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COMPARISON OF
ORTHOGONAL
CURVILINEAR
COORDINATE SYSTEMS
FOR MINIMUM ERROR

BDX-613-184

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Final Report

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ABSTRACT

Nineteen types of curvilinear coordinate systems are examined to assess their usefulness in designing and constructing precision measuring machines. Ten systems are shown to have smaller errors in locating a point in space than those in an orthogonal rectangular system for certain parameter values and in certain locations in space. Three systems are shown to have errors exactly equal to those in the rectangular system. Coordinate systems constructed by rotating a basic family of two dimensional curves about one of the axes do not appear to produce smaller errors than in the original system.

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SUMMARY

Measuring machines that could be based on some orthogonal curvilinear coordinate system other than the usual orthogonal rectangular system might contain smaller inherent errors resulting from the shape of the coordinate surfaces. To investigate this possibility, 19 types of orthogonal curvilinear coordinate systems were examined and the errors from these systems compared to corresponding errors in an orthogonal rectangular system.

Errors analyzed were those involved in locating or mislocating a point in space resulting from small errors in the magnitudes of the three coordinate values. The analysis included definition of the coordinate systems, derivation of the transformation equations, error analysis with a computer, and synthesis of a basic mechanism for generating one of the coordinate systems.

It was concluded that 10 systems have errors smaller than those in an orthogonal rectangular system for certain values of the parameters and in certain x, y, z locations in space. Three other systems have errors exactly equal to those in the rectangular system for all locations in space.

The commonly known cylindrical and spherical systems do not have inherent errors smaller than the rectangular system, and, in the general case, the errors are larger. Constructing a spherical type system from a cylindrical type by rotating the basic family of curves about one of the axes does not appear to produce errors smaller than in the rectangular system if the cylindrical system does not originally contain smaller errors.

Future work in this area of investigation should begin with design and fabrication of real machine mechanisms to generate some of the curvilinear coordinates for study of practical problems.

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INTRODUCTION

The size of manufactured products is frequently checked by precision coordinate measuring machines. These machines are designed around a particular coordinate system, the most common being the orthogonal rectangular system. In this system, the linkages, slides and pivots are arranged so that the sensing probe (or the part) can be moved independently in any one of three mutually orthogonal directions (x , y , z coordinates). Other common coordinate systems used in precision measuring machines are the cylindrical system and the spherical system.

Although these three are the most commonly used coordinate systems, essentially an infinite number could be used in measuring machines. In general, these systems are known as curvilinear coordinate systems and include curved coordinate surfaces such as paraboloids, ellipsoids and hyperboloids rather than the usual planes and cylinders. The most general systems are the non-orthogonal versions in which the intersection of the surfaces is constant at a definite angle other than a right angle.

The extreme complexity of the mathematics of non-orthogonal types caused these systems to be excluded from consideration in this study. The work reported was confined to the evaluation of selected orthogonal coordinate systems for possible application to precision measuring machines.

The basis of comparison for each of the 19 types of orthogonal coordinate systems considered was the error in locating a point in space because of small errors in the magnitudes of the three coordinate values. The study was made on a theoretical basis to determine whether any of these systems would result in a smaller point location error than the commonly used rectangular system.

The complete analysis of an actual measuring machine, taking into account mechanical and environmental properties, is beyond the scope of this project.

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DISCUSSION

COORDINATE SYSTEM THEORY

Non-Orthogonal Curvilinear Systems

In general, a coordinate system consists of three families of surfaces (trajectories) with each surface of a family intersecting all other surfaces in the other two families at a constant angle θ other than 90 degrees over the entire grid (Figure 1). Each of the surfaces in a family are identified by a consecutive number determined from a mathematical scheme that is purely arbitrary. These coordinate identification numbers can then be used to uniquely locate a point in space, although they do not directly indicate the distances to that point from the origin. In Figure 1, the two families of curves are identified by the two sets of numbers designated u and v , which would be analogous to x and y in the familiar rectangular system.

If the distances to a point are required, these can be determined mathematically from the arbitrary but known scheme of number assignment and the equations defining the shapes of the coordinate surfaces.

The equations defining the coordinate surfaces are usually in terms of the familiar orthogonal rectangular coordinates x , y , z , thus introducing the concept of transformation equations. Transformation equations allow the coordinates of a point in space to be computed in one coordinate system when its coordinates are known in another system. For example, the coordinates of a point in Figure 1 are related to those in an orthogonal rectangular system by the following general transformation equations:

$$\begin{array}{ll} x = f_1(u, v) & u = F_1(x, y) \\ y = f_2(u, v) & v = F_2(x, y) \end{array} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

To evaluate Equations (1) or (2), the functions f and F must be known, and would be available since the coordinate system surfaces would be ultimately defined in terms of rectangular coordinates.

Although the procedure for constructing families of non-orthogonal (oblique) trajectories is very elementary, the mathematics usually leads to equations that are very difficult to evaluate or use in other relationships.

Figure 1. Typical Non-Orthogonal Curvilinear Coordinate System in Two dimensions. The angle θ between tangents is constant over the entire net.

However, the basic technique, together with an example, is given in Appendix A. Because of this complexity, evaluation of non-orthogonal cases is considered beyond the scope of this report and will be left for possible future study.

Orthogonal Curvilinear Systems

If the intersecting coordinate surfaces in a curvilinear coordinate system cross each other at right angles everywhere in the grid, the system is said to be orthogonal. Such a system is shown in Figure 2 where the third dimension is again suppressed for convenience. The identification numbers assigned to the families of surfaces are determined in much the same arbitrary manner as described in the non-orthogonal systems.

In three dimensions, the rigorous method becomes prohibitively complex, involving the simultaneous solution of three non-linear differential equations. Although no further analysis of this approach will be given, the fundamental form of the equations appears in Appendix B where the general test for orthogonality is described. An easier way of forming three-dimensional orthogonal systems is by taking a two-dimensional orthogonal system and (1) translating it in the direction of the z axis making a cylindrical type system or (2) rotating it about the x or y axis making a spherical type system. Throughout this report the terms "cylindrical type" and "spherical type" refer to a general classification, rather than the usual cylindrical and polar systems described in most texts.

In two dimensions, orthogonal trajectories of a given system of curves are found by replacing dy/dx in the differential equation of the given system by $-dx/dy$ and solving the resulting differential equation. In addition, the parameter in the given system of curves defining each member must be eliminated prior to the substitution of the negative reciprocal of the slope.

As an example, consider the family of parabolas

$$y = vx^2 \tag{3}$$

where v is the parameter describing each member of the family. The differential equation for this family is

$$\frac{dy}{dx} = 2vx$$

Eliminating the parameter v between these two equations gives

$$\frac{dy}{dx} = \frac{2y}{x}$$

The differential equation of the orthogonal set is found by replacing $\frac{2y}{x}$ by its negative reciprocal giving

$$\frac{dy}{dx} = -\frac{x}{2y}$$

Separating the variables and integrating, the general solution is obtained in the form

$$x^2 + 2y^2 = u \quad (4)$$

where u is the arbitrary constant of integration. This equation clearly represents a family of ellipses, with u being the parameter defining the members (Figure 11).

The transformation equations may now be obtained by algebraic manipulation of Equations (3) and (4) and will be of the form shown in Equations (1) and (2). They are

$$\left. \begin{aligned} v &= \frac{y}{x} \\ u &= x^2 + 2y^2 \end{aligned} \right\} \quad (5)$$

$$\left. \begin{aligned} x &= \frac{1}{2v} \sqrt{-1 + \sqrt{1 + 8uv^2}} \\ y &= \frac{1}{4v} (-1 + \sqrt{1 + 8uv^2}) \end{aligned} \right\} \quad (6)$$

If it were not known that Equations (5) and (6) represented orthogonal sets of curves, a test for orthogonality may be made using Equation (6) (See Appendix B).

As mentioned previously, three dimensional systems may be readily constructed from two-dimensional systems by either a translation (cylindrical) or a rotation (spherical). To make the system represented by Equation (6) a cylindrical system, it is only necessary to add the z dimension as follows:

Figure 2. Typical Orthogonal Curvilinear Coordinate System in Two Dimensions. The angle between tangents is 90° at all points in the net.

$$\begin{aligned}
 x &= \frac{1}{2v} \sqrt{-1 + \sqrt{1 + 8uv^2}} \\
 y &= \frac{1}{4v} (-1 + \sqrt{1 + 8uv^2})
 \end{aligned} \tag{7}$$

$$z = w$$

The three coordinate parameters in the curvilinear system are now u , v , w . The u surfaces are cylindrical ellipses, the v surfaces are cylindrical parabolas, and the w surfaces are planes parallel to the xy plane.

To make it into a spherical system, the curves of Equation (6) are rotated about the x axis which is then relabeled the z axis. The new x and y axes are the cosine and sine projections of the original y axis. The results

$$\begin{aligned}
 x &= \frac{1}{4v} (-1 + \sqrt{1 + 8uv^2}) \cos w \\
 y &= \frac{1}{4v} (-1 + \sqrt{1 + 8uv^2}) \sin w \\
 z &= \frac{1}{2v} \sqrt{-1 + \sqrt{1 + 8uv^2}}
 \end{aligned} \tag{8}$$

The u surfaces are now ellipsoids, the v surfaces are doughnut-shaped paraboloids, and the w surfaces are planes passing through the z axis. A similar set of equations may be constructed by revolving the curves of Equation (6) about the y axis.

ERROR ANALYSIS THEORY

The type of error used in the analysis of the various coordinate systems was the error in the location of a point in space due to small errors in the measured values of the coordinate parameters (u , v , w).

Consider a general curvilinear coordinate system having coordinates u, v, w and related to the ordinary rectangular coordinate system (x, y, z) by the transformation equations

$$\begin{aligned} x &= f_1(u, v, w) & u &= F_1(x, y, z) \\ y &= f_2(u, v, w) & v &= F_2(x, y, z) \\ z &= f_3(u, v, w) & w &= F_3(x, y, z) \end{aligned} \quad (9)$$

which are known. Let

$$r = xi + yj + zk \quad (10)$$

be the position vector to the point P as shown in Figure 3. The i, j, k are unit vectors in the x, y, z system. If a small change du , which will be viewed as the error in u , is introduced into Equation (10), the terminal point of r will move through a distance dr in the direction of the tangent to the u curve. A similar movement will result from errors dv and dw . Now if all three errors are introduced simultaneously the resultant value of dr is given by

$$dr = \frac{\partial r}{\partial u} du + \frac{\partial r}{\partial v} dv + \frac{\partial r}{\partial w} dw \quad (11)$$

where the partial derivatives may be evaluated from Equations (9) and (10). Since dr in Equation (11) is a vector, its magnitude ds is found from the dot product of itself in the relationship

$$ds^2 = dr \cdot dr \quad (12)$$

The value ds is the distance a point will shift in space when its coordinates are altered a tiny amount, and thus is the desired error in the location of this point. It remains only to evaluate ds and express it in terms of known quantities.

For convenience let

$$\alpha_1 = \frac{\partial r}{\partial u} \quad \alpha_2 = \frac{\partial r}{\partial v} \quad \alpha_3 = \frac{\partial r}{\partial w} \quad (13)$$

Figure 3. Error in Position of Point P.

The evaluation of Equation (12) then becomes

$$\begin{aligned}
 ds^2 = & \alpha_1 \cdot \alpha_1 du^2 + \alpha_1 \cdot \alpha_2 du dv + \alpha_1 \cdot \alpha_3 du dw \\
 & + \alpha_2 \cdot \alpha_1 dv du + \alpha_2 \cdot \alpha_2 dv^2 + \alpha_2 \cdot \alpha_3 dv dw \\
 & + \alpha_3 \cdot \alpha_1 dw du + \alpha_3 \cdot \alpha_2 dw dv + \alpha_3 \cdot \alpha_3 dw^2
 \end{aligned} \tag{14}$$

The nine dot products of the α terms are the components of a covariant tensor of rank two known as the metric tensor. Equation (14) is perfectly general and applies to any system whether orthogonal or non-orthogonal. Due to symmetry, the equation really only contains six terms as follows:

$$\begin{aligned}
 ds^2 = & \alpha_1 \cdot \alpha_1 du^2 + \alpha_2 \cdot \alpha_2 dv^2 + \alpha_3 \cdot \alpha_3 dw^2 \\
 & + 2 \alpha_1 \cdot \alpha_2 du dv + 2 \alpha_1 \cdot \alpha_3 du dw + 2 \alpha_2 \cdot \alpha_3 dv dw
 \end{aligned} \tag{15}$$

A further reduction of Equation (14) may be achieved since only orthogonal systems are of interest. This is done by noting that in an orthogonal system the vector α_1 is perpendicular to the vector α_2 and therefore the dot product $\alpha_1 \cdot \alpha_2$ will be zero. Similarly for the other two dot products, $\alpha_1 \cdot \alpha_3 = 0$ and $\alpha_2 \cdot \alpha_3 = 0$. Thus Equation (15) reduces to

$$ds^2 = \alpha_1 \cdot \alpha_1 du^2 + \alpha_2 \cdot \alpha_2 dv^2 + \alpha_3 \cdot \alpha_3 dw^2 \tag{16}$$

From vector analysis, when a vector is dotted against itself there results the square of the absolute magnitude of the vector. Denoting this magnitude by the symbol h , the dotted products become

$$\begin{aligned}
 \alpha_1 \cdot \alpha_1 &= \left| \frac{\partial \mathbf{r}}{\partial u} \right|^2 = h_1^2 \\
 \alpha_2 \cdot \alpha_2 &= \left| \frac{\partial \mathbf{r}}{\partial v} \right|^2 = h_2^2 \\
 \alpha_3 \cdot \alpha_3 &= \left| \frac{\partial \mathbf{r}}{\partial w} \right|^2 = h_3^2
 \end{aligned} \tag{17}$$

reducing Equation (16) to

$$ds^2 = h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2 \tag{18}$$

It should be noted that evaluation of the h terms from Equation (17) is given by the following.

$$\begin{aligned} h_1^2 &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ h_2^2 &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \\ h_3^2 &= \left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2 \end{aligned} \quad (19)$$

which in turn is evaluated from the known transformation Equation (9).

Although Equations (18) and (19) constitute the solution for the desired error, an additional refinement is necessary to facilitate the analysis. When the coordinate parameter u, v, or w, is an angular number, the error is usually considered to be a fixed amount regardless of the size of the angle. In this case, Equation (18) is satisfactory since the errors du, dv, dw are levels of error and do not depend on the magnitude of the coordinates. However, when the coordinate parameter is a linear type number, the error is usually considered to be proportional to the size of the coordinate number. For this case, Equation (18) may be modified to be

$$ds^2 = h_1^2 u^2 \left(\frac{du}{u}\right)^2 + h_2^2 v^2 \left(\frac{dv}{v}\right)^2 + h_3^2 w^2 \left(\frac{dw}{w}\right)^2 \quad (20)$$

where the errors are now du/u, dv/v, and dw/w, and are expressed as the fractional part of the level such as parts per million, percentage in decimal form, etc. In the actual survey of the coordinate systems, the coordinate parameters were mixed so that some of the terms were of the kind in Equation (18) and some were of the kind in Equation (20). Furthermore, in a few of the systems, where it was felt that the angular type of coordinate number could conceivably have either the angular or the linear type of error, an analysis was made for both possibilities.

