RULES FOR CONSTRUCTING GENERALIZATIONS OF THE 
VENEZIANO N-POINT FUNCTION*

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ABSTRACT

Rules are presented for constructing N-point functions in dual resonance models with non-linear trajectories. The planar tree graph five-point function is discussed and an explicit expression for the N-point function is given.

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Recently, Coon\textsuperscript{1} showed that the Veneziano four-point function\textsuperscript{2} is a special limiting case of a one parameter family of functions, all of which possess (a) meromorphy in $s$ and $t$, (b) polynomial residues, (c) Regge asymptotic behavior in the $s$ and $t$ channels, and generally non-linear trajectories. A linear combination of functions of this type may even be the most general\textsuperscript{3} four-point function with the above properties.

In this paper we give rules\textsuperscript{4} for constructing corresponding generalizations of the Veneziano $N$-point functions of Bardakci and Ruegg\textsuperscript{5}, Chan and Tsou\textsuperscript{6}, Goebel and Sakita\textsuperscript{7}, and others\textsuperscript{8}. The rules turn out to be very simple and produce an amplitude which in a certain limit reduces to the Veneziano $N$-point function. This generalization, like the Veneziano formula itself, is a possible candidate for the Born term in a theory containing an infinite number of particles of arbitrarily high spin. Since the behavior of loop diagrams in such theories is not in general satisfactory, it is clearly of interest to have a more general class of Born terms upon which to base the construction of the higher order diagrams.

We first summarize the basic properties\textsuperscript{1} of the four-point function $B_4(s,t)$ found by Coon:

$$B_4(s,t) = \frac{G(\sigma \tau)}{G(\sigma)G(\tau)} = \prod_{l=0}^{\infty} \frac{(1-\sigma q^l)}{(1-\sigma q^l) (1-\tau q^l)}$$

(1)

where $q$ is a parameter, $0 < q < 1$,
\[ \sigma \equiv as+b, \quad \tau \equiv ct+d \quad (2) \]

and

\[ G(\sigma) = \prod_{\ell=0}^{\infty} (1-\sigma \ell) \quad (3) \]

is an entire function. The positions \( s_j \) of the s-channel poles of \( B_4(s,t) \) are given by the solutions of the equations

\[ \sigma = as+b = q^{-j}, \quad j = 0,1,2, \ldots \quad (4) \]

for \( s \). The residue of the \( j \)-th pole \( s_j \) is a polynomial in \( t \) of order \( j \). The s-channel trajectory function \( \alpha(s) \) is therefore obtained by setting \( j = \alpha(s) \) in Eq.(4). This gives

\[ \alpha(s) = - \frac{\log \sigma}{\log q} = - \frac{\log (as+b)}{\log q} \quad (5) \]

By making a partial fraction expansion of Eq.(1), it is easy to see that \( B_4(s,t) \) can be represented either as a sum over s-channel poles or as a sum over t-channel poles. That is, it satisfies duality.

One can also obtain the following expansion of the infinite product representation (1) in a double power series in \( \sigma \) and \( \tau \) (and hence in \( s \) and \( t \)):

\[ B_4(s,t) = \sum_{n,m=0}^{\infty} \frac{\sigma^n}{f_n} q^{nm} \frac{\tau^m}{f_m} \quad (6) \]

where \( f_n = (1-q) (1-q^2) \cdots (1-q^n) \).

Eq. (6) is valid for \( |\sigma| < 1 \) and \( |\tau| < 1 \). If we set \( \tau = 0 \) in Eqs. (1) and (6) we obtain the expansion
Without the $q_{nm}^n$ factor in Eq. (6), the $n$ and $m$ sums would have decoupled and by Eq. (7) we would have obtained $\left[ G(\sigma)G(\tau) \right]^{-1}$ which is just the product of the denominator factors of Eq. (1). The residues of the poles in $\sigma$ would then have had poles in $\tau$, in contrast with Eq. (1) where the residues of the poles in $\sigma$ are polynomials in $\tau$. Thus, the coupling factor $q_{nm}^n$ in Eq. (6) prevents simultaneous poles in $\sigma$ and $\tau$ but allows poles in either $\sigma$ or $\tau$ alone. We thus call $q_{nm}^n$ a simultaneous pole killer or "duality factor". With the above guide we can write down almost immediately the generalization of Eq. (6) to the $N$-point function $B_N$ corresponding to a given set of Feynman graphs without closed loops (planar graphs for example). This single function $B_N$ must possess the pole and residue structure corresponding to the given set of Feynman diagrams.

