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A CONTROL STRATEGY FOR PLASMA
EQUILIBRIUM IN A TOKAMAK

R. V. Miskell

August 1975

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OAK RIDGE, TENNESSEE

prepared for the U.S. ENERGY RESEARCH AND DEVELOPMENT ADMINISTRATION
under U.S. GOVERNMENT Contract W-7405 eng 26

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Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, Virginia 22161
Price: Printed Copy \$5.45; Microfiche \$2.25

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Date of Issue: August 29, 1975
Distribution Category: UC-20

Report Number: Y-1988

A CONTROL STRATEGY FOR PLASMA EQUILIBRIUM IN A TOKAMAK

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Adapted from a dissertation that was submitted to The University of Tennessee in partial fulfillment of the requirements for the Degree Doctor of Philosophy.

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Prepared for the US Energy Research
and Development Administration
Under US Government Contract W-7405-eng-26

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ABSTRACT

Dynamic control of the plasma position within the torus of a TOKAMAK fusion device is a significant factor in the development of nuclear fusion as an energy source. This investigation develops a state variable model of a TOKAMAK thermonuclear device, suitable for application of modern control theory techniques.

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SUMMARY

Dynamic control of the plasma position within the torus of a TOKAMAK fusion device is a significant factor in the development of nuclear fusion as an energy source. This investigation develops a state variable model of a TOKAMAK thermonuclear device, suitable for application of modern control theory techniques.

The model considers eddy currents in the conducting shell surrounding the torus and the classical Shafranov equilibrium equation. Equations necessary to characterize the operating conditions of a TOKAMAK are cast in state variable form. Two control variables are selected: the vertical field current and the plasma temperature. The figure of merit chosen minimizes the shift of the plasma within the torus and considers position perturbations necessary to maintain the dense and hotter portions of the plasma profile in the center of the torus; ie, overcome uneven poloidal fields due to the toroidal geometry. The model uses a Kalman filter to estimate unmeasured state variables, and uses the second variation of the calculus of variations to maintain an optimal control path.

The model was tested with data obtained from an operational TOKAMAK system, and the study showed that online modern control theory techniques can be applied to TOKAMAK systems to attain the desired plasma equilibrium positions. Model tests indicate that the use of the second variation and Kalman filter algorithms provide corrective action for online control when the operating system, TOKAMAK, exhibits dynamic properties different than the simulation model. The state variable description developed in the work is applicable to optimal control algorithms with figures of merit other than those used, and will be very useful once dynamic relationships between injection heating and plasma temperatures and shifts are developed.

INTRODUCTION

Today, there is a growing public awareness that most present energy sources are either deficient in availability or have potential harmful environmental effects. This, coupled with the recent progress in experimental efforts in controlled nuclear fusion, has enhanced interest in useful power from controlled thermonuclear fusion reactions.

The basic equations describing the energy release in a thermonuclear reaction and the corresponding fuel creation are: (1,2)

In Equation 1, 14.1 MeV of the released energy is carried off in the neutron which is slowed down in a lithium blanket, and the kinetic energy is liberated as useful heat. The neutron is captured in lithium-6 which constitutes 7.5 atom percent of natural lithium. Only at ion temperatures and densities of 10 keV and 10^{14} nuclei/cm³, respectively, is an adequate cross section of the D-T reaction attained. In regions of kinetic temperatures of one keV or more and high densities, substances of low mass number are in a state of matter consisting of positively charged nuclei (ions), an equivalent number of negative electrons, and neutral particles. Matter in this state is referred to as a "plasma."

The excessive temperatures required to confine and maintain the plasma prohibit the plasma from directly contacting a confining material. The confining materials could melt and lose their confinement capabilities, and the molten material particles would enter the plasma as impurities, cool the plasma, and reduce the plasma's energy content. Confinement of the plasma with magnetic fields is an important fundamental approach for controlled nuclear fusion.

There has been major activity in both the United States and in the Soviet Union on controlled fusion experiments which entail confining the plasma with magnetic fields in a doughnut-shape machine. These types of machines are called TOKAMAKs. (3,4)

A schematic of a TOKAMAK-type machine which utilizes magnetic fields around a doughnut-shape case in which the plasma is confined is shown in Figure 1. This type of machine is designated as a diffuse pinch as opposed to a theta pinch or z pinch. The particles migrate around the torus in a spiraling manner. The velocity component parallel to the plane of the torus is due to the large toroidal coils, while the component perpendicular to the toroidal field is due to the field that is created about the current in the plasma—the poloidal field. The ohmic heating coils are the primary windings of a fundamental transformer, and the plasma is the secondary winding. The toroidal field is many times greater than the poloidal field, and it is this relationship that separates the TOKAMAK class of machines from other pinch machines.

It is extremely important in a TOKAMAK device to maintain the plasma equilibrium position in the center of the torus. As previously mentioned, if the plasma strikes the shell

of the torus, the integrity of the case is weakened and the energy content of the plasma is lost. There are three primary forces which cause the plasma to drift toward the outside of the shell: (5) One force is that resulting from the plasma diamagnetic currents crossed with the toroidal field, B_θ . Since B_θ is $\sim \frac{1}{r}$, the toroidal field is stronger on the inside of the torus than on the outside, and a net force in the $+r$ direction results (see Figure 2). A second force is created by the poloidal field, B_ϕ , crossed with the plasma conduction current, J_θ . Since B_ϕ is stronger on the inside of the plasma loop than on the outside, there is again a net force in the $+r$ direction. The third primary force is the pressure force of the plasma which tends to extend the plasma loop like an inflated balloon.

Figure 1. TYPICAL TOROIDAL-DIFFUSE-PINCH MACHINE.

The TOKAMAK machines are pulsed-type systems; ie, the various magnetic coils are pulsed with a current over a period of time, and the fields containing and maintaining the plasma last only a finite period of time—today, approximately 100 ms. Since the magnitude of the

currents and the fields vary with time, eddy currents (also called image currents) are induced in the shell surrounding the plasma. These eddy currents create fields which counteract the outward directed forces of the plasma that were itemized before. The shell, however, is not a perfect conductor because construction, diagnostic, and maintenance requirements necessitate both longitudinal and transverse gaps. A transverse gap also is required in the shell casing to prevent its acting as another secondary winding.

Figure 2. PLASMA SHIFT WITHIN THE TORUS.

The primary control of the plasma equilibrium position is attained by the introduction of a vertical field which is initiated by the vertical field coils parallel to the ohmic heating coils⁽⁶⁻¹⁰⁾ (see Figure 1). The current magnitude of the vertical field can be optimally varied to maintain this plasma equilibrium position.

The shell casing and gap effects on the position of the plasma have been mentioned. Other effects on the plasma position include magnetic surface variation and the injection process.

The magnetic center of the plasma is not coincident with the geometric or mass center of the plasma; consequently, this positional variation with time must be considered in a control

strategy. The reason for the nonconcentricity is the toroidal shape of the casing, and the plasma carrying a longitudinal current enclosed within any of the magnetic surface is in an equilibrium condition much as the entire plasma is with respect to the torus casing.⁽³⁾

Another shift of the plasma that must be considered is that due to the injection of neutral particles. These neutrals are injected into the plasma at kinetic energies in the order of 20 keV. The purpose of this injection is to transfer the kinetic energy of the neutral particles to ions in the plasma to increase the temperature of the plasma. The injection of this kinetic energy imparts momentum to the plasma and causes the plasma loop to extend outwards.

The primary contributions to the plasma shift that currently are reported in the literature have been noted. Others exist, but they are either minor or theory has not been advanced to a point of justifying additional research, at least, until the full understanding and control of the major shift parameters have been accomplished.

Optimization of a control strategy must be based on a figure of merit by which the optimization can be measured. As mentioned before, maintaining the plasma equilibrium position is necessary to prevent damage to the containment vessel and to prevent cooling of the outer perimeter of the plasma due to the influx of impurities from the contacted vessel. The total plasma current remains somewhat constant after the initial transient condition as it is a function of the ohmic current which is kept approximately constant during operation. As the periphery of the plasma cools, the cross-sectional area of the plasma decreases and the current density increases. Increasing the plasma current density increases the potential of plasma instability which is a phenomenon that literally destroys the useful function of the plasma. Instead of a smooth loop, the plasma becomes an oscillating series of kinks and discontinuities, and the energy content is lost. Decreasing the plasma cross section reduces the energy content of the plasma. This decrease, in turn, reduces the potential thermal efficiency of a controlled fusion reactor.

Another consideration in the figure of merit is the effect the equilibrium position has on the q factor of the machine. The q factor of a TOKAMAK device is defined as:

$$q = \frac{r}{R} \frac{B_{\theta}}{B_{\phi}} \quad (3)$$

The greater the q factor, the more stable the plasma. Current machines employ q factors of 2 to 10. In Equation 3, r and R are constant and B_{ϕ} cannot be decreased without decreasing the plasma current which will lower the plasma temperature. The manner in which q can be increased is by increasing B_{θ} . Incremental increases in B_{θ} are attained with larger toroidal field coils and associated power supplies. This increase is additional expense; hence, it is desirable to increase q to maintain plasma stability, but it is desirable to reduce q for economic reasons.

The classical method to determine the location of the plasma within the casing during operation of a TOKAMAK is with magnetic field pickup probes which detect the outermost magnetic surfaces of the plasma. The location of these probes is shown in Figure 3. Because of the toroidal geometry of the plasma, the magnetic center of the plasma cross section is

not coincident with its geometric center. The various magnetic surfaces of the plasma cross section can be approximated roughly by the set of circles shown in Figure 3. This noncentricity causes the plasma shift described on Page 7.

Figure 3. TORUS CROSS SECTION.

These pickup probes are the feedback of the closed-loop control system to maintain plasma equilibrium. The probes will detect fields other than those of the plasma. The fields created by the vertical field transformer core saturation field leakage and leakage from the toroidal fields must be considered. In a control strategy, these fields which affect the feedback can be handled as noise. The general approximation of the noise can be attained by conducting experiments with a plasma; ie, energize all the related fields to determine the effect on the pickup probes.

Dynamic online feedback control of the plasma equilibrium position in a TOKAMAK will be developed in this work, and modern control theory techniques will be employed in the

development of the control scheme. The necessary equations which characterize the operating conditions of a TOKAMAK will be cast into state-variable form, and appropriate control variables will be selected. To adequately model the system, correlation will be determined between signals which detect the outermost plasma surface and Thomson scattering experiments which determine the dense and hotter sections of the plasma profiles.

Since all of the state variables which describe the TOKAMAK model cannot be measured with the plasma position feedback device, a Kalman filter will be used to optimally estimate those state variables which cannot be determined online. The Kalman filter will also be used to consider both uncertainties in the state variables and feedback measurement noise.

The experimental data used in this investigation were obtained on the ORMAK (Oak Ridge TOKAMAK) in operation at the Oak Ridge National Laboratory, operated by the Union Carbide Corporation's Nuclear Division for the United States Energy Research and Development Administration.

A CONTROL STRATEGY FOR PLASMA EQUILIBRIUM IN A TOKAMAK

ORMAK AND PRESENT FEEDBACK SYSTEMS

Oak Ridge TOKAMAK

The ORMAK machine, from which all the experimental data used in this work was obtained, is a low-aspect-ratio TOKAMAK. The major radius of the torus is 79.5 cm and the minor radius is 23.5 cm; thus, the aspect ratio is 79.5/23.5 or approximately 3.4, which is low in comparison to other TOKAMAKs currently in operation. The toroidal magnetic field, B_θ , is varied from 14 to 18 kilogauss. Ohmic heating power is supplied by a regulated power supply which produces a flat-top current pulse with a typical duration of 100 ms after a capacitor bank breaks down the plasma and establishes the current. The values of the plasma current, I_p , are varied from 60 to 160 kiloamps. Thus, the safety factor, q , (see Equation 3) varies from 10 to 3.5.

Average electron densities, n_e , measured with a microwave interferometer, range from 0.5 to $4 \times 10^{13} \text{ cm}^{-3}$. Central electron temperatures, $T_e(0)$, measured by Thomson scattering, vary from 200 to 1200 eV (electron volts) and central ion temperatures, $T_i(0)$, measured with parallel and perpendicular charge-exchange analyzers, range between 150 and 350 eV.

The vertical field, B_z , is varied from 150 to 200 gauss. The field presently is controlled by a function generator which is made up of a series of RC networks and, by trial and error, tuned to control the current to attain the desired field. There is a shunt feedback on this present vertical field control. A schematic of the control is shown in Figure 4.(a) The feedback scheme described in this work would initially augment or perturb the function generator. The algorithm could be implemented to control approximately 25% of the available vertical field coil current. With successful operation, the new control could replace the function generator.

The ohmic heating of the ORMAK is approximately 250 kW. Four energetic neutral beams of hydrogen or deuterium are used on the ORMAK to supplement this ohmic heating. Two of the beams are each approximately 28 kV, 100 kW; the other two which were just installed are each 30 kV, 150 kW. These beams are directed approximately tangential to the magnetic axis of the plasma with beam radii of about six centimeters. The injectors are oppositely arranged, providing beam injection both in the same direction (co-injection) and in the opposite direction (counter-injection) to the plasma current. Ohmic heating and neutral beam injection are then the two most common ways of increasing the energy of the plasma.