In the overall survey, each coordinate system was compared to a rectangular one to determine if it contained smaller errors, i. e. smaller values of ds, than in the rectangular system. This comparison was made by forming the ratio R of the error in the curvilinear system to that in the rectangular system. Using the form of error in Equation (20) this became

$$R = \sqrt{\frac{h_1^2 u^2 \left(\frac{du}{u}\right)^2 + h_2^2 v^2 \left(\frac{dv}{v}\right)^2 + h_3^2 w^2 \left(\frac{dw}{w}\right)^2}{1^2 x^2 \left(\frac{dx}{x}\right)^2 + 1^2 y^2 \left(\frac{dy}{y}\right)^2 + 1^2 z^2 \left(\frac{dz}{z}\right)^2}} \quad (21)$$

where the h terms in the rectangular system were replaced by their values of unity. To properly compare general curvilinear coordinate systems with a rectangular one, it is necessary to view Equation (21) in terms of two physically different measuring machines, where one machine, based on the x, y, z coordinates, is represented by the denominator and the other machine, based on the u, v, w coordinates, is represented by the numerator. Since different machines are involved, the error terms $dx/x, dy/y, dz/z$ are completely independent and unrelated to the other system error terms $du/u, dv/v, dw/w$. It is essential to note this independence because if these errors are viewed as dependent, then, by means of the transformation equations (9), it can be shown that the numerator can be derived from the denominator making R unity for all systems. These two sets of errors are dependent only if the two coordinate systems are superimposed and interconnected. In any given measuring machine, however, there is only one coordinate system, not two that are interconnected. Therefore, the errors of a system must be considered to be independent of those in all other systems.

All six of the error terms in Equation (21) could, in the general case, have a different magnitude. Allowing all six to be different introduces an unnecessary complication, so it was assumed throughout the entire analysis that all linear type errors would be of the same magnitude and all angular type errors would be of the same magnitude. In those systems then where all errors are of the linear type, the final expression for R becomes

$$R = \sqrt{\frac{h_1^2 u^2 + h_2^2 v^2 + h_3^2 w^2}{x^2 + y^2 + z^2}} \quad (22)$$

In the remaining cases where the errors are both linear and angular, the final expression for R is of the form (for a system where the u and v coordinates are angular)

$$R = \sqrt{\frac{h_1^2 \tau^2 + h_2^2 \tau^2 + h_3^2 w^2}{x^2 + y^2 + z^2}} \quad (23)$$

where $du = dv$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{dw}{w}$$

$$\tau = \frac{du}{dx/x} = \frac{dv}{dx/x}$$

If the w coordinate was angular and the others linear, the terms would be rearranged accordingly. With regard to these errors, it is evident from Equation (23) that it is only necessary to specify the ratio τ of the angular to the linear error and not the levels of the errors themselves.

Equation (22) and that shown typically by Equation (23) are the final error equations used in the survey. They represent the ratio of the error in the location of a point expressed in curvilinear coordinates to the error in the location of the same point in space expressed in rectangular coordinates. Maintaining the same point was achieved merely by expressing all quantities except τ in terms of x, y, z , which is readily done by means of the transformation Equation (9). A complete analysis of a system would then include evaluating Equations (22) and (23) at every point in space, or at least at a sufficient quantity to determine all significant trends.

COORDINATE SYSTEMS IN SURVEY

A total of 19 coordinate systems, all orthogonal, were examined in the survey. They are listed in Table 1 along with the code names used in the computer programs (Appendix C). Coordinate systems 4 and 13, are basically the same types as 3 and 12 respectively. The only difference is the different scheme for choosing the coordinate parameters. Of the remaining 17 systems, seven are spherical types constructed by rotating the corresponding cylindrical types. These rotated systems are 2, 5, 7, 10, 14, 16, and 17. Although these seven are treated as separate systems different from their cylindrical counterparts, they are basically the same in the sense that the same types of curves are used in both sets. Since the basic curves used in 15 are the same as in 6, there were then nine fundamental types of orthogonal trajectories examined. They are:

1. Circles and radial lines,
2. Self-orthogonal confocal parabolas,
3. Self-orthogonal circles,
4. Ellipses and parabolas,
5. Confocal ellipses and hyperbolas,
6. Cardioids,
7. Isosceles hyperbolas,
8. Elliptical cones and spheres, and
9. Ellipsoids and elliptical hyperboloids.

Systems 18 and 19 are unique in that they were not generated as cylindrical or spherical types from two-dimensional curves but were generated as three-dimensional curvilinear surfaces directly from the equations. From this viewpoint they might be loosely termed true three-dimensional coordinate systems.

Table 1. Coordinate Systems Examined in Survey

No.	Computer Code	Name	Figure
1	CYL	Cylindrical	4
2	SPH	Spherical	5
3	PARCYL	Parabolic Cylindrical	6
4	PARALT	Parabolic Cylindrical, alternate	7
5	PARLOD	Paraboloidal	8
6	CYLCIR	Coaxial Cylindrical Circles	9
7	COXSPH	Coaxial Spheroidal	10
8	ELLPAR	Elliptical Parabolic Cylindrical	11
9	HYPER	Isosceles Hyperbolic Cylindrical	12
10	HYPSPH	Isosceles Hyperboloidal	13
11	CARD	Cardioidic Cylindrical	14
12	BIPOL	Bipolar	15
13	BIPALT	Bipolar, alternate	16
14	TOROID	Toroidal	17
15	ELLCYL	Elliptic Cylindrical	18
16	PROSPH	Prolate Spheroidal	19
17	OBLSPH	Oblate Spheroidal	20
18	CONCAL	Conical	21
19	ELPSOD	Ellipsoidal	22

An abbreviated sketch of each of the 19 systems is shown in Figures 4 through 22 and the corresponding equations are given on adjacent pages. All sketches are in two dimensions for simplicity except 21 and 22 which are shown isometrically for lack of an easier way. The coordinates of all systems are designated by (u, v, w) , some of which may be of the linear (L) type, others the angular (A) type. To readily distinguish between the two types, the L and A designations, such as (L, A, L) , are also included with the description of each system and match the (u, v, w) coordinates by the order shown.

Figure 4. Cylindrical Coordinates, System 1 (CYL). The traces of the vertical cylinders u and the vertical planes v on the xy plane are shown. The third set of surfaces are planes w parallel to the xy plane.

Equations for Cylindrical Coordinates (Figure 4)

(u, v, w)
(L, A, L)

$$\begin{aligned}x &= u \cos v \\y &= u \sin v \\z &= w\end{aligned}$$

$$u = \sqrt{x^2 + y^2}$$

$$v = \tan^{-1} \frac{y}{x}$$

$$w = z$$

$$h_1^2 = 1$$

$$h_2^2 = u^2$$

$$h_3^2 = 1$$

Figure 5. Spherical Coordinates, System 2 (SPH). The above curves are revolved about the y axis, which is then relabeled the z axis, creating spheres u and cones v. The third set of surfaces are planes w through the z axis.

Equations for Spherical Coordinates (Figure 5)

(u, v, w)
(L, A, A)

$$\begin{aligned}x &= u \cos v \cos w \\y &= u \cos v \sin w \\z &= u \sin v\end{aligned}$$

$$\begin{aligned}u &= \sqrt{x^2 + y^2 + z^2} \\v &= \tan^{-1} \frac{z}{\sqrt{x^2 + y^2}} \\w &= \tan^{-1} \frac{y}{x}\end{aligned}$$

$$h_1^2 = 1$$

$$h_2^2 = u^2$$

$$h_3^2 = u^2 \cos^2 v$$

Figure 6. Parabolic Cylindrical Coordinates, System 3 (PARCYL). Traces of the parabolic cylinders u and v on the xy plane are shown. The third set of surfaces are planes w parallel to the xy plane.

Equations for Parabolic Cylindrical Coordinates (Figure 6)

(u, v, w)
(L, L, L)

$$\begin{aligned}x &= 1/2 (u^2 - v^2) \\y &= u v \\z &= w\end{aligned}$$

$$u = \frac{y}{\sqrt{-x + \sqrt{x^2 + y^2}}}$$

$$v = \sqrt{-x + \sqrt{x^2 + y^2}}$$

$$w = z$$

$$h_1^2 = u^2 + v^2$$

$$h_2^2 = u^2 + v^2$$

$$h_3^2 = 1$$

Figure 7. Parabolic Cylindrical Coordinates, Alternate, System 4 (PARALT). This system differs from that in Figure 6 only in the parameter number. Here u equals the square of the value for the u used in Figure 6. The w planes are unaltered and are parallel to the xy plane.

Equations for Parabolic Cylindrical Coordinates, alternate (Figure 7)

(u, v, w)

(L, L, L)

$$x = (u-v)/2$$

$$y = \sqrt{u v}$$

$$z = w$$

$$u = x + \sqrt{x^2 + y^2}$$

$$v = \sqrt{x^2 + y^2} - x$$

$$w = z$$

$$h_1^2 = \left(1 + \frac{v}{u}\right) / 4$$

$$h_2^2 = \left(1 + \frac{u}{v}\right) / 4$$

$$h_3^2 = 1$$

Figure 8. Paraboloidal Coordinates, System 5 (PARLOD). The above curves are revolved about the x axis, which is then relabeled the z axis, creating the paraboloids u and v. The third set of surfaces are planes w through the z axis.

Equations for Paraboloidal Coordinates (Figure 8)

(u, v, w)
(L, L, A)

$$x = uv \cos w$$

$$y = uv \sin w$$

$$z = (u^2 - v^2)/2$$

$$u = \frac{\sqrt{x^2 + y^2}}{v}$$

$$v = \sqrt{-z + \sqrt{x^2 + y^2 + z^2}}$$

$$w = \tan^{-1} \frac{y}{x}$$

$$h_1^2 = u^2 + v^2$$

$$h_2^2 = u^2 + v^2$$

$$h_3^2 = u^2 v^2$$

Figure 9. Coaxial Cylindrical Circle Coordinates, System 6 (CYLCIR). The traces of the u and v cylinders on the xy plane are shown. The third set of surfaces are planes w parallel to the xy plane.

Equations for Coaxial Cylindrical Circle Coordinates (Figure 9)

(u, v, w)

(L, L, L)

$$x = \frac{2 u v^2}{u^2 + v^2}$$

$$y = \frac{2 v u^2}{u^2 + v^2}$$

$$z = w$$

$$u = \frac{x^2 + y^2}{2x}$$

$$v = \frac{x^2 + y^2}{2y}$$

$$w = z$$

$$h_1^2 = \frac{4 v^4}{(u^2 + v^2)^2}$$

$$h_2^2 = \frac{4 u^4}{(u^2 + v^2)^2}$$

$$h_3^2 = 1$$

Figure 10. Coaxial Spheroidal Coordinates, System 7 (COXSPH). The above curves are revolved about the x axis, which is then relabeled the z axis, creating the toroids v and the spheres u. The third set of surfaces are planes w through the z axis.

Equations for Coaxial Spheroidal Coordinates (Figure 10)

(u, v, w)
(L, L, A)

$$x = \frac{2 v u^2}{u^2 + v^2} \cos w$$

$$y = \frac{2 v u^2}{u^2 + v^2} \sin w$$

$$z = \frac{2 u v^2}{u^2 + v^2}$$

$$u = \frac{1}{2z} \left(\frac{y^2}{\sin^2 w} + z^2 \right)$$

$$v = \frac{uz}{y} \sin w$$

$$w = \tan^{-1} \frac{y}{x}$$

$$h_1^2 = \frac{4 v^4}{(u^2 + v^2)^2}$$

$$h_2^2 = \frac{4 u^4}{(u^2 + v^2)^2}$$

$$h_3^2 = \frac{4 v^2 u^4}{(u^2 + v^2)^2}$$

Figure 11. Elliptical Parabolic Cylindrical Coordinates, System 8 (ELLPAR). The traces of the elliptical cylinders u and the parabolic cylinders v on the xy axis are shown. The third set of surfaces are planes w parallel to the xy plane.

Equations for Elliptical Parabolic Cylindrical Coordinates (Figure 11)

(u, v, w)

(L, L, L)

$$x = \frac{1}{2v} \sqrt{A-1} \quad A = \sqrt{1 + 8uv^2}$$

$$y = \frac{A-1}{4v}$$

$$z = w$$

$$u = x^2 + 2y^2$$

$$v = \frac{y}{x^2}$$

$$w = z$$

$$h_1^2 = \frac{1}{4x^2 A^2} + \frac{v^2}{A^2}$$

$$h_2^2 = \left(\frac{u}{Axv} - \frac{x}{v} \right)^2 + \left(\frac{2u}{A} - x^2 \right)^2$$

$$h_3^2 = 1$$

Figure 12. Isosceles Hyperbolic Cylindrical Coordinates, System 9 (HYPER). The traces of the hyperbolic cylinders u and v on the xy plane are shown. The third set of surfaces are planes w parallel to the xy plane.

Equations for Isosceles Hyperbolic Cylindrical Coordinates (Figure 12)

(u, v, w)

(L, L, L)

$$x = \sqrt{(A + u)/2} \quad A = \sqrt{u^2 + 4v^2}$$

$$y = \frac{v}{\sqrt{(A + u)/2}}$$

$$z = w$$

$$u = x^2 - y^2$$

$$v = xy$$

$$w = z$$

$$h_1^2 = \frac{1}{4A}$$

$$h_2^2 = \frac{v^2 + \left(A - \frac{v^2}{x^2}\right)^2}{A^2 x^2}$$

$$h_3^2 = 1$$

Figure 13. Isosceles Hyperboloidal Coordinates, System 10 (HYPSPH). The above curves are revolved about the y axis, which is then relabeled the z axis, creating the hyperboloids of revolution u and v. The third set of surfaces are planes w through the z axis.

Equations for Isosceles Hyperboloidal Coordinates (Figure 13)

(u, v, w)

(L, L, A)

$$x = \sqrt{(A+u)/2} \cos w$$

$$A = \sqrt{u^2 + 4v^2}$$

$$y = \sqrt{(A+u)/2} \sin w$$

$$z = \frac{v}{\sqrt{(A+u)/2}}$$

$$u = x^2 + y^2 - z^2$$

$$v = z \sqrt{x^2 + y^2}$$

$$w = \tan^{-1} \frac{y}{x}$$

$$h_1^2 = \frac{x^2 + y^2 + z^2}{4A^2}$$

$$h_2^2 = \frac{4v^2}{A^2 (A+u)^2} \left[x^2 + y^2 + \left(\frac{A}{z} - z \right)^2 \right]$$

$$h_3^2 = x^2 + y^2$$

Figure 14. Cardioidic Cylindrical Coordinates, System 11 (CARD). The traces of the cardioid cylinders u and v on the xy plane are shown. The third set of surfaces are planes w parallel to the xy plane.

Equations for Cardioidic Cylindrical Coordinates (Figure 14) (u, v, w)
 (L, L, L)

$$x = \frac{2 uv (u - v)}{(u + v)^2}$$

$$y = \frac{4 uv \sqrt{uv}}{(u + v)^2}$$

$$z = w$$

$$u = \frac{x^2 + y^2}{-x + \sqrt{x^2 + y^2}}$$

$$v = \frac{x^2 + y^2}{x + \sqrt{x^2 + y^2}}$$

$$w = z$$

$$h_1^2 = \frac{4 v^3}{(u + v)^3}$$

$$h_2^2 = \frac{4 u^3}{(u + v)^3}$$

$$h_3^2 = 1$$

Figure 15. Bipolar Coordinates, System 12 (BIPOL). The traces of the cylinders u and v on the xy plane are shown. The third set of surfaces are planes w parallel to the xy plane.

