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Errata

1. The sentence after Eq. (II.7) should read:
"where the product in the numerator is to be set equal to 1 if $2-n + 1 + 1 > 2$ and ... "
2. Insert the following sentence as a separate paragraph after the statement of lemma 4:
" A corollary is that if $h_n^0(n) \geq 0$ as one of the w_j 's varies between $-w_t$ and $+w_t$ and the remaining w_j 's are held fixed appropriately, there are at least two poles in the N/D solution not contained in (II.1)."

BOUNDARY CONDITIONS FOR A PARTIAL-WAVE AMPLITUDE^{*}A. P. Balachandran[†]

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ABSTRACT

A partial-wave amplitude is frequently subject to constraints which specify its values at a given set of points. This happens, for instance, when we insist on its correct threshold behavior. We investigate such constraints in this paper and derive a class of inequalities which are necessary conditions for the existence of such an amplitude.

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I. Introduction

Recent investigations of partial-wave dispersion relations^{1,2} have focussed attention on the existence of solutions for these equations satisfying given boundary conditions. Such conditions may specify the asymptotic behavior of the spectral functions and *then* lead to theorems^{1,2} similar to the one proved by Pomeranchuk³ in a different context. Or they may specify the values of the amplitude at known points in the finite complex plane. This is the case, for example, when an amplitude for orbital angular momentum ℓ is required to have an ℓ -fold zero at threshold. Literature on some aspects of this problem exists^{1,2} and the present work may be regarded as an extension of the results contained therein.

For pedagogical reasons, in section II, we begin with a simple situation where the amplitude has two zeros whose positions are subject to some constraints. Certain inequalities are derived involving moments of the spectral function for the unphysical cut and of the inelasticity coefficient and a formalism is developed which expresses these inequalities in a concise form. The distinction between the original dispersion relation and the N/D

¹G. Frye and R. L. Warnock, Phys. Rev. 130, 478 (1963).

²A. P. Balachandran and F. N. von Hippel (to be published).

³I. Ia. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 725 (1958) [Translation: Soviet Phys. - JETP 7, 499 (1958)]

equation is then utilized to obtain inequalities for bound state parameters.

In section III, we consider an amplitude which has an arbitrary, but finite, number of zeros and prove the associated inequalities. These results can be sharpened if physical information regarding the behavior of the phase shifts over some range of energies can be obtained from experiment and the way to do this is discussed next. As in section II, we can also derive inequalities for bound state parameters. These are seen to imply some restrictions on the positions of the bound state poles. Finally, all these results are generalized to boundary conditions which specify the values of the amplitude at some known points. These should prove useful particularly when unstable particles are present in the system.

The material in these two sections refers only to the scattering of a spin zero by a spin half particle since it presents features not shared by a spin zero system. The latter is treated briefly in the final section. The scattering of particles of arbitrary spin is not studied in this work, but it appears to involve no serious difficulty.

In the appendix, the algebraic properties of a certain function are analyzed and used to simplify our previous results somewhat. This is important since the inequalities of section III calls for the knowledge of the zeros of a function of several real variables and may not be suitable for numerical work. However, these simplified inequalities do not seem to exhaust the contents of the original one.

II. Spin Zero - Spin Half Scattering - An Example

Let f_{l+} denote the partial-wave amplitude for orbital angular momentum l and total angular momentum $J = l + 1/2$. If the center-of-mass energy w is chosen as the complex variable, $f_{l+}(w)$ satisfies the dispersion relation

$$\begin{aligned}
 \operatorname{Re} f_{l+}(w) &= \operatorname{Re} \left\{ \frac{\eta_{l+}(w) \exp(2i\delta_{l+}(w)) - 1}{2ik} \right\} \\
 &= \frac{\eta_{l+}(w) \sin 2\delta_{l+}(w)}{2k} \\
 &= \frac{1}{\pi} \int_U dw' \frac{\phi_{l+}(w')}{w' - w} + \frac{P}{\pi} \int_{w_I}^{\infty} \frac{dw'}{2k'} \frac{1 - \eta_{l+}(w')}{w' - w} \\
 &\quad - \frac{1}{\pi} \int_{-w_I}^{-\infty} \frac{dw'}{2k'} \frac{1 - \eta_{(l+1)-}(-w')}{w' - w} \\
 &\quad + \frac{P}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} \left[\frac{\eta_{l+}(w') \sin^2 \delta_{l+}(w')}{w' - w} - \frac{\eta_{(l+1)-}(w') \sin^2 \delta_{(l+1)-}(w')}{w' + w} \right], \\
 &\qquad\qquad\qquad w \geq w_t \qquad\qquad\qquad (\text{II.1})
 \end{aligned}$$

where $\eta_{l+}(w)$ and $\delta_{l+}(w)$ are the inelasticity coefficients and real parts of the phase shifts for $f_{l+}(w)$, w_t and w_I are the scattering and inelastic thresholds, $\phi_{l+}(w)$ is the spectral function for the unphysical cut U and k is the center-of-mass

momentum. The function $\varphi_{\ell+}(w)$ will contain δ -functions if there are poles in (II.1). We are required to solve for $\delta_{\ell+}(w)$ and $\delta_{(\ell+1)-}(w)$ when $\varphi_{\ell+}(w)$, $\eta_{\ell+}(w)$ and $\eta_{(\ell+1)-}(w)$ are given and will therefore, as far as possible, state our results entirely in terms of these known functions. In (II.1), we have assumed the absence of any contribution from the circle at infinity. In addition, in this work, we will also assume the existence of asymptotic expansions for I_1 (see below), the η 's, the $\sin \delta$'s and their analogues for the spin zero system. These two assumptions seem to be intimately related and have not been proved hitherto.