The ORMAK has two vacuum chambers. The outer chamber is enclosed by a one-inch aluminum shell. This shell, as will be described in a subsequent section (Page 16), assists in maintaining the plasma equilibrium position. The inner shell, called the liner, is a gold-plated

(a) In mid February 1975, a hardwired feedback control, based upon the plasma position, was installed in the ORMAK.

torus which serves as the main chamber for the plasma. It is this chamber that is charged with hydrogen gas prior to each discharge. The radius of the liner is 23.5 cm. In one section of the liner, there is a protective ring called the "limiter". This limiter, made of tungsten, diverts the swirling plasma from the walls of the liner. The purity of the liner is extremely important. When the plasma strikes the liner, desorption takes place; that is, particles on the surface of the liner, usually carbon and oxygen, leave the liner and enter the plasma stream. These elements are impurities and they reduce the temperature of the plasma which, in turn, reduces the potential energy of the plasma. Figure 5 is a schematic of the ORMAK.

Figure 4. VERTICAL FIELD CONTROL ON THE ORMAK.

Introductory Physics

The gas kinetic pressure of the plasma fundamentally is made up of the electrons and ions and can be expressed as $n_e(r) T_e(r) + n_i(r) T_i(r)$, which indicates spatial dependence. The net rate of movement of the charge, that is, the ion and electron movement, is equal to the plasma current density, $\vec{j}$. In the steady state, the force on all the particles in a unit volume (ie, the force density) is balanced by the rate of momentum transfer which is equal to the gradient of the pressure. Therefore:

$$\frac{1}{C} (\vec{j} \times \vec{B}) = \nabla p. \quad (4)$$

For the electric field which does not vary with time, Maxwell's equation can be written:

$$\nabla \times \vec{B} = \frac{1}{C} (4\pi \vec{j}). \quad (5)$$

Figure 5. CROSS SECTION OF THE ORMAK.

Combining Equations 4 and 5:

$$\frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B} = \nabla p . \quad (6)$$

In general:

$$(\nabla \times \vec{B}) \times \vec{B} = \frac{1}{2} \nabla B^2 + (\vec{B} \cdot \nabla) \vec{B} , \quad (7)$$

and in a magnetic field in which the lines of force are straight and parallel, the last term on the right-hand side is zero. Equations 6 and 7 can be expressed as:

$$p + \frac{B^2}{8\pi} = \text{constant} . \quad (8)$$

The quantity $B^2/8\pi$, which has the units of energy/volume, is the energy density of the magnetic field. Since energy/volume is equivalent to force/area, $B^2/8\pi$ is also regarded as the magnetic pressure of the field. It follows from Equation 8 that the sum of the kinetic and magnetic pressures is constant in a plasma if the toroidal geometry is ignored, and a straight conductor is assumed. If the plasma is completely confined by an external magnetic field of strength B_ϕ , the pressure at the outside must fall to zero, so that the constant in Equation 8 is equal to $B_\phi^2/8\pi$; that is:

$$p + \frac{B^2}{8\pi} = \frac{B_\phi^2}{8\pi}. \quad (9)$$

Hence, an external magnetic field, B_ϕ , can confine a plasma having a kinetic pressure, p , and a contained field, B , where B_ϕ , B , and p are related by Equation 9. It is convenient to express the kinetic pressure of the particles in a plasma in terms of its ratio to the external magnetic pressure. This dimensionless ratio is designated β_ϕ "beta poloidal", that is:

$$\beta_\phi = \frac{8\pi p}{B_\phi^2}. \quad (10)$$

It is desirable to increase β_ϕ since this results in a higher plasma pressure or energy content. Too great a value of β_ϕ , however, will result in plasma instability since the plasma pressure will not be constrained.

The internal inductance per unit length of the plasma column can be expressed as:

$$\ell_i = \frac{\int B_\phi^2 \rho d\rho d\phi}{\pi a^2 B^2}, \quad (11)$$

where the quasicylindrical coordinate system of Figure A-1, Appendix A, is used. If the plasma column axis is taken as the coordinate axis, the distribution of the poloidal field on the outermost toroidal magnetic surface of the plasma column has the form:⁽¹¹⁾

$$B_\phi(a, \phi) = B \left(1 + \frac{a}{R} \Lambda \cos\phi \right), \quad (12)$$

where Λ is the coefficient of asymmetry of the poloidal field and is determined by:

$$\Lambda = \frac{8\pi \bar{p}}{B^2} + \frac{\ell_i}{2} - 1. \quad (13)$$

The value, Λ , is a measure of the energy content of the plasma and poloidal field generated by the plasma column. The internal inductance of the plasma is mainly a function of the geometry of the cross section of the plasma column. It depends upon the flux surfaces and, consequently, there is little control of this parameter in a TOKAMAK.

Equation 13 is a very common parameter in the plasma physics literature. This expression will be used in a later section (Page 23) when the state variables of the system are derived.

Present Feedback Systems

Feedback control to maintain plasma equilibrium has been employed on TOKAMAK machines. Among the notable ones described in the literature^(12,13) are the ATC (Adiabatic Toroidal Compressor) at Princeton, the TFR (TOKAMAK de Fontenay aux Roses) in France, the CLEO at the Culham Laboratory in the United Kingdom, and the TO-1 at the Kurchatov Institute in the Soviet Union. Brief discussions of the reported feedback approaches used on the devices follow.

ATC - A set of four pickup coils is used on the ATC, which has no conducting shell, to detect the position of the plasma. A precalculated equilibrium position is maintained during the discharge by continuously monitoring the position, comparing it to the desired position, and applying a corrective vertical field.

TFR - The TFR, which has a conducting shell, uses a set of pickup coils to simultaneously detect the flux from the plasma and the flux created by the image currents in the shell. The difference in these fluxes is determined, and a vertical field proportional to the difference is created. The vertical field control coils are outside the shell and the compensating field passes through the gaps in the shell. The need to pass the compensating field through the shell causes control difficulty because only at the gaps is the control field predominate.

CLEO - The CLEO device, which does not employ a conducting shell, uses a set of probes and servo control to continuously monitor and maintain the plasma position in accordance with an algorithm determined experimentally. The algorithm used in the control loop assumes a constant value for $\beta_\phi + \ell_i/2$. The radial position of the plasma is detected by coils outside the vacuum vessel, and these coils provide input to a 24-kW power amplifier which drives and supplements the vertical field. The position of the detector coils outside the shell reduces the sensitivity of the pickup.

TO-1 - The feedback device used on the Soviet Union's TO-1 machine is quite similar to that used on the TFR. The TO-1, however, does not have a conducting shell. Sets of pickup coils similar to those used on the TFR detect the flux from the discharge, and a vertical field is applied to balance this discharge field. The magnitude of the applied vertical field is proportional to the radial position of the plasma.

Absolute Feedback

All of the aforementioned feedback devices operate in common in that the absolute value of the plasma shift at which the equilibrium takes place is determined offline. A predetermined position is attained and held, but the system is not dynamic in that the plasma equilibrium position is not continuously monitored and repositioned.

The feedback and optimal control strategy developed in this work will employ an algorithm that continuously monitors the absolute position of the plasma and applies a corrective vertical field control online. The absolute position of the plasma is determined by continuously solving the classical Shafranov equilibrium equation which is developed in Appendix A. This work will differ from past reported techniques in that an absolute rather than incremental control will be applied.

MAGNETIC AXIS OFFSET

Offset Calculation

The pickup loops which detect the plasma position measure the outermost magnetic surface of the plasma. Because of the toroidal geometry, the magnetic center of the plasma is not coincident with the geometric center. The equation evaluating this offset is:⁽¹¹⁾

$$\delta(\rho) = \int_0^\rho \frac{r}{R} \lambda(r) dr, \quad (14)$$

where:

$$\lambda(r) = \frac{2\mu_0}{B_\phi^2(r)} \left[\bar{p}(r) - p(r) \right] + \frac{l_i(r)}{2}. \quad (15)$$

Equation 15 is derived in Appendix B.

It was necessary to solve Equation 14 and apply the results to the signal obtained from the pickup probes so that some correlation with the Thomson scattering experiments on the ORMAK would be attained. These scattering experiments were conducted by M. Murakami⁽¹⁴⁾ to attain plasma profiles and to determine experimentally the effect that the vertical field has on the plasma equilibrium position.

Expressions for $B_\phi(r)$, $p(r)$, and $l_i(r)$ are required to solve Equation 15. Now:

$$\oint \vec{B}(r) \cdot d\vec{\ell} = \mu_0 \int \vec{J}(r) \cdot d\vec{A}, \quad (16)$$

so:

$$B_\phi(r) = \frac{\mu_0 J_0}{r} \int_0^r J'(r') r' dr', \quad (17)$$

where:

$$J(r) = J_0 J'(r).$$

The plasma current, I_p , can be expressed as:

$$I_p = \int_0^a \vec{J}(r) \cdot d\vec{A}, \quad (18)$$

or:

$$I_p = 2\pi J_0 \int_0^a J'(r) r dr. \quad (19)$$

From Equations 17 and 18:

$$B_\phi(r) = \frac{\mu_0 I_p}{2\pi r} \frac{\int_0^r J'(r') r' dr'}{\int_0^a J'(r') r dr'}. \quad (20)$$

Now:

$$p(r) = n_e(r) T_e(r) + n_i(r) T_i(r). \quad (21)$$

The following density and temperature profiles for the electrons and ions were assumed:

$$n(r) = n(0) \left[1 - \left(\frac{r}{a} \right)^n \right], \quad (22)$$

and:

$$T(r) = T(0) \left[1 - \left(\frac{r}{a} \right)^m \right]. \quad (23)$$

The average pressure across the plasma is:

$$\bar{p}(r) = \frac{1}{\pi r^2} \int_0^r p(r') 2\pi r' dr'. \quad (24)$$

Thus, from Equations 15, 20, 21, 22, and 23:

$$\lambda(r) = \frac{2\mu [n_e(0) T_e(0) + n_i(0) T_i(0)]}{B_\phi^2(r)} \left[\frac{2}{r^2} \int \left[1 - \left(\frac{r}{a} \right)^n \right] \left[1 - \left(\frac{r}{a} \right)^m \right] r dr - \right.$$

$$\left[1 - \left(\frac{r}{a}\right)^n\right] \left[1 - \left(\frac{r}{a}\right)^m\right] + \frac{\int_0^r B_\phi^2(r) r dr}{r^2 B_\phi^2(r)}; \text{ and, from Equation 20:} \quad (25)$$

$$\lambda(r) = \frac{2\pi r^2 [n_e(0)T_e(0) + n_i(0)T_i(0)] \left[\int_0^a J'(r) r dr \right]^2}{10^{-7} I_p^2 \left[\int_0^r J'(r') r' dr' \right]^2}$$

$$\left\{ 2 \left[\frac{1}{2} - \frac{r^n}{(n+2)a^{n+2}} - \frac{r^m}{(m+2)a^{m+2}} + \frac{r^{m+n}}{(m+n+2)a^{m+n+2}} \right] - \right.$$

$$\left. \left[1 - \frac{r^n}{a^n} - \frac{r^m}{a^m} + \frac{r^{m+n}}{a^{m+n}} \right] \right\} +$$

$$\frac{\int_0^r \frac{1}{r'} \left[\int_0^{r'} J'(r'') r'' dr'' \right]^2 dr'}{\left[\int_0^r J'(r') r' dr' \right]^2} \quad (26)$$

Thus, from Equation 14:

$$\delta(\rho) = \frac{4\pi K^2 [n_e(0)T_e(0) + n_i(0)T_i(0)]}{R 10^{-7} I_p^2} \times$$

$$\int_0^\rho \frac{\left[\frac{r^3}{2} - \frac{r^{n+3}}{(n+2)a^{n+2}} - \frac{r^{m+3}}{(m+2)a^{m+2}} + \frac{r^{(m+n+3)}}{(m+n+2)a^{m+n+2}} \right] dr}{\left[\int_0^r J'(r') r' dr' \right]^2}$$

$$\frac{2\pi K^2 [n_e(0)T_e(0) + n_i(0)T_i(0)]}{R 10^{-7} I_p^2} \times$$

$$\begin{aligned}
& \int_0^{\rho} \frac{\left[r^3 - \frac{r^{n+3}}{a^n} - \frac{r^{m+3}}{a^m} + \frac{r^{m+n+3}}{a^{m+n}} \right] dr}{\left[\int_0^r J'(r') r' dr' \right]^2} + \\
& \frac{1}{R} \int_0^{\rho} \frac{\int_0^r \frac{1}{r'} \left[\int_0^{r'} J''(r'') r'' dr'' \right]^2 dr'}{\left[\int_0^r J'(r') r' dr' \right]^2} dr, \quad (27)
\end{aligned}$$

where:

$$K^2 = \left[\int_0^a J'(r) r dr \right]^2$$

Experimental Results

There are two basic types of experimental discharges run on the ORMAK. These are designated as Type A and Type B. The two types of shots exhibit different sets of profiles for the electron density and the electron temperature. A torus filling pressure of 4.8×10^{-4} torr results in a Type A shot, while a filling pressure of 4.4×10^{-4} torr results in a Type B shot.

Profile data from Type A and Type B discharges were obtained from Thomson scattering experiments and the data were fit with Equations 22 and 23. The final choice of the exponents m and n is listed in Table 1. The current density profiles were assumed to have the geometric dependence of the electron profiles raised to the 3/2 power.