Equations for Bipolar Coordinates (Figure 15)

(u, v, w)
(A, L, L)

$$x = \frac{a \sinh v}{\cosh v - \cos u}$$

$$y = \frac{a \sin u}{\cosh v - \cos u}$$

$$z = w$$

$$u = \tan^{-1} \frac{2ay}{x^2 + y^2 - a^2}$$

$$v = \tanh^{-1} \frac{2ax}{x^2 + y^2 + a^2}$$

$$w = z$$

$$h_1^2 = \frac{a^2}{(\cosh v - \cos u)^2}$$

$$h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}$$

$$h_3^2 = 1$$

Figure 16. Bipolar Coordinates Alternate, System 13 (BIPALT). The traces of the cylinders u and v on the xy plane are shown. The third set of surfaces are planes w parallel to the xy plane. This system differs from that in Figure 15 only in the parameter numbers.

Equations for Bipolar Coordinates, Alternate (Figure 16)

(u, v, w)
(L, L, L)

$$x = \frac{a^2}{u - vA}$$

$$A = \sqrt{\frac{u^2 - a^2}{v^2 + a^2}}$$

$$y = \frac{a^2}{\frac{u}{A} - v}$$

$$z = w$$

$$u = \frac{x^2 + y^2 + a^2}{2x}$$

$$v = \frac{x^2 + y^2 - a^2}{2y}$$

$$w = z$$

$$h_1^2 = \frac{x^4}{CBa^4} \left[(uv - AB)^2 + a^4 \right]$$

$$h_2^2 = \frac{x^4 A^4}{CBa^4} \left[(uv - AB)^2 + a^4 \right]$$

$$h_3^2 = 1$$

$$B = v^2 + a^2$$

$$C = u^2 - a^2$$

Figure 17. Toroidal Coordinates, System 14 (TOROID). The above curves are revolved about the y axis, which is then relabeled the z axis, creating the toroids v and the spheres u . The third set of curves are planes w through the z axis.

Equations for TOROIDAL Coordinates (Figure 17) (u, v, w)
 (A, L, A)

$$x = \frac{a \sinh v \cos w}{\cosh v - \cos u}$$

$$y = \frac{a \sinh v \sin w}{\cosh v - \cos u}$$

$$z = \frac{a \sin u}{\cosh v - \cos u}$$

$$u = \tan^{-1} \frac{2az}{x^2 + y^2 + z^2 - a^2}$$

$$v = \ln \left(W + \sqrt{W^2 - 1} \right)$$

$$w = \tan^{-1} \frac{y}{x}$$

$$W = \frac{a \sin u}{z} + \cos u$$

$$h_1^2 = \frac{a^2}{(\cosh v - \cos u)^2}$$

$$h_2^2 = \frac{a^2}{(\cosh v - \cos u)^2}$$

$$h_3^2 = \frac{a^2 \sinh^2 v}{(\cosh v - \cos u)^2}$$

Figure 18. Elliptic Cylindrical Coordinates, System 15 (ELLCYL). The traces of the elliptic cylinders u and the hyperbolic cylinders v on the xy plane are shown. The third set of surfaces are planes w parallel to the xy plane.

Equations for Elliptic Cylindrical Coordinates (Figure 18)

(u, v, w)
(L, A, L)

$$\begin{aligned}x &= a \cosh u \cos v \\y &= a \sinh u \sin v \\z &= w\end{aligned}$$

$$u = \tanh^{-1} \left[\frac{y}{x \tan v} \right]$$

$$v = \tan^{-1} \sqrt{\frac{1}{A} - 1}$$

$$w = z$$

$$A = \frac{G - \sqrt{G^2 - 4a^2 x^2}}{2a^2}$$

$$G = x^2 + y^2 + a^2$$

$$h_1^2 = a^2 (\sinh^2 u + \sin^2 v)$$

$$h_2^2 = a^2 (\sinh^2 u + \sin^2 v)$$

$$h_3^2 = 1$$

Figure 19. Prolate Spheroidal Coordinates, System 16 (PROSPH). The above curves are revolved about the x axis, which is then relabeled the z axis, creating the ellipsoids of revolution u and the hyperboloids of revolution v . The third set of surfaces are planes w through the z axis.

Equations for Prolate Spheroidal Coordinates (Figure 19)

(u, v, w)

(L, A, A)

$$x = a \sinh u \sin v \cos w$$

$$y = a \sinh u \sin v \sin w$$

$$z = a \cosh u \cos v$$

$$u = \tanh^{-1} \left[\frac{y}{z \tan v \sin w} \right]$$

$$v = \tan^{-1} \sqrt{\frac{+B + \sqrt{B^2 + 4y^2 z^2 \sin^2 w}}{2 z^2 \sin^2 w}}$$

$$w = \tan^{-1} \frac{y}{x}$$

$$B = y^2 + (a^2 - z^2) \sin^2 w$$

$$h_1^2 = a^2 (\sin^2 v + \sinh^2 u)$$

$$h_2^2 = a^2 (\sin^2 v + \sinh^2 u)$$

$$h_3^2 = a^2 \sin^2 v \sinh^2 u$$

Figure 20. Oblate Spheroidal Coordinates, System 17 (OBLSPH). The above curves are revolved about the y axis, which is then relabeled the z axis, creating the ellipsoids of revolution u and the hyperboloids of revolution v . The third set of surfaces are the planes w through the z axis.

Equations for Oblate Spheroidal Coordinates (Figure 20)

(u, v, w)
(L, A, A)

$$\begin{aligned}x &= a \cosh u \cos v \cos w \\y &= a \cosh u \cos v \sin w \\z &= a \sinh u \sin v\end{aligned}$$

$$u = \tanh^{-1} \left[\frac{z \sin w}{y \tan v} \right]$$

$$v = \tan^{-1} \sqrt{\frac{-B + \sqrt{B^2 + 4y^2 z^2 \sin^2 w}}{2y^2}}$$

$$w = \tan^{-1} \frac{y}{x} \quad B = y^2 - (a^2 + z^2) \sin^2 w$$

$$h_1^2 = a^2 (\sinh^2 u + \sin^2 v)$$

$$h_2^2 = a^2 (\sinh^2 u + \sin^2 v)$$

$$h_3^2 = a^2 \cos^2 v \cosh^2 u$$

Figure 21. Conical Coordinates, System 18 (CONCAL) Only one surface from each of the three families u , v , w are shown. They are the set that intersect at the point $P(4, 2, 5)$. Both cones may be extended to infinity.

Equations for Conical Coordinates (Figure 21) (u, v, w)
 (L, L, L)

$$x = \frac{u v w}{ab}$$

$$y = \frac{u}{a} \sqrt{\frac{(v^2 - a^2)(w^2 - a^2)}{(a^2 - b^2)}}$$

$$w^2 < a^2 < v^2 < b^2$$

$$z = \frac{u}{b} \sqrt{\frac{(v^2 - b^2)(w^2 - b^2)}{(b^2 - a^2)}}$$

$$u = \sqrt{R}$$

$$P = x^2 (a^2 + b^2) + a^2 z^2 + b^2 y^2$$

$$R = x^2 + y^2 + z^2$$

$$v = \sqrt{\frac{+P + \sqrt{P^2 - 4R(a^2 b^2 x^2)}}{2R}}$$

$$w = \sqrt{\frac{+P - \sqrt{P^2 - 4R(a^2 b^2 x^2)}}{2R}}$$

$$h_1^2 = 1$$

$$h_2^2 = \frac{u^2 (w^2 - v^2)}{(v^2 - a^2)(v^2 - b^2)}$$

$$h_3^2 = \frac{u^2 (v^2 - w^2)}{(w^2 - a^2)(w^2 - b^2)}$$

Figure 22. Ellipsoidal Coordinates, System 19 (ELPSOD). Only one surface from each of the three families u , v , w are shown. They are the set that intersect at the point $P(4, 2, 5)$. All hyperboloids may be extended to infinity.

Equations for Ellipsoidal Coordinates (Figure 22)

(u, v, w)
(L, L, L)

$$x = \sqrt{\frac{(a^2 - u)(a^2 - v)(a^2 - w)}{(a^2 - b^2)(a^2 - c^2)}}$$

$$y = \sqrt{\frac{(b^2 - u)(b^2 - v)(b^2 - w)}{(b^2 - a^2)(b^2 - c^2)}}$$

$$u < c^2 < v < b^2 < w < a^2$$

$$z = \sqrt{\frac{(c^2 - u)(c^2 - v)(c^2 - w)}{(c^2 - a^2)(c^2 - b^2)}}$$

u = Smallest root of cubic equation $Q^3 + AQ^2 + BQ + C = 0$

v = Middle root of cubic equation. (Solution for cubic equation on next page.)

w = Largest root of cubic equation

$$A = x^2 + y^2 + z^2 - (a^2 + b^2 + c^2)$$

$$B = a^2 c^2 + b^2 c^2 + a^2 b^2 - x^2 (b^2 + c^2) - y^2 (a^2 + c^2) - z^2 (a^2 + b^2)$$

$$C = b^2 c^2 x^2 + a^2 c^2 y^2 + a^2 b^2 z^2 - a^2 b^2 c^2$$

$$h_1^2 = \frac{(v - u)(w - u)}{4(a^2 - u)(b^2 - u)(c^2 - u)}$$

$$h_2^2 = \frac{(w - v)(u - v)}{4(a^2 - v)(b^2 - v)(c^2 - v)}$$

$$h_3^2 = \frac{(u - w)(v - w)}{4(a^2 - w)(b^2 - w)(c^2 - w)}$$

Equations for Ellipsoidal Coordinates (Continued)

Solution for cubic equation:

$$G = \frac{A^2}{9} - \frac{B}{3}$$

$$F = \frac{A^3}{27} - \frac{AB}{6} + \frac{C}{2}$$

$$\theta = \frac{1}{3} \cos^{-1} \left[-\sqrt{\frac{F^2}{G^3}} \right]$$

$$\text{If } F = 0 \quad D = 2\sqrt{G}$$

$$\text{If } F \neq 0 \quad D = 2 \frac{|F|}{F} \sqrt{G}$$

Roots:

$$Q_1 = D \cos \theta - \frac{A}{3}$$

$$Q_2 = D \cos (\theta - 120) - \frac{A}{3}$$

$$Q_3 = D \cos (\theta + 120) - \frac{A}{3}$$

DISCUSSION OF RESULTS

The 19 coordinate systems were compared by a complete analysis of Equation (22) and the type of equation shown as Equation (23). The variables in these equations are x , y , z , and the error ratio τ . Preliminary studies showed that R would not be the same at all points in space for a given system. It therefore appeared necessary to examine these equations for all possible combinations of the four variables. The task was further complicated, in some cases, by the presence of the constants a , b , or c which define the geometrical pattern of the system. These constants would also have to be varied sufficiently to give a clear picture of their effect.

A considerable reduction in the amount of work was brought about by the discovery that R would not change if x , y , z , a , b , and c were multiplied by an arbitrary constant. This meant that in those systems not having an a , b , or c all R values along a given radial line were the same. Therefore, only one point on each radial line need be examined instead of all points in space. For those systems having a single constant, a , it was then possible to explore all possibilities by examining all points in space but for only one arbitrarily chosen value of a . Different values of the constant a merely shifts the entire pattern in space along with a . For system 18, the only one with two constants (a and b), the equation for R can be expressed in terms of the ratio b/a . Again it was only necessary to check one point on each radial line, but with various values of b/a . For system 19, the only one with three constants (a , b and c), a single arbitrary value of a was chosen but all points in space had to be examined as well as all allowable values of b and c . In the entire survey of all systems no reduction in effort was possible with the error ratio τ .

Since radial lines played a significant role in the analysis, the survey was accomplished by examining the points on an equilateral triangularly shaped plane oriented as shown in Figure 23. Symmetry of all systems allowed the work to be confined to the positive quadrant. A plane such as the one shown in Figure 23 will intercept all radial lines in the quadrant. An array of points on the plane will then represent a group of these radial lines. A sample of 91 equally spaced points (Figure 23) was found to be sufficient to indicate the significant variations in R throughout space. The x , y , z coordinates of these points are

$$\begin{aligned} x &= B (A + J - 1) \\ y &= B (A + R - J) \\ z &= B (A + N - R) \end{aligned} \tag{24}$$

where B is the projected spacing between points, $A = 0.001$ for all systems except 18 where, because of round-off errors 0.01 had to be used, $N = 13$ the number of rows of points, R is the row number beginning at the z axis, and J is the point number in the row beginning at the yz plane. By varying the value of B , the plane was shifted to different locations in space but the 91 points always remained on the same radial lines.

It will be noticed that the peripheral points in the array lie very close to but not exactly on the coordinate planes. They were placed slightly off the coordinate planes to avoid indeterminate solutions yet close enough to give a clear indication of the patterns on the planes.

Orientation of Plane $x + y + z = k$

Enlarged top view of plane showing location of the 91 points.

Figure 23. Equilateral Triangularly-Shaped Plane.

All computations in the survey were done on an IBM-360 computer with a remote control terminal. The programs were written in CPS language and are in Appendix C. There is a main program called RADCUR and 19 subroutines; one for each coordinate system.

A detailed discussion of the results from each system together with a sample computer printout of the triangular array of R values follows. The types of information collected in the survey were (1) where and under what conditions do regions of $R < 1$ exist, indicating systems superior to a rectangular one, and (2) what is the overall pattern of R values from the origin to infinity.

System 1, CYL (Table 2)

In this system, R is constant for each radial line. Therefore, each triangular array of R values gives the pattern for the entire quadrant.

Since all values of R lie between 1 and $\sqrt{1 + \tau^2}$, there are no regions of $R < 1$. At $\tau = 0$, $R = 1$ over the entire quadrant, and at larger values of τ , R ranges from 1 along the z axis to $\sqrt{1 + \tau^2}$ throughout the xy plane (Table 2). At the origin R is indeterminate for $\tau > 0$.

Values of R are symmetrical about the $x = y$ plane, and are slightly lower on the plane than elsewhere. There are no sharp peaks or dips anywhere in the system.

CYL gives exactly the same values of R as system 7, COXSPH.

System 2, SPH (Table 3)

In this system, R is constant for each radial line. Therefore, each triangular array of values gives the pattern for the entire quadrant.

Since all R values lie between $\sqrt{1 + \tau^2}$ and $\sqrt{1 + 2\tau^2}$, there are no regions of $R < 1$. At $\tau = 0$, $R = 1$ over the entire quadrant, and at larger values of τ , R ranges from $\sqrt{1 + \tau^2}$ along the z axis to $\sqrt{1 + 2\tau^2}$ throughout the xy plane as shown in Table 3. At the origin R is indeterminate for $\tau > 0$.

Values of R are symmetrical about the $x = y$ plane, and are slightly lower on the plane than elsewhere. There are no sharp peaks or dips anywhere in the system.

