - \frac{1}{w_+^{2m+2p}(x_0)} D_{-(2m+2p+1), 2n+2(\sigma w_+(x_0))}^{(1)} \right] \end{aligned} \right\} \quad (65)
\end{aligned}$$

where

$$\begin{aligned} b_z^2 &= (z'-z)^2 + (z' y_0 - y)^2 \\ c_z^2 &= (z' y_0 - y)^2 \\ w_z^2 (z' x_0) &= (z'-z)^2 + (z' x_0 - x)^2 + (z' y_0 - y)^2. \end{aligned} \quad (66)$$

The general integral of interest in Eq. (65) has the form

$$\begin{aligned} I_{n,m,p}^{+,z} &= (-1)^m \int_{-z_0}^{z_0} dz' \\ &\times \frac{[(z'-z)^2 + (z' y_0 - y)^2]^m (z' y_0 - y)^{2p}}{[(z'-z)^2 + (z' x_0 - x)^2 + (z' y_0 - y)^2]^{m+p}} \\ &\times D_{-(2m+2p+1), 2n+2}^{(1)} \\ &\times (u[(z'-z)^2 + (z' x_0 - x)^2 + (z' y_0 - y)^2]^{1/2}). \end{aligned} \quad (67)$$

Using this definition Eq. (65) becomes

$$\begin{aligned} \frac{4\pi}{q_0} \phi(\vec{r}) &= \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \frac{\Gamma(n+1/2)}{n!} \frac{\Gamma(m+1/2)}{m!} \\ &\times \binom{n}{p} (I_{n,m,p}^{--} - I_{n,m,p}^{+-} - I_{n,m,p}^{-+} + I_{n,m,p}^{++}). \end{aligned} \quad (68)$$

If we define the new variables

$$\begin{aligned} u^2(z') &\equiv (z'-z)^2 + d_{z,z}^2, \\ d_{z,z}^2 &\equiv (z' x_0 - x)^2 + (z' y_0 - y)^2, \\ e_z^2 &\equiv (z' x_0 - x)^2, \end{aligned} \quad (69)$$

then Eq. (67) takes the form

$$\begin{aligned} I_{n,m,p}^{+,z} &= \int_{-z_0}^{z_0} dz' \frac{[u^2(z') - e_z^2]^m}{u^{2m+2p}(z')} \\ &\times D_{-(2m+2p+1), 2n+2}^{(1)}(ou(z')). \end{aligned} \quad (70)$$

Using the binomial theorem, Eq. (70) can be expressed

$$\begin{aligned} I_{n,m,p}^{+,z} &= \sum_{l=0}^m \binom{m}{l} (-e_z^2)^l \int_{-z_0}^{z_0} dz' \frac{1}{u^{2p+2l}(z')} \\ &\times D_{-(2m+2p+1), 2n+2}^{(1)}(ou(z')). \end{aligned} \quad (71)$$

Changing variables using Eq. (69) we find

$$\frac{\partial z'}{\partial u} = \frac{1}{(1 + d_{z,z}^2/u^2)^{1/2}}, \quad (72)$$

which when substituted into Eq. (71) yields

$$\begin{aligned} I_{n,m,p}^{+,z} &= \sum_{l=0}^m \binom{m}{l} (-e_z^2)^l \int_{u(-z_0)}^{u(z_0)} du \\ &\times \frac{D_{-(2m+2p+1), 2n+2}^{(1)}(ou)}{u^{2p+2l} (1 - d_{z,z}^2/u^2)^{1/2}}. \end{aligned} \quad (73)$$

Now using the expansion for $1/(1 - d_{z,z}^2/u^2)^{1/2}$ given by Eq. (16), Eq. (73) becomes

$$\begin{aligned} I_{n,m,p}^{+,z} &= \frac{1}{\sqrt{\pi}} \sum_{l=0}^m \sum_{q=0}^{\infty} \frac{\Gamma(q+1/2)}{q!} \binom{m}{l} (-e_z^2)^l \\ &\times (-d_{z,z}^2)^q \cdot I_{n,m,p,q,l}^{+,z} - I_{n,m,p,q,l}^{+,z} \end{aligned} \quad (74)$$

where

$$I_{n,m,p,q,l}^{+,z} = \frac{1}{u(z_0)^{2p+2q+2l-1}} J_{n,m,p,q,l}^{+,z} \quad (75)$$

and

$$J_{n,m,p,q,l}^{+,z} \equiv \int_1^{\infty} dt \frac{D_{-(2m+2p+1), 2n+2}^{(1)}[ou(z_0)t]}{t^{2p+2q+2l}}. \quad (76)$$

