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COMPARISON OF A SIMPLE THEORETICAL
TREATMENT OF CRITICAL ARRAYS OF FISSIONABLE
UNITS WITH EXPERIMENT

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CRITICAL ARRAYS OF FISSIONABLE UNITS WITH EXPERIMENT

by

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ABSTRACT

Calculations of the critical sizes of cubic arrays of interacting fissionable units are compared with critical experiments. The units were of two types: vessels containing 5 liters of an aqueous solution of highly enriched uranium and cylinders of highly enriched uranium metal. The arrays were surrounded by various thicknesses of hydrogenous reflectors. Agreement between calculation and experiment is reasonably good when consideration is given to the simplicity of the method of calculation. Curves are presented for computing critical sizes of cubic arrays of 8, 27, 64, or 125 identical units as a function of the albedo of the reflector surrounding the array and of the reactivity of an individual unit.

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COMPARISON OF A SIMPLE THEORETICAL TREATMENT OF CRITICAL ARRAYS OF FISSIONABLE UNITS WITH EXPERIMENT

INTRODUCTION

Over the past several years a simple method has been developed at the Savannah River Laboratory for computing the interaction in groups of fissionable units⁽¹⁻³⁾. Application of the method to critical experiments performed with solutions of U^{235} confined within pairs of parallel slabs, within groups of three parallel slabs, within pairs of perpendicular slabs, within groups of cylinders containing from two to seven cylinders, and within a slab parallel to a reflecting concrete wall has shown the method to be reasonably accurate and generally conservative in the sense that critical systems are generally calculated to be somewhat supercritical⁽¹⁾. Until recently, however, the only large arrays studied experimentally to which the method could be applied were cubic arrays of composite metal spheres equivalent in reactivity (when isolated) to 20 kg of uranium containing 93.5% U^{235} ⁽⁴⁾. Over-all source multiplications were measured as the arrays were built toward 27 units ($3 \times 3 \times 3$). The maximum multiplication attained was about 14. Application of the method to these experiments resulted in approximately a 5% underestimate of k_{eff} when allowance was made for the difference between the actual units and the 20-kg uranium spheres for which the calculations were made.⁽²⁾

Recently^(5,6), two series of critical experiments have been performed by Thomas with large, essentially cubic arrays of units. In his first series of experiments, 5-liter units of an aqueous solution of U^{235} containing 384.3 g/l were arranged in reflected and unreflected arrays ranging from 8 to 125 units in size. In his second series, the units were cylinders of uranium (93.2% U^{235}) metal having diameters of 11.506 cm and lengths of 10.770 or 13.462 cm, and studies were made of reflected and unreflected 8- and 27-unit arrays of cylinders of both sizes. In the present report the method for calculating interactions is applied to these recent experiments.

SUMMARY AND CONCLUSIONS

Tables III and VI show a comparison of calculations with critical experiments performed with reflected and unreflected arrays of cylinders of uranium solution and cylinders of uranium metal. The method of computing interaction has reasonable accuracy when consideration is given to its

simplicity. The chief trends shown in the comparisons are increases in the values of k_{eff} calculated for actually critical arrays as the size of the array increases and as the thickness of the surrounding reflector increases.

In practical applications of the method, small unreflected arrays calculated to have $k_{eff} \leq 0.9$ will be subcritical by an adequate margin of safety provided care is taken in the calculations not to underestimate the reactivity of individual units. Greater values of k_{eff} are acceptable for large arrays or for arrays surrounded by thick reflectors provided the reflector albedos are chosen in a manner that gives values consistent with those employed in the present calculations.

As an aid to computations for arrays, curves are plotted in Figures 1-4 giving the albedo of the environment of a unit in cubic arrays of 8, 27, 64, and 125 spherical units as a function of the ratio of sphere diameter to center-to-center spacing and of the albedo of the reflector. The curves are based on an evaluation of the interaction integral for a pair of equal spheres, which is plotted in Figure A-2. Programs were written in FORTRAN for evaluating the integral and computing the points used in plotting Figures 1-4; the calculations were carried out on an IBM 704.

DISCUSSION

DESCRIPTION OF METHOD OF CALCULATING INTERACTION

Only a brief description of the method for computing interaction will be given here, since it has been well described in the literature.⁽¹⁻³⁾ Attention is focused on each individual unit, and the fractions of the neutrons emitted by it (per unit time) that reach each of the other units in the group are computed. Clearly, when all units have been considered in this manner the number of neutrons entering each unit per unit time is obtained in terms of the total numbers emitted by each of the other units. Since the ratio of the neutron current emitted by a critical unit to the current entering the unit is determined by the dimensions of the unit and by the materials of which it is composed, a set of homogeneous simultaneous equations in the emitted currents can be obtained, which, in the case of identical units, has the ratio of the entering current to the emitted current as an eigenvalue. From the eigenvalue the critical composition of the identical units can be calculated; or a critical dimension can be calculated if the dimension has not been involved in calculating the interchange of neutrons between units.