Table 2. Values of R for CYL, System 1 ($\tau = 10$)

Z											
1.000											
1.350			1.350								
2.202		1.722		2.202							
3.317		2.611		2.611		3.317					
4.583		3.811		3.481		3.811		4.583			
5.838		5.173		4.688		4.688		5.173		5.898	
7.141		6.553		6.060		5.860		6.060		6.553	
8.198		7.730		7.396		7.142		7.142		7.396	
8.198		7.730		7.396		7.142		7.142		7.396	
9.000		8.761		8.510		8.307		8.226		8.307	
9.000		8.761		8.510		8.307		8.226		8.307	
9.539		9.425		9.299		9.183		9.110		9.110	
9.539		9.425		9.299		9.183		9.110		9.110	
9.857		9.816		9.769		9.723		9.688		9.674	
9.857		9.816		9.769		9.723		9.688		9.674	
10.009		10.001		9.992		9.982		9.974		9.969	
10.009		10.001		9.992		9.982		9.974		9.969	
10.050		10.050		10.050		10.050		10.050		10.050	
10.050		10.050		10.050		10.050		10.050		10.050	

y

x

Table 3. Values of R for SPH, System 2 ($t = 10$)

Z											
10.050											
10.091				10.091							
10.240			10.147		10.240						
10.536		10.335		10.335		10.536					
11.000		10.701		10.589		10.701		11.000			
11.610		11.259		11.044		11.044		11.259		11.610	
12.288		11.956		11.693		11.591		11.693		11.956	
12.931		12.676		12.438		12.288		12.288		12.438	
13.453		13.295		13.131		13.000		12.949		13.000	
13.820		13.742		13.656		13.577		13.528		13.528	
14.041		14.012		13.980		13.948		13.923		13.914	
14.148		14.143		14.136		14.130		14.124		14.120	
14.177		14.177		14.177		14.177		14.177		14.177	

y

x

The pattern of R values is essentially the same as in CYL but with the levels slightly higher in SPH.

System 3, PARCYL (Table 4)

In this system, R is constant for each radial line and since τ is absent, one triangular printout is sufficient to describe the entire system.

Since all R values lie between 1 and 2, there are no regions of $R < 1$. Values of R range from 1 along the z axis to 2 throughout the xy plane as shown in Table 4. At the origin R is indeterminate.

Values of R are symmetrical about the $x = y$ plane, and are slightly lower on the plane than elsewhere. There are no sharp peaks or dips anywhere in the system.

System 4, PARALT (Table 5)

In this system, R is constant for each radial line and since τ is absent, one triangular printout is sufficient to describe the entire system.

Since $R = 1$ for all conditions, there are no regions where $R < 1$.

This system was included to show the effect of using a different scheme for assignment of the coordinate parameter numbers. Comparison of Tables 4 and 5 shows that changing the method of numbering the coordinate parameters does alter the R values and can make them smaller.

System 5, PARLOD (Table 6)

In this system, R is constant for each radial line, therefore, each triangular array of R values gives the pattern for the entire quadrant.

Since all values of R lie between 2 and $\sqrt{4 + \tau^2}$, there are no regions of $R < 1$. At $\tau = 0$, $R = 2$ over the entire quadrant, and at larger values of τ , R ranges from 2 along the z axis to $\sqrt{4 + \tau^2}$ throughout the xy plane as shown in Table 6. At the origin R is indeterminate for $\tau > 0$.

Values of R are symmetrical about the $x=y$ plane, and are slightly lower on the plane than elsewhere. There are no sharp peaks or dips anywhere in the system.

Rotation of PARCYL to create PARLOD does not alter the basic pattern of R values but does increase the overall level.

Table 4. Values of R for PARCYL, System 3

Z											
1.000											
1.012			1.012								
1.056		1.029		1.056							
1.140		1.084		1.084		1.140					
1.265		1.186		1.155		1.186		1.265			
1.419		1.332		1.276		1.276		1.332		1.419	
1.581		1.503		1.439		1.414		1.439		1.503	
1.728		1.670		1.616		1.581		1.581		1.616	
1.844		1.809		1.773		1.744		1.732		1.744	
1.924		1.907		1.888		1.871		1.860		1.860	
1.971		1.965		1.958		1.951		1.946		1.944	
1.994		1.993		1.991		1.990		1.989		1.988	
2.000		2.000		2.000		2.000		2.000		2.000	
y											
x											

Table 6. Values of R for PARLOD, System 5

Z											
2.000											
2.196		2.196									
2.802		2.442		2.802							
3.742		3.134		3.134		3.742					
4.899		4.186		3.888		4.186		4.899			
6.147		5.456		4.998		4.998		5.456		6.147	
7.348		6.778		6.302		6.111		6.302		6.778	
8.379		7.980		7.596		7.349		7.349		7.596	
9.165		8.931		8.685		8.435		8.406		8.485	
9.165		8.931		8.685		8.435		8.406		8.485	
9.695		9.583		9.459		9.345		9.273		9.273	
9.695		9.583		9.459		9.345		9.273		9.273	
10.008		9.967		9.922		9.876		9.841		9.828	
10.008		9.967		9.922		9.876		9.841		9.828	
10.158		10.150		10.141		10.131		10.123		10.119	
10.158		10.150		10.141		10.131		10.123		10.119	
10.198		10.198		10.198		10.198		10.198		10.198	
10.198		10.198		10.198		10.198		10.198		10.198	

y

x

System 6, CYLCIR (Table 7)

In this system, R is constant for each radial line and since τ is absent, one triangular printout is sufficient to describe the entire system.

Since $R = 1$ for all conditions, there are no regions where $R < 1$.

System 7, COXSPH (Table 8)

In this system, R is constant for each radial line, therefore, each triangular array of R values gives the pattern for the entire quadrant.

Since all values of R lie between 1 and $\sqrt{1 + \tau^2}$, there are no regions of $R < 1$. At $\tau = 0$, $R = 1$ over the entire quadrant, and at larger values of τ , R ranges from 1 along the z axis to $\sqrt{1 + \tau^2}$ throughout the xy plane as shown in Table 8. At the origin R is indeterminate for $\tau > 0$.

Values of R are symmetrical about the $x = y$ plane, and are slightly lower on the plane than elsewhere. There are no sharp peaks or dips anywhere in the system.

COXSPH gives exactly the same R values as system 1, CYL. Rotation of CYLCIR to create COXSPH alters the pattern and increases the level of the values.

System 8, ELLPAR (Table 9)

In this system, R is constant for each radial line and since τ is absent, one triangular printout is sufficient to describe the entire system.

Since all values of R lie between $1/2$ and 1, the region of $R < 1$ covers all space and therefore the system is everywhere superior to a rectangular one.

$R = 1$ along the z axis, $1/2$ along the x and y axes, and slightly higher than $1/2$ throughout the xy plane as shown in Table 9. The R values in the xz and yz planes are symmetrical about the $x = y$ plane, but the interior values are not. Values on the xz plane side of the $x = y$ plane are slightly larger than the corresponding values on the other side. At the origin, R is indeterminate.

There are no sharp peaks or dips anywhere in the system. The R array is almost identical to that in HYPER except HYPER is symmetrical and has slightly lower interior values. Along the xz and yz planes the R values are the same in both systems.

Table 7. Values of R for CYLCIR, System 6

Z												
1.000												
1.000		1.000										
1.000			1.000									
1.000				1.000								
1.000					1.000							
1.000						1.000						
1.000							1.000					
1.000								1.000				
1.000									1.000			
1.000										1.000		
1.000											1.000	
1.000												1.000
1.000												

x

y

Table 8. Values of R for COXSPH, System 7 ($\tau = 10$)

Z											
1.000											
1.350 1.350											
2.202 1.722 2.202											
3.317 2.611 2.611 3.317											
4.583 3.811 3.481 3.811 4.583											
5.898 5.173 4.688 4.688 5.173 5.898											
7.141 6.553 6.060 5.860 6.060 6.553 7.141											
8.198 7.790 7.396 7.142 7.142 7.396 7.790 8.198											
9.000 8.761 8.510 8.307 8.226 8.307 8.510 8.761 9.000											
9.539 9.425 9.299 9.183 9.110 9.110 9.183 9.299 9.425 9.539											
9.857 9.816 9.769 9.723 9.688 9.674 9.688 9.723 9.769 9.816 9.857											
10.009 10.001 9.992 9.982 9.974 9.969 9.969 9.974 9.982 9.992 10.001 10.009											
10.050 10.050 10.050 10.050 10.050 10.050 10.050 10.050 10.050 10.050 10.050 10.050											

y

x

Table 9. Values of R for ELLPAR, System 8

Z												
1.000												
.997 .997												
.985 .993 .985												
.962 .979 .980 .962												
.922 .943 .962 .952 .922												
.864 .900 .924 .927 .903 .864												
.791 .830 .863 .880 .871 .834 .791												
.710 .745 .781 .809 .816 .794 .751 .710												
.632 .659 .691 .722 .742 .738 .708 .665 .632												
.570 .586 .609 .636 .661 .673 .661 .629 .592 .570												
.528 .536 .550 .570 .593 .612 .617 .601 .570 .542 .528												
.506 .509 .517 .531 .551 .571 .585 .583 .564 .537 .514 .506												
.500 .502 .507 .518 .534 .553 .570 .577 .570 .549 .524 .506 .500												
y												x

System 9, HYPER (Table 10)

In this system, R is constant for each radial line and since τ is absent, one triangular printout is sufficient to describe the entire system.

Since all R values lie between $1/2$ and 1 , the region of $R < 1$ covers all space and therefore the system is everywhere superior to a rectangular one.

$R = 1$ along the z axis and $1/2$ throughout the xy plane as shown in Table 10. The entire array is symmetrical about the $x = y$ plane. R values on the plane are slightly higher than elsewhere. At the origin, R is indeterminate. There are no sharp peaks or dips anywhere in the system.

The R array is almost identical to that in ELLPAR except ELLPAR is not symmetrical and has slightly larger interior values. Along the xz and yz plane the R values are the same in both systems.

System 10, HYPSPH (Tables 11 and 12)

In this system, R is constant for each radial line, therefore, each triangular array of R values gives the pattern for the entire quadrant.

Since all values of R lie between $1/2$ and $\sqrt{(1/4) + \tau^2}$, a region of $R < 1$ does exist. At $\tau = 0$ this region covers all space and R is everywhere equal to $1/2$. As τ increases, the region shrinks and recedes to the z axis line. The R values in the region range from $1/2$ to 1 with the z axis always being $1/2$ (Tables 11 and 12). It appears that τ must become essentially infinite for the region to shrink to the z axis line. However, for the region to have any significant size τ must be less than 10 .

All R values are symmetrical about the $x = y$ plane. At the origin, $R = 1/2$ for $\tau = 0$ but is indeterminate for $\tau > 0$. Values in the xy plane are $\sqrt{(1/4) + \tau^2}$. There are no sharp peaks or dips anywhere in the system.

Rotation of HYPER to create HYPSPH does not produce smaller R values and for certain values of τ generates larger values.

System 11, CARD (Table 13)

In this system, R is constant for each radial line and since τ is absent, one triangular printout is sufficient to describe the entire system. As shown in Table 13, $R = 1$ throughout all space. The system is therefore not better than a rectangular one.

Table 10. Values of R for HYPER, System 9

Z												
1.000												
.997 .997												
.985 .993 .985												
.962 .978 .978 .962												
.922 .948 .957 .948 .922												
.864 .898 .918 .918 .898 .864												
.791 .828 .856 .866 .856 .828 .791												
.710 .743 .773 .791 .791 .773 .743 .710												
.632 .657 .681 .700 .707 .700 .681 .657 .632												
.570 .584 .599 .612 .621 .621 .612 .599 .584 .570												
.528 .534 .540 .546 .551 .553 .551 .546 .540 .534 .528												
.506 .507 .509 .510 .511 .512 .512 .511 .510 .509 .507 .506												
.500 .500 .500 .500 .500 .500 .500 .500 .500 .500 .500 .500 .500												
y												x

Table 11. Values of R for HYPSPH, System 10 ($\tau = 1$)

Z											
.500											
.508 .508											
.537 .519 .537											
.592 .555 .555 .592											
.671 .621 .601 .621 .671											
.767 .712 .678 .678 .712 .767											
.866 .818 .779 .764 .779 .818 .866											
.955 .920 .887 .856 .856 .887 .920 .955											
1.025 1.004 .982 .964 .957 .964 .982 1.004 1.025											
1.072 1.062 1.051 1.041 1.034 1.034 1.041 1.051 1.062 1.072											
1.101 1.097 1.093 1.089 1.086 1.084 1.086 1.089 1.093 1.097 1.101											
1.114 1.114 1.113 1.112 1.111 1.111 1.111 1.111 1.112 1.113 1.114 1.114											
1.118 1.118 1.118 1.118 1.118 1.118 1.118 1.118 1.118 1.118 1.118 1.118											

y

x

Table 12. Values of R for HYPSPH, System 10 ($\tau = 50$)

Z																		
.500																		
4.558		4.558																
9.822		7.025		9.822														
15.822		12.072		12.072		15.822												
22.369		18.392		16.680		18.392		22.369										
29.067		25.384		22.905		22.905		25.384		29.067								
35.359		32.384		29.887		28.875		29.887		32.384		35.359						
40.689		38.629		36.646		35.360		35.360		36.646		38.629	40.689					
44.723		43.521		42.260		41.234		40.828		41.234		42.260	43.521	44.723				
47.436		46.862		46.230		45.645		45.279		45.279		45.645	46.230	46.862	47.436			
49.031		48.825		48.593		48.362		48.183		48.114		48.183		48.362	48.593	48.825	49.031	
49.797		49.756		49.710		49.663		49.622		49.597		49.597		49.622	49.663	49.710	49.756	49.797
50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002	50.002
y																		x

System 12, BIPOL (Tables 14 through 17)

In this system, distances throughout the quadrant may be expressed in terms of the constant a . It was only necessary to use a single value for a in the survey, which in this case was 10.

There were two different surveys carried out on this system, one with the coordinate parameter u considered to be of an angular type, and the other with u as a linear type.

Angular: A region of $R \leq 1$ does exist in this system because the R values range from 0 to infinity. For $\tau = 0$, R lies between 0 and 1 over the entire quadrant. At the origin, R is indeterminate, but along the y axis $R = 0$ and in the xz plane $R = 1$. Except in the general vicinity of $x = a$ the values are symmetrical about the $x = z$ plane, with values on the plane slightly lower than elsewhere (Table 14). The pattern of R values is the same at the origin as at distances much greater than a .

As τ increases from zero, the infinite sized region of $R \leq 1$ shrinks and recedes toward the point $(a, 0, 0)$ where R approaches 1 as τ approaches infinity (Table 15). This region is reduced to a vertical spike of negligible volume directly over the point $(a, 0, 0)$ for $\tau > 20$.

For $\tau > 0$, R appears to be infinite at the origin. At distances much smaller than a , $R \gg 1$ and the pattern is symmetrical about the $x = y$, $x = z$, and $y = z$ planes with a maximum in the center region of the quadrant. At the distances much larger than a , the R values range from 1 along the z axis to values much greater than 1 throughout the rest of the quadrant. The pattern here is symmetrical about the $x = y$ plane. In both regions there are no sharp peaks or dips in the pattern.

Linear: In this system, since u is considered a linear type measurement, τ is absent. The R value pattern is then fixed for the entire quadrant when expressed in terms of the constant a .

At the origin, R appears to be infinite. At distances much smaller than a , $R \gg 1$ and the pattern is symmetrical about the $x = y$, $x = z$, and $y = z$ planes with a maximum in the center region of the quadrant. There are no sharp peaks or dips in the pattern.

There is a region of $R \leq 1$ present and it lies approximately between the limits of $7.8 a \rightarrow x \rightarrow \infty$, $0 \rightarrow y \rightarrow x$, and $0 \rightarrow z \rightarrow \infty$ (Table 16). It is confined to the half of the quadrant on the x side of the $x = y$ plane. At the point $(a, 0, 0)$, $R = 0$.

