42-1733

LA-6073-MS

Informal Report

UC-34a

Reporting Date: August 1975

Issued: September 1975

RECEIVED
SEP 10 1975
LASL

Higher Order Corrections to Energy Levels of Muonic Atoms

by

G. A. Rinker, Jr.

R. M. Steffen*

*Visiting Staff Member, LASL. Purdue University, Lafayette, IN 47907

An Affirmative Action/Equal Opportunity Employer

UNITED STATES
ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION
CONTRACT W-7405-ENG. 38

In the interest of prompt distribution, this report was not edited by the Technical Information staff.

**Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22151
Price: Printed Copy \$4.00 Microfiche \$2.25**

This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights.

NOTICE
This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Energy Research and Development Administration, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.

HIGHER ORDER CORRECTIONS TO ENERGY LEVELS OF MUONIC ATOMS

by

G. A. Rinker, Jr., and R. M. Steffen

ABSTRACT

In order to facilitate the analysis of muonic x-ray spectra, the results of numerical computations of all higher order quantum electrodynamical corrections to the energy levels of muonic atoms are presented in tabular and graphical form. These corrections include the vacuum polarization corrections caused by emission and re-absorption of virtual electron pairs to all orders, including "double-bubble" and "cracked-egg" diagrams. An estimate of the Delbrücke scattering-type correction is presented. The Lamb-shift (second- and fourth-order vertex) corrections have been calculated including the correction for the anomalous magnetic moment of the muon. The relativistic nuclear motion (or recoil) correction as well as the correction caused by the screening of the atomic electrons is presented in graphs. For the sake of completeness a graph of the nuclear polarization as computed on the basis of Chen's approach has been included. All calculations were made with a two-parameter Fermi distribution of the nuclear charge density.

1. INTRODUCTION

With presently available instrumentation the absolute energies of muonic x-ray transitions can be determined with a relative accuracy of about 2×10^{-5} , if adequate calibration sources are available in the energy region of interest. Before nuclear parameters can be extracted from muonic x-ray experiments, however, it is important to assess all the higher order corrections to the muonic energy levels and to be aware of the uncertainties with which these corrections can, at present, be computed.

In this work numerical values of all important energy corrections $\Delta B(nlj)$ to muonic states are presented in the form of graphs. The computer code MUON that was used for the calculations was originally written by J. G. Wills to compute muon binding energies in the field of a realistic nuclear charge distribution. The corrections discussed in this work have been incorporated into this computer program by G. A. Rinker, Jr.

The values of $\Delta B(nlj)$ refer to corrections of the binding energies $B(nlj)$ of the states $|nlj\rangle$ of a muonic atom, i.e., $B(nlj) = mc^2 - E(nlj)$, where the energies $E(nlj) < mc^2$. The notation nlj refers to the description of the quantum state of the muon in the non-relativistic limit.

Transition energies (muonic x-ray energies) for a transition from a state $|n_i l_i j_i\rangle$ to a final state $|n_f l_f j_f\rangle$ are given by

$$\Delta E(i \rightarrow f) = B(n_f l_f j_f) - B(n_i l_i j_i) \quad . \quad (1)$$

We write the binding energy in the form

$$\begin{aligned} B(nlj) = & B^0(nlj) + \Delta B_{VP}(nlj) + \Delta B_{LS}(nlj) \\ & + \Delta B_R(nlj) + \Delta B_{NP}(nlj) + \Delta B_{ES}(nlj) \end{aligned} \quad (2)$$

where $B^0(nlj)$ is the "uncorrected" Dirac binding energy, calculated for a bare finite-size nucleus without electrons. In all the computations a Fermi charge distribution

$$\rho(r) = \frac{\rho_0}{1 + \exp\left(\frac{r - c}{a}\right)} \quad (3)$$

has been assumed for the nucleus with

$$c = (1.183 A^{1/3} - 0.414) \text{ fm} \quad (4)$$

and

$$a = 0.55 \text{ fm} \quad (5)$$

where c is approximately the radius at which $\rho(c) = 0.50 \rho(0)$ and $t = (4 \ln 3)a \approx 2.417 \text{ fm}$ is the approximate distance over which $\rho(r)$ falls from 90% to 10% of its value at $r = 0$. As compared to more realistic charge distributions, the relative error of using the distribution (3) - (5) is of the order of 10^{-3} for the largest correction (the Uehling polarization) discussed here, and is negligibly small for the other corrections.

