RULES FOR CONSTRUCTING GENERALIZATIONS OF THE
VENEZIANO N-POINT FUNCTION*

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ABSTRACT

Rules are presented for constructing N-point functions in dual resonance models with non-linear trajectories. The planar tree graph five-point function is discussed and an explicit expression for the N-point function is given.

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Recently, Coon¹ showed that the Veneziano four-point function² is a special limiting case of a one parameter family of functions, all of which possess (a) meromorphy in s and t , (b) polynomial residues, (c) Regge asymptotic behavior in the s and t channels, and generally non-linear trajectories. A linear combination of functions of this type may even be the most general³ four-point function with the above properties.

In this paper we give rules⁴ for constructing corresponding generalizations of the Veneziano N -point functions of Bardakci and Ruegg⁵, Chan and Tsou⁶, Goebel and Sakita⁷, and others⁸. The rules turn out to be very simple and produce an amplitude which in a certain limit reduces to the Veneziano N -point function. This generalization, like the Veneziano formula itself, is a possible candidate for the Born term in a theory containing an infinite number of particles of arbitrarily high spin. Since the behavior of loop diagrams in such theories is not in general satisfactory, it is clearly of interest to have a more general class of Born terms upon which to base the construction of the higher order diagrams.

We first summarize the basic properties¹ of the four-point function $B_4(s, t)$ found by Coon:

$$B_4(s, t) = \frac{G(\sigma\tau)}{G(\sigma)G(\tau)} \equiv \prod_{\ell=0}^{\infty} \frac{(1-\sigma\tau q^{\ell})}{(1-\sigma q^{\ell})(1-\tau q^{\ell})} \quad (1)$$

where q is a parameter, $0 < q < 1$,

$$\sigma \equiv as+b, \quad \tau \equiv ct+d \quad (2)$$

and

$$G(\sigma) = \prod_{\ell=0}^{\infty} (1-\sigma q^{\ell}) \quad (3)$$

is an entire function. The positions s_j of the s -channel poles of $B_4(s,t)$ are given by the solutions of the equations

$$\sigma \equiv as+b = q^{-j}, \quad j = 0, 1, 2, \dots \quad (4)$$

for s . The residue of the j -th pole s_j is a polynomial in t of order j . The s -channel trajectory function $\alpha(s)$ is therefore obtained by setting $j = \alpha(s)$ in Eq.(4). This gives

$$\alpha(s) = - \frac{\log \sigma}{\log q} = - \frac{\log (as+b)}{\log q} \quad (5)$$

By making a partial fraction expansion of Eq.(1), it is easy to see that $B_4(s,t)$ can be represented either as a sum over s -channel poles or as a sum over t -channel poles. That is, it satisfies duality.

One can also obtain the following expansion of the infinite product representation (1) in a double power series in σ and τ (and hence in s and t)⁹:

$$B_4(s,t) = \sum_{n,m=0}^{\infty} \frac{\sigma^n}{f_n} q^{nm} \frac{\tau^m}{f_m} \quad (6)$$

where $f_n = (1-q)(1-q^2) \dots (1-q^n)$.

Eq. (6) is valid for $|\sigma| < 1$ and $|\tau| < 1$. If we set $\tau = 0$ in Eqs. (1) and (6) we obtain the expansion

$$\frac{1}{G(\sigma)} \equiv \frac{1}{\prod_{n=0}^{\infty} (1 - \sigma q^n)} = \sum_{n=0}^{\infty} \frac{\sigma^n}{f_n} \quad (7)$$

Without the q^{nm} factor in Eq.(6), the n and m sums would have decoupled and by Eq.(7) we would have obtained $[G(\sigma)G(\tau)]^{-1}$ which is just the product of the denominator factors of Eq.(1). The residues of the poles in σ would then have had poles in τ , in contrast with Eq.(1) where the residues of the poles in σ are polynomials in τ . Thus, the coupling factor q^{nm} in Eq.(6) prevents simultaneous poles in σ and τ but allows poles in either σ or τ alone. We thus call q^{nm} a simultaneous pole killer or "duality factor". With the above guide we can write down almost immediately the generalization of Eq.(6) to the N -point function B_N corresponding to a given set of Feynman graphs without closed loops (planar graphs for example). This single function B_N must possess the pole and residue structure corresponding to the given set of Feynman diagrams.