Table 15. Values of R for BIPOL, System 12 (u: angular, $\tau = 2$, $B = 1$)

Z												
1.302												
1.352				1.343								
1.415			1.404		1.373							
1.490		1.479		1.443		1.381						
1.575		1.568		1.529		1.456		1.356				
1.665		1.664		1.629		1.549		1.428		1.285		
1.752		1.758		1.731		1.652		1.517		1.343		1.158
1.827		1.839		1.822		1.752		1.615		1.421		1.197 .976
1.887		1.899		1.889		1.852		1.707		1.510		1.263 .996 .746
1.934		1.939		1.930		1.884		1.776		1.595		1.349 1.064 .762 .476
1.971		1.965		1.951		1.910		1.819		1.662		1.439 1.164 .857 .522 .196
2.003		1.985		1.962		1.920		1.841		1.710		1.518 1.274 .991 .684 .371 .358
2.033	2.004	1.971	1.925	1.853	1.742	1.581	1.374	1.130	.864	.602	.452	.573
y												x

At distances much larger than a , the R values on the x side of the $x = y$ plane are increasing toward 1 and on the y side are decreasing toward 1 (Table 17). The entire quadrant becomes 1 for distances greater than about $30a$.

In comparing the linear case to the angular it was found that for $\tau = 0$ the patterns were somewhat different and that the region of $R < 1$ was smaller for the linear case. However, for $\tau \gg 0$, the linear case is superior since this region essentially vanishes at large values of τ in the angular case.

System 13, BIPALT (Table 18)

In this system τ is absent since all coordinate parameters are linear, so the pattern of R values is fixed for the entire quadrant when expressed in terms of the constant a .

At the origin and at distances much smaller than a , $R = 1$. There is a region of $R < 1$ present on the y side of the $x = y$ plane (Table 18). This region extends from distances of about $a/100$ where $R = 1$ in a point sized volume on the y axis to distances of about $100a$ where R again is 1 but the volume has enlarged and occupies the entire half of the quadrant on the y side.

At the point $(0, a, 0)$, $R = 0$. From this point, in a direction toward the $x = y$ plane, R increases until it is 1 throughout the entire $x = y$ plane.

At the point $(a, 0, 0)$, $R = \infty$. From this point, in a direction toward the $x = y$ plane, R decreases to the value of 1 on the $x = y$ plane.

At distances greater than a , the R values on the x side of the $x = y$ plane are decreasing toward 1 and on the y side are increasing toward 1. The entire quadrant becomes 1 for distances greater than about $100a$.

As with the case of system 4, PARALT, this system (BIPALT) was included to show the effect of using a different scheme for assignment of the coordinate parameter numbers. Comparison of BIPALT with BIPOL shows that the pattern can be changed and that the region of $R < 1$ will be altered.

Table 17. Values of R for BIPOL, System 12 (u: linear, B = 15)

Z											
1.004											
1.003 .998											
1.003 1.000 .997											
1.003 1.002 .998 .997											
1.004 1.003 1.000 .997 .996											
1.004 1.004 1.002 .998 .996 .996											
1.004 1.004 1.003 1.000 .997 .996 .996											
1.004 1.005 1.004 1.002 .998 .996 .995 .996											
1.004 1.004 1.004 1.003 1.000 .997 .996 .996 .996											
1.003 1.004 1.004 1.003 1.001 .999 .997 .996 .996 .997											
1.003 1.003 1.004 1.003 1.002 1.000 .998 .997 .996 .997 .997											
1.002 1.003 1.003 1.003 1.002 1.001 .999 .998 .997 .997 .997 .998											
1.002 1.002 1.003 1.003 1.002 1.001 1.000 .999 .998 .997 .997 .998 .998											

y

x

Table 18. Values of R for BIPALT, System 13

Z											
1.000											
1.000			1.000								
		.997	1.000	1.003							
		.985	.997	1.004	1.022						
		.951	.981	1.000	1.025	1.087					
		.885	.940	.981	1.023	1.101	1.265				
		.781	.860	.931	1.000	1.102	1.298	1.661			
		.644	.738	.837	.940	1.073	1.296	1.707	2.448		
		.488	.585	.701	.835	1.000	1.244	1.661	2.453	4.101	
		.332	.418	.543	.695	.884	1.140	1.536	2.245	3.825	9.052
		.196	.260	.386	.547	.745	1.000	1.361	1.947	3.105	6.569 6933.992
		.131	.141	.259	.419	.613	.853	1.173	1.645	2.452	4.204 9.930 10.471
		.180	.135	.201	.337	.511	.726	1.000	1.378	1.955	2.963 4.974 7.429 5.543
y											
x											

System 14, TOROID (Tables 19 through 21)

In this system, distances throughout the quadrant may be expressed in terms of the constant a . It was only necessary to use a single value for a in the survey, which was 10.

Two different surveys were conducted, one with the coordinate parameter u considered to be an angular type, and the other with u as a linear type. The parameter w was not given two interpretations, because it was felt that w could never be anything but an angular type measurement.

Angular: The entire quadrant is a region of $R \leq 1$ for $\tau = 0$ and is symmetrical about the $x = y$ plane. For $\tau = 0$, R is indeterminate at the origin. At distances much smaller than a , $R = 0$ along the z axis, and 1 throughout the xy plane with interior values lying between these limits. At distances in the vicinity of a , the R values, except on the z axis, are lower. Here, the pattern appears to dip to zero at the two points $(a/5, a, 0)$ and $(a, a/5, 0)$. (See Table 19.) At distances much greater than a , the R values increase back to the exact same values they were at distances much smaller than a .

As τ increases, the infinite sized region of $R \leq 1$ shrinks and recedes to the circular line $x^2 + y^2 = a^2$, $z = 0$ and to the point $(0, 0, a)$ where R becomes 1 when τ becomes 1.

For $\tau = 0$, $R = \infty$ at the origin. At distances much smaller than a , $R \gg 1$ and the pattern is symmetrical about the $x = y$, $x = z$, and $y = z$ planes with a slight maximum in the center of the array. At distances much greater than a , the R values are much greater than 1, the symmetry is of the same type, but the center of the array is a minimum. At these large distances, the R values are directly proportional to the distance.

Linear: Even though u is here used as a linear measurement, τ is still present due to the parameter w . For $\tau = 0$, $R = \infty$ at the origin. For distances much smaller than a , $R \gg 1$ and the pattern is symmetrical about the $x = y$, $x = z$ and $y = z$ planes with a slight maximum in the center of the array. There are no sharp peaks or dips in the pattern.

For $\tau = 0$, a region of $R \leq 1$ exists shaped somewhat like twin cones. The apexes of the twin cones are at the points $(0.768 a, 0, 0)$ and $(0, 0.768 a, 0)$, and the region spreads out approximately in a radial fashion with increasing x , y , and z distances (Table 20). At a distance approximately equal to a the cones begin to merge at the center of the xy plane and at larger distances they have filled the lower three-fourths of the quadrant between the $x + y = 3$ plane and xy plane (Table 21). The values of R in this region range from zero on the circular line $x^2 + y^2 = a^2$, $z = 0$

Table 19. Values of R for TOROID, System 14 (u : angular, $\tau = 0$, $B = 1$)

z																																							
.000																																							
			.090			.090																																	
				.195		.140		.195																															
					.311		.239		.239		.311																												
						.431		.359		.327		.359		.431																									
							.542		.483		.441		.441		.483		.542																						
								.626		.592		.557		.542		.557		.592		.626																			
									.668		.662		.649		.636		.636		.649		.662		.668																
										.653		.676		.689		.693		.693		.693		.689		.676		.653													
											.579		.621		.661		.688		.701		.701		.688		.661		.621		.579										
												.455		.493		.556		.610		.643		.655		.643		.610		.556		.493		.455							
													.363		.299		.358		.448		.514		.546		.546		.514		.448		.358		.299		.363				
														.440		.299		.091		.191		.323		.391		.413		.391		.323		.191		.091		.299		.440	
															y																				x				

Table 21. Values of R for TOROID, System 14 (u: linear, $\tau = 0$, $B = 10$)

Z											
1.000											
1.005						1.005					
1.006				1.006				1.006			
1.006			1.007			1.007			1.006		
1.005		1.007		1.007		1.007		1.005			
1.003		1.005		1.006		1.006		1.005		1.003	
1.000		1.002		1.003		1.004		1.003		1.002	
.997		.998		.999		1.000		1.000		.999	
.995		.995		.995		.995		.995		.995	
.994		.993		.992		.992		.992		.992	
.994		.993		.992		.991		.990		.989	
.995		.994		.992		.991		.990		.990	
.995		.995		.994		.993		.992		.991	
.995		.995		.994		.993		.992		.991	
.995		.995		.994		.993		.992		.991	
.995		.995		.994		.993		.992		.991	
.995		.995		.994		.993		.992		.991	
.995		.995		.994		.993		.992		.991	
.995		.995		.994		.993		.992		.991	
.995		.995		.994		.993		.992		.991	
 	
.995											

to 1 at the two apexes and at distances greater than about $40a$. In the remaining upper one-fourth of the quadrant, the R values decrease from ∞ as the distance from the origin increases until at about $40a$ R becomes 1. Beyond about $40a$, the entire quadrant has $R = 1$.

As τ increases, the region of $R < 1$ shrinks and recedes toward the circular line $x^2 + y^2 = a^2$, $z = 0$ where R becomes 1 when τ becomes approximately 1.

For $\tau > 0$, $R = \infty$ at the origin and the pattern is the same as for $\tau = 0$. At distances much greater than a , the array of R values approaches a fixed pattern with $R = 1$ along the z axis, $R \approx \tau$ throughout the xy plane, and values between these two limits in the interior.

Rotation of RIPOL to create TOROID changes the pattern of R values but makes essentially no improvement in the level of the R values.

A comparison between the linear and angular cases shows that the patterns are somewhat different and the region of $R < 1$ is slightly smaller in the linear case.

System 15, ELLCYL (Tables 22 and 23)

In this system, distances throughout the quadrant may be expressed in terms of the constant a . It was only necessary to use $a = 10$ in the survey.

Again, two different surveys were conducted. One was with the parameter v considered as an angular measurement, the other with v as a linear measurement.

Angular: For $\tau = 0$, R at the origin is indeterminate. At distances much smaller than a , $R = 0$ on the x axis, 1 along the yz plane, and values between these two limits in the interior. The pattern is symmetrical about the $y = z$ plane with no sharp peaks or dips. At distances about equal to a , the region of $R < 1$ has reduced to about the one-half of the quadrant lying on the x side of the xy plane (Table 22). At greater distances, the region has shrunk to a sharp vertical spike directly over the point $(a, 0, 0)$. $R = 0$ at the point $(a, 0, 0)$ and 1 at the tip of the vertical spike over this point. At distances much greater than a , R ranges from 1 along the z axis to values greater than 1 along the xy plane. The pattern is now symmetrical about the $x = y$ plane and the R values, except the 1 on the z axis which is stable, appears to only increase asymptotically toward a fixed array with increasing distance. There are no sharp spikes or dips in these outer regions.

Table 22. Values of R for ELLCYL, System 15 (v : angular, $\tau = 0$, $B = 1$)

Z											
1.000											
1.000 .996											
1.001 .995 .981											
1.003 .995 .976 .949											
1.010 .997 .973 .937 .894											
1.026 1.005 .972 .926 .870 .814											
1.054 1.024 .980 .919 .847 .773 .707											
1.093 1.056 1.002 .927 .834 .735 .648 .581											
1.140 1.099 1.040 .956 .846 .721 .602 .508 .447											
1.190 1.147 1.088 1.004 .888 .746 .598 .466 .369 .316											
1.238 1.195 1.139 1.060 .951 .811 .651 .490 .349 .246 .196											
1.282 1.241 1.188 1.117 1.020 .893 .742 .579 .419 .279 .173 .205											
1.323 1.282 1.232 1.169 1.085 .976 .843 .694 .541 .399 .283 .242 .344											

y

x

As τ increases from zero, the region of $R < 1$ shrinks and recedes toward the point $(a, 0, 0)$ where R becomes 1 when τ appears to become infinite. Due to mathematical limitations, it was not determined if the sharp vertical spike directly over the point $(a, 0, 0)$ actually vanished for $\tau = \infty$.

For $\tau > 0$, $R = \infty$ at the origin. At distances much smaller than a , $R \gg 1$ and the pattern is symmetrical about the $x = y$, $x = z$, and $y = z$ planes with a slight maximum in the center of the array. There are no sharp peaks or dips. At distances much greater than a , $R = 1$ along the z axis and for $\tau \gg 1$, $R \approx \tau$ along the xy plane with values between these two limits in the interior. The pattern is symmetrical about the $x = y$ plane if $B \gg \tau$. With the exception of the R value on the z axis, the R values increase asymptotically toward a fixed array with increasing distance for a given value of τ .

Linear: With the interpretation that v is linear, τ is absent here and the pattern of R values is fixed for the entire quadrant when expressed in terms of the constant a .

At the origin, $R = \infty$ and at distances much smaller than a , $R \gg 1$ with the pattern being symmetrical about the $x = y$, $x = z$, and $y = z$ planes. There is a slight maximum in the center of the pattern but with no sharp peaks or dips anywhere in the array. At distances approaching the value of a , symmetry no longer exists.

A region of $R < 1$ exists between the approximate limits of $a/2 \rightarrow x \rightarrow 3a/2$, $0 \rightarrow y \rightarrow 3a/4$, and $0 \rightarrow z \rightarrow \infty$ (Table 23). The region appears to contain a sharp vertical spike directly over the point $(a, 0, 0)$. This spike possibly extends to infinity in the z direction at which point $R = 1$.

The entire region of $R < 1$ contains R values ranging from zero to 1, with the zero occurring at the point $(a, 0, 0)$.

At distances much greater than a , R ranges from 1 along the z axis to values greater than 1 along the xy plane. The pattern of R values appears to increase asymptotically toward a fixed array with increasing distance, and is very nearly symmetrical about the $x = y$ plane. There are no sharp spikes or dips in these outer regions.

In comparing the linear case with the angular for $\tau = 0$, it was found that whereas the patterns of R values were roughly similar, the region of $R < 1$ was smaller in the linear analysis. However, for $\tau \gg 0$, the linear case is superior since in the angular case the region of $R < 1$ essentially vanishes at these high levels of τ .

Table 23. Values of R for ELLCYL, System 15 (v : linear, $\tau = 0$, $B = 1$)

Z											
1.647											
1.744 1.657											
1.862 1.764 1.641											
1.998 1.894 1.753 1.587											
2.144 2.043 1.890 1.697 1.487											
2.285 2.196 2.045 1.832 1.582 1.331											
2.402 2.334 2.198 1.982 1.703 1.401 1.124											
2.482 2.435 2.324 2.123 1.836 1.497 1.161 .879											
2.521 2.490 2.402 2.229 1.958 1.609 1.233 .887 .621											
2.525 2.499 2.428 2.285 2.046 1.717 1.334 .949 .611 .378											
2.505 2.478 2.415 2.294 2.092 1.802 1.443 1.057 .687 .369 .196											
2.472 2.439 2.379 2.273 2.102 1.856 1.541 1.184 .822 .487 .222 .205											
2.435 2.396 2.334 2.238 2.091 1.884 1.616 1.304 .974 .658 .393 .261 .344											
y											x

System 16, PROSPH (Tables 24 and 25)

In this system, distances throughout the quadrant may be expressed in terms of the constant a . It was only necessary to use $a = 10$ in the survey.

Again two different surveys were conducted. One was with the parameter v considered as an angular measurement, the other with v as a linear measurement. The parameter w was considered to only be an angular type.