Table 1
DENSITY, TEMPERATURE, AND CURRENT-DENSITY PROFILES

Type A Discharge			Type B Discharge		
Electron Density	$n_e(o)$	$\left[1 - \frac{r}{a} \right]^{0.97}$	$n_e(o)$	$\left[1 - \frac{r}{a} \right]^{1.17}$	
Electron Temperature	$T_e(o)$	$\left[1 - \frac{r}{a} \right]^{1.06}$	$T_e(o)$	$\left[1 - \frac{r}{a} \right]^{2.54}$	
Ion Density	$n_i(o)$	$\left[1 - \frac{r}{a} \right]^{0.97}$	$n_i(o)$	$\left[1 - \frac{r}{a} \right]^{1.17}$	
Ion Temperature	$T_i(o)$	$\left[1 - \frac{r}{a} \right]^{0.97}$	$T_i(o)$	$\left[1 - \frac{r}{a} \right]^{1.17}$	
Current Density	$J(o)$	$\left[1 - \frac{r}{a} \right]^{1.06} \cdot 1.5$	$J(o)$	$\left[1 - \frac{r}{a} \right]^{2.54} \cdot 1.5$	

To solve Equation 26, a computer code, which uses gaussian integration, was prepared.⁽¹⁵⁾ Offsets were computed for various radii up through the liner radius minus offset values determined from the probe signal. Offsets for both types of discharge at various times into the discharge were computed.

Figure 6 shows a typical magnetic surface shift. A comparison of the shifts determined from the pickup probes, the offsets determined by Murakami's Thomson scattering experiments (as shown in Figure 7), the difference, and the magnetic axis offsets are shown in Table 2.

Figure 6. FLUX SURFACES. (Type A Discharge, 18 ms)

The m and n exponents used in Equations 21 and 22 were varied from 1 to 4 to determine the sensitivity of the offset value to the density and temperature profiles. The effect of the variations on the offset is shown in Table 3.

In all cases shown in Table 2, the laser shift measurement and the position of the magnetic axis are in agreement to within less than one centimeter. This relationship is true even though both the laser and the shift signal data were averaged over several shots. This agreement implies that the laser is measuring density and temperature maxima which are in the magnetic axis.

The nonconcentricity of the flux surfaces due to toroidal curvature can be observed from Figure 6. The center-most surfaces are offset 1.5 - 2.5 cm. This offset depends mostly on the magnitude of β_ϕ , as shown by the geometry variations in Table 3; and, thus, is surprisingly insensitive to profile changes. The agreement between the two measurements increases the confidence on the applicability of the Shafranov theory for the low-aspect-ratio TOKAMAK and the use of the optimal feedback control to maintain the plasma position. The Δ_N value in the figure of merit expressed in Equation 70 is an offset which is based on the correlation between two types of measurements.

Figure 7. PROFILES DETERMINED FROM THOMSON SCATTERING EXPERIMENTS.

Table 2
PLASMA OFFSETS
(From Data of 2/19/74 and 2/20/74 Shots)

Type of Discharge	Time into Discharge (ms)	Offset from Probe Signal (cm)	Offset from Thomson Scattering (cm)	Difference (cm)	Magnetic Axis Offset (cm)
A	18	- 1.24	1.0	2.24	2.18
A	33	- 0.55	2.0	2.55	2.64
A	48	- 0.30	1.2	1.5	2.38
A	63	- 0.47	1.0	1.47	2.1
B	18	- 1.67	0	1.67	1.72
B	33	- 1.23	0	1.23	1.47
B	48	- 1.05	1.0	2.05	1.50
B	63	- 0.87	1.0	1.87	1.38

Table 3
EFFECTS OF EXPONENTS FOR DENSITY AND
TEMPERATURE PROFILES ON SHIFT MAGNITUDE

Type A Discharge(1)			Type B Discharge(1)		
m	n	Shift	m	n	Shift
1	1	2.336	1	1	1.625
1	4	2.500	1	4	1.677
4	1	2.492	4	1	1.426
4	4	2.196	4	4	1.332

(1) Typical discharges at 33 ms into the shot.

STATE VARIABLES

Eddy Currents in Shell

Referring to Figure 8, the field directly due to the vertical field coils can be expressed as follows: (16 - 18)

$$A_c^i = \sum_{m=1}^{\infty} A_m(t) \left(\frac{\rho}{a} \right)^m \cos m\theta \quad \text{for } \rho < a, \quad (28)$$

and:

$$A_c^0 = \sum_{m=1}^{\infty} B_m(t) \left(\frac{a}{\rho} \right)^m \cos m\theta \quad \text{for } \rho > a. \quad (29)$$

Figure 8. FIELD INTENSITIES DUE TO FIELD COILS AND SHELL IMAGE CURRENTS USING THIN-SHELL APPROXIMATION.

The boundary condition, $B_{\rho}^i = B_{\rho}^0$, at $\rho = a$ requires:

$$A_m(t) = B_m(t), \quad (30)$$

and the tangential boundary condition, $H_{\phi}^0 - H_{\phi}^i = i_c$, where i_c is the surface density of the coll current, requires:

$$\mu_0 i_c = \sum_{m=1}^{\infty} A_m \left(\frac{2m}{a} \right) \cos m\theta. \quad (31)$$

The vector potential due to the eddy (or image) currents in the shell can be expressed as:

$$A_i^i = \sum_{m=1}^{\infty} C_m(t) \left(\frac{\rho}{b} \right)^m \cos m\theta \quad \text{for } 0 < \rho < b, \quad (32)$$

and:

$$A_i^0 = \sum_{m=1}^{\infty} D_m(t) \frac{b}{\rho}^m \cos m\theta \quad \text{for } \rho > b. \quad (33)$$

The previous similar boundary conditions give:

$$C_m(t) = D_m(t),$$

and:

$$\mu_0 i_i = \sum_{m=1}^{\infty} C_m(t) \frac{2m}{b} \cos m\theta, \quad (34)$$

where i_i is the surface density of the image current.

Faraday's induction law relates the total vector potential to the image density as follows:

$$- \left[\frac{d}{dt} (A_c^0 + A_i^i) \right]_{\rho=b} = \frac{n i_i db d\theta}{db d\theta} = \frac{n i_i}{d},$$

which can be expressed as:

$$\frac{dC_m(t)}{dt} + \frac{2mn}{\mu_0 b d} C_m(t) = - \left(\frac{a}{b} \right)^m \frac{dA_m(t)}{dt}. \quad (35)$$

Taking the Laplace transform of Equation 35 and transposing terms:

$$C_m(s) = \left(\frac{a}{b}\right)^m \left(\frac{\alpha_m}{s+\alpha_m} - 1\right) A_m(s) + \frac{1}{s+\alpha_m} C_m(0) + \left(\frac{a}{b}\right)^m \frac{1}{s+\alpha_m} A_m(0), \quad (36)$$

where:

$$\alpha_m = \frac{2mn}{\mu_0 db}.$$

The inverse Laplace transform of Equation 36 is:

$$C_m(t) = \left(\frac{a}{b}\right)^m \alpha_m \int_0^t A_m(t') e^{-\alpha_m(t-t')} dt' - \left(\frac{a}{b}\right)^m A_m(t) + \left(\frac{a}{b}\right)^m A_m(0) e^{-\alpha_m t} + C_m(0) e^{-\alpha_m t}. \quad (37)$$

The vector potential inside the vertical field coil is, accordingly: (19)

$$A^i = A_c^i + A_i^i, \text{ and since:}$$

$$\vec{B}_\rho^i = \nabla \times A^i,$$

$$\begin{aligned} \vec{B}_\rho^i = & \frac{1}{\rho} \sum_{m=1}^{\infty} m \left\{ A_m(t) \left[\left(\frac{\rho}{a}\right)^m - \left(\frac{a\rho}{b^2}\right)^m \right] + \right. \\ & \left. \left(\frac{a\rho}{b^2}\right)^m \left[\alpha_m \int_0^t A_m(t') e^{-\alpha_m(t-t')} dt' + A_m(0) e^{-\alpha_m t} \right] + \right. \\ & \left. \left(\frac{\rho}{b}\right)^m C_m(0) e^{-\alpha_m t} \right\} \sin m\theta. \end{aligned} \quad (38)$$

Since there are no image currents at time $t = 0$, the last two terms in Equation 38 are zero.

Since $B(t)$ is proportional to the vertical field current, $I(t)$, Equation 38 can be represented as:

$$B(t) = C_0 I(t) + \sum_{i=1}^n C_i e^{-\tau_i t} \int_0^t I(t') e^{\tau_i t'} dt'. \quad (39)$$

The i-through-n terms on the right side of Equation 39 can be represented by differential equations using the convolution theorem:

$$B(t) = I(t) * e^t = \int_0^t I(t') e^{(t-t')} dt' = e^t \int_0^t I(t') e^{-t'} dt'. \quad (40)$$

The first term on the right-hand side of Equation 39 represents the steady-state component of the field, while the remainder of the terms reflect the contribution of the various image currents induced in the shell.

From experimental studies on the ORMAK I, Colchin⁽⁸⁾ has determined that the field can be satisfactorily represented by a series of first-order equations, and he has determined the magnitudes of the various coefficients and time constants. Therefore, the control field can be represented by the following equation:

$$B(t) = K_7 I(t) + K_{g\tau 1} e^{-\tau_1 t} \int_0^t I(t') e^{\tau_1 t'} dt' + K_{g\tau 2} e^{-\tau_2 t} \int_0^t I(t') e^{\tau_2 t'} dt'. \quad (41)$$

The state vectors⁽²⁰⁾ which describe the vertical field control are:

$$\dot{X}_1 = -C_1 X_1 + K_8 U_1, \quad (42)$$

and:

$$\dot{X}_2 = -C_2 X_2 + K_9 U_1, \quad (43)$$

where U_1 is the input variable and is equal to the vertical field coil current, I , and K_8 and K_9 equal $K_{g\tau 1}$ and $K_{g\tau 2}$, respectively.

Shafranov Equilibrium Equation

The other fundamental equation necessary to describe the behavior of the system is the equation which relates the time dependence of the plasma position with the control field or the vertical field current. This equation is defined by Mukhovatov and Shafranov⁽¹⁰⁾ as:

$$\frac{d}{dt} I_p \left[\Delta(t) - \Delta_0 + \frac{cb^2}{2I_p} h_i \right] = \frac{cb^2}{2\tau} (H_{\perp 0} - h_i). \quad (44)$$

In Equation (44), Δ_0 is the offset due to an ideally conducting shell. The expression is derived in Appendix A and is:

$$\Delta_0 = \frac{b^2}{2R} \left[\ln \frac{b}{a} + \left(1 - \frac{a^2}{b^2} \right) \left(\Lambda + \frac{1}{2} \right) \right]. \quad (45)$$

$H_{\perp 0}$ is the transverse magnetic field necessary to maintain plasma equilibrium. When the plasma is inside the ideally conducting shell, this field is generated automatically by the image currents in the shell which are created by the Δ_0 movement. The field is equal to:

$$H_{\perp 0} = \frac{I_p}{cR} \left(\ln \frac{8R}{a} + \Lambda - \frac{1}{2} \right). \quad (46)$$

The field, h_i , is the true additional (control) balancing field inside the shell, and it is generated by the vertical field coils.

It was mentioned in the **INTRODUCTION** that because of construction, operation, and maintenance reasons, the shell of the torus will have various longitudinal and transverse gaps. These gaps permit leakage of various other fields into the shell. These other fields include those created by the ohmic heating coils and toroidal field coils, and the field due to core saturation. Colchin⁽²¹⁾ has conducted experiments on ORMAK to determine the magnitude of the gap field. In this study, the gap field will be treated as a percentage of the field at the gap. A "gap factor" of 5% will be used.

The term Λ , in Equations 45 and 46, is defined by Equation 13. Since Λ is a function of the electron and ion temperatures, and these temperatures can be varied with the input of high-energy neutral particles, Λ will be considered a control variable and be time dependent. Equation 44 can therefore be expanded to:

$$\begin{aligned} \dot{I}_p \Delta(t) - \dot{I}_p \frac{b^2}{2R} \ln \frac{b}{a} - \dot{I}_p \frac{b^2}{2R} \left(1 - \frac{a^2}{b^2} \right) \Lambda - \dot{I}_p \frac{b^2}{4R} \left(1 - \frac{a^2}{b^2} \right) + \\ I_p \dot{\Delta}(t) - I_p \frac{b^2}{2R} \left(1 - \frac{a^2}{b^2} \right) \dot{\Lambda} + \frac{cb^2 \dot{B}_{\perp}}{2} = \\ \frac{b^2 I_p}{2\tau R} \left(\ln \frac{8R}{a} - \frac{1}{2} \right) + \frac{b^2 I_p}{2\tau R} \Lambda - \frac{cb^2}{2\tau} B_{\perp} \end{aligned} \quad (47)$$

Equation 47 can be written as:

$$\dot{\Delta} - K_2 \dot{\Lambda} = K_1 \Delta + K_3 \Lambda - K_4 \dot{B}_\perp - K_5 B_\perp + K_6, \quad (48)$$

where:

$$K_1 = \frac{\dot{l}_p}{l_p}, \quad (49)$$

$$K_2 = \frac{b^2 - a^2}{2R}, \quad (50)$$

$$K_3 = K_1 K_2 + \frac{b^2}{2\tau R}, \quad (51)$$

$$K_4 = \frac{cb^2}{2l_p}, \quad (52)$$

$$K_5 = \frac{K_4}{\tau}, \quad (53)$$

and:

$$K_6 = K_1 \left[\frac{b^2}{2R} \ln \frac{b}{a} + \frac{b^2 - a^2}{4R} \right] + \frac{b^2}{2\tau R} \left(\ln \frac{8R}{a} - \frac{1}{2} \right). \quad (54)$$

Equation 48 can be expressed as:

$$\begin{aligned} \dot{X}_3 = & (K_4 C_1 - K_5) X_1 + (K_4 C_2 - K_5) X_2 - K_1 X_3 - (K_1 K_2 - K_3) \Lambda + \\ & (-K_4 K_8 - K_4 K_9 - K_5 K_7 + K_1 K_4 K_7) I + K_6, \end{aligned} \quad (55)$$

where:

$$\Delta = X_3 + K_2 \Lambda - K_4 K_7 I. \quad (56)$$

A simulation diagram from Equations 42, 43, and 56 is shown in Figure 9. The outputs from the various integrators are the state variables.