To simplify the calculations a number of approximations are made:

(1) Only one energy group of neutrons is usually considered, i.e., the same spatial distribution is assumed for neutrons of all energies. This approximation is a good one for bare arrays, but is not so good for reflected arrays where the neutron spectrum differs considerably in unit and in reflector.

(2) The angular distribution of emitted neutrons is taken to be proportional to the cosine of the angle between the direction of emission and the normal to the element of surface through which the neutrons are emitted. This is a zero-order approximation that in many cases⁽³⁾ appears fairly good.

(3) The emitted neutron current is assumed to be uniform over the entire surface of each unit (including reflectors). This approximation is nonconservative and is worst at small separations. For a plane surface it tends to compensate for ignoring the outward bias that actually exists in the angular distribution. In the case of slabs, little complication is introduced by permitting the currents at the two principal surfaces to differ, but for cylinders or spheres a nonuniform current distribution greatly complicates the calculations.

(4) In arrays of units, complete shielding, such as occurs when units are directly in line, is allowed for; but partial shielding of units by intermediate units is ignored until the fractions of the neutrons reaching successively more distant units in an infinite array formed by extending the finite array in question total unity, after which all more distant units are taken to be completely blocked from view. This approximation is conservative, but the degree of conservatism varies with the size of the array and with the spacings within the array.

(5) Within a unit in a critical array, the spatial variation of the neutron flux Φ is assumed to satisfy the wave equation

$$\nabla^2\Phi + B^2\Phi = 0 \quad (1)$$

where B^2 is the material buckling.

(6) The neutron currents leaving (j^+) and entering (j^-) the surface of a unit are assumed to be given by

$$j^\pm = \frac{\Phi}{4} \mp \frac{\nabla\Phi}{6\Sigma_{tr}} \quad (2)$$

where Σ_{tr} is the transport cross section and the flux and its gradient are evaluated at the surface. As indicated above, the equations for an interacting system involve the ratio

$$\beta = \frac{j^-}{j^+} \quad (3)$$

This ratio may be regarded alternatively as the albedo that the environment surrounding a unit must have if the unit (and hence the system) is to be critical, or as the surface multiplication of a critical unit. Its reciprocal (β^{-1}) is the albedo of the unit, which for units with positive material bucklings, is greater than unity. The choice of the proper average value of Σ_{tr} (to use in computing β) may present some difficulty, particularly when only one energy group is employed; but if the critical size of an unreflected unit of the same composition as the interacting units is known from experiment, Σ_{tr} can be calculated from the requirement that $j^- = 0$. By means of this approach, β can be expressed in terms of the material buckling (B^2), the bare extrapolation distance (S_0), and the extrapolation distance (S) of the critical interacting unit. For an infinite slab

$$\beta = \frac{\sin B(S - S_0)}{\sin B(S + S_0)} \quad (4)$$

and for a sphere

$$\beta = \frac{\frac{\pi - BS}{1 + (\pi - BS) \cot BS} - \frac{\pi - BS_0}{1 + (\pi - BS_0) \cot BS_0}}{\frac{\pi - BS}{1 + (\pi - BS) \cot BS} + \frac{\pi - BS_0}{1 + (\pi - BS_0) \cot BS_0}} \quad (5)$$

Solution of an interaction problem involves calculating the fractions ρ of the neutrons emitted by a unit that reach all the other units in the group. According to approximations 2 and 3 this fraction for a pair of units is

$$\rho = \frac{\iint \frac{\cos\phi \cos\phi' dS dS'}{\pi R^2}}{\int dS} \quad (6)$$

where R is the distance between two elements of surface on the two units, ϕ and ϕ' are the angles between R and the normals to the two elements, and the integration in the numerator is over the surfaces of the two units and in the denominator is over the surface of the emitter. Equation 6 has been evaluated⁽¹⁻³⁾ for a number of cases. Its evaluation for curved surfaces is difficult and approximations have been suggested⁽¹⁻³⁾ for pairs of cylinders and spheres involving square cylinders and cubes. The cube approximation was employed⁽⁷⁾ in a preliminary analysis of the Thomas experiments. (For purposes of analysis, the Thomas experiments were assumed to have been performed with regular cubic arrays of spheres.) It appreciably overestimates ρ at small spacings and thus tends to compensate for approximation 3. At large spacings it slightly underestimates ρ . In the present work, however, it was decided to evaluate Equation 6 for pairs of spheres (see Appendix) rather than to employ the cube approximation.

