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DUAL LOOP TERMS AND COUNTERTERMS BY MEANS OF THEIR PROJECTIVE GROUP+

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ABSTRACT

A unified treatment of dual loop term and counterterm by means of their projective group is presented for the one loop case. The significance of parameters in this method of linear differential equations is discussed for multi loop terms.

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A global approach to multiloop diagrams has been described by Lovelace[1] and Alessandrini.[2] The p-loop integral is given in terms of an automorphic function for a Fuchsian group of the second kind. This function conformally maps the fundamental region (whose shape depends on p) on to the upper half plane in a one-to-one manner. An earlier and different approach, using harmonic oscillator operators, was presented by GNSS[3] who were able to go a step further and obtain a new integral, called the counterterm, which could be subtracted from the infinite loop integral, the difference being finite and renormalized.[4] Later, Kaku and Scherk[5] described in the two-loop case the mechanism of the touching of isometric circles which gave the leading divergent contribution. It has remained an open question: what is the connection of this counterterm to the underlying Fuchsian group? Knowing this may indicate the uniqueness of a given renormalization method and give its p-loop generalization in global form.

We present here a group-theoretical interpretation of the counterterm which keeps the same Fuchsian group as its infinite loop integral. We are able to demonstrate by this method the N.S.[4] counterterm (for $p = 1$) as the limiting function of a certain sequence; we can only conjecture that a similar limiting process will yield the p-loop counterterms.

It is known that one can generate automorphic functions for Fuchsian groups by the method of second order differential equations.[6] This technique is alternative to the one of Abelian integrals. Let y_1

and y_2 be two particular solutions of the 2^d order d.e.

$$y'' + p(\zeta)y' + q(\zeta) = 0, \tag{1}$$

and consider the variation of the ratio

$$z(\zeta) \equiv \frac{y_1(\zeta)}{y_2(\zeta)} \tag{2}$$

under a circuit of the independent complex variable ζ around a singular point of the differential equation. $y_1(\zeta)$ becomes $ay_1(\zeta) + by_2(\zeta)$ and y_2 becomes $cy_1(\zeta) + dy_2(\zeta)$, where a, b, c, d are independent of ζ. Thus, we find

$$z \to \tilde{z} = \frac{az+b}{cz+d} = T(z) \tag{3}$$

which is a projective transformation, an element of a Fuchsian group associated with the differential equation (monodromic group). Further developments lead to the Schwarzian derivative

$$[z] \equiv \frac{d}{d\zeta}\left(\frac{z''}{z'}\right) - \frac{1}{2}\left(\frac{z''}{z'}\right)^2 \tag{4}$$

which satisfies

$$[z] = 2q - \frac{1}{2}p^2 - \frac{dp}{d\zeta}, \tag{5}$$

a rational function. $z(\zeta)$ performs a conformal mapping determined by the parameters in the given functions

$$p(\zeta) = \frac{1 - a' - a''}{\zeta - e_1} + \ldots + \frac{1 - \tau' - \tau''}{\zeta - e_t}, \tag{6}$$

$$q(\zeta) = \frac{1}{(\zeta-e_1)\ldots(\zeta-e\ )} \left\{ \frac{a'a''(e_1-e_2)\ldots(e_1-e_t)}{\zeta-e_t} + \ldots \right. \quad (7)$$

$$\left. \ldots + \frac{\tau'\tau''(e_t-e_1)\ldots(e_t-e_s)}{\zeta-e_t} + G_{n-4}(\zeta) \right\} \quad ;$$

and satisfies

$$[z] = \frac{1}{(\zeta-e_1)\ldots(\zeta-e_t)} \left\{ \frac{1-a^2}{2} \cdot \frac{(e_1-e_2)\ldots(e_1-e_t)}{(\zeta-e_1)} + \ldots \right.$$

$$\left. + \frac{1-\tau^2}{2} \cdot \frac{(e_t-e_1)\ldots(e_t-e_s)}{\zeta-e_t} + g_{n-4}(\zeta) \right\} . \quad (8)$$