2. VACUUM POLARIZATION CORRECTIONS

The vacuum-polarization correction $\Delta B_{VP}(nlj)$ is caused by emission and re-absorption of virtual electron-positron pairs in the field of the nucleus, an effect which causes a modification of the effective electro-magnetic potential in which the muon moves (like in a dielectric medium!).

We divide the vacuum polarization into four parts:

$$\Delta B_{VP} = \Delta B_{VPI} + \Delta B_{VPII} + \Delta B_{VPIII} + \Delta B_{VPIV} \quad (6)$$

The four parts which will be discussed separately correspond to different types of Feynman diagrams.

2.1 $\Delta B_{VPI}(n\ell j)$, Uehling Term

The lowest order term of this correction is represented by Diagram a), which represents the correction of order $\alpha(Z\alpha)$, or the second-order correction. [The order of the correction is defined here as the number of vertices (= powers of e in the process) minus two. The "normal" (lowest order interaction) has two vertices and is of zero order.]

The "Uehling" vacuum polarization correction is computed by adding a vacuum-polarization potential energy $V_{Ue}(r)$, the Uehling potential, to the electrostatic Coulomb energy of the nucleus (35 Ue, 55 Sc, 62 Fo, 68 Bar):

$$V_{Ue}(r) = + \frac{2Z\alpha e^2}{3r} \lambda_e \int_0^\infty \rho(r') r' dr' \left[K(r+r') - K(|r-r'|) \right] \quad (7)$$

The nuclear charge distribution $\rho(r)$ is here normalized to unity $\int \rho(r) d^3r = 1$.

The function $K(x)$ is defined by the integral

$$K(x) = \int_1^\infty \frac{(1 + \frac{1}{2y^2})(y^2 - 1)^{1/2}}{y^3} e^{-2yx/\lambda_e} dy \quad (8)$$

with λ_e the reduced electron Compton wave length. This function is evaluated as in (71 Ri).

After adding the potential energy $V_{Ue}(r)$ to the nuclear Coulomb potential the Dirac equation is solved numerically. The ladder iterations of the $\alpha Z \alpha$ term are included in these calculations [Diagrams b), c), etc.]. The Uehling correction $\Delta B_{VPI}(n\ell j)$, which increases the binding energy, is by far the most significant higher order correction to the binding energy of muonic states, being as large as $\Delta B_{VPI}(1s_{1/2}) = + 75 \text{ keV}$ for $Z = 90$.

The correction ΔB_{VPI} is relatively easy to evaluate and the numerical results are accurate within an eV. For the analysis of muonic x-ray data it is necessary to compute ΔB_{VPI} for each level involved using the same nuclear charge distribution that is used to compute the zero-order electrostatic potential in the data-fitting procedure.

The Uehling vacuum-polarization correction, as computed with the two-parameter Fermi charge distribution of Eqs. (3) - (4), is presented in Table 1 for some selected values of Z and A .

Figure 1 shows a plot of $\Delta B_{VPI}(n\ell j)$ for the $1s_{1/2}$, $2p_{1/2}$, $2p_{3/2}$, $2d_{3/2}$ and $4f_{5/2}$ muonic states as a function of Z . The (reading) accuracy of these curves is clearly not sufficient for the analysis of muonic x-ray data; the curves are presented here only as a guide.

In addition to the values presented in Table 1, we have attempted to reproduce some of the $\Delta B_{VPI}(n\ell j)$ results of Engfer et al. (74 En), using their quoted charge distributions. Discrepancies as large as 40 eV were found for the low-lying muonic states of Pb. In view of the extensive numerical checks that we made using different computational procedures, we believe that the results quoted in Table 1 are accurate to within about 1 or 2 eV for the quoted charge distributions.

TABLE 1

Uehling Vacuum Polarization Corrections (in eV) Computed with a Two-Parameter Fermi Distribution of the Nuclear Charge ($c = 1.183 A^{1/3} - 0.414$ fm, $a = 0.55$ fm).