The positions of the poles in $B_N$ are then determined as follows: if $p_L^2$ is the momentum of the particle associated with an internal line $L$ of one of the diagrams, then $B_N$ must possess poles in the variable $p_L^2$ at values determined from the equation

$$\sigma_L(p_L^2) \equiv a_L p_L^2 + b_L = q^{-j} , \ j = 0,1,2,\cdots \quad (8)$$

where $a_L$ and $b_L$ are constants and $q$ is the same parameter.
that appeared in $B_4$. This guarantees that the mass spectrum determined from $B_N$ is consistent with that deduced from $B_4$.

We can construct a function $B_N^{\text{sing}}$ with all the above poles by forming the product

$$B_N^{\text{sing}} = \prod \frac{1}{G[\sigma_L(p_L^2)]}$$

(9)

where $G(\sigma)$ is the infinite product defined by Eq. (3). The product (9) is taken over all internal lines $L$ which carry distinct momenta in the given set of diagrams. (There are $\frac{1}{2}N(N-3)$ such lines in the planar tree diagrams.) $B_N^{\text{sing}}$ is not a satisfactory candidate for $B_N$ because it contains simultaneous poles in all the variables. If we insert the expansion (7) for $1/G$ into (9) we obtain a multiple power series in the variables $\sigma_L(p_L^2)$:

$$B_N^{\text{sing}} = \sum_{n_1, n_2, \ldots n_r} \prod_{L} \frac{(\sigma_{L_1})^{n_1}}{f_{n_1}} \frac{(\sigma_{L_2})^{n_2}}{f_{n_2}} \cdots \frac{(\sigma_{L_r})^{n_r}}{f_{n_r}}$$

(10)

which is similar to our expansion (6) for $B_4$ except for the absence of $q^{n_in_j}$ "pole killing factors". We want to construct a function $B_N$ which has no simultaneous poles in the variables $p_{L_1}^2$ and $p_{L_j}^2$ if there is no Feynman diagram in the given set which contains the pair of internal lines $L_1$ and $L_j$.

Using the analogy with Eq. (6) for $B_4$ we can construct such a function $B_N$ from Eq. (10) by the following simple rule:
RULE: For each pair of variables $p_{L_i}^2$ and $p_{L_j}^2$ for which there exists no Feynman diagram containing both lines $L_i$ and $L_j$, introduce a "pole killing factor" $q^{n_i n_j}$ under the multiple sum in Eq. (10).

We take the power series defined by the above rule as our $N$-point function $B_N$. Using Eq. (7), it can be shown that the function $B_N$ constructed from the above rule does possess the correct pole structure. Furthermore, $B_N$ possesses distinct partial fraction expansions each of which corresponds to the pole and residue structure of one of the Feynman diagrams of the given set which are thus said to be related by duality.

Since our rule was not arrived at deductively, our expression for $B_N$ may not be unique. However, the rule for $B_N$ gives the direct and natural extension of Eq. (6) for $B_4$ and $B_4$ was arrived at deductively. Therefore, it is natural to use this rule as the first guess for $B_N$. It may of course be the only acceptable rule aside from generalizations\(^{10}\) which yield formulas analogous to Veneziano satellites.

We should emphasize that the above rule gives the correct pole structure for planar as well as non-planar sets of tree graphs. In the case of planar tree graphs it yields the following explicit formula\(^{11}\) for $B_N$: 
\[ B_N = \sum_{\text{all } i=1}^{\infty} \prod_{j=i+1}^{N-1} \prod_{k=i+1}^{N-1} \sigma_{ij} \prod_{l=i+1}^{N-1} q^{n_{ij} n_{kl}} \]  

(11)

where

\[ \sigma_{ij} = a_{ij} (p_i + p_{i+1} + \cdots + p_j)^2 + b_{ij} \]  

(12)

except for \( \sigma_{1,N-1} = 0 \) which is introduced to make Eq. (12) more compact.