Here, a', a" are the exponents and a = a' - a" the exponent-difference of y_1 and y_2 belonging to the first singularity e_1, while G_{n-4} and g_{n-4} are polynomials of degree at most n-4 which contain n-3 arbitrary so called accessory parameters. The enumeration of arbitrary constants in the differential equation neatly matches that of the Mobius transformation generators T(z). For p loops we have p generators with 3p-6 independent real constants. The differential equation contains n = 4p singular points, one for each intersection of an isometric circle in the z-plane with the real axis (fundamental circle). We note that the fundamental polygon (i.e. the upper half of the fundamental region) for p loops or p T_α generators[2] contains 4p sides since it corresponds to the double of a surface with p holes, which is a torus with p handles. The topological symbol for this torus is $a_1 b_1 a^{-1} b^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_p b_p a_p^{-1} b_p^{-1}$ with 4p letters. The real constants in the differential equation are enumerated as follows: (i) 4p exponent differences of the two solutions y_1 and y_2 at the 4p singular points; (ii) 4p-3 cross ratios for locations of 4p singular points on the real axis; and (iii) 4p-3 accessory parameters. Now the exponent differences (i) must be 1/2 so that the isometric circles intersect the real axis at right angles; and the isometric circles come in pairs of equal radii, a fact which changes 4p to 3p in (ii). We can always arrange things so that the 3p-3 Riemann moduli or constants in the p generators T_a are given by the isometric circle positions (ii), leaving free the accessory parameters (iii). What now is the significance of the accessory parameters? Consider the fundamental region of the Fuchsian group. If, for example, $p = 2$, a possible fundamental region R_0 is shown in Fig. 1. We now mention the significant difference between a general automorphic function $\zeta(z)$, defined as the inverse of (2), and the one defined as the exponential of the Abelian integrals $\omega(z)$ usually considered. Both $\zeta(z)$ and $\exp(\omega(z))$ will yield a one-to-one mapping of R_0 on the upper half ζ plane; but with $\zeta(z)$ we can choose the accessory parameters so that the mapping is many-to-one. That is, the n-sheeted covering R_{on} of R_0 (Fig. 1) can be mapped by $\zeta(z)$ on the upper half plane. (It may be possible to do this with $\omega(z)$, but we cannot say at present anything about it.) Now the one-to-one mapping yields the p-loop integrand.[1] We now show that the ∞-to-one mapping yields the counterterm in the case $p = 1$. Let us consider the d.e.

$$\frac{d^2y}{d\zeta^2} + \frac{1}{2}\left(\frac{1}{\zeta-e_1} + \frac{1}{\zeta-e_2} + \frac{1}{\zeta-e_3}\right)\frac{dy}{d\zeta} + \frac{By(\zeta)}{(\zeta-e_1)(\zeta-e_2)(\zeta-e_3)} = 0 \qquad (9)$$

which contains the accessory parameter B. We can bring it into the form

$$\frac{d^2y}{d\tilde{z}^2} + DB\, y(\tilde{z}) = 0 \qquad (10)$$

by introducing a general elliptic integral

$$\frac{d\zeta}{d\tilde{z}} = \sqrt{D(\zeta-e_1)(\zeta-e_2)(\zeta-e_3)} \qquad (11)$$

whose periods are defined by the singular points $e_1 = \zeta(0)$, $e_2 = \zeta(-\omega_1)$ and $e_3 = \zeta(\omega_1 - i\omega_2)$. Choosing as the 2 independent solutions the exponential forms we find for the ratio z of eq. 2,

$$z = \exp(-2i\sqrt{BD}\tilde{z}). \qquad (12)$$

for the kinds of groups involved here it has been shown[7] that the accessory parameter is quantised to have the values (with $\omega_2 = \pi$) $BD = -k^2/4$, where k is an odd integer. Then the mapping of the lower half ζ-plane with e_i on the real axis is into a rectangle $(o, -\omega_1, i\pi-\omega_1, i\pi)$ in the $\tilde{z}$ plane by (11). This in turn by (12) is mapped into the upper half of an annulus for $k = 1$, or into additional annuli attached to this for higher values of k, which cover the region multiply. This displays the role of the accessory parameter (fig. 2).

Since, however, the radii of these annuli will also vary with k we are forced to change the positions of the e_i in the ζ-plane simultaneously into e_{ik} in order to keep the radii fixed at 1 and K respectively as required by the Möbius group (K is the multiplier in the generator[8]).

This can be achieved by setting $\omega_1 = -\frac{1}{2}(\ln K)/k$.

If for the sake of tranparency we make the particular choice $e_1=0$, $e_2=1$, $e_3=(a_1 - a_3)/a_2 - a_3)$, $D = 4(a_2 - a_3)$, then we get a very simple elliptic function in terms of the Weierstrass function $\wp(\tilde{z}) = \wp(\tilde{z}\ |\omega_1\ i\pi)$

$$\zeta(\tilde{z}) = \frac{a_1 - a_3}{\wp(\tilde{z}) - a_3}\ , \tag{12}$$

where a_1, a_2, a_3 are the values of $\wp$ at the arguments $\tilde{z} = \omega_1$, $\omega_1 + i\pi$, $i\pi$, respectively.

On using homogeneity relations[9] this function becomes

$$\zeta(z) = \frac{a_{1k} - a_{3k}}{\wp(\frac{\ln z}{\ln K}\ \mid \frac{1}{2}, \frac{ik\ \pi}{\ln K}) - a_{3k}} \tag{13}$$

where a_{1k} and a_{3k} are now the values of $\wp(u\ |1/2,\ ik\pi/\ln K)$ at $u = 1/2$ and $ik\pi/\ln K$, respectively. This automorphic function (13) will be single-valued in the z-plane for $k = 1$ and multiple-valued for all higher odd values of k, and it can be viewed as the automorphic master-function from which both loop term and counterterm derive: For $k = 1$ this will give the automorphic part $\exp\{\omega_{z_i z_o}(z) + \omega_{z_i z_o^*}(z)\}$ in Lovelace's formulation of the one-loop amplitude, up to α_0-dependent factors and with $z_i = E_1'$, $z_o = z_o^* = E_\infty'$ of fig. 2. To show this one has merely to use the expansion of an automorphic function in terms of θ_1-functions of the variable Φ (first Abelian integral)[11] and to compare it to the well known representation of $\wp$ in the same functions.