Z	A	1s _{1/2}	2s _{1/2}	2p _{1/2}	2p _{3/2}	3d _{3/2}	3d _{5/2}	4f _{5/2}	4f _{7/2}	5g _{7/2}
10	20	1426	185	123	123	17	17	3	3	0
15	31	3692	523	407	401	71	71	15	15	4
20	40	6950	1060	913	895	181	180	45	44	12
25	55	10905	1779	1687	1639	363	357	97	97	29
30	64	15519	2678	2761	2656	620	613	179	178	57
	68	15413	2676	2761	2656	620	613	179	178	57
35	81	20264	3735	4163	3966	974	959	296	293	101
40	90	25512	4965	5911	5582	1429	1402	449	445	159
45	103	30719	6311	8010	7505	1995	1948	646	640	238
50	116	35962	7778	10462	9735	2683	2609	890	879	337
	120	35710	7742	10449	9730	2683	2609	890	879	337
55	133	40905	9318	13223	12248	3502	3387	1183	1167	462
60	146	46069	10998	16297	15041	4460	4289	1530	1505	606
65	159	51129	12776	19640	18083	5564	5321	1934	1897	779
70	172	56121	14645	23212	21346	6824	6489	2399	2346	982
75	187	60840	16571	26939	24778	8248	7798	2926	2855	1214
80	200	65634	18608	30842	28388	9842	9248	3522	3427	1477
	204	65262	18538	30759	28333	9841	9247	3522	3427	1477
82	204	67638	19467	32464	29888	10528	9868	3780	3672	1591
	208	67259	19394	32375	29829	10527	9868	3781	3673	1592
90	232	74233	22789	38792	35853	13551	12591	4930	4764	2104
92	238	76059	23666	40417	37388	14377	13331	5247	5063	2248
100	253	83727	27473	47272	43843	17978	16535	6647	6377	2878
	257	83311	27381	47127	43737	17970	16534	6647	6378	2878

2.2 $\Delta B_{\text{VPII}}(n\ell j)$, Källen-Sabry Term

This term is a fourth-order correction, i.e., of order $(Z\alpha)\alpha^2$. The prescription for its calculation has first been given by Källen and Sabry (55 KÄ). The term consists of a reducible part a) ("double-bubble" diagram) and three irreducible parts b), c), and d) ("cracked-egg diagrams").

a)

b)

c)

d)

Ze

μ

The expressions of Källén and Sabry (55 Kå) can be reduced to a potential $V_{21}(r)$ (see 72 Bå) which must be folded with the finite charge distribution so that the finite size of the nucleus is taken into account. Figure 2 shows the "Källén" correction $B_{VP\text{II}}(n\ell j)$ calculated with the charge distribution of Eq. (3). The vacuum polarization correction $B_{VP\text{II}}(n\ell j)$ is positive, i.e., it increases the binding energy of the muon.

2.3 $\Delta B_{VP\text{II}}, \alpha(Z\alpha)^{n \geq 3}$ term

In deriving the Uehling potential (7) it was assumed that the virtual electron and positron are propagating freely. The influence of the nuclear Coulomb field on the electron and positron wave functions can be represented by the Feynman diagrams of type:

The distortion of the electron and positron wave functions by the nuclear Coulomb field can be interpreted in terms of a repulsion of the virtual positron and an attraction of the virtual electron which gives rise to a partial shielding of the nuclear charge and hence to a lowering of the binding energy. Hence, $\Delta B_{VP\text{III}}$ is negative. The contribution of the diagrams $\alpha(Z\alpha)^3 + \alpha(Z\alpha)^5 + \alpha(Z\alpha)^7 + \dots$ can again be represented by the vacuum-polarization potentials $V_{13}(r) + V_{15}(r) + V_{17}(r) + \dots = V_{1n \geq 3}(r)$. The potential $V_{1n \geq 3}(r)$ is repulsive for all r as expected from the preceding discussion.

Fig. 1. "Uehling" vacuum-polarization correction.

Fig. 3. $\alpha(2\alpha)^{n-3}$ vacuum-polarization correction.

Fig. 2. "Källén-Sabry" vacuum-polarization correction.

Fig. 4. "Lamb-shift" correction.