It is convenient to combine the first three integrals and write them as

$$I_1 = \frac{P}{\pi} \int_V dw' \frac{\psi_{\ell+}(w')}{w' - w}, \quad w \geq w_t \quad (\text{II.2})$$

where V now runs over all the three cuts and $\psi_{\ell+}(w)$ takes on appropriate values in the different ranges. If

$$I_2 = \frac{1}{\pi} \int_V dw' \psi_{\ell+}(w') < \infty \quad (\text{II.3}),$$

the following sum rule is true:

$$I_2 + \frac{1}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} \left[\eta_{\ell+}(w') \sin^2 \delta_{\ell+}(w') + \eta_{(\ell+1)-}(w') \sin^2 \delta_{(\ell+1)-}(w') \right] = 0 \quad (\text{II.4})$$

For, if it were not true, the coefficient of $1/w$ on the left-hand side of (II.1) will not vanish as $w \rightarrow \infty$ along the unitarity cut, that is, $\eta_{\ell+}(\infty) \neq 0$ and $\delta_{\ell+}(\infty) \neq n\pi/2$ for any integer n . But, then, $\eta_{\ell+}(\infty) \sin^2 \delta_{\ell+}(\infty) \neq 0$ and the last integral will behave like $\frac{\ln w}{w}$ for large w . Since the left-hand side is $O(1/w)$ for large w , this is a contradiction and proves (II.4).

Now suppose that $f_{\ell+}(w)$ is required to have zeros at two points z_1 and z_2 which do not lie in either of the intervals $w_t < z_j < \infty$ and $-\infty < z_j < -w_t$. If z_1 is complex, the real analyticity of $f_{\ell+}(w)$ implies that $z_2 = z_1^*$. By hypothesis, the function

$$g_{\ell+}^1(w) = \frac{f_{\ell+}(w)}{(w - z_1)(w - z_2)} \quad (\text{II.5})$$

may have no poles at z_1 and z_2 and therefore,

$$\begin{aligned} \frac{\eta_{\ell+}(w) \sin^2 \delta_{\ell+}(w)}{2k \prod_{j=1}^2 (w - z_j)} &= \frac{P}{\pi} \int_V dw' \frac{\eta_{\ell+}(w')}{(w' - w) \prod_{j=1}^2 (w' - z_j)} \\ &+ \frac{P}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} \left[\frac{\eta_{\ell+}(w') \sin^2 \delta_{\ell+}(w')}{(w' - w) \prod_{j=1}^2 (w' - z_j)} \right. \\ &\quad \left. - \frac{\eta_{(\ell+1)-}(w') \sin^2 \delta_{(\ell+1)-}(w')}{(w' + w) \prod_{j=1}^2 (w' + z_j)} \right], \quad w \geq w_t \quad (\text{II.6}) \end{aligned}$$

It is easily seen that the first integral exists even when z_j falls on U . Let

$$h_n^i(2) = \frac{1}{\pi} \int_V dw' \frac{\Psi_{l+}(w')}{\prod_{j=1}^2 (w' - z_j)} w'^i \prod_{j=2-n+i+1}^2 (w' - w_j),$$

$$n \leq 2 \quad (\text{I.7})$$

where the product is to be set equal to 1 if $2-n+i+1 > 2$ and the w_j are complex variables in a domain to be specified later. The integrals $h_2^1(2)$ are assumed to exist which is true if I_2 does.

(This restriction will be removed at the end of section III.)

The left-hand side of (II.6) is $O(1/w^3)$ for large positive w and so

$$h_0^0(2) + \frac{1}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} \left[\frac{\eta_{l+}(w') \sin^2 \delta_{l+}(w')}{\prod_{j=1}^2 (w' - z_j)} + \frac{\eta_{(l+1)-}(w') \sin^2 \delta_{(l+1)-}(w')}{\prod_{j=1}^2 (w' + z_j)} \right]$$

$$\equiv h_0^0(2) + j_+^0 + j_-^0 = 0,$$

$$h_1^1(2) + \frac{1}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} w' \left[\frac{\eta_{l+}(w') \sin^2 \delta_{l+}(w')}{\prod_{j=1}^2 (w' - z_j)} - \frac{\eta_{(l+1)-}(w') \sin^2 \delta_{(l+1)-}(w')}{\prod_{j=1}^2 (w' + z_j)} \right]$$

$$\equiv h_1^1(2) + j_+^1 - j_-^1 = 0,$$

$$h_2^2(2) + \frac{1}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} w'^2 \left[\frac{\eta_{l+}(w') \sin^2 \delta_{l+}(w')}{\prod_{j=1}^2 (w' - z_j)} + \frac{\eta_{(l+1)-}(w') \sin^2 \delta_{(l+1)-}(w')}{\prod_{j=1}^2 (w' + z_j)} \right]$$

$$\equiv h_2^2(2) + j_+^2 + j_-^2 = 0 \quad (\text{II.8})$$

The last equation is the analogue of (II.4). These sum rules immediately lead to inequalities for the known functions $h_1^1(2)$.