After the fractions ρ_{ji} are calculated, they are employed in equations of the form

$$\begin{aligned} (\rho_{11} - \beta_1)j_1^+ + \rho_{12}j_2^+ + \dots + \rho_{1n}j_n^+ &= 0 \\ \rho_{21}j_1^+ + (\rho_{22} - \beta_2)j_2^+ + \dots + \rho_{2n}j_n^+ &= 0 \\ \rho_{n1}j_1^+ + \rho_{n2}j_2^+ + \dots + (\rho_{nn} - \beta_n)j_n^+ &= 0 \end{aligned} \quad (7)$$

For units with convex surfaces, $\rho_{ii} = 0$. For identical units, $\rho_{ij} = \rho_{ji}$, and approximations 1, 3, 5, and 6 require that β_i be the same for each. The number of equations (but not the number of ρ_{ji} 's) can be reduced markedly by taking account of the symmetry. For example, a 5 x 5 x 5 cubic array containing 125 identical units has only 10 symmetrically different locations; hence the number of equations can be reduced to 10. Since for a pair of spheres of equal size, ρ is a function only of the ratio of sphere diameter to center-to-center separation, β as obtained from Equations 7 is a function only of this same ratio, of the size of the array, and of the albedo of the reflector surrounding the array.

A code was written in FORTRAN for the IBM 704 to calculate the ρ_{ji} 's and to solve Equations 7 for regular cubic arrays containing 8, 27, 64, and 125 spheres. Extension to larger arrays or to rectangular arrays is straightforward. Both unreflected and reflected arrays

are considered. The reflectors are all cubic and each face is separated by half a lattice unit from the plane on which the centers of the adjacent units lie (essentially the same separation as was employed in the Thomas experiments).. Besides being applied to the Thomas experiments, the code was used to compute critical values of β for arrays of 8, 27, 64, and 125 identical units with the reflector albedos calculated for the Thomas experiments for the entire range of ratios of sphere diameter to center-to-center separation from 0 to 1. The results of the calculations are given in Figures 1-4.

FIG. 1 CRITICAL ALBEDO β PROVIDED BY OTHER UNITS AND REFLECTOR IN AN 8-UNIT CUBIC ARRAY

FIG. 2 CRITICAL ALBEDO β PROVIDED BY OTHER UNITS AND REFLECTOR IN A 27-UNIT CUBIC ARRAY

FIG. 3 CRITICAL ALBEDO β PROVIDED BY OTHER UNITS AND REFLECTOR IN A 64-UNIT CUBIC ARRAY

FIG. 4 CRITICAL ALBEDO β PROVIDED BY OTHER UNITS AND REFLECTOR IN A 125-UNIT CUBIC ARRAY

SOLUTION EXPERIMENTS

The Thomas^(5,6) experiments with solution were performed in an aluminum Unistrut framework that could be adjusted to vary the spacing between units. Each unit consisted of an 8-inch-diameter, 7.5-inch-high (outside dimensions) cylindrical vessel made of 1/4-inch-thick "Plexiglas"* and containing 5.0000 ± 0.0003 liters of aqueous uranyl nitrate solution. The solution contained 415 g of uranium per liter, and the specific gravity was 1.555. The uranium contained 92.6% U^{235} . Paraffin and "Plexiglas" reflectors of various thicknesses were employed around the arrays, separated from neighboring units by one-half the surface-to-surface separation between units. In the experiments the critical value of the common surface-to-surface separation between adjacent units was found as a function of array size and of the thickness of the reflector.

A number of simplifying approximations were made in order to apply the method of calculating interaction to these experiments. The cells surrounding each unit were approximated by cubes of the same volume, hence the actual array was approximated by one having the same average density of material, but having equal center-to-center separations between adjacent units in all three directions. The units were assumed to be 5.0-liter spheres enclosed by 1/4-inch-thick "Plexiglas" and to have an outer diameter of 8.85 inches. The "Plexiglas" was assumed to be equivalent to the same volume of H_2O and was homogenized with the solution to give an effective concentration of 349 g of uranium per liter. The material buckling of this homogenized solution was calculated to be 0.0306 cm^{-2} , a value which, with bare and water-reflected extrapolation distances of 2.6 and 5.8 cm, is consistent with experimental data for similar solutions⁽²⁾. The migration area required for determining k_{eff} was calculated to be 32.3 cm^2 . The "Plexiglas" and paraffin reflectors were assumed to be equivalent to the same thickness of H_2O . Their albedos as a function of thickness were estimated⁽²⁾ from experimental data to have the values in Table I.

TABLE I

Albedo of Water Reflector as a Function of Thickness

Thickness, in.	Albedo
0.5	0.206
1.5	0.416
3.0	0.528
6.0	0.550

*Trademark of the Rohm and Haas Co.

The experiments^(5,6) to which the interaction method was applied are summarized in Table II, together with corresponding values of β obtained from solutions of Equations 7. The value of β calculated from the composition and diameter of a unit (i.e., with $B^2 = 0.0306 \text{ cm}^{-2}$, $S_0 = 2.6 \text{ cm}$, and $S = \frac{\pi}{B} - 1.27 \times 8.85 = 6.72 \text{ cm}$) by (5) is 0.446. Besides comparing this value of β with those of Table II, there are several other ways to compare theory and experiment. For example, the solution concentration might be altered until β as calculated by (5) equaled the values of Table II.