Equation 55 can be expressed in the following state variable notation:

$$\begin{aligned} \dot{X}_3 = & (K_4 C_1 - K_5) X_1 + (K_4 C_2 - K_5) X_2 - K_1 X_3 + \\ & (-K_4 K_8 - K_4 K_9 - K_5 K_7 + K_1 K_4 K_7) U_1 - \\ & (K_1 K_2 - K_3) U_2 + K_6, \end{aligned} \quad (57)$$

where U_1 and U_2 are the vertical field current and Δ , respectively.

Figure 9. STATE-VARIABLE MODEL OF A TOKAMAK.

The state vectors and state model of the system and the input are therefore:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -C_1 & 0 & 0 \\ 0 & -C_2 & 0 \\ K_4C_1-K_5 & K_4C_2-K_5 & K_1 \end{bmatrix} \times \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} K_8 & 0 \\ K_9 & 0 \\ K_{10} & K_{11} \end{bmatrix} \times \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad (58)$$

where:

$$K_{10} = (-K_4K_8 - K_4K_9 + K_1K_4K_7 - K_5K_7), \text{ and} \quad (59)$$

$$K_{11} = (K_1K_2 - K_3), \quad (60)$$

and the constants are defined in Equations 41 through 43 and 49 through 55.

The output from $A(t)$ matrix in Figure 10 is input to an observation matrix. This observation will be denoted by $H(t)$, and the output variables will be denoted by the vector $\vec{Y}$.

Figure 10. OPTIMAL CONTROL SYSTEM TO MAINTAIN THE PLASMA EQUILIBRIUM POSITION.

Feedback Signal

The primary feedback transducers to detect the plasma position are the pickup probes shown in Figure 8. These pickup probes detect the outermost magnetic field of the plasma; ie, B_ϕ . The probes are in the equatorial plane of the torus and are situated 180° apart in the shell. The difference and sum of the probe signals are determined, and the ratio of these is obtained. Denoting the sum of the signals as U^+ and the difference as U^- , the ratio can be expressed as:

$$\frac{U^-}{U^+} = \frac{B_\phi(\phi = 0^\circ) - B_\phi(\phi = 180^\circ)}{B_\phi(\phi = 0^\circ) + B_\phi(\phi = 180^\circ)}$$

From Equation A-11, Appendix A:

$$B_\phi = \frac{2I}{c\rho} + \frac{1}{CR} \left[\left(\Lambda + \frac{1}{2} \right) \left(1 + \frac{a^2}{d^2} \right) + \ln \frac{d}{a} - 1 + \frac{2R\Delta}{d^2} \right] \cos\phi + B_\phi^i \cos\phi + B_g \cos\phi, \quad (61)$$

where B_{ϕ}^i is the field generated by the vertical field coil, and B_g is the field at the gap. The former is handled in the observation matrix; the latter can be ignored since the probes are located a distance from are gaps in the shell.

Therefore:

$$\frac{U^-}{U^+} = \frac{d}{dR} \left[\left(\Lambda + \frac{1}{2} \right) \left(1 + \frac{a^2}{d^2} \right) + \ln \frac{d}{a} - 1 + \frac{2R\Delta}{d^2} \right]. \quad (62)$$

Now:

$$\Delta = \frac{b^2}{2R} \left[\ln \frac{b}{a} + \left(\Lambda + \frac{1}{2} \right) \left(1 - \frac{a^2}{b^2} \right) - \frac{cb^2}{2I_p} \left(B_{\perp} + \alpha B_g \right) \right], \quad (63)$$

where α is the gap factor.

From Equations 62 and 63:

$$\begin{aligned} \frac{U^-}{U^+} = \frac{d}{dR} \left\{ \frac{b^2}{b^2 - a^2} \left[\frac{a^2 + d^2}{d^2} \right] \left[\frac{2R\Delta}{b^2} + \frac{CR}{I_p} \left(B_{\perp} + \alpha B_g \right) - \ln \frac{b}{a} \right] + \right. \\ \left. \ln \frac{d}{a} - 1 + \frac{2R\Delta}{d^2} \right\}. \end{aligned} \quad (64)$$

Equation 64 is the observation vector of the system and can be expressed as:

$$\vec{Y} = C_{10}\Delta + C_{11}B_{\perp} + C_{12}, \quad (65)$$

where:

$$C_{10} = \left[\frac{1}{b^2 - a^2} \left(\frac{a^2 + d^2}{d} \right) + \frac{1}{d} \right], \quad (66)$$

$$C_{11} = \frac{C}{2I_p} \left(\frac{b^2}{b^2 - a^2} \right) \left(\frac{a^2 + d^2}{d} \right) (1 + \alpha), \quad (67)$$

and:

$$C_{12} = \frac{d}{2R} \ln \frac{d}{a} - \frac{d}{2R} - \frac{1}{2R} \left(\frac{b^2}{b^2 - a^2} \right) \left(\frac{a^2 + d^2}{d} \right) \ln \frac{b}{a}, \quad (68)$$

From Equations 65 and 68, the observation vector can be written as:

$$Y' = \begin{bmatrix} C_{11} & C_{11} & C_{10} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} (C_{11}K_7 - C_{10}K_4K_7) \\ C_{10}K_2 \end{bmatrix} \begin{bmatrix} U_1 \\ I_2 \end{bmatrix}, \quad (69)$$

where:

$$Y' = Y + C_{12}.$$

Figure of Merit

The strategy of the proposed optimal control system will use a figure of merit that will minimize the plasma deviation from a desired absolute offset. The desired offset may be to maintain the maximum plasma temperature and density in the center of the torus rather than the center of the outermost flux surface detected by the pickup probes. The figure of merit is, therefore:

$$J = \min \frac{1}{2} \int_{t_0}^{t_f} (\Delta - \Delta_N)^2 dt, \quad (70)$$

where Δ_N is the desired offset.

Figure 10 shows the block diagram of the optimal control system to be applied to a TOKAMAK. A second-variation approach will be used, and a Kalman filter technique will be employed to estimate optimal states and to treat the noise attributed to signal detection.

The Hamiltonian can be expressed as: (12,22)

$$H[\vec{X}(t), \vec{U}(t), \vec{\lambda}(t), t] = \phi[\vec{X}(t), \vec{U}(t), t] + \vec{\lambda}^T(t) f[\vec{X}(t), \vec{U}(t), t], \quad (71)$$

or:

$$\begin{aligned}
H[\vec{X}(t), \vec{U}(t), \vec{\lambda}(t), t] = & \frac{1}{2} (X_3 - K_4 K_7 U_1 + K_2 U_2 - \Delta_N)^2 + \\
& \lambda_1 (-C_1 X_1 + K_8 U_1) + \lambda_2 (-C_2 X_2 + K_9 U_1) + \lambda_3 [(K_4 C_1 - K_5) X_1 + \\
& (K_4 C_2 - K_5) X_2 - K_1 X_3 + (-K_4 K_8 - K_4 K_9 - K_5 K_7 + K_1 K_4 K_7) U_1 - \\
& (K_1 K_2 - K_3) U_2] .
\end{aligned} \tag{72}$$

Now:

$$\frac{\partial H}{\partial U_i} = 0 , \tag{73}$$

$$\dot{\vec{\lambda}}_i = - \frac{\partial H}{\partial X_i} , \tag{74}$$

and:

$$\dot{\vec{X}}_i = \frac{\partial H}{\partial \lambda_i} . \tag{75}$$

From Equations 72 through 75:

$$\begin{aligned}
\frac{\partial H}{\partial U_1} = & -K_4 K_7 [X_3 - K_4 K_7 U_1 + K_2 U_2 - \Delta_N] + K_8 \lambda_1 + K_9 \lambda_2 + \\
& [-K_4 K_8 - K_4 K_9 - K_5 K_7 + K_1 K_4 K_7] \lambda_3 = 0 ,
\end{aligned} \tag{76}$$

$$\frac{\partial H}{\partial U_2} = K_2 [X_3 - K_4 K_7 U_1 + K_2 U_2 - \Delta_N] - (K_1 K_2 - K_3) \lambda_3 , \tag{77}$$

$$\dot{\lambda}_1 = +C_1 \lambda_1 - (K_4 C_1 - K_5) \lambda_3 , \tag{78}$$

$$\dot{\lambda}_2 = +C_2 \lambda_2 - (K_4 C_2 - K_5) \lambda_3 , \tag{79}$$

$$\dot{\lambda}_3 = - (X_3 - K_4 K_7 U_1 + K_2 U_2 - \Delta_N) + K_1 \lambda_3 , \tag{80}$$

$$\dot{X}_1 = -C_1 X_1 + K_8 U_1 , \tag{81}$$

$$\dot{X}_2 = -C_2 X_2 + K_9 U_1, \quad (82)$$

and:

$$\begin{aligned} \dot{X}_3 = & (K_4 C_1 - K_5) X_1 + (K_4 C_2 - K_5) X_2 - K_1 X_3 + \\ & (-K_4 K_8 - K_4 K_9 - K_5 K_7 + K_1 K_4 K_7) U_1 - \\ & (K_1 K_2 - K_3) U_2. \end{aligned} \quad (83)$$

At the beginning of the discharge, the state vector is given by:

$$\vec{X}(t_0) = \begin{bmatrix} 0 \\ 0 \\ -.61 \end{bmatrix}_{t=t_0}, \quad (84)$$

and, at the termination of the discharge, the adjoined state vector is:

$$\lambda(t_f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{t=t_f}. \quad (85)$$

The optimal control conditions specified by Equations 76 and 77 must be satisfied. To accomplish this, a time profile of the adjoined vector component, $\lambda_3(t)$, is assumed to solve Equations 78 and 79 with the constraining relations:

$$\frac{K_8}{K_1 K_7} \lambda_1 + \frac{K_9}{K_4 K_7} \lambda_2 + K_{10} \lambda_3 = \frac{K_{11}}{K_2} \lambda_3, \quad (86)$$

and the final conditions specified by Equation 85.

Once a satisfactory adjoined vector is attained, a relationship between U_1 and U_2 can be obtained from either Equation 76 or Equation 77. A time profile for U_2 is then assumed. This profile is based upon experimental results. An optimal trajectory for U_1 is then obtained:

$$U_1 = \frac{1}{K_4 K_7} X_3 - \frac{K_{11}}{K_2 K_4 K_7} \lambda_3 + \frac{K_2}{K_4 K_7} U_2 - \frac{1}{K_4 K_7} \Delta_N. \quad (87)$$

Let:

$$K_{20} = \frac{K_{11}}{K_2 K_4 K_7} \lambda_3 + \frac{K_2}{K_4 K_7} U_2 + \frac{1}{K_4 K_7} \Delta_N, \quad (88)$$

$$K_{21} = \frac{1}{K_4 K_7}, \quad (89)$$

$$K_{22} = K_{10} K_{21} - K_1, \quad (90)$$

and:

$$K_{23} = K_{10} K_{20} + K_{11} U_2. \quad (91)$$

Therefore, Equations 81 through 83 can be expressed in the following format:

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} -C_1 & 0 & K_8 K_{21} \\ 0 & -C_2 & K_9 K_{21} \\ L_3 & L_4 & K_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} -K_8 K_{20} \\ -K_9 K_{20} \\ -K_{23} + K_6 \end{bmatrix}. \quad (92)$$

SECOND VARIATION AND STATE ESTIMATION

Neighboring Optimal Control

Addition of the second variation⁽²³⁾ of the Hamiltonian with respect to the state and control variables of the system results in a direct second-order method which achieves an improved convergence rate near the optimum trajectory or control law. Equations 76 through 85 represent the nominal optimal control of the system. It is desired to determine a "gain" matrix which will rapidly perturb the control variables to attain the optimal trajectory in the event of a deviation.

Referring to Figure 11, the actual time varying state variables of the system are compared to the nominal state variables. The differences are multiplied by the gain matrix, $\vec{\Lambda}(t)$, to attain the optimal corrective control action to be applied to the time varying control variables. The new control variable is the summation of the nominal control variable and the optimal corrective control action.