Thus,

$$h_0^0(2) < 0 ,$$

$$h_2^2(2) < 0 \quad (\text{II.9}).$$

The equalities $h_0^0(2) = 0$, $h_2^2(2) = 0$ are possible if and only if there is no scattering and can therefore be ignored. Also, since

$$h_0^0(2) = -(j_+^0 + j_-^0) > -\frac{1}{w_t^2} (j_+^2 + j_-^2) = \frac{1}{w_t^2} h_2^2(2) \quad (\text{II.10}) ,$$

we have

$$h_2^2(2) - w_t^2 h_0^0(2) < 0 \quad (\text{II.11}).$$

In deriving (II.9) and (II.10), we have used the positive definiteness of the integrands of $j_{\pm}^{0,2}$. There is one further inequality which is implied by (II.8). The integral

$$\frac{1}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} (w' - \lambda)^2 \frac{\eta_{l+}(w') \sin^2 \delta_{l+}(w')}{\prod_{j=1}^2 (w' - z_j)} \quad (\text{II.12})$$

is positive for any real λ . So too is ~~the~~ the corresponding integral for $(l+1)-$. That is,

$$\lambda^2 j_{+}^0 - 2\lambda j_{+}^1 + j_{+}^2 > 0 \quad (\text{II.13})$$

for any real λ . Therefore,

$$(j_{\pm}^1)^2 < j_{\pm}^0 \cdot j_{\pm}^2 \quad (\text{II.14})$$

It follows that

$$\begin{aligned} [h_1^1(2)]^2 &= (j_+^1 - j_-^1)^2 < (j_+^1)^2 + (j_-^1)^2 + j_+^0 j_-^2 + j_+^2 j_-^0 \\ &< j_+^0 j_+^2 + j_-^0 j_-^2 + j_+^0 j_-^2 + j_+^2 j_-^0 \\ &= h_0^0(2) h_2^2(2) \end{aligned}$$

i.e.

$$[h_1^1(2)]^2 < h_0^0(2) \cdot h_2^2(2) \quad (\text{II.15})$$

It has been observed previously^{1,2} that if the analytic continuation of I_1 plus its appropriate imaginary part has zeros at γ_1 and γ_2 , (II.1) has no solution with these zeros. This is a special case of our results for then,

$$\frac{I_1}{\prod_{j=1}^2 (w - \gamma_j)} = \frac{P}{\pi} \int_V dw' \frac{\psi_{l+}(w')}{(w' - w) \prod_{j=1}^2 (w' - \gamma_j)} \quad (\text{II.16})$$

Since $I_1 / \prod_{j=1}^2 (w - \gamma_j)$ is $O(1/w^2)$ for large w , (II.16) tells us that

$h_0^0(2) = h_1^1(2) = 0$. This is inconsistent with (II.9) and (II.15) and proves the assertion.

These inequalities are by no means equivalent to the set that

can be written down by successively assuming $f_{\ell+}(w)$ to have a zero first at y_1 and then at y_2 . For instance, if $y_1 = y_2$, this will merely give an equation for $f_{\ell+}(w)$ to have a simple zero at y_1 while the preceding set corresponds to its having a double zero at that point.

Let us derive these results in a different way. Construct the function

$$g_{\ell+}^2(w) = f_{\ell+}(w) \prod_{j=1}^2 \frac{(w-w_j)}{(w-y_j)} \quad (\text{II.17})$$

where the w_j are complex variables. This gives, instead of (II.4),

$$\begin{aligned} h_2^0(2) + \frac{1}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} \left[\eta_{\ell+}(w') \sin^2 \delta_{\ell+}(w') \prod_{j=1}^2 \frac{(w'-w_j)}{(w'-y_j)} \right. \\ \left. + \eta_{(\ell+)-}(w') \sin^2 \delta_{(\ell+)-}(w') \prod_{j=1}^2 \frac{(w'+w_j)}{(w'+y_j)} \right] = 0 \end{aligned} \quad (\text{II.18})$$

It is readily verified that Eqs. (II.8) and (II.18) imply each other. Eq. (II.18) shows that

$$h_2^0(2) < 0 \quad (\text{II.19})$$

if either a) $w_1 = w_2$ and takes any real value or b) $-w_t \leq w_1 \leq w_t$, $i = 1, 2$. Since

$$h_2^0(2) = w_1 w_2 h_0^0(2) - (w_1 + w_2) h_1^1(2) + h_2^2(2) \quad (\text{II.20}),$$

a) gives

$$w_1^2 h_0^0(2) - 2w_1 h_1^1(2) + h_2^2(2) < 0 \quad (\text{II.21})$$

for any real w_1 . Therefore,

$$\begin{aligned} [h_1^1(2)]^2 &< h_0^0(2) h_2^2(2) , \\ h_0^0(2) &< 0 , \\ h_2^2(2) &< 0 \end{aligned} \quad (\text{II.22})$$

To reduce b), observe that for fixed w_2 , $h_2^0(2)$ is a linear function of w_1 and will be negative for $-w_t \leq w_1 \leq w_t$ if it is negative at the end points $w_1 = \pm w_t$. That is,

$$(\pm w_t) w_2 h_0^0(2) - (\pm w_t + w_2) h_1^1(2) + h_2^2(2) < 0 \quad (\text{II.23})$$

This is in turn a linear function of w_2 and so should be negative for $w_2 = \pm w_t$. Only the sign combination $w_1 = -w_2$ gives a result different from (II.21) and this reads

$$-w_t^2 h_0^0(2) + h_2^2(2) < 0 \quad (\text{II.24})$$

Eqs. (II.22) and (II.24) are identical with the previous set and are the necessary and sufficient conditions for (II.19) to be true. These are also the necessary conditions for the existence of solutions of (II.1) with zeros at γ_1 and γ_2 . The simplicity and generality of the formalism which leads to (II.19) will turn out to be very useful in section III where a complete reduction of the analogue of this equation into its component inequalities does not seem feasible.

Eq. (II.18) tells us that $h_2^0(2) < 0$ when $w_2 = w_1^*$ and assumes any value. This is not a new equation though and is implied by (II.21) as a simple calculation shows.