For $k = \infty$ on the other hand the masterfunction (13) becomes[10] the counterterm of GNSS[3] (with our special choice of parameters)

$$\zeta_\infty = \sin^2(\pi \frac{\ln z}{\ln K}). \qquad (14)$$

This is therefore seen to be an ∞-valued automorphic function of the same Moebius group. The crucial relation is $B_k = - e_{3k} \ln^2 K/16\pi^2$, between the accessory parameter of the corresponding d.e. and the regular sinular point e_{3k}, which moves to infinity as k increases. Its qualification as a counterterm derives immediately from the fact that the main term with $k = 1$ has exactly the same limiting function (14) for $\ln K \to 0$ (parabolic limit), since k and ln K appear in the same factor.

The generalisation to arbitrary positions of z_i and z_0, z_0^* will yield other rational functions of $\wp$. Similar ∞-valued functions also exist for all other multiply connected fundamental regions so that this method can be extended to investigate counterterms for multi-loop expressions.

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Figure Captions

Figure 1: Fundamental region R_0 in the two loop case after application of a Möbius tranformation. a, a^{-1}, b, b^{-1} are the isometric circles, c is the principle circle.

Figure 2: Twice covered half annulus for one loop case.

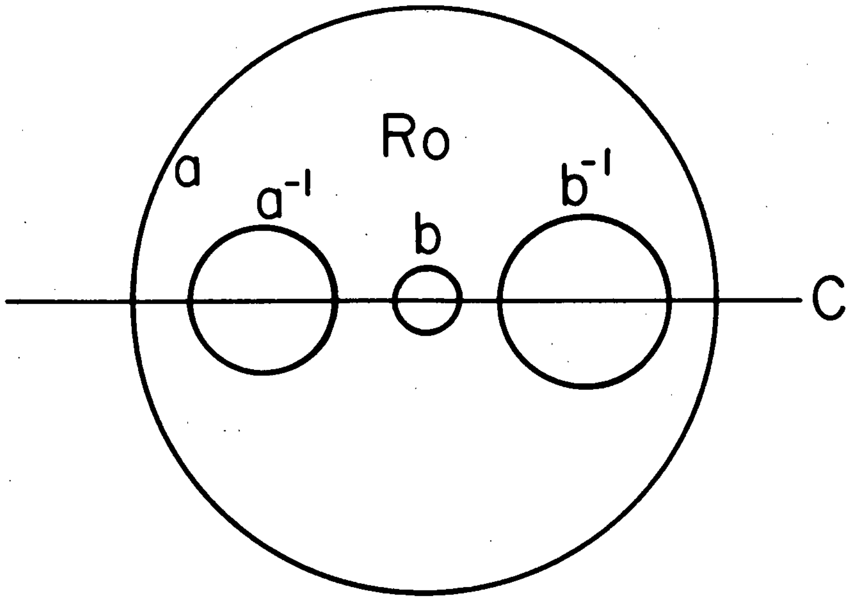

Fig. 1

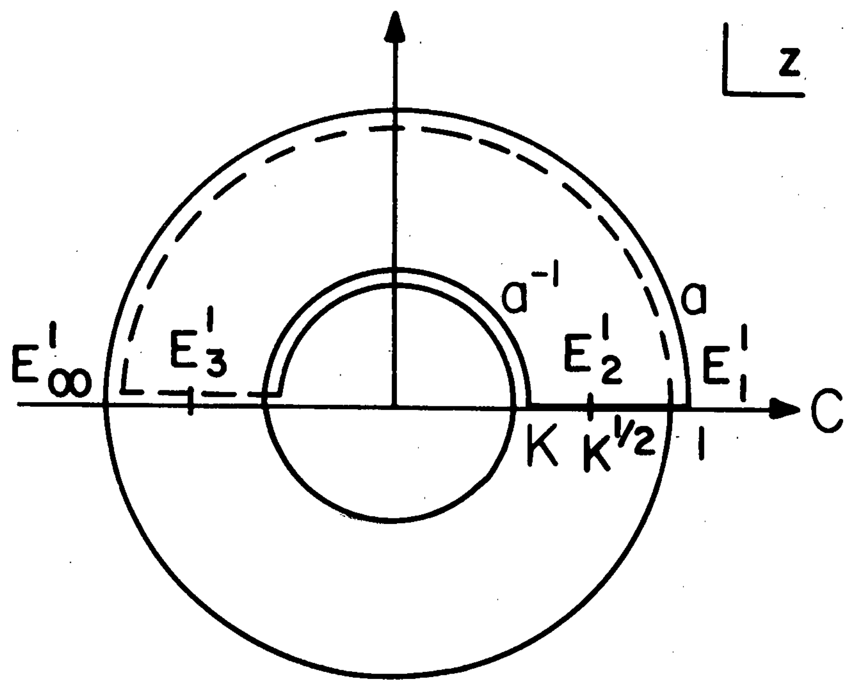

Fig.2