TABLE II

Calculated Values of the Critical Albedo β Provided for a Solution Unit by the Other Units and by the Reflector Surrounding the Array

Number of Units	Reflector Thickness, cm	Surface-to-Surface Separation, cm	Diameter/Average Pitch	Critical Albedo β
8	0	1.43	>1(a)	-
	1.27(b)	3.00	0.9824	0.4394
	1.27	3.28	0.9705	0.4324
	3.81	6.01	0.8390	0.4578
	7.62	8.48	0.7926	0.5018
	15.24	8.99	0.7786	0.5084
27	0	6.48	0.8527	0.4662
	1.27(b)	8.76	0.7848	0.4833
	1.27	9.02	0.7778	0.4769
	3.81	13.69	0.6696	0.4767
	15.24	16.53	0.6174	0.5130
64	0	10.67	0.7358	0.5077
125	0	14.40	0.6557	0.5291

(a) The equivalent spheres are in contact when the surface-to-surface separation is 2.5; hence no calculation of interaction was made for this case.

(b) This reflector was "Plexiglas". The other reflectors were paraffin.

However, the comparisons that will be made here consist (1) of comparing the critical separations of Table II with those read from Figures 1-4 that make $\beta = 0.446$, (2) of comparing the critical numbers of units with those obtained by plotting β as read from Figures 1-4 at the separations given in Table II against the cube root of the number of units, extrapolating the curve if necessary, and determining the number of units corresponding to a β of 0.446, and (3) of solving (5) for the values of S that give the values of β in Table II and using these values of S to compute values of

$$k_{\text{eff}} = \frac{1 + M^2 B^2}{1 + M^2 \frac{\pi^2}{(R+S)^2}}$$

relating theory and experiment. The second type of comparison will be made only for the unreflected arrays and for those surrounded by a 15.24-cm-thick reflector. These three comparisons of theory and experiment are shown in Table III.

TABLE III
Comparison of Calculations and Experiments
with Critical Arrays of Solution Units

Reflector Thickness, cm	Number of Units in a Critical Array		Average Pitch of Units in a Critical Array, cm		k_{eff} Calculated for Critical Array
	Exptl.	Calc.	Exptl.	Calc.	
0	8	-	<22.48 ^(a)	-	-
1.27 ^(b)	8	-	22.88	22.7	0.995
1.27	8	-	23.16	22.7	0.990
3.81	8	-	26.79	27.3	1.009
7.62	8	-	28.36	31.4	1.046
15.24	8	4.7	28.87	32.3	1.051
0	27	24.1	26.36	27.1	1.016
1.27 ^(b)	27	-	28.64	30.3	1.030
1.27	27	-	28.90	30.3	1.024
3.81	27	-	33.57	35.3	1.024
15.24	27	14.5	36.41	41.0	1.055
0	64	44.4	30.55	33.3	1.051
0	125	73.6	34.28	38.0	1.069

(a) The equivalent spheres (diameter 22.48 cm) would intersect in a cubic lattice; hence no calculation was made.

(b) "Plexiglas" reflector. The other reflectors were paraffin.

In general, agreement between calculations and experiment is reasonably good. There are trends toward increased k_{eff} as the size of the array increases and as the thickness of the reflector increases. The former trend may result from approximation (4) and perhaps from a decrease in the effect of approximation (3). From the practical point of view of computing safe arrays, it is gratifying that the calculations are for the most part conservative. Some of this conservatism, however, as discussed below, may stem from homogenizing the "Plexiglas" walls with the solution. The large degree of conservatism in the results calculated for arrays surrounded by thick reflectors may result in part from the choice of too large an albedo for the reflector but, as also discussed below, it is difficult to account for it all on this basis.

Strictly on a one-group basis, treating the vessel wall as a separate region increases the albedo at the outer surface of the unit required to make the unit critical by about 20%. Two-group calculations, however, indicate this effect to be much smaller. In contrast to the one-group calculations, they show the critical albedo at the surface of a 5-liter sphere of 415 g/l solution actually to be larger than the critical albedo at the outer surface of a 1/4-inch-thick layer of water surrounding the sphere. The critical albedo at the outer surface is, however, about 7% larger than that for the homogenized sphere. The result of this increase in the critical albedo is essentially a displacement of the relation between theory and experiment, roughly equivalent to subtracting 0.025 from each k_{eff} in the last column of Table III.