We have remarked that (II.19) is a necessary condition if (II.1) is to have solutions with zeros at y_1 and y_2 . However, even if (II.19) is not true, it may very well be that the N/D decomposition possesses solutions with these zeros, since these are not always solutions of (II.1). Suppose now that the N/D function exists and is not a solution of (II.1), either because (II.19) is not satisfied or for some other reason. The function D must then have zeros which give rise to poles in N/D not contained in (II.1). Therefore, if these poles are m in number and are at the points b_i ,

$$\begin{aligned} \text{Re } N/D &= \frac{\eta_{\ell+}(w) \sin 2\delta'_{\ell+}(w)}{2k} \\ &= I_1 + \sum_{i=1}^m \frac{\lambda_i}{w - b_i} + \frac{P}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} \left[\frac{\eta_{\ell+}(w') \sin^2 \delta_{\ell+}(w')}{w' - w} \right. \\ &\quad \left. - \frac{\eta_{(\ell+1)-}(w') \sin^2 \delta_{(\ell+1)-}(w')}{w' + w} \right], \quad w \geq w_t \end{aligned}$$

(II.25)

where the λ_i 's are the residues at the poles and the dash on $\delta'_{\ell+}$ and $\delta'_{(\ell+1)-}$ distinguishes these phase shifts from the ones in (II.1). Eq. (II.19) is now replaced by

$$h_2^0(z) - \sum_{i=1}^m \lambda_i \prod_{j=1}^n \frac{(b_i - w_j)}{(b_i - y_j)} < 0 \quad (\text{II.26})$$

These poles may be called bound states provided only that the b_i 's are not complex and the residues have the proper signs. Eq.

(II.26) is thus an inequality for bound state parameters in terms of the known function $h_2^0(2)$.

III: Spin Zero - Spin Half Scattering - General

The amplitude $f_{\ell+}(w)$ is required to have zeros at the n points $\gamma_1, \gamma_2, \dots, \gamma_n$. In the first instance, let us suppose that none of these lies in the intervals $w_t < \gamma_i < \infty$ and $-\infty < \gamma_i < -w_t$. If there are complex zeros, they always occur in conjugate pairs. As in section II, we define the functions

$$g_{\ell+}^2(w) = f_{\ell+}(w) \prod_{j=1}^n \frac{(w - w_j)}{(w - \gamma_j)} \quad (\text{III.1})$$

and

$$h_n^1(m) = \frac{1}{\pi} \int_V dw' \psi_{\ell+}(w') \frac{w'^i \prod_{j=m-n+i+1}^m (w' - w_j)}{\prod_{j=1}^m (w' - \gamma_j)}, \quad n \leq m \quad (\text{III.2})$$

where $h_n^1(n)$ is assumed to exist. This leads to the sum rule

$$h_n^0(n) + \frac{1}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} \left[\eta_{l+}(w') sm^2 \delta_{l+}(w') \prod_{j=1}^n \frac{(w' - w_j)}{(w' - z_j)} \right. \\ \left. + \eta_{(l+1)-}(w') sm^2 \delta_{(l+1)-}(w') \prod_{j=1}^n \frac{(w' + w_j)}{(w' + z_j)} \right]$$

(III.3)

For any real a and b with $a \leq b$, we will say that $\{w_j\}$ is contained in $D(a,b)$ and write $\{w_j\} \in D(a,b)$ if some pairs of w_j are equal and assume any real value and the rest satisfy the inequality $a \leq w_j \leq b$. Similarly, $\{z_j\} \in E(a,b)$ if the z_j fall in neither of the intervals $-\infty < z_j < a$ and $b < z_j < \infty$.

Eq. (II.3) is now seen to imply the following theorem:

Theorem 1

Let Eq. (II.1) be required to have a solution with zeros at $z_1, z_2, \dots, z_n$ where $\{z_j\} \in E(-w_t, w_t)$. Then, a necessary condition for the existence of such a function is that

$$h_n^0(n) < 0 \quad (\text{III.4})$$

when $\{w_j\} \in D(-w_t, w_t)$.

With a slight modification, this theorem can also be used in situations where, for instance, the function vanishes like a square root at some point.¹

If some of these zeros lie on the unitarity cuts, let z_r denote the one among these with the largest positive affix and z_l the one with the smallest negative affix. Further, let $|z_l| \geq w_t$. If we rewrite (III.3) in the form

$$\begin{aligned}
 h_n^0(n) + \frac{1}{\pi} \left[\int_{w_t}^{z_r} + \int_{z_l}^{\infty} \right] \frac{dw'}{k'} \eta_{l+}(w') \sin^2 \delta_{l+}(w') \prod_{j=1}^n \frac{(w' - w_j)}{(w' - z_j)} \\
 + \frac{1}{\pi} \left[\int_{w_t}^{|z_l|} + \int_{|z_l|}^{\infty} \right] \frac{dw'}{k'} \eta_{(l+1)-}(w') \sin^2 \delta_{(l+1)-}(w') \prod_{j=1}^n \frac{(w' + w_j)}{(w' + z_j)} \\
 = 0
 \end{aligned}
 \tag{III.5}$$

lemma 1 is seen to follow:

Lemma 1.