The vacuum polarization correction ΔB_{VPIII} has been calculated by Rinker and Wilets (75 Ri) to all orders n in $\alpha(Z\alpha)^n$ (n - odd) using the nuclear charge distribution (3). Figure 3 shows a plot of the results of these calculations.

2.4 The $\alpha^2(Z\alpha)^2$ Term: $\Delta B_{\text{VPIV}}(nlj)$

The fourth contribution to the vacuum polarization of significance, ΔB_{VPIV} is of order $\alpha^2(Z\alpha)^2$ and corresponds to virtual Delbrück scattering which is represented by the diagram

The contribution of this term has been accurately computed only for the $4f_{5/2}$ state of muonic Pb and was found to be very small $\Delta B_{\text{VPIV}}(\text{Pb}; 4f_{5/2}) = + 1.0 \pm 0.1$ eV by Wilets and Rinker (75 Wi). Two subsequent calculations of this term (75 Fu, 75 Su) are in good agreement with this result, while one (75 Ch) is in substantial disagreement. There is no apparent reason to prefer the disagreeing calculation to the other three mutually independent ones, so we conclude that the Wilets and Rinker result is the correct one. A rough estimate of this small correction is provided by replacing one of the nuclear sources in the $\alpha(Z\alpha)^3$ diagram by a muon source, yielding

$$\Delta B_{\text{VPIV}}(nlj) \approx - \frac{1}{Z} \Delta B_{\text{VPIII}}(nlj) \quad . \quad (9)$$

This estimate is sufficient for muonic x-ray analysis, since even for the $1s_{1/2}$ state of muonic U, the correction amounts to about $\Delta B_{\text{VPIV}}(\text{U}, 1s_{1/2}) \approx + 7$ eV, which is considerably smaller than the experimental errors.

It is fair to say that the total vacuum polarization correction $\Delta B_{VP}(n\ell j)$ can at present be computed with an accuracy of a few eV.

3. THE SELF-ENERGY OR LAMB-SHIFT CORRECTION $\Delta B_{LS}(n\ell j)$

We write the Lamb-Shift correction in the form

$$\Delta B_{LS} = \Delta B_{LSI} + \Delta B_{LSII} + \Delta B_{LSIII} + \Delta B_{VP\mu} \quad (10)$$

The main source of this correction, $\Delta B_{LSI}(n\ell j)$ is due to the virtual emission and re-absorption of photons according to the Feynman diagram

This is a second-order vertex correction of order $\alpha(Z\alpha)$. In addition the (non-physical) graphs

contribute to the renormalization for large virtual photon momenta of graph a).

The total correction $\Delta B_{LSI}(n\ell j)$, i.e., the second-order vertex correction of the self energy, corresponding to the lowest order diagrams is approximately (to first order in the field strength):

$$\Delta B_{LSI}(nlj) = - \frac{\alpha \hbar^2}{3\pi m_\mu c^2} \langle nlj | \nabla^2 V | nlj \rangle \left[\ln \left(\frac{m_\mu c^2}{2\Delta\epsilon} \right) + \frac{11}{24} + \frac{3}{8} \right] \\ - \frac{\alpha \hbar^2}{8\pi m_\mu c^2} \langle nlj | \frac{2}{r} \frac{\partial V}{\partial r} \vec{\sigma} \cdot \vec{L} | nlj \rangle . \quad (11)$$

The second term is the correction for the anomalous magnetic moment $(\alpha/2\pi)(\hbar/2m_\mu c)$ of the muon, which couples to the magnetic field $Ze \vec{L} \hbar/m_\mu c r^3$ as experienced by the "moving" muon. This coupling gives rise to the interaction energy

$$\frac{Ze^2 \hbar^3}{4\pi m_\mu c^2} \cdot \frac{1}{r^3} (\vec{L} \cdot \vec{S}) . \quad (12)$$