In order for calculations and experiments to agree in the case of the arrays surrounded by the 15.24-cm reflectors, the reflector albedo would have to be either about 0.5 or about 0.45, depending on whether the above correction for homogenization is or is not made. The latter albedo appears to be much too low but the former is not completely outside the realm of possibility. The albedo of an infinite water reflector for the 349 g/l homogenized solution assumed in the calculation is

$$\beta = \frac{\sin 0.0306 (5.8 - 2.6)}{\sin 0.0306 (5.8 + 2.6)} = 0.534$$

The discrepancy between this figure and 0.55, the value used in the calculations, results from difficulties in fitting the experimental data from which Table I was constructed. This lower figure might represent a better choice for a water reflector; on the other hand, the actual reflectors employed ("Plexiglas" and paraffin) have been shown to be better reflectors than water and hence would have higher albedos. In the calculations the actual surface area of the reflector was employed, and the albedo employed was that of an infinite slab. If the space enclosed by the reflector is approximated by a sphere with the same volume as the actual cube, the albedo of the reflector is reduced from 0.534 to 0.516 as the result of the finite radius of curvature^(2,3); but this effect is compensated by the smaller area of reflecting surface of the sphere as compared to the cube. As the result of moderation in the reflector it would seem that the albedo of the reflector for neutrons leaving the container walls should be somewhat lower than that for neutrons leaving solution, just as the albedo for neutrons re-entering the reflector should be lower than that for neutrons coming directly from a unit. No attempt was made to extend the two-group calculations to study these effects, however.

EXPERIMENTS WITH URANIUM CYLINDERS

Thomas performed two series of experiments with uranium cylinders^(5,6). In the first series the cylinders were 11.506 cm in diameter and 10.770 cm high. In the second series the cylinder heights were increased to 13.462 cm. The uranium was 93.2% U^{235} and had a density of 18.66 g/cm³. Critical reflected and unreflected arrays of 8 and 27 units were constructed. As in the solution experiments, the common surface-to-surface separation between adjacent units was adjusted to make the array critical. Various thicknesses of paraffin reflectors were employed, and the separation between reflector and adjacent units was one-half the surface-to-surface separation between units, just as in the solution experiments.

The same approximations were made in applying the theory to these experiments as were made for the solution experiments. The cylinders were considered to be spheres having the same volume with diameters of 12.88 and 13.88 cm. The arrays were approximated by ones having equal center-to-center spacings in all three directions and the same average densities of uranium as in the actual arrays. The paraffin reflectors were assumed to be equivalent to "Plexiglas" of the same thickness since data⁽⁸⁾ are available giving the effectiveness of "Plexiglas" as a reflector for uranium (18.7 g/cm³, 93.4% U^{235}) slabs as a function of reflector thickness. The albedo as a function of reflector thickness for infinite slabs is given in Table IV. These albedos were calculated by Equation 4 from a calculated⁽²⁾ material buckling of 0.08258 cm⁻² and from extrapolations to zero transverse geometric buckling of extrapolation distances obtained by equating this material buckling to the geometric bucklings of the slabs studied experimentally (Figure 5). The material buckling calculated⁽²⁾ for the uranium cylinders was 0.08204 cm⁻², which together with a bare extrapolation distance of 2.17 cm is consistent with a bare critical experiment performed with a uranium sphere. The migration area was calculated⁽²⁾ to be 15.7 cm².

TABLE IV

Albedo of "Plexiglas" in Contact with an
Infinite Slab of Uranium (93.4% U^{235})
as a Function of Thickness

Thickness of "Plexiglas" Reflector, cm	Albedo
0	0
1.27	0.175
2.54	0.332
3.81	0.447
7.62	0.660
15.24	0.719

FIG. 5 EXTRAPOLATION DISTANCE RELATING THEORY AND EXPERIMENTS WITH SLABS OF URANIUM (93.4% U^{235}) REFLECTED BY "PLEXIGLAS"

In Table V the experiments performed with the arrays are summarized and values of β obtained from Equations 7 are given. Solution of Equation 5 for β gives values of 0.371 and 0.309, respectively, for the smaller and larger units. Comparison between calculations and experiments is made in terms of average center-to-center spacing and of k_{eff} in Table VI. As in the case of the solution experiments for bare arrays, k_{eff} increases with the number of units, possibly as the result of the changing influence of approximations (3) and (4), and k_{eff} increases with the thickness of the reflector surrounding the array.