Let some of the zeros of $f_{l+}(w)$ lie on the unitarity cuts and let z_r denote the one which occurs farthest to the right on the right-hand unitarity cut and z_l the one which occurs farthest to the left on the left-hand unitarity cut. Then a necessary condition for the existence of such a function is that

$$\begin{aligned}
 h_n^0(n) + \frac{1}{\pi} \int_{w_t}^{z_r} \frac{dw'}{k'} \eta_{l+}(w') \sin^2 \delta_{l+}(w') \prod_{j=1}^n \frac{(w' - w_j)}{(w' - z_j)} \\
 + \frac{1}{\pi} \int_{w_t}^{|z_l|} \frac{dw'}{k'} \eta_{(l+1)-}(w') \sin^2 \delta_{(l+1)-}(w') \prod_{j=1}^n \frac{(w' + w_j)}{(w' + z_j)} < 0
 \end{aligned}
 \tag{III.6}$$

when $\{w_j\} \in D(\gamma_\ell, \gamma_r)$. If γ_r does not exist, the second integral in (III.6) is to be removed and the condition $\{w_j\} \in D(\gamma_\ell, \gamma_r)$ is to be replaced by $\{w_j\} \in D(\gamma_\ell, w_t)$. A similar change is to be made if γ_ℓ does not exist.

Eq. (III.6) is much weaker than (III.4) since it involves the phase shifts $\delta_{\ell+}(w)$ and $\delta_{(\ell+1)-}(w)$ which are not known a priori. For threshold zeros, $\gamma_j = \pm w_t$ and (III.4) and (III.6) give identical results as they should.

In physical situations, the inequality (III.4) is not the best possible one. Often, fairly detailed information regarding the behavior of the phase shifts $\delta_{\ell+}$ and $\delta_{(\ell+1)-}$ for some ranges of energies $w_{i+} - \Delta_{i+}$ to $w_{i+} + \Delta_{i+}$ ($i = 1, 2, \dots, p$) and $w_{i-} - \Delta_{i-}$ to $w_{i-} + \Delta_{i-}$ ($i = 1, 2, \dots, q$) can be obtained from experiment. If such is the case, write (III.3) as

$$\begin{aligned}
 & h_n^0(w) + \frac{1}{\pi} \sum_{l=1}^p \int_{w_{l+} - \Delta_{l+}}^{w_{l+} + \Delta_{l+}} \frac{dw'}{k'} \eta_{\ell+}(w') \sin^2 \delta_{\ell+}(w') \prod_{j=1}^n \frac{(w' - w_j)}{(w' - \gamma_j)} \\
 & + \frac{1}{\pi} \sum_{l=1}^q \int_{w_{l-} - \Delta_{l-}}^{w_{l-} + \Delta_{l-}} \frac{dw'}{k'} \eta_{(\ell+1)-}(w') \sin^2 \delta_{(\ell+1)-}(w') \prod_{j=1}^n \frac{(w' + w_j)}{(w' + \gamma_j)} \\
 & \equiv h_n^0(w) + \sum_{l=1}^p J_l + \sum_{l=1}^q K_l < 0
 \end{aligned}$$

(III.7)

where $\{w_j\} \in D(-w_t, w_t)$. This leads to lemma 2.

Lemma 2.

If the integrals J_i and K_i are known to have the minimum values $\min J_i$ and $\min K_i$ for $\{w_j\} \in D(-w_t, w_t)$ and (II.1) is to have a solution with zeros at $\{z_j\} \in E(-w_t, w_t)$, it is necessary that

$$h_n^0(n) + \sum_{i=1}^p \min J_i + \sum_{i=1}^q \min K_i < 0 \quad (\text{III.8})$$

when $\{w_j\} \in D(-w_t, w_t)$.

If any of the lower limits of J_i or K_i coincides with w_t , (III.8) can be improved by modifying the ranges of w_j as in lemma 2.

Eq. (III.8) corresponds to the existence of a particular type of solution for (II.1) while (III.4) corresponds to its having some solution. Lemma 2 should therefore prove more useful in practice since it contains more physics.

To get inequalities for bound state parameters, we proceed as in section II. The result is the next lemma.

Lemma 3.

Suppose that $\{z_j\} \in E(-w_t, w_t)$ and suppose further that an N/D solution with these zeros exists and has m poles which are not implied by (II.1). Let these poles be at the points b_i with residues λ_i . Then,

$$h_n^0(n) - \sum_{i=1}^m \lambda_i \prod_{j=1}^n \frac{(b_i - w_j)}{(b_i - z_j)} < 0 \quad (\text{III.9})$$

when $\{w_j\} \in D(-w_t, w_t)$.

Let us rewrite (III.9) in the form

$$h_n^0(n) - \sum_{i=1}^{m-1} \lambda_i \prod_{j=1}^n \frac{(b_i - w_j)}{(b_i - z_j)} - \lambda_m \prod_{j=1}^n \frac{(b_m - w_j)}{(b_m - z_j)} < 0 \quad (\text{III.10})$$

These equations are true regardless of whether the poles are bound states or ghosts. If they are, in fact, bound states, we have the further inequality $-w_t < b_a < w_t$. If

$$h_n^0(n) - \sum_{i=1}^{m-1} \lambda_i \prod_{j=1}^n \frac{(b_i - w_j)}{(b_i - z_j)} \geq 0 \quad (\text{III.11})$$

as one of the w_j 's, say w_k , is varied between two points in the interval $-w_t \leq w_k \leq w_t$ and the rest of the w_j 's are held fixed,

$$-\lambda_m \prod_{j=1}^n \frac{(b_m - w_j)}{(b_m - z_j)} < 0 \quad (\text{III.12})$$

in the same domain of the variables. Hence, b_m cannot lie in this range of w_k , for if it did, the factor $(b_m - w_k)$ will change

sign as w_k crosses b_m and (III.12) will be contradicted. For $m > 1$, (III.11) involves bound state parameters while initially, we are given only $h_n^0(n)$. The most interesting result therefore emerges when $m = 1$ which we state as a lemma.

Lemma 4.