The limit \( q \to 1 \) of \( B_N \) times a factor depending only on \( q \) has been shown\(^1\) to be the Veneziano \( N \)-point function\(^5-8\). For \( B_4 \), this is easy to verify. We choose the \( q \)-dependence of the constants \( a, b, c, d \) of Eq. (2) so that near \( q = 1 \),

\[ \sigma = 1 + (1-q) \sigma' + O[(1-q)^2] \]  

(13)

\[ \tau = 1 + (1-q) \tau' + O[(1-q)^2] \]  

(14)

where

\[ \sigma' = a's + b', \tau' = c't + d' \]  

(15)

then using Eq. (1) and the infinite product representation of the \( \Gamma \) function we find

\[ \lim_{q \to 1} (1-q)G(q)B_4(s,t) = \frac{\Gamma(-\sigma')\Gamma(-\tau')}{\Gamma(-\sigma'-\tau')} \]  

(16)
and from (5) we obtain

$$\lim_{q \to 1} \gamma(s) = \sigma'. \quad (17)$$

Thus, in the limit \( q \to 1 \), \( B_4 \) is proportional to the Veneziano four-point function with trajectory functions given by (15). The corresponding proof for \( B_N \) is much more involved since we do not have an infinite product representation for \( B_N \). We also note that the power series expansions (6) and (11) converge only for \( |\sigma|, |\tau|, |\sigma_{ij}| < 1 \) and hence from (13) and (14) we see that these representations do not exist in the Veneziano \((q \to 1)\) limit.

As a concrete example of our rule we construct the five-point function \( B_5 \) for the planar tree diagrams of Fig. 1. We can construct \( B_5 \) either from Eq. (11) or, more simply, directly from our rule. Noting that internal lines with momentum \( p_1 + p_2 \) and \( p_2 + p_3 \) never occur in the same diagram and exploiting cyclic symmetry we easily obtain

\[
B_5 = \sum_{\text{all } n_{ij} = 0}^{\infty} \frac{\sigma_{12}}{f} q^{n_{12}} n_{12} \frac{\sigma_{23}}{f} q^{n_{23}} n_{23} \frac{\sigma_{23}}{f} q^{n_{23} n_{34}}
\]

\[
\times \frac{\sigma_{34}}{f} q^{n_{34}} n_{12} \frac{\sigma_{45}}{f} q^{n_{45}} n_{23} \frac{\sigma_{51}}{f} q^{n_{51}} n_{34} \left( n_{45} n_{51} \frac{\sigma_{51}}{f} q^{n_{51}} n_{12} \left( \frac{\sigma_{12}}{f} q^{n_{12}} n_{12} \frac{\sigma_{23}}{f} q^{n_{23}} n_{23} \frac{\sigma_{23}}{f} q^{n_{23} n_{34}} \right) \right)
\]

(18)

where \( \sigma_{ij} = a_{ij} s_{ij} + b_{ij} = a_{ij} (p_i + p_j)^2 + b_{ij} \).
The sums in Eq. (18) converge for \( |\sigma_{ij}| < 1 \) and the relevant range \((0 < q < 1)\) of \( q \). Using Eq. (7), we can trivially do any one of the sums in Eq. (18) and verify that there are poles in each channel. Doing the \( n_{23} \) sum generates a factor

\[
\frac{1}{G(\sigma_{23} q^{n_{12}+n_{34}})}
\]

which has poles at \( \sigma_{23} = q^{-\ell} \) where \( \ell \geq n_{12}+n_{34} \). Thus, in the sum over \( n_{12} \) and \( n_{34} \) we find poles at \( \sigma_{23} = q^{-j} \) with \( j = 0,1,2,\ldots \). The pole at \( \sigma_{23} = 1 \) is contained only in the \( n_{12} = n_{34} = 0 \) terms of Eq. (18). The residue of the spin zero \( \sigma_{23} = 1 \) pole of \( B_5 \) can be evaluated using Eqs. (3) and (19). The surviving \( \sigma_{ij} \) dependence of the residue is easily seen to be

\[
\sum_{n_{45}=0}^{\infty} \sum_{n_{51}=0}^{\infty} \frac{\sigma_{45}^{n_{45}}}{q^{n_{45}}} \frac{n_{45}^{n_{51}}}{f^{n_{45}}} \frac{\sigma_{51}^{n_{51}}}{f^{n_{51}}}
\]

which is just a four-point function (6). This proves the consistency of our five-point and four-point functions. With similar techniques we can derive\(^{10} \) all other required properties of \( B_5 \) directly from the power series expansion (18). A similar discussion\(^{11} \) could be carried out for \( B_N \). However, in order to have a theory one has to specify a consistent set of rules for calculating loop diagrams. Only some steps in this program have been carried out\(^{12} \). If this work can be completed we will have a wider range of candidates for physically relevant dual resonance theories.
REFERENCES AND FOOTNOTES


3. See Ref. 10.

4. Mathematical details, satellite terms, properties of our N-point functions and the proof that in a certain limit our N-point functions become the Veneziano N-point functions will be presented in Refs. 10 and 11.


FIGURE CAPTION

Figure 1. Planar tree diagrams contained in the five-point function.
Figure 1