Angular: For $\tau = 0$, R at the origin is indeterminate. At distances much smaller than a , $R = 0$ along the z axis, 1 along the xy plane, and values between these two limits in the interior. The pattern is symmetrical about the $x = y$ plane with a minimum along the plane but with no sharp peaks or dips anywhere in the array.

There is a region of $R < 1$ that appears to be spherically shaped of radius a with its center at the point $(0, 0, a)$. (Table 24.) In this region, $R = 0$ near the origin and 1 along the periphery of the sphere. The pattern throughout the entire quadrant in the region near a is still symmetrical about the $x = y$ plane. At distances much greater than a , $R > 1$ and the pattern is symmetrical about the $x = y$, $x = z$, and $y = z$ planes with a minimum in the center. The values appear to be increasing asymptotically toward a fixed array with increasing distance. There are no sharp peaks or dips in the array.

As τ increases from zero, the region of $R < 1$ shrinks and recedes to the point $(0, 0, a)$ where R becomes 1 when τ reaches about $6.5a$.

For $\tau > 0$, $R = \infty$ at the origin. At distances much smaller than a , $R \gg 1$ and the pattern is symmetrical about the $x = y$, $x = z$, and $y = z$ planes with a maximum in the center. There are no sharp peaks or dips in the array. At distances much greater than a , $R > 1$ and very nearly the same over the entire array. The values appear to increase asymptotically with distance toward a fixed level. The pattern is symmetrical about the $x = y$ plane, with a minimum along the plane. When $\tau \gg a$ at these large distances, $R = \tau$ along the z axis, $R \approx \sqrt{2}\tau$ throughout the xy plane with values between these two limits in the interior. The pattern is symmetrical as previously mentioned.

Table 24. Values of R for PROSPH, System 16 (v: angular, $\tau = 0$, $B = 1$)

Z											
.344											
.242 .242											
.283 .201 .283											
.399 .292 .292 .399											
.541 .428 .382 .428 .541											
.694 .586 .517 .517 .586 .694											
.843 .748 .675 .647 .675 .748 .843											
.976 .899 .833 .794 .794 .833 .899 .976											
1.085 1.025 .972 .933 .919 .933 .972 1.025 1.085											
1.169 1.121 1.080 1.047 1.029 1.029 1.047 1.080 1.121 1.169											
1.232 1.192 1.158 1.130 1.112 1.106 1.112 1.130 1.158 1.192 1.232											
1.282 1.245 1.213 1.187 1.168 1.159 1.159 1.168 1.187 1.213 1.245 1.282											
1.323 1.286 1.254 1.227 1.207 1.195 1.190 1.195 1.207 1.227 1.254 1.286 1.323											

y

x

Linear: For $\tau = 0$, $R = \infty$ at the origin. At distances much smaller than a , $R \gg 1$ and the pattern is symmetrical about the $x = y$, $x = z$, and $y = z$ planes with a maximum in the center of the array. There are no sharp peaks or dips anywhere in the array.

There is a region of $R < 1$, that appears to be some what egg-shaped, extending from $0 \rightarrow x \rightarrow 0.6a$, $0 \rightarrow y \rightarrow 0.6a$ and $0.6a \rightarrow z \rightarrow 2a$ (Table 25). In this region, $R = 0$ at the point $(0, 0, a)$, increasing to 1 along the periphery.

The pattern of R values in the vicinity of a is symmetrical about the $x = y$ plane, with a maximum along the plane.

At distances much greater than a , $R \gg 1$ and the pattern is symmetrical about the $x = y$ plane with a minimum along the plane. The pattern appears to be increasing asymptotically toward a fixed array with increasing distance.

As τ increases from zero, the region of $R < 1$ shrinks and recedes toward the point $(0, 0, a)$ where R becomes 1 when τ becomes exceedingly large, possibly infinite. The region essentially has vanished for $\tau \approx 850a$.

For $\tau > 0$, $R = \infty$ at the origin and at distances much smaller than a , the R values and pattern are the same as for $\tau = 0$. At distances much greater than a , R along the z axis is greater than 1, and the same as for $\tau = 0$, with the remainder of the R values in the array also greater than 1. In these outer regions when $\tau \gg a$, $R = \tau$ along the xy plane with lower values in the interior. The pattern is symmetrical about the $x = y$ plane with a minimum along the plane. There are no sharp peaks or dips in the array.

A comparison of the linear case to the angular reveals that the patterns of R values from the two are somewhat similar except the region of $R < 1$ is smaller in the linear case.

System 17, OBLSPH (Table 26 and 27)

In this system, distances throughout the quadrant may be expressed in terms of the constant a . It was only necessary to use a single value of a in the survey, which was 10.

This is the remaining system in which two different surveys were conducted. As in PROSPH, v was considered as an angular measurement in the first case and as a linear measurement in the second. Again the parameter w was considered to only be an angular type.

Table 25. Values of R for PROSPH, System 16 (v: linear, $\tau = 0$, $B = 1$)

Z											
.344											
.261				.261							
.393			.281			.393					
.658		.521		.521		.658					
.974		.845		.792		.845		.974			
1.304		1.201		1.137		1.137		1.201		1.304	
1.616		1.555		1.509		1.491		1.509		1.555	
1.884		1.868		1.857		1.851		1.851		1.857	
2.091		2.112		2.139		2.161		2.170		2.161	
2.238		2.282		2.334		2.384		2.415		2.415	
2.334		2.387		2.450		2.514		2.565		2.584	
2.396		2.446		2.508		2.574		2.632		2.667	
2.435		2.479		2.532		2.591		2.647		2.688	
 	
2.435		2.479		2.532							

Angular: For $\tau = 0$, R is indeterminate at the origin. At distances much smaller than a , $R = 1$ along the z axis, zero throughout the xy plane with values between these two limits in the interior. The pattern is symmetrical about the $x = y$ plane with a maximum along the plane. There are no sharp peaks or dips in the array.

There is a region of $R \leq 1$ lying between the approximate limits of $0 \rightarrow x \rightarrow 3a$, $0 \rightarrow y \rightarrow 3a$, and $0 \rightarrow z \rightarrow a$. All R values in this region are between 0 and 1, with $R = 0$ in the xy plane for distances up to about a , and $R = 1$ at the periphery of the region in the z direction (Table 26).

At distances much greater than a , $R > 1$ and appears to be increasing asymptotically toward a fixed array with increasing distance. The pattern is symmetrical about the $x = y$, $x = z$, and $y = z$ planes with a slight minimum in the center of the array. There are no sharp peaks or dips anywhere in the array.

As τ increases from zero, the region of $R \leq 1$ shrinks and recedes toward the circular line $x^2 + y^2 = a^2$, $z = 0$ upon which R becomes 1 when $\tau = 1$.

For $\tau > 0$, $R = \infty$ at the origin. At distances much smaller than a , $R \gg 1$, and the pattern is symmetrical about the $x = y$, $x = z$, and the $y = z$ planes with a maximum in the center of the array. There are no sharp peaks or dips in the pattern. At distances much greater than a and τ in the vicinity of a , $R > 1$, with a symmetry of pattern like the one previously described. For $\tau \gg a$, R values range from τ along the z axis to $\sqrt{2}\tau$ along the xy plane with symmetry being only around the $x = y$ plane.

There is a slight minimum in this plane. In these outer regions, for all τ the R values increase asymptotically toward a fixed array with increasing distance. There are no sharp peaks or dips anywhere in the array.

Linear: For $\tau = 0$, $R = \infty$ at the origin. At distances much smaller than a , $R \gg 1$ and the pattern is symmetrical about the $x = y$, $x = z$, and $y = z$ planes with a maximum in the center of the array. There are no sharp peaks or dips anywhere in the array.

There is a region of $R \leq 1$ lying between the approximate limits of $0.7a \rightarrow x \rightarrow 2.5a$, $0.7a \rightarrow y \rightarrow 2.5a$, and $0 \rightarrow z \rightarrow 1.2a$ (Table 27). In this region, $R = 0$ at the points $(a, 0, 0)$ and $(0, a, 0)$ with values increasing toward 1 in directions away from these points.

Table 26. Values of R for OBLSPH, System 17 (v : angular, $\tau = 0$, $B = 1$)

Z											
1.323											
1.282						1.282					
1.232				1.239				1.232			
1.169			1.185			1.185			1.169		
1.085		1.113		1.124		1.113		1.085			
.976		1.014		1.038		1.038		1.014		.976	
.843		.886		.920		.933		.920		.886	
.694		.733		.770		.794		.794		.770	
.541		.567		.599		.625		.635		.625	
.399		.405		.424		.445		.458		.458	
.283		.259		.261		.273		.283		.287	
.242		.145		.121		.122		.127		.130	
.344		.193		.039		.000		.000		.000	

y

x

At distances much larger than a , $R > 1$ and is increasing asymptotically toward a fixed array with increasing distance. The array is symmetrical about the $x = y$ plane and appears to be approaching symmetry about the $x = z$ and $y = z$ planes also. There are no sharp peaks or dips anywhere in the array.

As τ increases from zero, the region of $R < 1$ shrinks and recedes toward the circular line $x^2 + y^2 = a^2$, $z = 0$ upon which R becomes 1 when $\tau = 1$.

For $\tau > 0$, $R = \infty$ at the origin. At distances much smaller than a , R is essentially the same as for $\tau = 0$. At distances much greater than a , $R > 1$ and increases asymptotically toward a fixed array with increasing distance. The pattern has the smallest value of R along the z axis, increasing to the maximum values along the xy plane. The array is symmetrical about the $x = y$ plane with a slight depression along the plane. For $\tau \gg a$, $R = \tau$ along the xy plane, decreasing to the constant value on the z axis. Symmetry here is the same as previously mentioned.

A comparison of the linear case to the angular reveals that the patterns of R values from the two are somewhat similar except the region of $R < 1$ is smaller in the linear case.

System 18, CONCAL (Table 28)

In this system, the two constants a and b are needed to specify the coordinate surfaces, and as mentioned previously these constants enter into the calculation of R only as the ratio b/a . Since τ is absent, the pattern of R values is fixed for the entire quadrant for given values of b/a .

The condition $b > a$ must always exist for there to be a solution. Therefore $b/a = 1$ is the lower limit of this ratio.

At $b/a = 1$, $R = \infty$ everywhere in space. At b/a slightly greater than 1 all $R \gg 1$ with $R = \infty$ throughout the xy and xz planes. A minimum exists along the radial lines near the x axis.

As b/a increases from 1 the R values over the entire quadrant decrease (Table 28). The pattern stays generally the same except the minimum shifts to the z axis where R becomes 1 when b/a becomes about 1000. Mathematical limitations prevented analysis for $b/a > 1000$.

For the conditions explored then, no symmetry in the pattern of R values nor any region of $R < 1$ was found.

Table 28. Values of R for CONCAL, System 18 (b/a = 2)

Z											
400.335											
3.387 356.242											
2.172 3.476 287.569											
1.756 2.010 2.845 189.489											
1.667 1.694 1.789 2.043 59.260											
1.736 1.672 1.632 1.592 1.537 101.217											
1.944 1.815 1.714 1.641 1.644 2.350 283.319											
2.329 2.134 1.973 1.858 1.839 2.132 3.955 473.218											
3.002 2.723 2.481 2.294 2.203 2.317 2.972 5.790 656.424											
4.223 3.820 3.459 3.156 2.946 2.897 3.147 4.078 7.657 822.788											
6.781 6.152 5.571 5.060 4.650 4.393 4.363 4.689 5.751 9.745 968.233											
14.600 13.319 12.121 11.035 10.104 9.390 8.963 8.894 9.267 10.342 14.054 1093.376											
1602.187 1471.089 1348.118 1234.807 1134.522 1051.734 991.601 958.733 955.609 981.745 1034.184 1108.752 1699.623											

y

x

System 19, ELPSOD (Table 29)

In this system, three constants (a , b , and c) are needed to specify the coordinate surfaces. Values of R will be constant along a given radial line for given values of a , b , and c . Therefore, for particular values of a , b , c , the entire quadrant must be explored. The constant a was fixed at 10 while b and c were varied over their full allowable range. The parameter τ was not present.

For a solution to exist, the condition $a > b > c$ must prevail. The constant c may then range from near zero to a , and b may range from c to a .

At the origin, $R = \infty$ for all a , b , c . A region of $R < 1$ exists for all c in the range from $c \ll a$ to $c \leq 0.6a$, and for all b in the range from $c \leq b \leq 0.6a$. The region is approximately cone-shaped with the apex near the point $(a, 0, 0)$ and lying with its axis about parallel to and near the x axis (Table 29). The value of R appears to be zero or near zero at the apex and 1 along the surface of the cone, with the interior values lying between these limits. Values of R in this cone appear to increase asymptotically toward a fixed array with increasing distance.

The cone is at its maximum size when b is near c . For small c as b increases toward its upper limit, the cone shrinks and recedes toward the approximate point $(0.8a, 0, 0)$ where $R = 1$ when b reaches the upper limit. For larger c , the initial size of the cone is smaller and as c increases to its upper limit the cone finally vanishes. When c is near its upper limit, the vanishing point appears to move out to around $(1.2a, 0, 0)$ and the cone no longer extends to infinity.

For all a , b , c , R appears to be infinitely large throughout the xz and yz planes except in the vicinity of the point $(a, 0, 0)$ which has just been previously discussed. Along the xy plane, R ranges from less than 1 to infinity, depending on the values of b and c and whether or not the point lies within the cone.

At distances much greater than a and in regions outside that where the cone would, for certain combinations of b and c , exist, $R > 1$ and is decreasing asymptotically toward a fixed array with increasing distance. The pattern is still generally similar to that shown in Table 29. In the region where the cone would exist, $R > 1$ but increasing asymptotically toward a fixed array.

Table 29. Values of R for ELPSOD, System 19 (a = 10, b = 5, c = 1, B = 10)

Z											
8742.531											
5638.953 5701.718											
5263.594 7.309 5054.102											
4998.078 5.339 5.104 4302.150											
4363.082 4.775 2.981 4.027 3442.806											
4873.224 4.577 2.418 2.122 3.057 2497.088											
5030.916 4.603 2.241 1.624 1.548 2.098 1517.572											
5324.118 4.810 2.248 1.486 1.192 1.105 1.237 579.050											
5730.427 5.162 2.372 1.503 1.123 .937 .837 .776 246.693											
6224.685 5.624 2.573 1.605 1.165 .940 .825 .795 1.004 915.599											
6785.028 6.165 2.827 1.755 1.258 .997 .862 .816 .890 1.414 1417.697											
7394.947 6.766 3.120 1.940 1.324 1.083 .917 .840 .847 1.003 1.738 1767.306											
8046.010 228.393 205.839 182.182 157.576 132.792 109.252 88.756 72.870 62.382 57.228 56.805 1990.819											

y

x

MECHANICAL GENERATION OF CURVILINEAR COORDINATES

A measuring machine can be made to function with any curvilinear coordinate system by incorporating a suitable choice of linkages, slides, pivots, etc., assembled in such a way that the transformation equations will be satisfied. An error analysis of a mechanical apparatus of this kind involving the tolerances in the slides, pivots, etc., is beyond the scope of this study.

However, in order to illustrate the basic approach in designing a measuring machine of this type, a typical case will be discussed.