Let $\{z_j\} \in E(-w_t, w_t)$. Then, if there is one bound state and

$$h_n^0(n) \geq 0 \quad (\text{III.13})$$

for some range of one of the w_j 's in the gap between the unitarity cuts: when the remaining w_j 's are held fixed in $D(-w_t, w_t)$, the bound state does not lie in this range.

We can now consider a related boundary condition where the values of the function are specified at some set of points $z_j (j = 1, 2, \dots, n)$. Let $f_{l+}(z_j) = a_j$. As before, construct the function

$$g_{l+}^2(w) = f_{l+}(w) \prod_{j=1}^n \frac{(w-w_j)}{(w-z_j)} \quad (\text{III.14})$$

This gives

$$\begin{aligned} h_n^0(n) - \sum_{i=1}^n a_i \frac{\prod_{j=1}^n (z_i - w_j)}{\prod_{j \neq i} (z_i - z_j)} \\ + \frac{1}{\pi} \int_{w_t}^{\infty} \frac{dw'}{k'} \left[\eta_{l+}(w') \sin^2 \delta_{l+}(w') \prod_{j=1}^n \frac{(w' - w_j)}{(w' - z_j)} + \eta_{(l+)-}(w') \sin^2 \delta_{(l+)-}(w') \prod_{j=1}^n \frac{(w' + w_j)}{(w' + z_j)} \right] \\ = 0 \quad (\text{III.15}) \end{aligned}$$

and implies theorem 2.

Theorem 2.

Let $f_{\ell+}(w)$ be required to assume the values a_j at the points $z_j (j=1, 2, \dots, n)$ where $\{z_j\} \in E(-w_t, w_t)$. Then, if such a function is to exist, it is necessary that

$$h_n^o(n) - \sum_{i=1}^n a_i \frac{\prod_{j=1}^n (z_i - w_j)}{\prod_{j \neq i} (z_i - z_j)} < 0 \quad (\text{III.16})$$

when $\{w_j\} \in D(-w_t, w_t)$.

Eq. (III.16) should prove useful if there are unstable particles present in the system which are identified with poles on some unphysical sheet. As an example, consider the continuation of $f_{\ell+}(w)$ through the elastic unitarity cut (assuming that it exists). This continuation is given by

$$f_{\ell+}^{\text{II}}(w) = \frac{f_{\ell+}(w)}{1 + 2ik f_{\ell+}(w)} \quad (\text{III.17})$$

Therefore, if there is a pole on this sheet at $w = z_j$,

$$f_{\ell+}(z_j) = - \frac{1}{2ik_j} \quad (\text{III.18})$$

where $k_j = k(y_j)$. In practice, y_j can be fixed by requiring a resonance of specified mass and width at least when the width is small. Eq. (III.18) then determines one of the a_j . Further, $f_{l+}(y_j^*) = a_j^*$. When these values are inserted into (III.16), if the inequality is violated, we may conclude that (II.1) has no solution which is such that $f_{l+}(y_j) = a_j$.

If the integral I_2 in (II.3) diverges, let

$$I_1 \sim -\frac{1}{w} x_{l+}(w) \quad (\text{III.19})$$

for large positive w . Then, it is known^{4,2} that

$$x_{l+}(w) \sim -\frac{1}{\pi} \int_{w_t}^w \frac{dw'}{k'} \left[\eta_{l+}(w') \sin^2 \delta_{l+}(w') + \eta_{(l+1)-}(w') \sin^2 \delta_{(l+1)-}(w') \right] \quad (\text{III.20})$$

for large positive w . Therefore,

$$\begin{aligned} x_{l+}(w) &< 0, \\ |x_{l+}(w)| &\leq \frac{4}{\pi} l_n w \end{aligned} \quad (\text{III.21})$$

for all sufficiently large positive w . Further,

$$h_{n-2}^0(n) < 0 \quad (\text{III.22})$$

These results are summarized in the next theorem.

⁴G.F. Chew and S. Mandelstam, Nuovo Cimento 19, 752 (1961).

Theorem 3

If (III.19) is true, where $|\chi_{\ell+}(w)| \rightarrow \infty$ as $w \rightarrow \infty$, then (II.1) has a solution with zeros at $\{z_j\} \in E(-w_t, w_t)$ only if

$$\chi_{\ell+}(w) < 0,$$

$$|\chi_{\ell+}(w)| \leq \frac{4}{\pi} \ell \ln w$$

for all sufficiently large positive w and

$$h_{n-2}^0(n) < 0 \quad (\text{III.23})$$

when $\{w_j\} \in D(-w_t, w_t)$.

The statement $|\chi_{\ell+}(w)| \leq \frac{4}{\pi} \ell \ln w$ can in fact be strengthened and reference 2 should be consulted for a detailed discussion of this point.

Theorems 2 and 3 can be easily extended to cover the situations implied in lemmas 1 to 4.