The positions of the poles in B_N are then determined as follows: if p_L is the momentum of the particle associated with an internal line L of one of the diagrams, then B_N must possess poles in the variable p_L^2 at values determined from the equation

$$\sigma_L(p_L^2) \equiv a_L p_L^2 + b_L = q^{-j}, \quad j = 0, 1, 2, \dots \quad (8)$$

where a_L and b_L are constants and q is the same parameter

that appeared in B_4 . This guarantees that the mass spectrum determined from B_N is consistent with that deduced from B_4 . We can construct a function B_N^{sing} with all the above poles by forming the product

$$B_N^{\text{sing}} = \prod_{[L]} \frac{1}{G[\sigma_L(p_L^2)]} \quad (9)$$

where $G(\sigma)$ is the infinite product defined by Eq. (3). The product (9) is taken over all internal lines L which carry distinct momenta in the given set of diagrams. (There are $\frac{1}{2}N(N-3)$ such lines in the planar tree diagrams.) B_N^{sing} is not a satisfactory candidate for B_N because it contains simultaneous poles in all the variables. If we insert the expansion (7) for $1/G$ into (9) we obtain a multiple power series in the variables $\sigma_L(p_L^2)$:

$$B_N^{\text{sing}} = \sum_{n_1, n_2, \dots, n_r}^{\infty} \frac{(\sigma_{L_1})^{n_1}}{f_{n_1}} \frac{(\sigma_{L_2})^{n_2}}{f_{n_2}} \dots \frac{(\sigma_{L_r})^{n_r}}{f_{n_r}} \quad (10)$$

which is similar to our expansion (6) for B_4 except for the absence of $q^{n_i n_j}$ "pole killing factors". We want to construct a function B_N which has no simultaneous poles in the variables $p_{L_i}^2$ and $p_{L_j}^2$ if there is no Feynman diagram in the given set which contains the pair of internal lines L_i and L_j .

Using the analogy with Eq. (6) for B_4 we can construct such a function B_N from Eq. (10) by the following simple rule:

RULE: For each pair of variables $p_{L_i}^2$ and $p_{L_j}^2$ for which there exists no Feynman diagram containing both lines L_i and L_j , introduce a "pole killing factor" $q^{n_i n_j}$ under the multiple sum in Eq. (10).

We take the power series defined by the above rule as our N -point function B_N . Using Eq. (7), it can be shown that the function B_N constructed from the above rule does possess the correct pole structure. Furthermore, B_N possesses distinct partial fraction expansions each of which corresponds to the pole and residue structure of one of the Feynman diagrams of the given set which are thus said to be related by duality.

Since our rule was not arrived at deductively, our expression for B_N may not be unique. However, the rule for B_N gives the direct and natural extension of Eq. (6) for B_4 and B_4 was arrived at deductively. Therefore, it is natural to use this rule as the first guess for B_N . It may of course be the only acceptable rule aside from generalizations¹⁰ which yield formulas analogous to Veneziano satellites.

We should emphasize that the above rule gives the correct pole structure for planar as well as non-planar sets of tree graphs. In the case of planar tree graphs it yields the following explicit formula¹¹ for B_N :

$$B_N \equiv \sum_{\substack{\text{all} \\ n_{ij}=0}}^{\infty} \prod_{i=1}^{N-2} \prod_{j=i+1}^{N-1} \frac{\sigma_{ij}^{n_{ij}}}{f_{n_{ij}}} \prod_{k=i+1}^j \prod_{\ell=i+1}^{N-1} \frac{q^{n_{ij}n_{k\ell}}}{q^{n_{ij}n_{k\ell}}} \quad (11)$$

where

$$\sigma_{ij} = a_{ij} (p_i + p_{i+1} + \dots + p_j)^2 + b_{ij} \quad (12)$$

except for $\sigma_{1,N-1} = 0$ which is introduced to make Eq. (12) more compact.