The control algorithm is one of minimizing:

$$J = \frac{1}{2} \int_{t_0}^{t_f} [\Delta X^T(t) Q(t) \Delta X(t) + \Delta U^T(t) M^T(t) \Delta X(t) +$$

$$\Delta X^T(t)M(t)\Delta U(t) + \Delta U^T(t)R(t)\Delta U(t)] dt . \quad (93)$$

The system is represented by the following expressions:

$$\vec{\Delta X} = A(t)\vec{\Delta X}(t) + B(t)\vec{\Delta U}(t) , \quad (94)$$

and:

$$\vec{\Delta X}(t_0) = \vec{\Delta X}_0 , \quad (95)$$

where:

$$\vec{A}(t) = \frac{\partial f[\vec{\hat{X}}(t)_1 \vec{\hat{U}}(t)_1 t]}{\partial \vec{\hat{X}}(t)} , \quad (96)$$

and:

$$\vec{B}(t) = \frac{\partial f[\vec{\hat{X}}(t)_1 \vec{\hat{U}}(t)_1 t]}{\partial \vec{\hat{U}}(t)} . \quad (97)$$

Figure 11. LAMBDA MATRIX CONTROL SYSTEM.

The matrices which make the gain matrix are the $\vec{R}(t)$, $\vec{Q}(t)$, and $\vec{M}(t)$ matrices which are determined by evaluating various derivatives of the Hamiltonian about the optimal trajectory. The matrices are:

$$\vec{R}(t) = \frac{\partial^2 H}{\partial \hat{U}^2(t)}, \quad (98)$$

$$\vec{Q}(t) = \frac{\partial^2 H}{\partial \hat{X}^2(t)} \quad (99)$$

and:

$$\vec{M}(t) = \frac{\partial^2 H}{\partial \hat{U}(t) \partial \hat{X}(t)} \quad (100)$$

The optimum closed loop control is:

$$\begin{aligned} \Delta \vec{U}(t) &= -\vec{R}^{-1}(t) [\vec{M}(t) + \vec{B}^T(t) \vec{P}(t)] \Delta \vec{X}(t) \\ &= \vec{\Lambda}(t) \Delta \vec{X}(t), \end{aligned} \quad (101)$$

where:

$$\begin{aligned} \dot{\vec{P}} &= [-\vec{A}^T(t) + \vec{M}(t) \vec{R}^{-1}(t) \vec{B}^T(t)] \vec{P}(t) + \vec{P}(t) [-\vec{A}(t) + \vec{B}(t) \vec{R}^{-1}(t) \vec{M}^T(t)] + \\ &\quad \vec{P}(t) \vec{B}(t) \vec{R}^{-1}(t) \vec{B}^T(t) \vec{P}(t) - \vec{Q}(t) + \vec{M}(t) \vec{R}^{-1}(t) \vec{M}^T(t). \end{aligned} \quad (102)$$

From Equations 58 and 72, the following matrices are determined:

$$\vec{R}(t) = \begin{bmatrix} (\kappa_4 \kappa_7)^2 & 0 \\ 0 & \kappa_2^2 \end{bmatrix}, \quad (103)$$

$$\vec{Q}(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad (104)$$

$$\vec{M}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -K_4K_7 & K_2 \end{bmatrix}, \quad (105)$$

$$\vec{A}(t) = \begin{bmatrix} -C_1 & 0 & 0 \\ 0 & -C_2 & 0 \\ L_3 & L_4 & -K_1 \end{bmatrix}, \quad (106)$$

and:

$$\vec{D}(t) = \begin{bmatrix} K_8 & 0 \\ K_9 & 0 \\ K_{10} & K_{11} \end{bmatrix}. \quad (107)$$

Now:

$$\vec{R}^{-1}(t) = \begin{bmatrix} \frac{1}{(K_4K_7)^2} & \\ 0 & \frac{1}{K_2^2} \end{bmatrix}, \quad (108)$$

and:

$$-\vec{A}^T(t) = \begin{bmatrix} +C_1 & 0 & -L_3 \\ 0 & +C_2 & -L_4 \\ 0 & 0 & K_1 \end{bmatrix}; \quad (109)$$

therefore, from Equations 102 through 109:

$$\begin{bmatrix} \dot{P}_{11} & \dot{P}_{12} & \dot{P}_{13} \\ \dot{P}_{21} & \dot{P}_{22} & \dot{P}_{23} \\ \dot{P}_{31} & \dot{P}_{32} & \dot{P}_{33} \end{bmatrix} = \begin{bmatrix} +C_1 & 0 & -L_3 \\ 0 & +C_2 & -L_4 \\ -\frac{K_8}{K_4K_7} - \frac{K_9}{K_4K_7} \left(\frac{K_{11}}{K_2} - \frac{K_{10}}{K_4K_7} \right) + K_1 \end{bmatrix} \times \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} +$$

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{33} & P_{33} \end{bmatrix} \times \begin{bmatrix} +C_1 & 0 & -\frac{K_8}{K_4 K_7} \\ 0 & +C_2 & -\frac{K_9}{K_4 K_7} \\ -L_3 & -L_4 & \left(-\frac{K_{10}}{K_4 K_7} + \frac{K_{11}}{K_2} + K_1\right) \end{bmatrix} +$$

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{22} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \times \begin{bmatrix} \frac{K_8^2}{K_4^2 K_7^2} & \frac{K_8 K_9}{K_4^2 K_7^2} & \frac{K_8 K_{10}}{K_4^2 K_7^2} \\ \frac{K_8 K_9}{K_4^2 K_7^2} & \frac{K_9^2}{K_4^2 K_7^2} & \frac{K_9 K_{10}}{K_4^2 K_7^2} \\ \frac{K_8 K_{10}}{K_4^2 K_7^2} & \frac{K_9 K_{10}}{K_4^2 K_7^2} & \frac{K_{10}^2}{K_4^2 K_7^2} + \frac{K_{11}^2}{K_2^2} \end{bmatrix} \times \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} +$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (110)$$

And, from Equation 101:

$$\Lambda(t) = \begin{bmatrix} 0 & 0 & \frac{1}{K_4 K_7} \\ 0 & 0 & \frac{1}{K_2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{K_8}{K_4^2 K_7^2} & \frac{K_9}{K_4^2 K_7^2} & \frac{K_{10}}{K_4^2 K_7^2} \\ 0 & 0 & \frac{K_{11}}{K_2^2} \end{bmatrix} \times \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \quad (111)$$

Equation 110 is the Riccati matrix equation. This equation must be solved for the P matrix in order to obtain the gain matrix $\Lambda(t)$ in Equation 111. The final conditions of the variables of the P matrix are $P(T_f) = 0$. Since it is desired to end up on the optimal trajectory, the Riccati equation is integrated backward in time from the final condition on the optimal trajectory.

A computer code using Runge-Kutta method was prepared to solve the nine first-order nonlinear differential equations described in Equation 110. The coefficients used in the solution were from data obtained from ORMAK discharges. The gain matrix is determined from the solution of the P matrix.

Optimal State Estimation with a Kalman Filter

In order to continually apply online corrections to keep the system on the optimal trajectory, it is necessary to know the values of the deterministic state variables of the system at the sampling times. The measurement noise is considered in the model to compensate for inaccuracies in the representation of the vertical field by Equations 42, 43, and 57. Noise associated with the pickup probe signals (Equation 62) is also considered. A noise error value of 10% is assumed.

In order to attain accurate estimates for the noise contamination state variables, a Kalman filter approach⁽²⁴⁾ will be employed. Figure 12 shows the message model and optimum state estimator.

The expressions used in the Kalman filter approach are:

$$\dot{\vec{X}} + \vec{F}(t)\vec{X}(t) + \vec{K}(t) [\vec{Z}(t) - \vec{H}(t)\vec{X}(t)] + \vec{L}(t)\vec{U}(t), \quad (112)$$

$$\vec{K}(t) = \vec{P}(t)\vec{H}^T(t)\vec{R}^{-1}(t), \quad (113)$$

and:

$$\dot{\vec{P}}(t) = \vec{F}(t)\vec{P}(t) + \vec{P}(t)\vec{F}^T(t) - \vec{P}(t)\vec{H}^T(t)\vec{R}^{-1}(t)\vec{H}(t)\vec{P}(t) + G(t)Q(t)G^T(t). \quad (114)$$

Figure 12. MESSAGE MODEL AND OPTIMUM STATE ESTIMATOR.

In these expressions, $\hat{\vec{X}}$ is the best estimate of the state variables $\vec{X}$ in the least mean square error sense. $\vec{H}$ is the observation matrix from Equation 49, and $\vec{F}(t)$ is the transition matrix for the deterministic state variables (Equation 58). $\vec{Z}(t)$ is the observation vector; that is, the actual values of the observed state variables. $\vec{Q}(t)$ and $\vec{R}(t)$ are the covariance matrices of the process and measurement noise, $\vec{P}(t)$ is the estimated mean square error associated with the deterministic state variables, and $\vec{K}(t)$ is the Kalman gain.

The Kalman gain is determined by assuming initial values for the best estimate of the state variables and the mean square error, $\vec{P}(t)$, of these state variables. The covariance matrices, $\vec{Q}(t)$ and $\vec{R}(t)$, will be approximated by the covariances, σ_w^2 and σ_v^2 .

The procedure for determining the optimal state estimator is to solve Equation 114 using the initial conditions:

$$\vec{X}(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0.61 \end{bmatrix}, \quad (115)$$

and:

$$\vec{P}(t_0) = 0. \quad (116)$$

The covariance, $\vec{P}(t)$, obtained from Equation 114 is substituted into Equation 113 to solve for the Kalman gain. This gain is then used to solve Equation 112 for the optimal states.

The observation matrix from Equation 69 is:

$$\vec{H}(t) = [C_{11} \ C_{11} \ C_{10}] \quad (117)$$

The observation vector, $\vec{Z}(t)$, is, in the case of the ORMAK system, a scalar quantity which is the signal on the pickup probes, Equation 64.

Equation 114 can be expressed as:

$$\begin{bmatrix} \dot{P}_{11} & \dot{P}_{12} & \dot{P}_{13} \\ \dot{P}_{21} & \dot{P}_{22} & \dot{P}_{23} \\ \dot{P}_{31} & \dot{P}_{32} & \dot{P}_{33} \end{bmatrix} = \begin{bmatrix} -C_1 & 0 & 0 \\ 0 & -C_2 & 0 \\ L_3 & L_4 & -K_1 \end{bmatrix} \times \begin{bmatrix} P_{11} & P_{11} & P_{11} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{33} & P_{33} \end{bmatrix} +$$

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \times \begin{bmatrix} -C_1 & 0 & L_3 \\ 0 & -C_2 & L_4 \\ 0 & 0 & -K_1 \end{bmatrix} - \frac{1}{2\sigma_v^2} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

$$\begin{bmatrix} H_1^2 & H_1 H_2 & H_1 H_3 \\ H_1 H_2 & H_2^2 & H_2 H_3 \\ H_1 H_3 & H_2 H_3 & H_3^2 \end{bmatrix} \times \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} + \sigma_w^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (118)$$

where:

$$H_1 = C_{11}C_1, \quad (119)$$

$$H_2 = C_{11}C_2, \quad (120)$$

and:

$$H_3 = C_{10}. \quad (121)$$

Equation 118 is again a Riccati matrix equation. A computer code similar to that used to solve Equation 120 was used to solve Equation 118. Contrary to the solution of Equation 120, for which the final conditions were known and which was integrated backward in time, the initial conditions of Equation 118 are known, and it is integrated forward.

The differential equation for the optimal estimated states is, therefore:

$$\begin{bmatrix} \dot{\hat{X}}_1 \\ \dot{\hat{X}}_2 \\ \dot{\hat{X}}_3 \end{bmatrix} = \begin{bmatrix} -(C_1 + K_1 H_1) & -K_1 H_2 & -K_1 H_3 \\ -K_2 H_1 & -(C_2 + K_2 H_2) & -K_2 H_3 \\ (L_3 - K_3 H_1) & (L_4 - K_3 H_2) & -(K_1 + K_3 H_3) \end{bmatrix} \times \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} +$$

$$Z \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} + \begin{bmatrix} K_8 & 0 \\ K_9 & 0 \\ K_{10} & K_{11} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad (122)$$

where the K values are the components of the K vector obtained from Equation 113.

RESULTS AND CONCLUSIONS

Experimental Data

All the modeling theory developed in the preceding two sections was tested using experimental data and operating parameters from the ORMAK. The vertical field resulting from the vertical field coil current, U_1 , was compared to that measured in typical ORMAK runs. Also, the plasma shift (see Equation 56), predicted by the model, was compared to the actual plasma shift in the ORMAK as a result of changes in the vertical field.

The average discharge time on the ORMAK is approximately 100 ms. The rapidly rising plasma-current value during the first 10 ms of the discharge (about 9 MA per second) and the fall off after 60 ms (about 1.2 MA per second) create notable transients at both ends of the shot. The results reported in this chapter reflect testing the model between 10 and 60 ms.

As mentioned earlier (Page 19), there are two basic types of discharges in ORMAK, Type A and Type B. The Type A discharge is noted by its peaked density and temperature profiles. These higher profiles result in a higher plasma pressure and, accordingly, higher β_ϕ and Λ values. The majority of the results presented are based upon Type A runs. The type of discharge does not have any effect on the theory of the model.

The majority of the elements in the various matrices throughout the work are time dependent since they are a function of the plasma current. The major variables used in the study are tabulated in Table 4 as a function of time.