Consider System 13, (BIPALT). The transformation equations are given on page 52 and a sketch of the coordinate curves is shown in Figure 16. A satisfactory solution would be a mechanism that generated the numbers u , v , w when the measuring probe was positioned at a point x , y , z in space. Figure 24 shows a simplified sketch of one arrangement that would achieve this. In the figure AA and BB are rigid rods forming the orthogonal x and y coordinate axes. Angle CPD is also exactly 90 degrees and is formed by a rigid rod pivoting on the slider at P and sliding in the two sliders at E and F. The rod KHJ is a rigid T-shaped rod sliding at E, P, and G. Angle HMK is a right angle. RN is a straight rigid rod pivoting at M, the midpoint. Rods PN and RG are equal in length and serve to keep M exactly midway between P and G. The stationary pivot point G is located a distance a along the x axis from the origin. In this particular example, the mechanism is only intended to cover the positive quadrant. Furthermore, as in other systems, the range of coverage would be restricted to those regions where $R < 1$.

The measuring probe or stylus is located at P. When the probe is positioned at some point, as shown in Figure 24, this point will always lie at the intersection of the two orthogonal coordinate curves u and v . The values of u and v will now automatically be indicated by the distances of F and E from the origin on suitable scales as shown. The third dimension, w , which in this case is in the z direction, could be achieved by merely translating the entire mechanism vertically, possible in a similar manner to that used on conventional rectangular type measuring machines. Thus, by means of this mechanism the location of a point in space may be specified in terms of the curvilinear coordinates u , v , w .

From a practical viewpoint, however, the u , v , w are somewhat meaningless since these values don't directly appear on the usual mechanical drawing. All dimensions are customarily given in x , y , z coordinates. Therefore, it becomes necessary to convert the u , v , w values into x , y , z values, which may be done by means of the transformation equations on page 59. It would seem at first here that the effort has been

Figure 24. Two Dimensional Mechanism for Generation of Curvilinear Coordinates (u , v).

circular due to this conversion and that the entire usefulness of curvilinear coordinates has been nullified. However, further reflection shows that this is not really true. By using a mechanism that directly generates the u , v , w numbers, the error associated with the point will be the error arising from errors in u , v , w , even though the location is finally converted to x , y , z numbers. This conversion introduces essentially no additional error because it can be done on a calculator or computer to more significant figures than the overall accuracy required by the measurement. Therefore, an advantage is gained in the inherent accuracy of a measured point by building measuring machines around those coordinate systems found in this report to contain regions of $R < 1$.

Artobolevskii's book "Mechanisms for the Generation of Plane Curves" listed in the Bibliography is an excellent source of linkage mechanisms that will produce the types of curves found in the coordinate systems in this report.

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CONCLUSIONS AND RECOMMENDATIONS

The following conclusions were drawn from the study of the 19 orthogonal coordinate systems listed in Table 1:

- Orthogonal coordinate systems having inherent errors smaller than in the ordinary rectangular one do exist.
- Of the 19 systems examined, 10 displayed smaller inherent errors, under certain conditions, than exist in a rectangular system. These systems (numbered 8, 9, 10, 12, 13, 14, 15, 16, 17, and 19) would be somewhat complex in hardware. One of the simplest, as a typical example, is system 12, Bipolar, a set of cylinders intersecting parallel planes shown on Page 56.
- Three systems studied have inherent errors exactly equal to those in a rectangular system throughout all space (numbered 4, 6, and 11). A typical example is system 6, Coaxial Cylindrical Circle, which is shown on Page 44.
- Two systems were found which had alternates defined by a variation in one parameter u which effected a change in the pattern of errors. These systems are shown as numbers 3 and 4, Parabolic Cylindrical, and numbers 12 and 13, Bipolar.
- The commonly known cylindrical and spherical systems (numbered 1 and 2) are not inherently better than the rectangular system.
- Coordinate systems constructed by rotating a basic family of two-dimensional curves about one of the axes do not have a better error pattern; i. e., if a cylindrical type has no errors smaller than the rectangular system, the corresponding spherical type will not.
- In every system not requiring constants to specify the orientation of the coordinate surfaces, the ratio of the error in the given system to that in a rectangular one (R) is constant along a given radial line. Thus, all space may be surveyed by an examination at only one distance.
- In those systems containing τ , which is the ratio of the angular error to the linear error, the region where $R < 1$ will be the largest when $\tau = 0$ and will decrease as τ increases.

- In five systems (12, 14, 15, 16, and 17), in addition to the regular analysis, one of the coordinate parameters normally considered an angular type number was changed to be a linear type and another analysis made. It was found that in all five cases the region of $R < 1$ was smaller for the linear cases. For systems 12 and 15, the linear case will be superior to the angular for $\tau \gg 0$ because at these high levels of τ the region of $R < 1$ essentially vanishes. In systems 15, 16, and 17 the patterns were somewhat similar between the linear and angular cases but differed for 12 and 14.

If additional effort is to be made on this type of study, it is recommended that mechanisms similar to that shown in the section on Mechanical Generation of Curvilinear Coordinates, Page 117, be devised for the remainder of the coordinate systems. Real machines could be constructed and a calibration made on each to determine the overall uncertainty of the complete device. Additional coordinate systems, including non-orthogonal types, might be investigated to find possibly better systems. Finally, the point concept in the error analysis could be replaced by the line concept and the surveys rerun.

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APPENDIX A

DERIVATION OF NON-ORTHOGONAL (OBLIQUE) TRAJECTORIES

The following discussion is based on two-dimensional systems only. Three-dimensional theory is beyond the scope of this analysis.

$$\text{Let } F(x, y, u) = 0 \quad (1)$$

represent a one-parameter family of curves where u is the parameter specifying each member of the family. The desired family of curves is a set wherein each member intersects every curve of Equation (1) at a constant angle $\alpha \neq 90^\circ$. Such a family is termed oblique trajectories and combined with Equation (1) is the basic set of curves in non-orthogonal coordinate systems.

The differential equation of the family of curves in Equation (1) is

$$\frac{dy}{dx} = f(x, y) \quad (2)$$

after elimination of the parameter u . The tangent line at the point (x, y) will have an angle of inclination θ given by

$$\theta = \tan^{-1} [f(x, y)] \quad (3)$$

as shown in Figure A-1. The non-orthogonal trajectory v , intersecting

Figure A-1. Oblique Trajectory and the Included Angle α .

u at (x, y) at an angle α , will have an angle of inclination ϕ equal to

$$\phi = \theta + \alpha \quad (4)$$

Using a trigonometric identity with Equation (4), the $\tan \phi$ becomes

$$\tan \phi = \frac{\tan \theta + \tan \alpha}{1 - \tan \theta \tan \alpha} \quad (5)$$

Eliminating θ by using Equation (3) and replacing $\tan \phi$ with dy/dx , Equation (5) reduces to

$$\frac{dy}{dx} = \frac{f(x, y) + \tan \alpha}{1 - f(x, y) \tan \alpha} \quad (6)$$

which is the differential equation of the family of oblique trajectories. The parameter v specifying each member, will be introduced as the constant of integration when Equation (6) is integrated. Thus, a family of oblique trajectories may be obtained by inserting the derivative of the equation of the original family into Equation (6) where the term $f(x, y)$ appears and solving the resulting expression.

As an example, let it be required to find a family of oblique trajectories that intersect the family of straight lines $y = ux$ at an angle of 45° . The differential equation is

$$\frac{dy}{dx} = u \quad (7)$$

Eliminating u gives

$$\frac{dy}{dx} = \frac{y}{x} \quad (8)$$

The right side of Equation (8), y/x , will now replace the term $f(x, y)$ in Equation (6) giving

$$\frac{dy}{dx} = \frac{\frac{y}{x} + \tan 45^\circ}{1 - \frac{y}{x} \tan 45^\circ} \quad (9)$$

It remains only to solve this differential equation, which reduces to

$$\frac{dy}{dx} = \frac{x+y}{x-y} \quad (10)$$

Since it is homogeneous, the substitution $y = rx$ may be made yielding

$$r + x \frac{dr}{dx} = \frac{1+r}{1-r} \quad (11)$$

which may be simplified to

$$\frac{r-1}{r^2+1} dr = -\frac{1}{x} dx \quad (12)$$

Integration of Equation (12) yields

$$(1/2) \ln (r^2 + 1) - \tan^{-1} r = -\ln |x| - \ln |v| \quad (13)$$

where the arbitrary constant of integration was set equal to $-\ln |v|$ for convenience. Equation (13) reduces to

$$\ln \left[v^2 x^2 (r^2 + 1) \right] - 2 \tan^{-1} r = 0 \quad (14)$$

Replacing r by y/x , the family of oblique trajectories is obtained in the form

$$\ln \left[v^2 (x^2 + y^2) \right] - 2 \tan^{-1} \frac{y}{x} = 0 \quad (15)$$

Although Equation (15) represents a solution to the problem, the complete analysis requires the formation of the transformation equations, which are not readily attainable from Equation (15).

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APPENDIX B

TEST FOR OTHOGONALITY

It is required to determine if a curvilinear coordinate system has mutually orthogonal coordinate surfaces. Let it be assumed that the transformation equations

$$\left. \begin{aligned} x &= f_1(u, v, w) \\ y &= f_2(u, v, w) \\ z &= f_3(u, v, w) \end{aligned} \right\} \quad (1) \quad \left. \begin{aligned} u &= F_1(x, y, z) \\ v &= F_2(x, y, z) \\ w &= F_3(x, y, z) \end{aligned} \right\} \quad (2)$$

that relate the curvilinear system (u, v, w) to the orthogonal rectangular system (x, y, z) are known.

The orthogonality will be checked at an arbitrarily chosen point P that is intersected by the three coordinate surfaces u, v, w . Each pair of these surfaces intersect in curves called coordinate curves, and if these curves intersect orthogonally at P then the coordinate surfaces do likewise.

To check the orthogonality of these coordinate curves, which will also be specified by u, v, w , a vector in the direction of the tangent to each of these curves will be constructed at P . This is done by forming the position vector r from the origin to P as shown in Figure B-1. The vector r is given by

$$r = xi + yj + zk \quad (3)$$

where i, j, k are the mutually orthogonal unit vectors in the rectangular system.

Figure B-1. Tangent Vectors to the u, v Curves.
(Third dimension suppressed for simplicity)

From elementary vector theory, the partial derivative of $\mathbf{r}$ with respect to u will be a vector in the direction of the tangent to u at P . Taking the derivative of $\mathbf{r}$ in Equation (3), there results

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} \quad (4)$$

Similarly, for the other two coordinates

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k} \quad (5)$$

$$\frac{\partial \mathbf{r}}{\partial w} = \frac{\partial x}{\partial w} \mathbf{i} + \frac{\partial y}{\partial w} \mathbf{j} + \frac{\partial z}{\partial w} \mathbf{k} \quad (6)$$

The partial derivatives on the right side of these equations may be evaluated using Equation (1).

Again from vector theory, if the dot product of two vectors is zero, then the angle between them is exactly 90° . The condition for mutual orthogonality will then be that the dotted products of all possible pairs of Equations (4), (5), and (6) vanish. Letting L_1 , L_2 , L_3 represent these pairs, then

$$\left. \begin{aligned} L_1 &= \frac{\partial \mathbf{r}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} \\ L_2 &= \frac{\partial \mathbf{r}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial w} \\ L_3 &= \frac{\partial \mathbf{r}}{\partial w} \cdot \frac{\partial \mathbf{r}}{\partial u} \end{aligned} \right\} \quad (7)$$

Substituting Equations (4), (5), and (6) into (7) gives the final expressions (scalars) for the orthogonality test:

$$\begin{aligned} L_1 &= \left(\frac{\partial x}{\partial u} \right) \left(\frac{\partial x}{\partial v} \right) + \left(\frac{\partial y}{\partial u} \right) \left(\frac{\partial y}{\partial v} \right) + \left(\frac{\partial z}{\partial u} \right) \left(\frac{\partial z}{\partial v} \right) \\ L_2 &= \left(\frac{\partial x}{\partial v} \right) \left(\frac{\partial x}{\partial w} \right) + \left(\frac{\partial y}{\partial v} \right) \left(\frac{\partial y}{\partial w} \right) + \left(\frac{\partial z}{\partial v} \right) \left(\frac{\partial z}{\partial w} \right) \\ L_3 &= \left(\frac{\partial x}{\partial w} \right) \left(\frac{\partial x}{\partial u} \right) + \left(\frac{\partial y}{\partial w} \right) \left(\frac{\partial y}{\partial u} \right) + \left(\frac{\partial z}{\partial w} \right) \left(\frac{\partial z}{\partial u} \right) \end{aligned} \quad (8)$$

The system is mutually orthogonal if

$$L_1 = L_2 = L_3 = 0 \quad (9)$$

It can be seen then that to derive a three-dimensional orthogonal system in a completely rigorous manner, one would begin with Equations (8) and (9) and work backward to obtain Equations (1) and (2). Obviously this would be far too complicated since in general the differential equations would be non-linear.

A more satisfactory approach is to begin with a known system and use Equation (8) to check for orthogonality. For example, consider the system described by the transformation equations.

$$\left. \begin{aligned} x &= \frac{1}{2v} \sqrt{-1 + \sqrt{1 + 8uv^2}} \\ y &= \frac{1}{4v} \left(-1 + \sqrt{1 + 8uv^2} \right) \\ z &= w \end{aligned} \right\} \quad (10) \quad \left. \begin{aligned} u &= x^2 + 2y^2 \\ v &= y/x^2 \\ w &= z \end{aligned} \right\} \quad (11)$$

From the first set of these, the needed partial derivatives may be formed as follows:

$$\begin{aligned} \frac{\partial x}{\partial u} &= \frac{1}{2xA} & \frac{\partial y}{\partial u} &= \frac{v}{A} & \frac{\partial z}{\partial u} &= 0 \\ \frac{\partial x}{\partial v} &= \frac{u}{Axv} - \frac{x}{v} & \frac{\partial y}{\partial v} &= \frac{2u}{A} = x^2 & \frac{\partial z}{\partial v} &= 0 \\ \frac{\partial x}{\partial w} &= 0 & \frac{\partial y}{\partial w} &= 0 & \frac{\partial z}{\partial w} &= 1 \end{aligned}$$

where $A = \sqrt{1 + 8uv^2}$

Substituting into Equation (8) gives:

$$L_1 = \left(\frac{1}{2xA} \right) \left(\frac{u}{Axv} - \frac{x}{v} \right) + \left(\frac{v}{A} \right) \left(\frac{2u}{A} - x^2 \right) + (0)(0)$$

$$L_1 = \frac{u - x^2 A + 4x^2 uv^2 - 2x^4 Av^2}{2x^2 A^2 v}$$

For convenience let $K = 2x^2 A^2 v$. Using Equation (10) to eliminate x in the numerator:

$$L_1 = \frac{1}{K} \left[u - \frac{A^2}{4v^2} + \frac{A}{4v^2} + Au - u - \frac{A^3}{8v^2} + \frac{A^2}{4v^2} - \frac{A}{8v^2} \right]$$

$$L_1 = \frac{1}{8Kv^2} (A + Au 8v^2 - A^2)$$

$$\text{but } 8uv^2 = A^2 - 1$$

$$\therefore L_1 = \frac{1}{8Kv^2} (A + A^3 - A - A^3) = 0$$

$$L_2 = \left(\frac{u}{Axv} - \frac{x}{v} \right) (0) + \left(\frac{2u}{A} - x^2 \right) (0) + (0)(1) = 0$$

$$L_3 = (0)(0) + (0)\left(\frac{v}{A}\right) + (1)(0) = 0$$

Therefore the system represented by Equations (10) and (11) is orthogonal.