Using the representation for $D_{-(2m+2p+1), 2n+2}^{(1)}(out)$ given by Eq. (60) and defining $\Lambda = ou(z_0)$, Eq. 76 can be written as

$$J_{n,m,p,q,l}^{+,z} = K_1 + K_2 + K_3 + K_4 + K_5 + K_6 \quad (77)$$

where

$$K_1 \equiv \frac{\theta(2n+1-2m-2p)}{(2n+1)!(2n+1-2m-2p)} (-\Lambda)^{2n+1} \hat{K}_1, \quad (78)$$

$$K_2 \equiv \frac{-\theta(2n+1-2m-2p)(-\Lambda)^{2n+1}}{(2n+1)!(2n+1-2m-2p)} \hat{K}_2, \quad (79)$$

$$K_3 \equiv \theta(2m + 2p - 1 - 2n)(|2n - 2m - 2p| - 1)\hat{K}_3, \quad (80)$$

$$K_4 \equiv -\theta(2m + 2p - 1 - 2n)(|2n - 2m - 2p| - 1)\hat{K}_4, \quad (81)$$

$$K_5 \equiv \frac{\theta(2n + 1)}{(2n + 1)!} \sum_{k=1}^{2n+1} (2n + 1 - k)! (-\Lambda)^{k-1} \times \theta(k - 2m - 2p - 1)\hat{K}_5, \quad (82)$$

$$K_6 \equiv \frac{\theta(2n + 1)}{(2n + 1)!} \sum_{k=1}^{2n+1} (2n + 1 - k)! (-\Lambda)^{k-1} \times \theta(2 + k + 2m + 2p)\hat{K}_6, \quad (83)$$

and the $\hat{K}_i$ are given by

$$\hat{K}_1 \equiv \int_1^\infty dt \frac{\alpha_{2n+1-2m-2p}(\Lambda t)}{t^{2p+2q+2\ell-2n-1}}, \quad (84)$$

$$\hat{K}_2 \equiv \int_1^\infty dt \frac{E_1(\Lambda t)}{t^{2p+2q+2\ell-2n-1}}, \quad (85)$$

$$\hat{K}_3 \equiv \int_1^\infty dt \frac{E_1(\Lambda t)}{t^{2p+2q+2\ell}} \quad (86)$$

$$\hat{K}_4 \equiv \int_1^\infty dt \frac{E_2|n-m-p|(\Lambda t)}{t^{2p+2q+2\ell}}, \quad (87)$$

$$\hat{K}_5 \equiv \int_1^\infty dt \frac{t^{k-1} \alpha_{k-2-2m-2p}(\Lambda t)}{t^{2p+2q+2\ell}}, \quad (88)$$

$$\hat{K}_6 \equiv \int_1^\infty dt \frac{E_{|2m+2p+1|+1}(\Lambda t)}{t^{2p+2q+2\ell}}. \quad (89)$$

These integrals are readily evaluated using the techniques developed in Ref. 2. In particular, since

$$\alpha_n(z) = \int_1^\infty dt t^n e^{-zt} = \frac{n!}{z^{n+1}} e^{-z} e_n(z), \quad (90)$$

$$e_n(z) = \sum_{k=0}^n \frac{z^k}{k!}, \quad (91)$$

Eq. (84) becomes

$$\begin{aligned} \hat{K}_1 &= \frac{(2n + 1 - 2m - 2p)!}{\Lambda^{2n+1-2m-2p}} \sum_{k=0}^{2n+1-2m-2p} \frac{\Lambda^k}{k!} \int_1^\infty dt \frac{t^{k+2m}}{t^{2q+2\ell}} e^{-\Lambda t} \\ &= \frac{(2n + 1 - 2m - 2p)!}{\Lambda^{2n+1-2m-2p}} \sum_{k=0}^{2n+1-2m-2p} \frac{\Lambda^k}{k!} \\ &\quad \times [\theta(k + 2m - 2q - 2\ell) \alpha_{k+2m-2q-2\ell}(\Lambda) + \theta(2q + 2\ell - 2m - k) E_{2q+2\ell-2m-k}(\Lambda)] . \end{aligned} \quad (92)$$