The 3/8 term is also due to the anomalous magnetic moment of the muon (Fermi contact interaction in s states!). These terms can be quite accurately calculated. The "Bethe Logarithm"

$$\ln \left(\frac{m_\mu c^2}{2\Delta\epsilon} \right) = \frac{\sum_{\mu'} \langle \mu | \vec{p} | \mu' \rangle \langle \mu' | [V, \vec{p}] | \mu \rangle \ln \left(2 \frac{m_\mu c^2}{|E_\mu - E_{\mu'}|} \right)}{\frac{\hbar^2}{2} \langle \mu | \nabla^2 V | \mu \rangle} , \quad (13)$$

where $|\mu\rangle$ are the various muonic states, is, however, cumbersome to compute with good accuracy. Barrett et al. (68 Ba) replaced $\Delta\epsilon$ by the binding energy $B(\mu)$ of the muon state $|\mu\rangle$, using $\bar{\epsilon} = 2B(\mu)$ to produce a lower limit and $\underline{\epsilon} = B(\mu)/2$ to produce an upper limit of the value of the Bethe logarithm.

Bethe and Negele (68 Be) have derived rigorous upper and lower limits, which result in somewhat smaller errors than Barrett's approach. Better calculations of the Bethe logarithm which, in principle, should not present any serious difficulties, would be highly desirable for the analysis of muonic x-ray data.

For the $1s_{1/2}$ state of muonic Pb, the correction is: $\Delta B_{LSI}(1s_{1/2}) = -(3.208 \pm 0.187)$ keV (Bethe-Negele limits).

The vacuum polarization correction $\Delta B_{VP\mu}(n\ell j)$ due to the creation and re-absorption of virtual muon pairs represented by the graph

is also of order $\alpha(Z\alpha)$, and is usually included in the Lamb-shift correction for ease of computation, although it is of course a true vacuum polarization correction.

$$\Delta B_{VP\mu}(n\ell j) = + \frac{\alpha \hbar^2}{15\pi m_\mu^2 c^2} \langle n\ell j | \nabla^2 V | n\ell j \rangle \quad (14)$$

For the muonic Pb $1s_{1/2}$ state this correction amounts to $\Delta B_{VP\mu}(1s_{1/2}) = + 0.247$ keV, with a negligible error.

In view of the experimental accuracy with which muonic x-ray energies can be determined today, higher order vertex corrections to the Lamb shift must be considered.

The fourth-order vertex corrections $\Delta B_{LSII}(n\ell j)$ of order $\alpha^2(Z\alpha)$ that are represented by the physical diagrams

are very small, less than about 20 eV for the $s_{1/2}$ states in heavy muonic atoms. In view of the error in $\Delta B_{LSI}(n\ell j)$ the correction $\Delta B_{LSII}(n\ell j)$ will be neglected.

On the other hand, the fourth-order vertex correction of order $\alpha(Z\alpha)^2$, i.e., $\Delta B_{LSIII}(n\ell j)$ corresponding to the diagram

is not negligible. This term corresponds to the quadratic term in the field strength expansion and thus it is proportional to $\langle \mathcal{E}^2 \rangle$, where $\mathcal{E}$ is the electric field strength. The major contribution to this graph is

$$\Delta B_{LSIII}^{\max}(n\ell j) = - \frac{2\alpha\hbar^3 c}{3\pi m_\mu} \langle \mathcal{E}^2 \rangle \int_0^1 \frac{dz}{zm_\mu^2 c^4 + (1-z)p^2 c^2} . \quad (15)$$

For computational purposes, we take an average value of the local momentum over the muon wave function, i.e., $p^2 c^2 = \langle (E - V)^2 - m^2 c^4 \rangle$. For muonic Pb the correction is $\Delta B_{LSIII}^{\max}(1s_{1/2}) = - 0.306$ keV.

Since Eq. (15) represents only one contribution to this correction and cancellations usually occur, we regard it as an approximate upper limit. Hence we take

$$\Delta B_{LSIII}(n\ell j) \cong \left(\frac{1}{2} \pm \frac{1}{2} \right) \Delta B_{LSIII}^{\max}(n\ell j) \quad (16)$$

and write for the total Lamb-shift correction

$$\Delta B_{LS}(n\ell j) = \Delta B_{LSI}(n\ell j) + \frac{1}{2} \Delta B_{LSIII}(n\ell j) + \Delta B_{Vp\mu}(n\ell j) . \quad (17)$$

We assign, somewhat artificially, an error of

$$\delta[\Delta B_{LS}(n\ell j)] \cong \pm \sqrt{\delta\Delta B_{LSI}(n\ell j)^2 + \frac{1}{4} \Delta B_{LSIII}^{\max}(n\ell j)^2} \quad (18)$$

to the total Lamb-shift correction.