Optimal Trajectory

The results of the solution of Equation 92 are shown in Figures 13 and 14. Three test cases were run: (1) $\Delta_N = 0$; (2) $\Delta_N = + 2.0$ cm; and (3) $\Delta_N = - 2.0$ cm. The $\Delta_N = 0$ case (Case 1)

Table 4

PRIMARY VARIABLES USED IN THE CONTROL MODEL

Time (sec)	Plasma Current (A)	C1, Equation 42 (sec ⁻¹)	C2, Equation 43 (sec ⁻¹)	Kg, Equation 42	Kg, Equation 43	τ , Equation 44 (sec ⁻¹)
0.010	74,000	50	0.877	0.125	0.035	1.14
0.015	79,000	50	0.877	0.125	0.035	1.14
0.020	83,000	50	0.877	0.125	0.035	1.14
0.025	85,000	50	0.877	0.125	0.035	1.14
0.030	87,000	50	0.877	0.125	0.035	1.14
0.035	88,000	50	0.877	0.125	0.035	1.14
0.040	89,000	50	0.877	0.125	0.035	1.14
0.045	89,000	50	0.877	0.125	0.035	1.14
0.050	89,000	50	0.877	0.125	0.035	1.14
0.055	89,000	50	0.877	0.125	0.035	1.14
0.060	89,000	50	0.877	0.125	0.035	1.14

Figure 13. VERTICAL FIELD CURRENT AS A FUNCTION OF TIME.

Figure 14. VERTICAL FIELD AS A FUNCTION OF TIME.

typifies retaining the plasma equilibrium condition in the center of the shell based upon the feedback signal from the pickup probes. Case 2 places a bias on the plasma to shift it toward the center of the torus and overcome the natural forces which tend to push the plasma out from the center of the torus. The corresponding vertical field coil current and vertical field have higher values to overcome this force. The third case (Case 3) shifts the plasma out from the torus center, the direction the plasma drifts. The current and field are understandably less than in the first case. From Figure 14 it is observed that field profiles from the three cases are somewhat similar, just vary in magnitude. The current profile in Figure 13 for the third case falls off faster than in the other two cases. This is due to the small eddy current field built up in the shell as the plasma moves outward. The field created by the coil current is, accordingly, diminished less rapidly than in the other cases where the eddy current's contribution is greater.

The nominal state and control variables for the optimal trajectory are listed in Table 5.

Neighboring Optimal Control

Components of the time-varying lambda gain matrices are shown in Table 6. These gain matrices would be stored in a file and applied to the difference between the nominal state variables listed in Table 5 and the optimal state estimations obtained with the Kalman filter approach.

STATE ESTIMATION WITH THE KALMAN FILTER

The solution of Equation 114 allows determination of the Kalman gain vector. With the data in Table 4, the Kalman gain vector was computed and the elements are listed in Table 7.

Table 5
NOMINAL STATE AND CONTROL VARIABLES
FOR OPTIMAL CONTROL TRAJECTORY
($\Delta_N = 0.0$)

Time (sec)	X ₁ (gauss)	X ₂ (gauss)	X ₃	U ₁ (A)	U ₂
0.010	1.803	0.5617	2.389	3,487	- 0.19
0.015	4.361	1.496	2.765	4,514	- 0.16
0.020	6.466	2.467	2.799	5,303	- 0.01
0.025	8.014	3.405	2.776	5,670	0.05
0.030	9.216	4.336	2.772	5,831	0.06
0.035	10.28	5.305	2.763	5,554	- 0.02
0.040	11.40	6.357	2.735	4,914	- 0.18
0.050	13.54	8.659	2.625	3,951	- 0.36
0.055	14.41	9.851	2.567	3,654	- 0.41
0.060	15.18	11.06	2.509	3,133	- 0.52

Table 6
LAMBDA GAIN MATRICES

Time (sec)	Elements					
	K ₁₁	K ₁₂	K ₁₃	K ₂₁	K ₂₂	K ₂₃
0.010	245.836	- 0.0019	10,867	- 0.0030	0	- 0.7464
0.015	15.226	- 0.0001	2,399	- 0.0031	0	- 0.7332
0.020	3.591	0	2,057	- 0.0026	0	- 0.7202
0.025	- 11.041	0.0001	1,473	- 0.0021	0	- 0.7073
0.030	- 10.018	0.0001	1,543	- 0.0017	0	- 0.6945
0.035	- 12.405	0.0001	1,395	- 0.0013	0	- 0.6817
0.040	- 9.753	0.0001	1,509	- 0.0009	0	- 0.6688
0.045	- 8.659	0	1,495	- 0.0005	0	- 0.6556
0.050	- 4.728	0	1,683	- 0.0003	0	- 0.6419
0.055	- 1.393	0	1,920	- 0.0001	0	- 0.6275
0.060	0	0	2,231	0	0	- 0.6116

Table 7
KALMAN GAIN VECTORS

Time (sec)	K_1	K_2	K_3
0.010	$1.19 \cdot 10^{-3}$	$2.80 \cdot 10^{-5}$	$1.59 \cdot 10^{-4}$
0.015	$1.74 \cdot 10^{-3}$	$5.02 \cdot 10^{-5}$	$1.71 \cdot 10^{-4}$
0.020	$2.06 \cdot 10^{-3}$	$7.22 \cdot 10^{-5}$	$1.74 \cdot 10^{-4}$
0.025	$2.26 \cdot 10^{-3}$	$9.40 \cdot 10^{-5}$	$1.73 \cdot 10^{-4}$
0.030	$2.38 \cdot 10^{-3}$	$1.15 \cdot 10^{-4}$	$1.73 \cdot 10^{-4}$
0.035	$2.46 \cdot 10^{-3}$	$1.37 \cdot 10^{-4}$	$1.73 \cdot 10^{-4}$
0.040	$2.50 \cdot 10^{-3}$	$1.58 \cdot 10^{-4}$	$1.73 \cdot 10^{-4}$
0.045	$2.53 \cdot 10^{-3}$	$1.79 \cdot 10^{-4}$	$1.73 \cdot 10^{-4}$
0.050	$2.55 \cdot 10^{-3}$	$2.00 \cdot 10^{-4}$	$1.73 \cdot 10^{-4}$
0.055	$2.56 \cdot 10^{-3}$	$2.21 \cdot 10^{-4}$	$1.73 \cdot 10^{-4}$
0.060	$2.56 \cdot 10^{-3}$	$2.41 \cdot 10^{-4}$	$1.73 \cdot 10^{-4}$

Model Simulation

The solutions obtained from the optimal trajectory code and the codes used to determine the lambda gain matrices and the Kalman gain vectors were combined in a code to simulate the complete control strategy developed in this work.

The figure of merits, Equation 70 with $\Delta_n = 0$, and the plasma shifts versus time for a typical Type A ORMAK discharge and for the optimal simulated model are plotted in Figures 15 and 16, respectively. The optimal simulated model incorporates the shell eddy current time constants derived from experimental data from

Type A ORMAK discharges. Comparison of the plots in Figures 15 and 16 indicates that the use of the optimal control model results in significant improvements in the control of the plasma position. Time constants which differed by 40% from the values used to obtain the optimal trajectory were used in the system, and the results are plotted in Figure 17. This figure indicates that the model is somewhat insensitive to changes in the actual time constants in the system compared to those assumed. Figure 18 shows the results of introducing an additional time constant into the system; that is, one that was not assumed when the optimal trajectory was determined. The model is again somewhat insensitive to this perturbation. The plots represent an average of time constants chosen between the values of the time constants used to compute the optimal trajectory.

The demands that the control strategy places on the control variables, vertical field current and Λ , are not excessive. The range is well within that presently available on the ORMAK.

TOKAMAK Controller

Implementation of the various components of the optimal controller discussed in this study would tax the computing power of a reasonable online controller. The consistent evaluation of the differential equations necessary to obtain the optimal trajectory, gain matrices, Kalman gain vector, and optimal estimated states presently would not be feasible for the short discharge time of present TOKAMAKs. Longer discharges with less frequent sampling times could remedy the situation.

Another solution which might be feasible would be to assume that the variables in Table 4 are quite typical for a series of discharges. The nominal state and control variables could be stored in a file along with the lambda gain matrix and the Kalman gain vector. Only the differential equation necessary to obtain the optimal estimated states would be solved online. The flow diagram of such an algorithm is shown in Figure 19.

The conclusions of this study are:

Figure 15. FIGURE OF MERIT AS A FUNCTION OF TIME FOR ORMAK DISCHARGES AND WITH AN OPTIMAL SIMULATED MODEL.

Figure 16. PLASMA SHIFT AS A FUNCTION OF TIME FOR ORMAK DISCHARGES AND WITH AN OPTIMAL SIMULATED MODEL.

Figure 17. PLASMA SHIFT AS A FUNCTION OF TIME WITH TIME CONSTANTS ALTERED IN THE OPTIMAL SIMULATED MODEL.

Figure 18. PLASMA SHIFT AS A FUNCTION OF TIME WITH AN ADDITIONAL TIME CONSTANT IN THE OPTIMAL SIMULATED MODEL.

Figure 19. FLOW DIAGRAM OF A TOKAMAK CONTROLLER.

1. An optimal online controller to maintain plasma equilibrium in a TOKAMAK is feasible.
2. Further refinement in the estimation of the state transition matrix should be undertaken.
3. The applicability of this work to TOKAMAKs without conducting shells should be studied since future systems may not have the stabilizing conducting shell for economic reasons.

APPENDIX A

EQUILIBRIUM FOR A PERFECTLY CONDUCTING SHELL

The shell casing of a TOKAMAK plays an important role in maintaining the equilibrium position of the plasma. The equilibrium force is generated by the eddy currents or image currents generated in the shell wall.^(25,26) If the torus curvature is small (where $R \gg b$), this equilibrium condition can be obtained without determining the field distribution in the plasma provided there is a region of low conductivity between the plasma and shell in which the current density of the plasma can be set equal to zero. The magnetic field at the surface of the plasma can be determined and Maxwell's equation can be used to find the magnetic field between the plasma and the wall. This field will coincide with one of the magnetic surfaces of the plasma, and the position of the plasma with respect to the wall can be determined, accordingly.^(27 - 29)

The equation of motion can be written:

$$\nabla p = \frac{1}{c} \vec{V} \cdot \vec{\nabla} \vec{V} + \frac{1}{c} [\vec{J} \times \vec{B}] . \quad (\text{A-1})$$

If the inertia term is neglected:

$$\nabla p = \frac{1}{c} [\vec{J} \times \vec{B}] . \quad (\text{A-2})$$

Maxwell's equations are:

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} , \quad (\text{A-3})$$

and:

$$\vec{\nabla} \cdot \vec{B} = 0 . \quad (\text{A-4})$$

Substituting the current density, $\vec{J}$, from Equation A-3 into Equation A-2 gives:

$$\nabla p = \frac{1}{4\pi} [(\vec{\nabla} \times \vec{B}) \times \vec{B}] .$$

A tensor is selected such that the plasma equilibrium condition can be written in the form of an expression in which the divergency of the selected tensor can be set equal to zero; ie:

$$\frac{\partial}{\partial X_i} T_{ik} = 0 . \quad (\text{A-5})$$

The chosen tensor is:

$$T_{ik} = \left(p + \frac{B^2}{8\pi} \right) \left(\delta_{ik} - \frac{B_i B_k}{B^2} \right) + \left(p - \frac{B^2}{8\pi} \right) \frac{B_i B_k}{B^2}, \quad (\text{A-6})$$

where δ_{ik} is the Kronecker Delta defined as:

$$\delta_{ik} = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

Making the substitution:

$$p_{\perp} = p + \frac{B^2}{8\pi}, \quad (\text{A-7})$$

and:

$$p_{\parallel} = p - \frac{B^2}{8\pi}. \quad (\text{A-8})$$

Equation A-6 can be written:

$$T_{ik} = p_{\perp} \left(\delta_{ik} - \frac{B_i B_k}{B^2} \right) + p_{\parallel} \frac{B_i B_k}{B^2}. \quad (\text{A-9})$$

In Equation A-9, $p_{\perp}$ is the pressure transverse to the magnetic field and $p_{\parallel}$ is the pressure along the magnetic field. The magnetic field, accordingly, by review of Equations A-7 and A-9, exerts a pressure in the transverse direction, but it produces a tension in the longitudinal direction.