Equation (95) is readily evaluated to be

$$\hat{K}_2 = D_{-(2p+2q+2\ell-2n-1),1}^{(1)}(\Lambda) \quad (93)$$

using Eq. (51). Similarly,

$$\hat{K}_3 = D_{-(2p+2q+2\ell),1}^{(1)}(\Lambda) \quad (94)$$

$$E_{2s}(z) = \frac{\theta(2s-1)}{(2s-1)!} e^{-z} \sum_{r=0}^{2s-1} (2s-1-r)! (-z)^{r-1} + \frac{(-z)^{2s-1}}{(2s-1)!} E_1(z) \quad (95)$$

for $s > 0$, and

Using the expansion

$$E_0(z) = \frac{e^{-z}}{z} \quad (96)$$

for $s = 0$, the integral of Eq. (87) becomes

$$\begin{aligned} \hat{K}_4 &= \frac{\delta_{n-m-p,0}}{\Lambda} \int_1^\infty dt \frac{e^{-\Lambda t}}{t^{2p+2q+2\ell+1}} + \frac{\theta(2|n-m-p|-1)}{(2|n-m-p|-1)!} \sum_{r=0}^{2|n-m-p|-1} (2|n-m-p|-1-r)! (-\Lambda)^{r-1} \\ &\times \int_1^\infty dt \frac{e^{-\Lambda t} t^{r-1}}{t^{2p+2q+2\ell}} + \frac{\theta(2|n-m-p|-1)}{(2|n-m-p|-1)!} (-\Lambda)^{2|n-m-p|-1} \int_1^\infty dt \frac{E_1(\Lambda t) t^{2|n-m-p|-1}}{t^{2p+2q+2\ell}} \\ &= \frac{\delta_{n-m-p,0}}{\Lambda} E_{2p+2q+2\ell+1}(\Lambda) + \frac{\theta(2|n-m-p|-1)}{(2|n-m-p|-1)!} \sum_{r=0}^{2|n-m-p|-1} (2|n-m-p|-1-r)! (-\Lambda)^{r-1} \\ &\times [\theta(r-1-2p-2q-2\ell) \alpha_{r-1-2p-2q-2\ell}(\Lambda) + \theta(2p+2q+2\ell-r+1) E_{2p+2q+2\ell-r-1}(\Lambda)] \\ &+ \frac{\theta(2|n-m-p|-1)}{(2|n-m-p|-1)!} D_{2|n-m-p|-1-2p-2q-2\ell,1}^{(1)}(\Lambda) \end{aligned} \quad (97)$$

where we have used Eqs. (56) - (60).

Equation (88) is evaluated similar to Eq. (84)

to be

$$\begin{aligned} \hat{K}_5 &= \frac{(k-2-2m-2p)!}{\Lambda^{k-2-2m-2p}} \sum_{r=0}^{k-2-2m-2p} \frac{\Lambda^r}{r!} \\ &\times [(r+1-k+2m-2q-2\ell) \alpha_{k-1+r-2p-2q-2\ell}(\Lambda) + \theta(2q+2\ell-2m+k-1-r) E_{2p+2q+2\ell-k+1-r}(\Lambda)] \end{aligned} \quad (98)$$

and finally, Eq. (89) is evaluated using Eqs. (95) and (96) as

$$\begin{aligned}
 \hat{K}_6 = & \frac{\theta(|2m+2p+1|+k)}{(|2m+2p+1|+k)!} \sum_{r=0}^{|2m+2p+1|+k} (|2m+2p+1|+k-r)! (-\Lambda)^{r-1} \int_1^{\infty} dt \frac{t^{r-1} e^{-\Lambda t}}{t^{2p+2q+2\ell}} \\
 & + \frac{(-\Lambda)^{|2m+2p+1|+k}}{(|2m+2p+1|+k)!} \int_1^{\infty} dt \frac{t^{|2m+2p+1|+k} E_1(\Lambda t)}{t^{2p+2q+2\ell}} \\
 = & \frac{\theta(|2m+2p+1|+k)}{(|2m+2p+1|+k)!} \sum_{r=0}^{|2m+2p+1|+k} (|2m+2p+1|+k-r)! (-\Lambda)^{r-1} \\
 & \times [\theta(r-1-2p-2q-2\ell) \alpha_{r-1-2p-2q-2\ell}(\Lambda) + \theta(2p+2q+2\ell+1-r) E_{2p+2q+2\ell+1-r}(\Lambda)] \\
 & + \frac{(-\Lambda)^{|2m+2p+1|+k}}{(|2m+2p+1|+k)!} D_{|2m+2p+1|+k-2p-2q-2\ell,1}^{(1)}(\Lambda) \quad (99)
 \end{aligned}$$

This completes the analytic solution for the finite box problem. Equations (92)-(99) when substituted into Eqs. (78)-(83) for K_ℓ , $\ell = 1, 2, \dots, 6$ define the function $J_{n,m,p,q,\ell}^{\pm}$ which is related to the function $I_{n,m,p}^{\pm}$ when substituted into Eq. (68) yields the solution for the flux $\phi(\vec{r})$. All the functions involved can be evaluated numerically using the techniques developed in Ref. 2.

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