The total Lamb-shift correction is plotted in Fig. 4. In some points the approximate errors are indicated.

4. RELATIVISTIC NUCLEAR MOTION (RECOIL) CORRECTION $\Delta B_R(n\ell j)$

The Dirac equation is solved by using the reduced muon mass

$$m_{\text{red}} = \frac{m_\mu M}{M + m_\mu} , \quad (19)$$

where m_μ is the muon rest mass and M is the rest mass of the nucleus.

Relativistic effects of the nuclear motion are not taken into account by using this reduced mass.

To first order in $\beta^2 = v^2/c^2$ and m_μ/M , there are two major contributions to $\Delta B_R(nlj)$; the kinetic energy is changed because of the relativistic nuclear motion and the moving nuclear charge distribution gives rise to a retardation of the electromagnetic potential (transverse field!). The correction $\Delta B_R(nlj)$, which is positive, has been computed by Barrett et al. (73 Ba) and by Friar and Negele (73 Fr) with similar results. The relativistic nuclear motion correction $\Delta B_R(nlj)$ as computed with the charge distribution (3) using the Friar-Negele technique is shown in Fig. 5.

5. ELECTRON-SCREENING CORRECTION $\Delta B_{ES}(nlj)$

The energies of muonic levels are slightly shifted due to the screening of the atomic electrons. An accurate calculation of $\Delta B_{ES}(nlj)$ is complicated by the fact that the muon cascading down from the $n \approx 14$ level, where it is usually captured, makes predominantly Auger transitions ejecting the atomic electrons, and thus the actual populations of the electron states are uncertain. As n decreases the interaction of the muon with the electron structure becomes less important and radiative transitions dominate for lower n 's. Fortunately, the $1s^2$ electrons contribute about 90 percent of the screening and by the time it is energetically possible to eject $1s_{1/2}$ electrons by Auger transitions, the transitions are predominantly radiative. Hence, in 90 percent of all cases the $1s_{1/2}$ shell can be considered complete. The screening of the field, in which the atomic electrons move, by the muon must also be taken into account. Elaborate calculations have shown that it is sufficient (to within a few eV) to approximate the electron structure by that of a nucleus with charge $Z-1$, so long as the muon principal quantum number n is less than ~ 10 .

Fig. 5. Relativistic-nuclear-motion correction.

Fig. 6. Electron-screening correction.

Fig. 7. Nuclear-polarization correction.

In order to account for the screening of the atomic electrons a screening potential $V_{scr}(r)$ may be added to the nuclear Coulomb potential and the vacuum polarization potentials, before the Dirac equation for the muon is solved. The screening potential energy is

$$V_{scr}(r) = \frac{4\pi e^2}{r} \int_0^r \rho_{el}(t) t^2 dt + 4\pi e^2 \int_r^\infty \rho_{el}(t) t dt \quad (20)$$

where $\rho_{el}(r)$ is the spherically averaged electron density of the atom, normalized to $4\pi \int_0^\infty \rho_{el}(r) r^2 dr = \text{number of electrons (73 Vo)}$.

The second term of Eq. (20) gives rise to a potential at $r = 0$ of the order of $V_{scr}(0) \approx 0.049Z^{4/3}$ keV. At larger radii r , $V_{scr}(r)$ is a slowly decreasing function of r . Hence, $|\Delta B_{ES}(n\ell j)|$ is smaller for large values of n , the largest correction occurring in the $1s_{1/2}$ state $[\Delta B_{ES}(n\ell j)]$. Thus the quantity

$$\Delta B'_{ES}(n\ell j) = \Delta B_{ES}(n\ell j) - \Delta B_{ES}(1s_{1/2}) \quad (21)$$

is always positive (or zero). In other words, the electron-screening correction reduces the transition energies of the muonic x rays [see Eq. (1)].

Since only transition energies between muonic levels are observed, it is sufficient to know the correction $\Delta B'_{ES}(n\ell j)$ of Eq. (21). Values of $\Delta B'(n\ell j)$ have been computed with the charge distribution (3) and for a complete atomic electron shell, using true Hartree-Fock electron densities (75 Ma). The results are shown in Fig. 6.