The limit $q \rightarrow 1$ of B_N times a factor depending only on q has been shown¹¹ to be the Veneziano N -point function⁵⁻⁸. For B_4 , this is easy to verify. We choose the q -dependence of the constants a, b, c, d of Eq. (2) so that near $q = 1$,

$$\sigma = 1 + (1-q) \sigma' + O[(1-q)^2] \quad (13)$$

$$\tau = 1 + (1-q) \tau' + O[(1-q)^2] \quad (14)$$

where

$$\sigma' = a's + b', \quad \tau' = c't + d'. \quad (15)$$

then using Eq. (1) and the infinite product representation of the Γ function we find

$$\lim_{q \rightarrow 1} (1-q) G(q) B_4(s, t) = \frac{\Gamma(-\sigma') \Gamma(-\tau')}{\Gamma(-\sigma' - \tau')} \quad (16)$$

and from (5) we obtain

$$\lim_{q \rightarrow 1} \alpha(s) = \sigma' \quad (17)$$

Thus, in the limit $q \rightarrow 1$, B_4 is proportional to the Veneziano four-point function with trajectory functions given by (15). The corresponding proof¹¹ for B_N is much more involved since we do not have an infinite product representation for B_N . We also note that the power series expansions (6) and (11) converge only for $|\sigma|, |\tau|, |\sigma_{ij}| < 1$ and hence from (13) and (14) we see that these representations do not exist in the Veneziano ($q \rightarrow 1$) limit.

As a concrete example of our rule we construct the five-point function B_5 for the planar tree diagrams of Fig. 1. We can construct B_5 either from Eq. (11) or, more simply, directly from our rule. Noting that internal lines with momentum $p_1 + p_2$ and $p_2 + p_3$ never occur in the same diagram and exploiting cyclic symmetry we easily obtain

$$B_5 = \sum_{\text{all } n_{ij}=0}^{\infty} \frac{\sigma_{12}^{n_{12}}}{f^{n_{12}}} q^{n_{12}n_{23}} \frac{\sigma_{23}^{n_{23}}}{f^{n_{23}}} q^{n_{23}n_{34}} \times \frac{\sigma_{34}^{n_{34}}}{f^{n_{34}}} q^{n_{34}n_{45}} \frac{\sigma_{45}^{n_{45}}}{f^{n_{45}}} q^{n_{45}n_{51}} \frac{\sigma_{51}^{n_{51}}}{f^{n_{51}}} q^{n_{51}n_{12}} \quad (18)$$

where $\sigma_{ij} \equiv a_{ij} s_{ij} + b_{ij} = a_{ij} (p_i + p_j)^2 + b_{ij}$.

The sums in Eq. (18) converge for $|\sigma_{ij}| < 1$ and the relevant range ($0 < q < 1$) of q . Using Eq. (7), we can trivially do any one of the sums in Eq. (18) and verify that there are poles in each channel. Doing the n_{23} sum generates a factor

$$\frac{1}{G(\sigma_{23} q^{n_{12}+n_{34}})} \quad (19)$$

which has poles at $\sigma_{23} = q^{-\ell}$ where $\ell \geq n_{12}+n_{34}$. Thus, in the sum over n_{12} and n_{34} we find poles at $\sigma_{23} = q^{-j}$ with $j = 0, 1, 2, \dots$. The pole at $\sigma_{23} = 1$ is contained only in the $n_{12} = n_{34} = 0$ terms of Eq. (18). The residue of the spin zero $\sigma_{23} = 1$ pole of B_5 can be evaluated using Eqs. (3) and (19). The surviving σ_{ij} dependence of the residue is easily seen to be

$$\sum_{n_{45}=0}^{\infty} \sum_{n_{51}=0}^{\infty} \frac{\sigma_{45}^{n_{45}}}{f_{n_{45}}} q^{n_{45}n_{51}} \frac{\sigma_{51}^{n_{51}}}{f_{n_{51}}} \quad (20)$$

which is just a four-point function (6). This proves the consistency of our five-point and four-point functions. With similar techniques we can derive¹⁰ all other required properties of B_5 directly from the power series expansion (18). A similar discussion¹¹ could be carried out for B_N . However, in order to have a theory one has to specify a consistent set of rules for calculating loop diagrams. Only some steps in this program have been carried out¹². If this work can be completed we will have a wider range of candidates for physically relevant dual resonance theories.

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FIGURE CAPTION

Figure 1. Planar tree diagrams contained in the five-point function.

Figure 1