TABLE V

Calculated Values of the Critical Albedo β Provided for a Metal Unit by the Other Units and by the Reflector Surrounding the Array

Number of Units	Size of Unit	Reflector Thickness, cm	Surface-to-Surface Separation, cm	Diameter/Avg. Pitch	Critical Albedo β
8	Small	0 ^(a)	2.217	0.9563	0.324
		0	2.248	0.9541	0.323
		1.27	3.678	0.8627	0.338
		2.54	5.710	0.7594	0.351
		3.81	8.207	0.6619	0.351
		7.62	11.509	0.5659	0.434
27	Small	15.24	11.986	0.5543	0.483
		0	6.363	0.7312	0.356
		1.27	8.547	0.6506	0.343
		3.81	14.764	0.4951	0.330
		7.62	18.720	0.4298	0.409
		15.24	19.147	0.4237	0.458
8	Large	0	3.543	0.8854	0.273
		1.27	5.423	0.7905	0.290
		3.81	11.532	0.5863	0.294
		7.62	15.697	0.4985	0.372
		15.24	16.378	0.4866	0.418
27	Large	0	8.494	0.6727	0.298
		1.27	11.323	0.5915	0.290
		3.81	19.606	0.4371	0.274

(a). The average unit weight was 20.805 kg compared with 20.960 kg for the other 8 unit arrays of small cylinders and 20.877 kg for the 27-unit arrays.

The reflector albedo of the 15.24-cm reflector that would give agreement between calculation and experiment is 0.596 for the small cylinders in the 8-unit array, 0.617 in the 27-unit array, and 0.590 for the large cylinders in the 8-unit array. Because of the small size of these arrays, consideration of the effective curvature of the reflector has more effect here than in the case of the solution experiments. For the 8-unit array of small cylinders reflected by

TABLE VI

Comparison of Calculations and Experiments
with Critical Arrays of Metal Cylinders

Reflector Thickness, cm	Number of Units in Array	Cylinder Size	Average Pitch in a Critical Array, cm		k_{eff} Calculated for Critical Array
			Exptl.	Calc.	
0	8	Small	13.47	12.6(a)	0.960
0			13.50	12.6(a)	0.959
1.27			14.93	14.1	0.972
2.54			16.96	16.3	0.982
3.81			19.46	18.7	0.982
7.62			22.76	25.9	1.056
15.24			23.24	29.2	1.102
0	27	Small	17.61	17.3	0.987
1.27			19.80	19.0	0.975
3.81			26.01	23.9	0.965
7.62			29.97	32.3	1.034
15.24			30.40	36.4	1.079
0	8	Large	15.68	14.8	0.971
1.27			17.56	16.9	0.984
3.81			23.67	22.9	0.988
7.62			27.84	32.1	1.051
15.24			28.52	36.4	1.091
0	27	Large	20.63	20.4	0.991
1.27			23.47	22.7	0.984
3.81			31.75	29.2	0.972

(a) This corresponds to a diameter/pitch ratio >1 and was obtained by extrapolating Figure 1 slightly.

15.24 cm of "Plexiglas" the sphere enclosing the same volume has a radius of 28.84 cm. The albedo at this finite radius is reduced from 0.719 at infinite radius to 0.637 if the extrapolation distance in a critical sphere of uranium reflected by 15.24 cm of "Plexiglas" is assumed to be 4.74 cm as it is for the infinite slab. If this approach is followed, however, the surface area of the equivalent spherical reflector rather than that of the cubic reflector should presumably be employed in computing the fraction of the neutrons leaving the reflector that reaches a unit. Since the sphere area is smaller than the cube area, the effect of the lower albedo is reduced. In this example, k_{eff} relating theory and experiment is reduced from 1.102 to 1.07. Conceivably, the albedo of the reflector for neutrons that have been reflected at least once is enough lower than for neutrons coming directly from a unit to give a further reduction in k_{eff} if the effect were considered, but considerably more elaborate calculations would be required to show this.

APPENDIX

INTERACTION INTEGRAL FOR SPHERES

The form of the integral in the numerator of Equation 6 is useful for demonstrating the reciprocity of the transmitting and receiving bodies, but it is not the simplest form to employ for evaluating the integral. The term $\frac{\cos \phi' dS'}{R^2}$ represents the element of solid angle $d\Omega$ subtended at an element of surface on the transmitter by an element of surface dS' on the receiver. It is more convenient to express $d\Omega$ in terms of coordinates with origin at the element of surface dS . Carlvik and Pershagen⁽⁹⁾ found this to be the case in evaluating Equation 6 for interacting infinite cylinders, and an extension of their procedures to spheres is employed here.

Because of symmetry it is sufficient to consider transmission only from points lying on a great circle of the transmitting sphere that is coplanar with the line joining the centers of the two interacting spheres. Integration over the azimuthal angle in the plane perpendicular to this line merely introduces a factor of 2π . In making a drawing to aid in visualizing the limits of the integration, it is convenient to place the great circle in the plane of the paper. Such a drawing is presented in Figure A-1 for two interacting spheres of equal size.

A ray from the transmitting area dS to the receiving sphere is represented by R . In general, R is not in the plane of the paper. The radius drawn to the point at which R intersects the receiver is represented by a'' . The radius drawn to the point at which the projection of R on the plane of the paper intersects the receiver is represented by a' . In magnitudes a , a' , and a'' are, of course, all equal and will be represented by a where only the magnitude is required. Except for a'' and R , all lines and angles shown in Figure A-1 are in the plane of the paper. In addition

FIG. A-1 INTERACTING SPHERES

to the angles shown, ϕ represents the angle R makes with the extension of a, θ represents the angle R makes with its projection, and ϵ represents the angle between c and R.