6. NUCLEAR POLARIZATION CORRECTION $B_{NP}(n\ell j)$

So far it has been assumed that the muon moves in the static Coulomb field of the nucleus and the various higher order corrections were taken

into account assuming a static field produced by an inert nucleus. In other words, it was assumed that the nucleus is not affected by the presence of the muon.

The nucleus, however, is not an inert structure; it is a complicated system of protons and neutrons with a large number of excitation modes. For instance, the muon charge density penetrates into the nuclear volume thus reducing the Coulomb repulsion inside the nucleus.

Classically speaking, this effect causes a shrinking of the nucleus and hence the binding energies of the muonic levels are increased. Quantum-mechanically speaking, the electromagnetic interaction between the muon and the nucleons in the nucleus induces virtual transitions to excited intermediate states of the muon-nucleus system. Obviously, an exact calculation of the nuclear polarization requires knowledge of the complete spectrum of excited nuclear states and their transition strengths, information which, in general, is not available.

The Hamiltonian for the muon-nucleus system can be expressed in the form

$$H = H_{\text{Nucleus}}^0 + H_{\mu}^0 + H_{N\mu} \quad , \quad (22)$$

where H_{Nucleus}^0 is the Hamiltonian of the isolated nucleus; H_{μ}^0 is the Hamiltonian of the free muon and $H_{N\mu}$ describes the electromagnetic interaction between the nucleus and the muon, i.e.,

$$H_{N\mu} = \int d^3\vec{r}_N \frac{\rho_{\text{op}}(\vec{r}_N)}{|\vec{r}_{\mu} - \vec{r}_N|} \quad , \quad (23)$$

where $\rho_{\text{op}}(\vec{r}_N)$ is the one-body charge density operator for the nucleus. The expectation value of $\rho_{\text{op}}(\vec{r}_N)$ in any nuclear state gives the electrostatic nuclear charge distribution for that state. It is convenient to write the interaction Hamiltonian in the form

$$H_{N\mu} = V(\vec{r}_\mu) + \delta H_{N\mu} \quad (24)$$

where

$$V(\vec{r}_\mu) = \langle 0 | H_{N\mu} | 0 \rangle \quad (25)$$

is the expectation value of $H_{N\mu}$ in the nuclear ground state. With the definitions of $\delta H_{N\mu}$ by Eqs. (24) and (25), $\delta H_{N\mu}$ gives no contribution to the energy in first-order perturbation theory. We expand $\delta H_{N\mu}$ in terms of multipole operators

$$\delta H_{N\mu} = \sum_{L=0}^{\infty} \delta H_L \quad (26)$$

with

$$\delta H_L = \int d^3\vec{r}_N \rho_{\text{op}}(\vec{r}_N) \sum_M \frac{4\pi}{2L+1} \frac{r_{<}^L}{r_{>}^{L+1}} Y_{LM}^*(\Omega_N) Y_{LM}(\Omega_\mu) - V(\vec{r}_\mu). \quad (27)$$

Since $V(\vec{r}_\mu)$ contains no nuclear coordinates, it does not contribute to transition matrix elements between different nuclear states and it is usually discarded after noting that $\langle 0 | \delta H_L | 0 \rangle = 0$, where $|0\rangle$ is the nuclear ground state.

The second-order correction to the energy $E(nlj)$ of the muonic atom in the state $|0; nlj\rangle$ is given by

$$\Delta E_{NP}(nlj) = \sum_{L, I\omega, n'l'j'} \frac{\langle 0; nlj | \delta H_L | I\omega; n'l'j' \rangle \langle I\omega; n'l'j' | \delta H_L | 0; nlj \rangle}{E_0 + E(nlj) - E_{I\omega} - E(n'l'j')} \quad (28)$$

where $I\omega$ are the quantum numbers of the nuclear states. The sum over the nuclear states $|I\omega\rangle$ excludes the nuclear ground state $|0\rangle$.