IV. Spin Zero - Spin Zero Scattering - General

In the previous sections, we were obliged to work in the w -plane due to the presence of kinematical singularities if $\Delta = w^2$ is chosen as the complex variable. There is, however, no such difficulty for a spinless system. Let

$$f_l(s) = \frac{\sqrt{s}}{k} \frac{\eta_l(s) e^{2i\delta_l(s)} - 1}{2i} \quad (\text{IV.1})$$

be the partial-wave amplitude for angular momentum l . It satisfies the dispersion relation

$$\begin{aligned} \frac{\sqrt{s}}{2k} \eta_l(s) \sin 2\delta_l(s) &= f_l(s_0) + \frac{s-s_0}{\pi} P \int_V ds' \frac{\psi_l(s')}{(s'-s_0)(s'-s)} \\ &+ \frac{s-s_0}{\pi} P \int_{s_t}^{\infty} ds' \frac{\frac{\sqrt{s'}}{k'} \eta_l(s') \sin 2\delta_l(s')}{(s'-s_0)(s'-s)}, \quad s \geq s_t \end{aligned}$$

(IV.2)

where $s_t = w_t^2$. The subtraction in (IV.2) may or may not be necessary according as whether the integral

$$I_1 = \frac{P}{\pi} \int_V ds' \frac{\psi_l(s')}{s'-s} \quad (\text{IV.3})$$

converges or diverges. In particular, for s -waves, its existence may imply that there are no arbitrary parameters in the system as has been suggested by Chew and Frautschi.⁵

⁵G.F. Chew and S.C. Frautschi, Phys. Rev. Letters 7, 394 (1961); 8, 41 (1962).

We may now define the integrals

$$h_n^z(m) = \frac{1}{\pi} \int_V ds' \frac{\psi_\ell(s')}{\prod_{j=1}^m (s' - \gamma_j)} s'^i \prod_{j=m-n+i+1}^m (s' - \sigma_j), \quad n \leq m-1 \quad (\text{IV.4})$$

and the domains $D(a,b)$ and $E(a,b)$ for $\{\sigma_j\}$ and $\{\gamma_j\}$ as in section III. This leads to theorem 4:

Theorem 4.

Let $f_\ell(s)$ be required to have zeros at the points γ_j ($j = 1, 2, \dots, n$) none of which lies in the interval $\sigma_t < \gamma_j < \infty$, that is, $\{\gamma_j\} \in E(-\infty, \sigma_t)$. Two cases are possible.

a) The integral $h_{n-1}^0(n)$ exists. Then, it is necessary that

$$h_{n-1}^0(n) < 0 \quad (\text{IV.5})$$

when $\{\sigma_j\} \in D(-\infty, \sigma_t)$.

b) The integral $h_{n-1}^0(n)$ does not exist. Let

$$\frac{p}{\pi} \int_V ds' \frac{\psi_\ell(s')}{s'(s' - s)} \sim -\frac{1}{s} \chi_\ell(s) \quad (\text{IV.6})$$

as $s \rightarrow \infty$. Then, it is necessary that

$$\begin{aligned} \chi_\ell(s) &< 0, \\ |\chi_\ell(s)| &\leq \frac{2}{\pi} \ln s \end{aligned}$$

for all sufficiently large positive s and

$$h_{n-2}^0(n) < 0$$

(IV.7)

when $\{\alpha_j\} \in D(-\infty, \alpha_t)$.

Unlike the w_j , the variables α_j are not bounded from below since (IV.2) has no analogue of the last integral in (II.1). This results, for instance, in the demand that h_{n-p}^i be negative for every $p \geq 1$ and $i \leq n-p$ while, previously, it was required to be negative only for every even p .

It is clear that in this case too, we can duplicate every result of section III by making a few suitable changes.

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Appendix

We wish to study the functions $h_n^1(m)$ and try to deduce a number of simpler equations from (III.4) and (IV.5). The other inequalities can be reduced similarly. Let us first prove the following identity for the $h_n^1(m)$ of (III.2):

$$h_n^1(m) = \sum_{k=0}^p (-1)^k \mathcal{C}(p, k) h_{n-k}^{1+p-k}(m)$$

(A.1)

The symbol $\mathcal{C}(p, k)$ denotes the sum over all combinations of the p variables $w_{m-n+i+1}, w_{m-n+i+2}, \dots, w_{m-n+i+p}$ taken k at a time and $\mathcal{C}(p, 0)$ is defined to be equal to one. The proof is by induction. For $p = 0$, (A.1) is a triviality. Since

$$h_{n-k}^{1+p-k}(m) = h_{n-k}^{1+p+1-k}(m) - w_{m-n+i+p+1} h_{n-k-1}^{1+p-k}(m)$$

(A.2)

we have,

$$\begin{aligned} h_n^1(m) &= h_n^{1+p+1}(m) - \mathcal{C}(p+1, 1) h_{n-1}^{1+p}(m) \\ &+ \sum_{k=2}^p (-1)^k \mathcal{C}(p, k) h_{n-k}^{1+p+1-k}(m) + \sum_{k=1}^p (-1)^{k+1} \mathcal{C}(p, k) w_{m-n+i+p+1} \\ &\quad \times h_{n-k-1}^{1+p-k}(m) \\ &= h_n^{1+p+1}(m) - \mathcal{C}(p+1, 1) h_{n-1}^{1+p}(m) + \sum_{k=2}^p (-1)^k \mathcal{C}(p, k) h_{n-k}^{1+p+1-k}(m) \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=2}^{p+1} (-1)^k c(p, k-1) w_{m-n+i+p+1} h_{n-k}^{i+p+1-k}(m) \\
& = \sum_{k=0}^{p+1} (-1)^k c(p+1, k) h_{n-k}^{i+p+1-k}(m) \quad (A.3)
\end{aligned}$$

because of the identity

$$c(p, k) + w_{m-n+i+p+1} c(p, k-1) = c(p+1, k) \quad (A.4)$$

Eq. (A.3) completes the induction.