With the foregoing definitions Equation 6 can be transformed as follows. The element of solid angle $d\Omega$ about R is

$$d\Omega = \frac{R \cos\theta \, d\alpha \, R \, d\theta}{R^2}$$

The element of surface dS integrated over the azimuthal angle is

$$dS = 2 \pi a^2 \sin\gamma \, d\gamma$$

In terms of α and θ

$$\cos\phi = \cos\theta \cos\alpha \quad (A-1)$$

Equation 6 then becomes

$$\rho = \frac{1}{2\pi} \int_0^{\pi/2} \int_{\alpha_1(\gamma)}^{\alpha_2(\gamma)} \int_{\theta_1(\gamma, \alpha)}^{\theta_2(\gamma, \alpha)} \sin\gamma \cos\alpha \cos^2\theta \, d\theta \, d\alpha \, d\gamma \quad (A-2)$$

The limits of integration can now be established in the following manner. By the law of sines

$$\frac{c}{\sin\gamma} = \frac{d}{\sin\delta} = \frac{a}{\sin(\gamma+\delta)} \quad (A-3)$$

and

$$\frac{c}{\sin\beta} = \frac{a}{\sin(\alpha+\delta)} \quad (A-4)$$

Thus
$$c = \frac{d \sin\gamma}{\sin\delta} = \frac{a \sin\gamma}{\sin(\gamma+\delta)} = \frac{a \sin\beta}{\sin(\alpha+\delta)}$$

and hence
$$d \sin\gamma \cos\delta + d \cos\gamma \sin\delta = a \sin\delta$$

$$d \sin\gamma \sin\alpha \cos\delta + d \sin\gamma \cos\gamma \sin\delta = a \sin\beta \sin\delta$$

From the first of this pair of equations

$$\cos\delta = \frac{(a - d \cos\gamma) \sin\delta}{d \sin\gamma}$$

Substitution of this expression into the second equation permits terms in $\sin\delta$ to be divided out with the result

$$d \sin(\alpha-\gamma) = a (\sin\alpha - \sin\beta) \quad (A-5)$$

Let (η_1, ξ_1, ζ_1) , (η_2, ξ_2, ζ_2) , and (η_3, ξ_3, ζ_3) be respectively the direction cosines of the projection of R on the plane of the paper, of R, and of c with respect to coordinate axes with origin at dS consisting of the extension of a, the line perpendicular to a lying in the plane of the paper, and the normal to the plane of the paper. Then

$$\eta_1 = \cos\alpha, \xi_1 = \sin\alpha, \zeta_1 = 0$$

$$\eta_2 = \cos\phi, \xi_2 = ? , \zeta_2 = \sin\theta$$

$$\eta_3 = \cos\delta, \xi_3 = \sin\delta, \zeta_3 = 0$$

The limiting values of $\theta = \theta_1$ occur when R is tangent to the receiver in which event $\cos\epsilon = \cos\epsilon_1 = \frac{\sqrt{c^2 - a^2}}{c}$. In terms of the direction cosines

$$\cos\epsilon_1 = -\cos\phi_1 \cos\delta + \xi_2 \sin\delta$$

where ϕ_1 and θ_1 specify the tangent ray. Eliminating $\cos\epsilon_1$ yields

$$\xi_2 = \frac{\cos\phi_1 \cos\delta + \frac{\sqrt{c^2 - a^2}}{c}}{\sin\delta}$$

In terms of the direction cosines

$$\cos\theta_1 = \cos\alpha \cos\phi_1 + \frac{\sin\alpha}{\sin\delta} \left(\cos\phi_1 \cos\delta + \frac{\sqrt{c^2 - a^2}}{c} \right)$$

Making use of A-1 and solving for $\cos\theta_1$ then gives

$$\cos\theta_1 = \frac{\sqrt{c^2 - a^2}}{c \cos(\alpha + \delta)}$$

By the law of cosines

$$c^2 = a^2 + d^2 - 2 ad \cos\gamma \quad (\text{A-6})$$

Making use of A-4, A-6, and A-5 in this order gives finally

$$\cos\theta_1 = \frac{\sqrt{d^2 - 2 ad \cos\gamma}}{d \cos(\gamma + \alpha) - a \cos\alpha}$$