Now, there is a vector identity:

$$\frac{\partial}{\partial X_i} X_k T_{ik} = T_{ik} \delta_{ik} + \frac{\partial T_{ik}}{\partial X_i}. \quad (\text{A-10})$$

In accordance with Equation A-5, the second term on the right-hand side of Equation A-10 will be zero at equilibrium. Integrating the remaining expression over the volume, V , bounded by the surface, S , the following is obtained:

$$\int_V T_{ik} dV = \oint_S T_{ik} X_k dS_i . \quad (A-11)$$

From Equation A-9:

$$\begin{aligned} \int_V \left[p_{\perp} \left(\delta_{ik} - \frac{B_i B_k}{B^2} \right) + p_{\parallel} \frac{B_i B_k}{B^2} \right] dv &= \oint_S p_{\perp} \delta_{ik} X_k dS_i + \\ &\oint_S \left(p_{\parallel} - p_{\perp} \right) \frac{B_i B_k}{B^2} X_k dS_i . \end{aligned} \quad (A-12)$$

Noting that $\delta_{ik} = 3$ and $\frac{B_i B_k}{B^2} = 1$, the left-hand side of Equation A-12 becomes:

$$\int_V \left[p_{\perp} (3-1) + p_{\parallel} \right] dv = \int_V \left(3p + \frac{B^2}{8\pi} \right) dv , \quad (A-13)$$

while the right-hand side of Equation A-12 can be expressed as:

$$\oint_S \left(p + \frac{B^2}{8\pi} \right) (\vec{r} \cdot d\vec{s}) + \oint_S \left(p - \frac{B^2}{8\pi} - p - \frac{B^2}{8\pi} \right) \frac{(\vec{B} \cdot \vec{r}) (\vec{B} \cdot d\vec{s})}{B^2} ,$$

or:

$$\oint_S \left(p + \frac{B^2}{8\pi} \right) (\vec{r} \cdot d\vec{s}) - \oint_S \frac{(\vec{B} \cdot \vec{r}) (\vec{B} \cdot d\vec{s})}{4\pi} . \quad (A-14)$$

Assuming that $\vec{B} \cdot d\vec{s} = 0$ on the magnetic surface, Equations A-13 and A-14 can be written as:

$$\int_V \left(3p + \frac{B^2}{8\pi} \right) dv = \oint_S \left(p + \frac{B^2}{8\pi} \right) (\vec{r} \cdot d\vec{s}) . \quad (A-15)$$

Using the geometry in Figure A-1, Equation A-15 can be expressed as:

$$\int_0^a \int_0^{2\pi} \int_0^{2\pi} \left(3\rho + \frac{B^2}{8\pi} \right) (R + \rho \cos \phi) d\theta \rho d\phi d\rho$$

$$= \int_0^{2\pi} \int_0^{2\pi} \left(\rho + \frac{B^2}{8\pi} \right) (R + \rho \cos \phi) (R \cos \phi + \rho) \rho d\phi d\theta . \quad (\text{A-16})$$

Figure A-1. PLASMA ELEMENT.

After performing the $d\theta$ integration:

$$\begin{aligned} & \int_0^a \int_0^{2\pi} \left(3\rho + \frac{B^2}{8\pi} \right) (R + \rho \cos\phi) \rho d\phi d\rho \\ &= \int_0^{2\pi} \left(\rho + \frac{B}{8\pi} \right) (R + \rho \cos\phi) (R \cos\phi + \rho) \rho d\phi . \end{aligned} \quad (\text{A-17})$$

Expanding Equation A-6:

$$\begin{aligned} T_{ik} = & \left(\rho + \frac{B^2}{8\pi} \right) \delta_{ik} - \rho \frac{B_1 B_k}{B^2} - \frac{B_i B_k}{8\pi} + \\ & \frac{B_i B_k}{B^2} - \frac{B_i B_k}{8\pi} , \end{aligned}$$

and:

$$T_{ik} = \left(\rho + \frac{B^2}{8\pi} \right) \delta_{ik} - \frac{B_i B_k}{4\pi} . \quad (\text{A-18})$$

Therefore, from Equation A-5, A-11, and A-18:

$$\oint_S \left[\left(\rho + \frac{B^2}{8\pi} \right) ds - \frac{\vec{B} \cdot (\vec{B} \cdot d\vec{s})}{4\pi} \right] = 0 . \quad (\text{A-19})$$

The projection of Equation A-19 on the equatorial plane is:

$$\oint \left(\rho + \frac{B^2}{8\pi} - \frac{B_\theta^2}{4\pi} \right) ds_\phi = 0 . \quad (\text{A-20})$$

Again, using the nomenclature of Figure A-1:

$$\int_0^{2\pi} \left(\rho + \frac{B^2}{8\pi} \right) (R + \rho \cos\phi) \cos\phi d\phi = 0 . \quad (\text{A-21})$$

It can be assumed that the toroidal field, B_θ , on the surface of the plasma and across the plasma is:

$$B_\theta(a, \phi) = B_e(a) \left(1 - \frac{a}{R} \cos \phi\right) \quad \frac{a}{R} \ll 1, \quad (\text{A-22})$$

where $B_e(a)$ is a vacuum toroidal field at $\rho = a$, $\phi = \pi/2$,

and:

$$B_\phi(\rho, \phi) = B_i(\rho) \left(1 - \frac{\rho}{R} \cos \phi\right), \quad (\text{A-23})$$

where $B_i(e)$ contains diamagnetic (toroidal field) effects.

The poloidal field is determined by assuming the components are analogous to those fields close to the loop of a filamentary circular loop.^(30 - 32) Referring to Figure A-2, the current density, $\vec{J}$, has a component in the θ direction:

$$J_\theta = I \delta(\cos \phi') \delta \frac{(r' - R)}{R}. \quad (\text{A-24})$$

The delta functions restrict the current flow to a ring of radius R . Since the current only has a θ components, the vector potential, $\vec{A}$, will have only a θ component. If, for calculation purposes, an observation point in the Z-X plane ($\theta = 0$) is chosen, then:

$$J_y = J_\theta \cos \phi'. \quad (\text{A-25})$$

Substituting Equations A-24 and A-25 into the general solution for $\vec{A}$ in the unbounded space:

$$A(x) = \frac{I}{C} \int \frac{J(x')}{|\vec{x} - \vec{x}'|} d^3x', \quad (\text{A-26})$$

the following expression for $A_\theta(\rho, \theta)$ is obtained:

$$A_\theta(\rho, \phi) = \frac{I}{CR} \int \rho'^2 d\rho' d\Omega' \frac{\cos \theta' \delta(\cos \phi') \delta(\rho' - R)}{|\vec{x} - \vec{x}'|}, \quad (\text{A-27})$$

where:

$$|\vec{x} - \vec{x}'| = \left[\rho^2 + \rho'^2 - 2\rho\rho' (\cos \phi \cos \phi' + \sin \phi \sin \phi' \cos \theta') \right]^{1/2}.$$

Integration over the delta function leaves the result:

$$A_{\theta}(\rho, \phi) = \frac{IR}{c} \int_0^{2\pi} \frac{\cos\theta' d\theta'}{(R^2 + \rho^2 - 2R\rho\sin\phi\cos\theta')^{1/2}} \quad (\text{A-28})$$

Equation A-28 can be expressed in terms of the complete elliptic integrals, K and E:

$$A_{\theta}(\rho, \phi) = \frac{4IR}{c\sqrt{R^2 + \rho^2 + 2R\rho\sin\phi}} \left[\frac{(2 - k)K(k) - 2E(k)}{K^2} \right], \quad (\text{A-29})$$

where:

$$K^2 = \frac{4R\rho\sin\phi}{R^2 + \rho^2 + 2R\rho\sin\phi}$$

The components of magnetic induction are:

$$\left. \begin{aligned} B_\rho &= \frac{1}{\rho \sin \phi} \frac{\partial}{\partial \phi} (\sin \phi A_\theta) \\ B_\phi &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\theta) \\ B_\theta &= 0 \end{aligned} \right\} \quad (A-30)$$

From Equations A-29 and A-30, the azimuthal field, B_ϕ , is obtained:

$$B_\phi = B_a \left(1 + \frac{a}{2R} \ln \frac{8R}{\rho} \cos \phi \right), \quad (A-31)$$

where:

$$B_a = \frac{2I}{ca}.$$

Setting:

$$\Lambda_\phi(\rho) = \frac{1}{2} \ln \frac{8R}{\rho},$$

Equation A-31 can be written:

$$B_\phi = B_a \left(1 + \frac{a}{R} \Lambda(a) \cos \phi \right). \quad (A-32)$$

From Equations A-17, A-21, and A-32, the following expressions are obtained:

$$3(\bar{p} - p_a) = \frac{3B_a^2}{8\pi} + \frac{B_e^2 - \bar{B}_i^2}{8\pi} - \frac{\bar{B}_\phi^2}{8\pi} + \frac{2\Lambda(a)B_a^2}{8\pi}, \quad (A-33)$$

and:

$$\bar{p} - p_a = \frac{B_a^2}{8\pi} - \frac{B_e^2 - \bar{B}_i^2}{8\pi} - \frac{B_\phi^2}{8\pi} + \frac{2\Lambda(a)B_a^2}{8\pi}, \quad (A-34)$$

where a bar over a quantity denotes an average taken over the cross section of the plasma, ie.

$$\bar{g} \equiv \frac{1}{\pi a^2} \int_0^a \int_0^{2\pi} g(\rho, \phi) \rho d\phi d\rho .$$

The pressure balance equation is obtained by subtracting Equation A-34 from Equation A-33:

$$\bar{p} = p_a + \frac{\bar{B}_i^2 - B_a^2}{8\pi} = \frac{B_a^2}{8\pi} . \quad (\text{A-35})$$

$\Lambda(a)$ is obtained by substituting Equation A-35 in Equation A-33 or Equation A-34:

$$\Lambda(a) = \frac{8\pi (\bar{p} - p_a)}{B_a^2} + \frac{\ell_i}{2} - 1 , \quad (\text{A-36})$$

where:

$$\frac{\ell_i}{B_a^2} = \frac{\bar{B}_i^2}{B_a^2} \text{ is the internal inductance of plasma per unit length of torus.}$$

Equations A-32 and A-36 apply to magnetic surfaces of the configuration being considered. The expressions are to be applied to the plasma surface, ie, that magnetic surface outside of which the current density vanishes, and the magnetic field is determined by the expression:

$$\mathbf{B} = \nabla \Psi .$$

Dotting each side of the expression:

$$\vec{\nabla} \cdot \mathbf{B} = \nabla^2 \Psi = 0 .$$

Using the nomenclature of Figure A-1:

$$\begin{aligned} \nabla^2 \Psi &= \frac{1}{r\rho} \left[\frac{\partial}{\partial \rho} \left(\rho r \frac{\partial \Psi}{\partial \rho} \right) + \frac{\partial}{\partial \phi} \left(\frac{r}{\rho} \frac{\partial \Psi}{\partial \phi} \right) \right] \\ &= \frac{\partial^2 \Psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Psi}{\partial \rho} + \frac{1}{r} \frac{\partial r}{\partial \rho} \frac{\partial \Psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \Psi}{\partial \phi^2} + \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{r}{\rho} \right) . \end{aligned}$$

Since $r = R + \rho \cos \phi$:

$$\frac{\partial r}{\partial \rho} = \cos \phi, \text{ and } \frac{\partial}{\partial \phi} \left(\frac{r}{\rho} \right) = -\sin \phi.$$

$$\vec{B} = \nabla \Psi,$$

$$\frac{\partial^2 \Psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \Psi}{\partial \phi^2} = \frac{\sin \phi}{r} \frac{\partial \Psi}{\partial \phi} - \frac{\cos \phi}{r} \frac{\partial \Psi}{\partial \rho}. \quad (\text{A-37})$$

Using Equation A-32 and $\vec{B} \cdot \vec{n} = 0$ as the boundary conditions for Equation A-37, the following expressions are obtained:

$$B_\rho = \frac{\partial \Psi}{\partial \rho}, \quad B_\phi = \frac{\partial \Psi}{\rho \partial \phi}, \quad (\text{A-38})$$

where:

$$\Psi = \Psi_0 \phi + \Psi_1 \sin \phi.$$

Here:

$$\Psi_0 = B_a a, \quad (\text{A-39})$$

and:

$$\Psi_1 = \frac{a}{2R} B_a a \left[(\Lambda + 1/2) \left(\frac{a}{\rho} + \frac{\rho}{a} \right) + \frac{\rho}{a} \left(\ln \frac{\rho}{a} - 1 \right) \right]. \quad (\text{A-40})$$

Therefore, the field components are determined from Equations A-38 through A-40:

$$B_\phi = B_a \frac{a}{\rho} + B_a \frac{a}{2R} \left[\left(1 + \frac{a^2}{\rho^2} \right) (\Lambda + 1/2) + \ln \frac{\rho}{a} - 1 \right] \cos \phi. \quad (\text{A-41})$$

$$B_\rho = B_a \frac{a}{2R} \left[\left(1 - \frac{a^2}{\rho^2} \right) (\Lambda + 1/2) + \ln \frac{\rho}{a} \right] \sin \phi, \quad (\text{A-42})$$

Consider Figure A-3, for $\Delta \ll b$, then the angle, $\alpha \approx \phi$, and $b^2 = \rho^2 + \delta_0^2 + 2\rho\delta_0 \cos \phi$. To first order in $\left(\frac{\delta_0}{b} \right)$, the equation for the wall is:

$$\rho = b - \delta_0 \cos \phi,$$

Figure A-3. SHELL CROSS SECTION.

and the unit normal vector to the wall (pointing into the shell) is:

$$\vec{n} = \left(1, 0, -\frac{\delta_0}{h} \sin \phi \right).$$

The displacement, Δ , of the plasma with respect to the wall is determined from the condition that the normal component of the magnetic field must vanish at the wall; ie, $\vec{b} \cdot \vec{n} = 0$. Keeping only terms that are first order in $\frac{b}{R}$, $\frac{a}{R}$, and $\frac{\delta_0}{b}$:

$$\vec{B} \cdot \vec{n} = B_{\rho}^1(b) - B_{\omega}^1(b) \frac{\delta_0}{b} \sin \phi = 0,$$

where the superscripts explicitly show the order of the components (B_{ρ}^1 = Equation A-42, but $B_{\phi}^0 = B_a \frac{a}{b}$ only).⁽⁵⁾

$$\vec{B} \cdot \vec{n} = \frac{a}{2R} B_a \left[\left(1 - \frac{a^2}{b^2} \right) (\Lambda + 1/2) + \ln \frac{b}{a} \right] \sin \phi -$$

$$\frac{a}{b} B_a \frac{\Delta}{b} \sin \phi = 0.$$

Therefore:

$$\Delta = \frac{b^2}{2R} \left[\ln \frac{b}{a} + \left(\Lambda + 1/2 \right) \left(1 - \frac{a^2}{b^2} \right) \right]. \quad (\text{A-43})$$

Equation A-43 is Shafranov's classic equilibrium equation. If the volume contains an external vertical field, $B_{\perp}$, perpendicular to the plane of the torus, the effective boundary condition at the wall surface becomes:

$$B_{\rho} - B_{\phi} \frac{\Delta}{b} \sin \phi = B_{\perp} \sin \phi. \quad (\text{A-44})$$

In this case, the displacement becomes:

$$\Delta = \frac{b^2}{2R} \left[\ln \frac{b}{a} + \left(\frac{8\pi p}{B_a^2} + \frac{l_i}{2} - \frac{1}{2} \right) \left(1 - \frac{a^2}{b^2} \right) \right] - b \frac{B_{\perp}}{B_b}, \quad (\text{A-45})$$

where:

$$B_b = \frac{2I}{cb}.$$

The ratio $\frac{B_{\perp}}{B_b}$ is positive if the external field, $B_{\perp}$, is in the field produced by the current, J , at the outer side of the plasma.