We may note two more useful identities. Thus,

$$h_n^i(m) = \sum_{k=0}^{2p} \alpha_{2p-k}^{2p} w_{m-n+i+1}^{2p-k} h_{n+k-2p}^{i+k}(m) \quad (A.5a)$$

where

$$\alpha_{2p-k}^{2p} = (-1)^k \frac{(2p)!}{k! (2p-k)!} \quad (A.5b)$$

if $w_{m-n+i+1} = w_{m-n+i+2} = \dots = w_{m-n+i+2p}$. Further,

$$h_n^i(m) = \sum_{k=0}^p \beta_{p-k}^p w_{m-n+i+1}^{2p-2k} h_{n-2p+2k}^{i+2k}(m) \quad (A.6a)$$

where

$$\beta_{p-k}^p = (-1)^{p-k} \frac{p!}{k!(p-k)!} \quad (\text{A.6b})$$

if $w_{m-n+l+1} = -w_{m-n+l+2} = +w_{m-n+l+3} = \dots = -w_{m-n+l+2p}$

Eqs. (A.5) and (A.6) can also be proved by induction. For instance, for (A.6a), this leads to a difference equation for β_{p-k}^p :

$$\beta_{p+1-k}^{p+1} + \beta_{p-k}^p - \beta_{p+1-k}^p = 0$$

with the boundary conditions

$$\beta_p^p = (-1)^p,$$

$$\beta_0^p = 1 \quad (\text{A.7})$$

for which (A.6b) is the solution.

It may be verified that the following inequalities are necessary and sufficient for (III.4) to be true:

$$[h_{n-1}^1(n)]^2 < h_{n-2}^0(n) h_n^2(n),$$

$$h_{n-2}^0(n) < 0,$$

$$h_n^2(n) < 0,$$

$$h_n^2(n) - w_t^2 h_{n-2}^0(n) < 0 \quad (\text{A.8})$$

Further reduction with such completeness appears difficult since it requires the location of zeros of polynomials of increasingly high order. We may, however, note the conditions implied by (A.8) when $|w_j| \leq w_t$. These are got by setting each of the $w_j = \pm w_t$. Let any member of this resultant class of functions be denoted by $j_n^i(m)$. The latter can be computed by using (A.1). Eq. (A.8) therefore leads to the inequalities

$$\begin{aligned} [j_{n-1-2p}^1(n-2q)]^2 &< j_{n-2-2p}^0(n-2q) \cdot j_{n-2p}^2(n-2q), \\ j_{n-2-2p}^0(n-2q) &< 0, \\ j_{n-2q}^2(n-2q) &< 0, \\ j_{n-2p}^2(n-2q) - w_t^2 j_{n-2-2p}^0(n-2q) &< 0 \end{aligned} \quad (A.9)$$

where $q = 0, 2, 4, \dots$ and $p = q, q+1, \dots$, integer $\left\{ \frac{1}{2}(n-2), \frac{1}{2}(n-3) \right\}$. When q is increased by two, every possible choice of two pairs of complex zeros conjugate to each other are to be removed in turn. This rule may require a word of explanation. We have remarked in section II that the inequality $h_n^0(n) < 0$ when $w_{j+1} = w_j^*$ is already implied by the corresponding inequality got by setting $w_{j+1} = w_j$ and letting them be real. We can therefore derive $h_{n-2q}^0(n-2q) < 0$ for any q from $h_n^0(n) < 0$ by substituting the values of the complex zeros for the appropriate w_j . But (A.9) with

$q = 0$ is necessary and sufficient only for that part of (III.4) where one pair of variables are equal and the rest are constrained by $|w_j| \leq w_t$. Thus, when $q = 0$, (A.9) enables us to remove one pair of zeros from (III.4), but not two pairs. We should therefore include a separate equation for $q = 2$. The situation then repeats itself and we have to include an equation for $q = 4$ too and so on. There is no point in associating q with a real zero, however, as such equations follow from (III.4) when one or more of the w_j coincide with these real points and hence are implicit in (A.9).

For a similar reason, we have not written the equations

$$j_{n-2p}^{(n-2q)} < 0 \text{ for } p \geq q + 1 \text{ in (A.9) as these are consequences of } j_{n-2-2p}^{(n-2q)} < 0.$$

Let us investigate (IV.5). The identities (A.1), (A.5) and (A.6) are still true if the w_j are replaced by Δ_j . Instead of (A.8), we find,

$$\begin{aligned} \left[h_{n-2}^1(n) \right]^2 &< h_{n-3}^0(n) h_{n-1}^2(n), \\ h_{n-3}^0(n) &< 0, \\ h_{n-1}^2(n) &< 0, \\ h_{n-2}^1(n) - \Delta_t h_{n-3}^0(n) &< 0 \end{aligned} \tag{A.10}$$

The following identities can now be verified:

$$h_n^1(m) \xrightarrow{\Delta_{m-n+l+1} \rightarrow -\infty} |\Delta_{m-n+l+1}| h_{n-1}^1(m),$$

$$\begin{aligned} h_n^1(m) &= h_n^1(m) \Big|_{\Delta_{m-n+l+1} = \Delta_{m-n+l+2} = \dots = \Delta_m = \Delta_t} \\ &= \sum_{p=0}^{n-i} (-1)^p \frac{(n-i)!}{p!(n-i-p)!} \Delta_t^p h_{n-p}^{n-p}(m) \end{aligned} \quad (A.11)$$

The last is a consequence of (A.1). The analogues of (A.9) therefore are,

$$\left[h_{n-2-p}^1(n-2q) \right]^2 < h_{n-3-p}^0(n-2q) h_{n-1-p}^2(n-2q),$$

$$h_{n-3-p}^0(n-2q) < 0,$$

$$h_{n-2q-1}^2(n-2q) < 0,$$

$$h_{n-2-p}^1(n-2q) - \Delta_t h_{n-3-p}^0(n-2q) < 0 \quad (A.12),$$

where $q = 0, 2, \dots$ and $p = 2q, 2q + 1, \dots, n-3$. As in (A.9), q is associated with the complex zeros.