Consideration of the other possible cases: namely, where the projection of R lies on the other side of c or on the other side of the extension of a leads to the same limits for θ . Performing the θ integration from $-\theta_1$ to θ_1 then reduces A-2 to

$$\rho = \frac{1}{2\pi} \int_0^{\pi/2} \int_{\alpha_1(\gamma)}^{\alpha_2(\gamma)} \left[\cos^{-1} \frac{\sqrt{d^2 - 2ad \cos \gamma}}{d \cos(\gamma - \alpha) - a \cos \alpha} \right. \\ \left. + \frac{\sqrt{d^2 - 2ad \cos \gamma} \sqrt{a^2(\sin^2 \alpha + \cos^2 \gamma) - (d \sin(\gamma - \alpha) + a \sin \alpha)^2}}{(d \cos(\gamma - \alpha) - a \cos \alpha)^2} \right] \\ \times (\cos \alpha \sin \gamma \, d\alpha \, d\gamma) \quad (A-7)$$

Examination of Figure A-1 shows that the upper α limit is $\pi/2$ if $\gamma \leq \cos^{-1}(2a/d)$. The lower limit is obtained by solving A-5 for α when $\beta = \pi/2$. When $\gamma > \cos^{-1}(2a/d)$, the upper α limit is similarly obtained from the analog of A-5 that applies when R is below c. Consideration also must be given to values of γ for which α is negative.

No way was discovered for completing the analytical evaluation of the integral so Gaussian quadrature was employed. The variables of integration were changed by letting $X = \sin \alpha$ and $Y = \cos \gamma$ and the following auxiliary functions were introduced:

$$\alpha = \frac{a}{d}$$

$$F(Z) = \sqrt{1 - Z^2}$$

$$S(Z) = 1 - 2\alpha Z$$

$$T(Z) = Z - \alpha$$

$$U(X, Y) = (T(Y) F(X) + X F(Y))^2$$

$$V(X, Y) = \tan^{-1} \sqrt{\frac{U(X, Y)}{S(Y)}} - 1$$

$$W(X, Y) = \frac{\sqrt{S(Y)} \sqrt{\alpha^2(X^2 + Y^2) - (F(X) F(Y) - XT(Y))^2}}{U(X, Y)}$$

In terms of these functions A-7 becomes

$$\rho = \frac{1}{2\pi} \int_0^1 \int_{X_1(Y)}^{X_2(Y)} (V(X,Y) + W(X,Y)) \, dX \, dY$$

Examination of the possible limits for X and Y shows that when $Y \leq 2\alpha$ the X integration is from

$$\frac{-\alpha T(Y) + F(Y) \sqrt{S(Y)}}{\alpha^2 + S(Y)}$$

to unity and that when $Y > 2\alpha$ the upper limit is

$$\frac{\alpha T(Y) + F(Y) \sqrt{S(Y)}}{\alpha^2 + S(Y)}$$

rather than unity.

A subroutine was written in FORTRAN for evaluating the double integral by Gaussian quadrature. Results obtained with 16 points for each integral (256 evaluations of the integrand) are given in Table A-1 and are plotted in Figure A-2. Some results were also obtained with 24 points for each integral and were found to agree well with the values of Table A-1; hence, these values are presumably fairly accurate. For large arrays many evaluations of Equation A-2 are required and considerable machine time is thereby consumed. Accordingly an 11-point Lagrange interpolation formula based on the ratios of values at 2α points at $0(0.1)1.0$ to the solid angles subtended by one sphere at the center of the other at these same points was employed. Agreement with Table A-1 at the midpoints of the intervals was good, the maximum error being 0.2% at $2\alpha = 0.95$.

The results given in Table A-1 were also compared with the average solid angle subtended by the surface of the receiver at the surface of the transmitter, i.e., with an evaluation of

$$\rho = \frac{1}{4\pi} \int_0^{\pi/2} \int_{\alpha_1(\gamma)}^{\alpha_2(\gamma)} \int_{\theta_1(\gamma, \alpha)}^{\theta_2(\gamma, \alpha)} \sin \gamma \cos \theta \, d\theta \, d\alpha \, d\gamma \quad (A-8)$$

as carried out by others^(10, 11).

TABLE A-1

Interaction Integral for Spheres

<u>Diameter/Separation</u>	<u>ρ</u>
0	0
0.05	0.00016
0.1	0.00063
0.15	0.00143
0.2	0.00257
0.25	0.00404
0.3	0.00586
0.35	0.00803
0.4	0.01057
0.45	0.01349
0.5	0.01681
0.55	0.02054
0.6	0.02470
0.65	0.02929
0.7	0.03434
0.75	0.03989
0.8	0.04606
0.85	0.05281
0.9	0.06020
0.95	0.06841
1.0	0.07820

FIG. A-2 FRACTION ρ OF NEUTRON CURRENT TRANSMITTED FROM ONE SPHERE TO ANOTHER

The differences are small and in the case of Reference (10) are in the expected direction with ρ as calculated from A-2 being greater than ρ as calculated by A-8. The values given in Reference (11) do not agree with those of Reference (10) and appear to be in error, particularly near $2\alpha = 1$; at $2\alpha = 1$ Reference (11) gives a larger value of ρ than that calculated here from A-2.

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