APPENDIX C

COMPUTER PROGRAMS USED IN ANALYSIS

A total of 20 computer programs were used in the survey and they are listed on the following pages. There is a main program called RADCUR and 19 subroutines, one for each system analyzed.

The programs were written in CPS language and executed on an IBM-360 computer by means of an IBM Model 2741 remote terminal.

In general, the symbols used in these computer programs apply only to the program in which they appear. There are a few exceptions, however, such as in the case of R, a, b, and c. These were used throughout the entire survey with the same unique meaning in every instance. The symbols X, Y, Z have the same meaning in all the computer programs and are the same as the x, y, z in the report. Finally, τ in the report is represented by r in the main computer program.


```

1.      /*Cylindrical coordinates*/;
2.      CYL:  PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.            USQ=X*X+Y*Y;
4.            WSQ=Z*Z;
5.            HU,HV=1;
6.            HV=USQ;
7.            DD=USQ+Z*Z;
8.            R=SQRT((HU*USQ+HV*rSQ+HW*WSQ)/DD);
9.            RETURN (R);
10.     END CYL;

```

```

1.      /*spherical coordinates*/;
2.      SPH:  PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.            USQ=X*X+Y*Y+Z*Z;
4.            V=ATAN(Z/SQRT(X*X+Y*Y));
5.            HU=1;
6.            HW=USQ*COS(V)**2;
7.            HV=USQ;
8.            R=SQRT((HU*USQ+HV*rSQ+HW*rSQ)/USQ);
9.            RETURN (R);
10.     END SPH;

```

```

1.      /*parabolic cylindrical coordinates*/;
2.      PARCYL: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.            USQ=SQRT(X*X+Y*Y)-X;
4.            VSQ=Y*Y/USQ;
5.            WSQ=Z*Z;
6.            HU,HV=USQ+VSQ;
7.            HW=1;
8.            DD=X*X+Y*Y+Z*Z;
9.            R=SQRT((HU*USQ+HV*VSQ+HW*WSQ)/DD);
10.     RETURN (R);
11.     END PARCYL;

```

```

1.      /* Parabolic cylindrical coordinates (alternate) */;
2.  PARALT: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      A=SQRT(X*X+Y*Y);
4.      U=A-X;
5.      V=A+X;
6.      WSQ=Z*Z;
7.      HU=(1+V/U)/4;
8.      HV=(1+U/V)/4;
9.      HW=1;
10.     DD=A*A+Z*Z;
11.     R=SQRT((HU*U+U+HV*V+V+HW*WSQ)/DD);
12.     RETURN (R);
13.     END PARALT;

```

```

1.      /*paraboloidal coordinates*/;
2.  PARLOD: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      USQ=SQRT(X*X+Y*Y+Z*Z)-Z;
4.      VSQ=(X*X+Y*Y)/USQ;
5.      HU,HV=USQ+VSQ;
6.      HW=USQ*VSQ;
7.      DD=X*X+Y*Y+Z*Z;
8.      R=SQRT((HU*USQ+HV*VSQ+HW*rSQ)/DD);
9.      RETURN (R);
10.     END PARLOD;

```

```

1.      /*coaxial cylindrical circular coordinates*/;
2.  CYLCIR: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      USQ=((X*X+Y*Y)/(2*X))**2;
4.      VSQ=USQ*X*X/(Y*Y);
5.      WSQ=Z*Z;
6.      A=(USQ+VSQ)**2;
7.      HU=4*VSQ*VSQ/A;
8.      HV=4*USQ*USQ/A;
9.      HW=1;
10.     DD=X*X+Y*Y+Z*Z;
11.     R=SQRT((HU*USQ+HV*VSQ+HW*WSQ)/DD);
12.     RETURN (R);
13.     END CYLCIR;

```

```

1.      /* Coaxial spheroidal coordinates */;
2. COXSPH: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      W=ATAN(Y/X);
4.      A=Y/SIN(W);
5.      USQ=((A*A+Z*Z)/(2*Z))**2;
6.      VSQ=USQ*Z*Z/(A*A);
7.      B=(USQ+VSQ)**2;
8.      HU=4*VSQ*VSQ/B;
9.      HV=4*USQ*USQ/B;
10.     HW=4*VSQ*USQ*USQ/B;
11.     DD=X*X+Y*Y+Z*Z;
12.     R=SQRT((HU*USQ+HV*VSQ+HW*rSQ)/DD);
13.     RETURN (R);
14.     END COXSPH;

```

```

1.      /* Parabolic cylindrical elliptical coordinates */;
2. ELLPAR: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      B=X*X;
4.      U=B+2*Y*Y;
5.      V=Y/B;
6.      E=V*V;
7.      D=1+8*U*E;
8.      A=SQRT(D);
9.      WSQ=Z*Z;
10.     HU=1/(4*B*D)+E/D;
11.     HV=(U/(A*X*V)-X/V)**2+(2*U/A-B)**2;
12.     HW=1;
13.     DD=B+Y*Y+Z*Z;
14.     R=SQRT((HU*U+HV*V+HW*WSQ)/DD);
15.     RETURN (R);
16.     END ELLPAR;

```

```

1.      /* Isosceles hyperbolic cylindrical coordinates */;
2. HYPER: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      B=X*X;
4.      U=(B-Y*Y)**2;
5.      V=(X*Y)**2;
6.      WSQ=Z*Z;
7.      A=SQRT(U+4*V);
8.      HU=1/(4*A);
9.      HV=(V+(A-V/B)**2)/(A*A*B);
10.     HW=1;
11.     DD=B+Y*Y+Z*Z;
12.     R=SQRT((HU*U+HV*V+HW*WSQ)/DD);
13.     RETURN (R);
14.     END HYPER;

```



```

1.      /* Isosceles hyperbolic spherical coordinates */;
2.  HYPSPH: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      C=X*X+Y*Y;
4.      V=Z*SQRT(C);
5.      U=C-Z*Z;
6.      A=SQRT(U*U+4*V*V);
7.      HU=(C+Z*Z)/(4*A*A);
8.      HV=4*V*V*(C+(A/Z-Z)**2)/(A*A*(A+U)**2);
9.      HW=X*X+Y*Y;
10.     DD=C+Z*Z;
11.     R=SQRT((HU*U*U+HV*V*V+HW*rSQ)/DD);
12.     RETURN (R);
13.     END HYPSPH;

```

```

1.      /* Cardioidic cylindrical coordinates */;
2.  CARD:  PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      A=X*X+Y*Y;
4.      U=A/(SQRT(A)-X);
5.      V=A/(SQRT(A)+X);
6.      B=4/(U+V)**3;
7.      WSQ=Z*Z;
8.      HU=B*V**3;
9.      HV=B*U**3;
10.     HW=1;
11.     DD=A+Z*Z;
12.     R=SQRT((HU*U*U+HV*V*V+HW*WSQ)/DD);
13.     RETURN (R);
14.     END CARD;

```

```

1.      /* Bipolar coordinates */;
2.  BIPOL: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      LET COSH(1)=(EXP(1)+EXP(-1))/2;
4.      a1=a*a;
5.      E=Y*Y;
6.      D=X*X;
7.      A=D+E+a1;
8.      B=2*a*X;
9.      V=LOG((A+B)/(A-B))/2;
10.     F=2*a*Y/(D+E-a1);
11.     IF F<0 THEN U=ATAN(F)+4*ATAN(1); ELSE U=ATAN(F);
12.     WSQ=Z*Z;
13.     HU,HV=a1/(COSH(V)-COS(U))**2;
14.     HW=1;
15.     DD=D+E+Z*Z;
16.     R=SQRT((HU*rSQ+HV*V*V+HW*WSQ)/DD);
17.     RETURN (R);
18.     END BIPOL;

```

```

1.      /* Bipolar (alternate) coordinates */;
2.  BIPALT: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      D=X*X+Y*Y;
4.      a1=a*a;
5.      U=(D+a1)/(2*X);
6.      V=(D-a1)/(2*Y);
7.      WSQ=Z*Z;
8.      B=V*V+a1;
9.      C=U*U-a1;
10.     A=SQRT(C/B);
11.     HU=X*X*X*X*((V*U-A*B)**2+a1*a1)/(C*B*a1*a1);
12.     HV=HU*A*A*A*A;
13.     HW=1;
14.     DD=D+Z*Z;
15.     R=SQRT((HU*U*U+HV*V*V+HW*WSQ)/DD);
16.     RETURN (R);
17.     END BIPALT;

```

```

1.      /* Toroidal coordinates */;
2.  TOROID: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      LET SINH(I)=((EXP(I)-EXP(-I))/2)**2;
4.      LET COSH(I)=(EXP(I)+EXP(-I))/2;
5.      W=ATAN(Y/X);
6.      B=(Y/SIN(W))**2;
7.      D=2*a*Z/(B-a*a+Z*Z);
8.      IF D<0 THEN U=ATAN(D)+4*ATAN(1); ELSE U=ATAN(D);
9.      A=a*SIN(U)/Z+COS(U);
10.     V=LOG(A+SQRT(A*A-1));
11.     C=(COSH(V)-COS(U))**2;
12.     HU,HV=a*a/C;
13.     HW=a*a*SINH(V)/C;
14.     DD=X*X+Y*Y+Z*Z;
15.     R=SQRT((HU*rSQ+HV*V*V+HW*rSQ)/DD);
16.     RETURN (R);
17.     END TOROID;

```

```

1.      /* Elliptic cylindrical coordinates */;
2.  ELLCYL: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      LET SINH(I)=((EXP(I)-EXP(-I))/2)**2;
4.      LET TAN(I)=SIN(I)/COS(I);
5.      a1=a*a;
6.      G=X*X+Y*Y+a1;
7.      A=(G-SQRT(G*G-4*a1*X*X))/(2*a1);
8.      V=ATAN(SQRT(1/A-1));
9.      B=X*TAN(V);
10.     U=.5*LOG((B+Y)/(B-Y));
11.     WSQ=Z*Z;
12.     HU,HV=a1*(SINH(U)+SIN(V)*SIN(V));
13.     HW=1;
14.     DD=X*X+Y*Y+WSQ;
15.     R=SQRT((HU*U*U+HV*rSQ+HW*WSQ)/DD);
16.     RETURN (R);
17.     END ELLCYL;

```

```

1.      /*prolate spheroidal coordinates*/;
2.  PROSPH: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      LET SINH(I)=((EXP(I)-EXP(-I))/2)**2;
4.      LET TAN(I)=SIN(I)/COS(I);
5.      a1=a*a;
6.      W=ATAN(Y/X);
7.      A=(Z*SIN(W))**2;
8.      B=Y*Y+(a1/(Z*Z)-1)*A;
9.      V=ATAN(SQRT((B+SQRT(B*B+4*Y*Y*A))/(2*A)));
10.     C=Z*TAN(V)*SIN(W);
11.     U=.5*LOG((C+Y)/(C-Y));
12.     D=SIN(V)*SIN(V);
13.     HU,HV=a1*(D+SINH(U));
14.     HW=a1*D*SINH(U);
15.     DD=X*X+Y*Y+Z*Z;
16.     R=SQRT((HU*U*U+HV*rSQ+HW*rSQ)/DD);
17.     RETURN (R);
18.     END PROSPH;

```

```

1.      /*oblate spheroidal coordinates*/;
2.  OBLSPH: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      LET SINH(I)=((EXP(I)-EXP(-I))/2)**2;
4.      LET COSH(I)=((EXP(I)+EXP(-I))/2)**2;
5.      LET TAN(I)=SIN(I)/COS(I);
6.      a1=a*a;
7.      W=ATAN(Y/X);
8.      A=(Z*SIN(W))**2;
9.      B=Y*Y-(a1/(Z*Z)+1)*A;
10.     V=ATAN(SQRT((-B+SQRT(B*B+4*Y*Y*A))/(2*Y*Y)));
11.     C=Y*TAN(V);
12.     U=.5*LOG((C+SQRT(A))/(C-SQRT(A)));
13.     HU,HV=a1*(SINH(U)+SIN(V)*SIN(V));
14.     HW=a1*COSH(U)*COS(V)*COS(V);
15.     DD=X*X+Y*Y+Z*Z;
16.     R=SQRT((HU*U+HV*rSQ+HW*rSQ)/DD);
17.     RETURN (R);
18.     END OBLSPH;

```

```

1.      /* Conical coordinates */;
2.  CONCAL: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      a1=a*a;
4.      b1=b*b;
5.      R=X*X+Y*Y+Z*Z;
6.      P=(a1+b1)*X*X+a1*Z*Z+b1*Y*Y;
7.      W=(P-SQRT(P*P-4*R*a1*b1*X*X))/(2*R);
8.      V=a1/(W*Y*Y*(a1-b1)/(X*X*b1*(a1-W))+1);
9.      U=X*X*a1*b1/(V*W);
10.     HU=1;
11.     HV=U*(W-V)/((V-a1)*(V-b1));
12.     HW=U*(V-W)/((W-a1)*(W-b1));
13.     DD=X*X+Y*Y+Z*Z;
14.     R=SQRT((HU*U+HV*V+HW*W)/DD);
15.     RETURN (R);
16.     END CONCAL;

```

```

1.      /*ellipsoidal coordinates*/;
2.  ELPSOD: PROCEDURE (X,Y,Z,rSQ,a,b,c);
3.      DECLARE S(3);
4.      a1=a*a;
5.      b1=b*b;
6.      c1=c*c;
7.      A1=a1+b1+c
8.      A2=a1*c1;
9.      A3=b1*c1;
10.     A4=a1*b1;
11.     A5=a1*b1*c1;
12.     A6=A2+A3+A4;
13.     A7=b1*c1;
14.     A8=a1*c1;
15.     A9=a1*b1;
16.     X1=X*X;
17.     Y1=Y*Y;
18.     Z1=Z*Z;
19.     A=X1+Y1+Z1-A1;
20.     B=A6-X1*A7-Y1*A8-Z1*A9;
21.     C=X1*A3+Y1*A2+Z1*A4-A5;
22.     G=A*A/9-B/3;
23.     F=A*A*A/27-A*B/6+C/2;
24.     K=-SQRT(F*F/(G*G));
25.     PP=2*ATAN(1);
26.     IF K=0 THEN E=PP; ELSE E=ATAN(SQRT(1-K*K)/K)+PP*(1-SIGN(K));
27.     T=E*30/PP;
28.     IF F=0 THEN D=1; ELSE D=SIGN(F);
29.     D1=D*2*SQRT(G);
30.     AA=A/3;
31.     S(1)=D1*COSD(T)-AA;
32.     S(2)=D1*COSD(T-120)-AA;
33.     S(3)=D1*COSD(T+120)-AA;
34.  L10: DO J=1 TO 3;
35.        KK=3-J;
36.  L20: DO I=1 TO KK;
37.        IF S(I)-S(I+1)<=0 THEN GO TO L1;
38.        TEMP=S(I);
39.        S(I)=S(I+1);
40.        S(I+1)=TEMP;
41.  L1:   END L20;
42.     END L10;
43.     U=S(1);
44.     V=S(2);
45.     W=S(3);
46.     HU=(V-U)*(W-U)/(4*(a1-U)*(b1-U)*(c1-U));
47.     HV=(W-V)*(U-V)/(4*(a1-V)*(b1-V)*(c1-V));
48.     HW=(U-W)*(V-W)/(4*(a1-W)*(b1-W)*(c1-W));
49.     DD=X1+Y1+Z1;
50.     R=SQRT((HU*U+HV*V+HW*W)/DD);
51.     RETURN (R);
52.  END ELPSOD;

```

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