The $L = 0$ monopole term, δH_0 , depends on the excitation energy of the, in general, unobserved, monopole excitations (breathing modes) of the nucleus. Because of the limited information on nuclear compressibility, etc., only an approximate calculation of this term is possible. The $L = 1$ or dipole term δH_1 depends on the, in general, known energy and strength of the giant-dipole term resonance state. For $L \geq 2$ the results are rather insensitive to the energies of the nuclear states involved. As an order of magnitude illustration the contributions to the nuclear polarization correction of the $1s_{1/2}$ states of muonic ^{208}Pb as computed by Skardhamar (70 Sk) are given: $\Delta B_{NP}(1s_{1/2}, \text{monopole breathing mode}) = 2.1 \text{ keV}$, $\Delta B_{NP}(1s_{1/2}, \text{isovector dipole}) = 1.7 \text{ keV}$, $\Delta B_{NP}(1s_{1/2}, \text{quadrupole and octupole shape vibrations}) = 1.1 \text{ keV}$.

Nuclear polarization corrections have been calculated by Cole (69 Co), Chen (70 Ch, 70 Che) and Skardhamar (70 Sk). The results of some of these calculations are shown in Fig. 7. The curves are only intended as a rough guide; the uncertainty of ΔB_{NP} is at least 30 percent.

The nuclear polarization correction for deformed nuclei requires a more refined procedure than outlined above. The nuclear polarization correction is obtained by extending the "model space" [usually spanned by the lowest-rotational-band states of the nucleus and the muon spin multiplet wave-functions $1s_{1/2}$, $(2p_{1/2}, 2p_{3/2})$, $(3d_{3/2}, 3d_{5/2})$ etc.] to include configurations up to several MeV excitation energy. This leads to a renormalization

of the intrinsic quadrupole moment Q_0 of the nuclear state and to a constant energy shift ΔE of the hyperfine components of a "model space" state, when a nonrelativistic treatment is used (70 Che). This energy shift ΔE corresponds to the nuclear polarization correction - $\Delta B_{NP}(nlj)$.

For a reliable analysis of muonic x-ray data, the nuclear polarization correction must be computed for each nucleus individually.

REFERENCES

- 35 Ue E. A. Uehling, Phys. Rev. 48, 55 (1935).
- 55 Kä B. Källén and A. Sabry, Dan. Met. Fys. Medd. 29, No. 17 (1955).
- 55 Sc J. Schwinger, Phys. Rev. 75, 651 (1955).
- 62 Fo K. W. Ford and J. G. Wills, Nucl. Phys. 75, 295 (1962)
- 68 Bar Barrett et al., Phys. Rev. 166, 1589 (1968).
- 68 Ba R. C. Barrett, Phys. Lett. 28B, 93 (1968).
- 68 Be H. A. Bethe and J. W. Negele, Nucl. Phys. A117, 575 (1968).
- 69 Co R. K. Cole, Phys. Rev. 177, 164 (1969).
- 70 Ch Min-Yi Chen, Phys. Rev. C1, 1167 (1970).
- 70 Che Min-Yi Chen, Phys. Rev. C1, 1176 (1970).
- 70 Sk H. F. Skardhamar, Nucl. Phys. A151, 154 (1970).
- 71 Ri G. A. Rinker, Jr., Ph.D. Thesis, University of California, Irvine, CA,
(unpublished).
- 72 Bl J. Blomqvist, Nucl. Phys. B48, 95 (1972).
- 73 Ba Barrett et al., Phys. Lett. 47B, 297 (1973).
- 73 Fr J. L. Friar and J. W. Negele, Phys. Lett. 46B, 5, (1973).
- 73 Vo P. Vogel, Phys. Rev. 7A, 63 (1973).
- 74 En R. Engfer et al., Atomic and Nucl. Data Tables 14, 509 (1974).
- 75 Fu D. Fujimoto, to be published.
- 75 Su M. K. Sundarisan and P. J. S. Watson, Int. Conference in High-Energy
Physics and Nuclear Structure, Santa Fe, NM, June 1975.
- 75 Ch Min-Yi Chen, Phys. Rev. Lett. 34, 341 (1975).
- 75 Ri G. A. Rinker and L. Wilets, Phys. Rev., to be published.
- 75 Wi L. Wilets and G. A. Rinker, Phys. Rev. Lett. 34, 339 (1975).
- 75 Ma J. B. Mann and G. A. Rinker, Jr., Phys. Rev. A11, 385 (1975).