There are two fundamental assumptions that are made when the shell is approximated as a perfect conductor: (1) the life of the plasma loop in the TOKAMAK is small in comparison to the skin time of the conducting shell, and (2) there are no gaps in the shell. The skin time of the conducting shell is that interval of time that the field penetrates the thickness of the shell. The skin time is defined as:

$$\tau_s = \frac{\mu_0 d^2}{2\eta}, \quad (\text{A-46})$$

where:

d = shell thickness

η = resistivity of the shell.

Therefore, if $t \ll \tau_s$ (where t is the plasma loop lifetime), the shell will provide the necessary equilibrium. Likewise, if the plasma lifetime considerably exceeds the life of leakage of the magnetic field through the wall of the conducting shell, the shell ceases to play a part in providing equilibrium for the plasma loop. This condition corresponds to the case when:

$$\tau_s \gg \frac{\mu_0 R d}{2\eta},$$

and the equilibrium must be attained solely by an externally applied field:

$$B_{\perp 0} = \frac{\mu_0 I_p}{4\pi R} \left(1 + \frac{8R}{d} + \beta_p + \frac{\ell_i - 3}{2} \right). \quad (A-47)$$

In the TOKAMAK devices in operation today, the lifetime of the plasma loop is an intermediate case where:

$$\frac{\mu_0 R d}{\eta} > \tau_s > \frac{\mu_0 d^2}{\eta}.$$

In the above range of values of t , the displacement is determined by the following equation:

$$\frac{d}{dt} \left[\frac{\mu_0 I_p}{2\pi b} \left(\frac{\Delta(t)}{b} - \frac{\Delta}{b} \right) \right] = \frac{1}{\tau_s} [B_{\perp 0} - B_{\perp}(t)]. \quad (A-48)$$

In the above equations, Δ is given by Equation A-43 and $B_{\perp 0}$ is given by Equation A-47.

APPENDIX B

MAGNETIC SURFACE OFFSET

Effects of the plasma pressure and current distribution on the magnetic surfaces in the plasma must be considered when determining the equilibrium conditions of the plasma.⁽⁵⁾ If the distribution of the quantities which characterize the equilibrium of the plasma in zeroth approximation (ie, in a cylindrical pinch) is known, the toroidal corrections to the plasma, magnetic field, and current density can be determined. In the discussion of these corrections, a coordinate system, $\rho \theta \phi$, in which the origin is the magnetic axis and not the center of the plasma will be used. This coordinate system is shown in Figure A-1. In this system, the equation for a magnetic surface with cross sectional radius, ρ_0 , is given by:

$$\rho = \rho_0 - \delta(\rho_0) \cos\phi. \quad (B-1)$$

Thus, the equation of the plasma surface ($\rho_0 = a$) is $\rho = a - \delta(a)\cos\phi$.

The toroidal correction (to the first order) to the poloidal field is $B_\phi(\rho, \phi)$. The toroidal correction to the azimuthal field on the magnetic surface $\rho = \rho_0 - \delta \cos\phi$ should be:

$$\delta B_\phi = B_\phi^1(\rho, \phi) + \frac{dB_\phi}{d\rho} \delta\rho = B_\phi^1(\rho_0, \phi) - \Delta(\rho_0) \cos\phi \frac{dB_\phi^0}{d\rho}. \quad (B-2)$$

From Equation A-32 it follows that:

$$\delta B_\phi(\rho_0, \phi) = B_\phi^0(\rho_0) \frac{\rho_0}{r} \Lambda(\rho_0) \cos\phi, \quad (B-3)$$

and, therefore:

$$B_\phi^1(\rho, \phi) = \left[B_\phi^0(\rho) \frac{\rho}{R} \Lambda(\rho) + \Delta(\rho) \frac{dB_\phi^0}{d\rho} \right] \cos\phi. \quad (B-4)$$

It is necessary to determine the toroidal corrections to ψ and p , where ψ is the poloidal flux through the XY plane of the torus and p is the plasma pressure. On magnetic surfaces, ψ and p are constant; therefore, $\delta\psi$ and δp both equal zero, thus:

$$\delta\psi = 0 = \psi'(\rho, \phi) + \frac{d\psi^0}{d\rho} \delta\rho \approx \psi'(\rho_0, \phi) + \frac{d\psi^0}{d\rho} \delta\rho, \quad (B-5)$$

which implies:

$$\psi'(\rho, \phi) \approx - \frac{d\psi^0}{d\rho} \delta\rho = \Delta(\rho) \frac{d\psi}{d\rho} \cos\phi. \quad (B-6)$$

Now, considering the equilibrium equations:

$$\begin{aligned}\nabla p &= \vec{J} \times \vec{B}, \\ \nabla \times \vec{B} &= \mu_0 \vec{J}, \text{ and} \\ \nabla \cdot \vec{B} &= 0,\end{aligned}$$

the poloidal field is defined as:

$$\vec{B}_p = B_\rho \vec{\epsilon}_\rho + B_\phi \vec{\epsilon}_\phi, \quad (\text{B-7})$$

where $\vec{\epsilon}_\rho$ and $\vec{\epsilon}_\phi$ are unit vectors.

Now, from Equation B-7:

$$\begin{aligned}\vec{B}_p = \nabla \times \vec{A} &= \frac{1}{\rho(R+\rho\cos\phi)} \left[\frac{\partial}{\partial\theta}(\rho A_\phi) - \frac{\partial}{\partial\theta}((R+\rho\cos\phi)A_\theta) \right] \vec{\epsilon}_\rho + \\ &\frac{1}{R+\rho\cos\phi} \left[\frac{\partial}{\partial\rho}((R+\rho\cos\phi)A_\theta) - \frac{\partial}{\partial\theta}A_\rho \right] \vec{\epsilon}_\phi.\end{aligned} \quad (\text{B-8})$$

Choosing $A_\rho = A_\phi = 0$ as a convenient formulation and remember that $r = R + \rho\cos\phi$,

$$B_\rho = -\frac{1}{r} \frac{1}{\rho} \frac{\partial}{\partial\phi} (r A_\theta),$$

and:

$$B_\phi = \frac{1}{r} \frac{\partial}{\partial\rho} (r A_\theta).$$

The quantity, Ψ , is defined as:

$$\Psi(r, z) = 2\pi r A_\theta(r, z) = \Psi(\rho, \phi) = 2\pi r A_\theta(\rho, \phi). \quad (\text{B-9})$$

Therefore:

$$B_\rho = -\frac{1}{2\rho r} \frac{1}{\rho} \frac{\partial\Psi}{\partial\phi}, \quad (\text{B-10})$$

and:

$$B_{\phi} = \frac{1}{2\pi r} \frac{\partial \Psi}{\partial \rho} . \quad (\text{B-11})$$

From Equation B-11:

$$B_{\phi}^0(\rho) = \frac{1}{2\pi r} \frac{\partial \Psi^0}{\partial \rho} ,$$

and, therefore:

$$\Psi^1(\rho, \phi) = 2 R B_{\phi}^0(\rho) \delta(\rho) \cos \phi . \quad (\text{B-12})$$

$$\text{Now, } \delta p = 0 = p^1(\rho, \phi) + \frac{dp^0}{d\rho} \delta \rho :$$

therefore:

$$p^1(\rho, \phi) = \delta(\rho) \frac{dp^0}{d\rho} \cos \phi . \quad (\text{B-13})$$

It should be noted that only the zero-order pressure and fields are necessary to compute the toroidal corrections. The magnetic surfaces to zero order are circular and concentric. In addition to B_{ϕ} , Ψ , and p , it is necessary to determine B_{ρ} , J_{ρ} , J_{α} , B_{θ} , J_{θ} , and $\delta(\rho)$.

Now, $\Psi = \Psi^0 + \Psi'$; and, since:

$$B_{\rho} = - \frac{1}{2\pi R} \frac{1}{\rho} \frac{\partial \Psi}{\partial \phi} ,$$

and:

$$\frac{\partial \Psi^0}{\partial \phi} = 0 ,$$

$$B_{\rho} = B'_{\rho}(\rho, \phi) ,$$

such that:

$$B'_{\rho}(\rho, \phi) = - \frac{1}{2\pi r} \frac{1}{\rho} \frac{\partial \Psi'}{\partial \phi} = \frac{\Delta(\rho)}{\rho} B_{\phi}^0(\rho) \sin \phi . \quad (\text{B-14})$$

The density components of the transverse current are expressed in complete analogy with the components B'_ρ and B'_ϕ of the transverse magnetic field, so that the toroidal corrections can be written by analogy with Equations B-4 and B-14.

Thus:

$$J'_\phi(\rho, \phi) = \left[J_\phi^0(\rho) \frac{\rho}{R} \Lambda(\rho) + \Delta(\rho) \frac{dJ_\phi^0}{d\rho} \right] \cos\phi, \quad (\text{B-15})$$

and:

$$J'_\rho(\rho, \phi) = \frac{\Delta(\rho)}{\rho} J_\phi^0(\rho) \sin\phi. \quad (\text{B-16})$$

In order to obtain the corrections to the toroidal plasma current density, J_θ , and magnetic field, B_θ , it is necessary to first write the corrections over the plasma surface ($\rho = \rho_0 - \Delta \cos\phi$).

From Equation A-23:

$$B_\theta(\rho, \phi) = B_\theta(\rho) \left(1 - \frac{\rho}{R} \cos\phi \right) \equiv B_\theta^0(\rho) \left(1 - \frac{\rho}{R} \cos\phi \right),$$

therefore:

$$\delta B_\theta \equiv B_\theta(\rho, \phi) - B_\theta^0(\rho) = - B_\theta^0(\rho) \frac{\rho}{R} \cos\phi. \quad (\text{B-17})$$

Defining:

$$\begin{aligned} \Delta^* &= \frac{\partial^2}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right), \\ \Delta^* \Psi &= - 2\pi r \mu_0 J_\theta = - 2\pi r \mu_0 \left[2\pi r \frac{dp}{d\Psi} + B \frac{dl}{d\Psi} \right] = \\ &= - 2\pi r \mu_0 \left[2\pi r \frac{dp}{d\Psi} + \frac{\mu_0}{4\pi r} \frac{dl^2}{d\Psi} \right]. \end{aligned} \quad (\text{B-18})$$

Let:

$$A(\Psi) = 2\pi \frac{dp}{d\Psi} \text{ and } B(\Psi) = \frac{\mu_0}{4\pi} \frac{dl^2}{d\Psi},$$

then, adding and subtracting $\frac{R^2 A(\Psi)}{R}$:

$$J_\theta = \left(\frac{R^2 A(\Psi) + B(\Psi)}{r} \right) + A(\Psi) \left(r - \frac{R^2}{r} \right). \quad (\text{B-19})$$

Letting $r = R + \rho \cos \phi$ to first order $\frac{\rho}{R}$:

$$J_\theta = \left(\frac{R^2 A(\Psi^0) + B(\Psi^0)}{R} \right) \left(1 - \frac{\rho}{R} \cos \phi \right) + A(\Psi^0) 2\rho \cos \phi,$$

or:

$$J_\theta \approx J_\theta^0(\rho) \left(1 - \frac{\rho}{R} \cos \phi \right) + 2 \frac{dp^0}{d\Psi} 2\rho \cos \phi.$$

Then, using:

$$\begin{aligned} 2\pi \frac{dp^0}{d\Psi} 2\rho \cos \phi &= 2\pi \frac{dp^0}{dp} 2\rho \cos \phi \left[\frac{d\Psi^0}{dp} \right]^{-1} \\ &= 4\pi \rho \cos \phi \frac{dp^0}{dp} \frac{1}{2\pi R B_\phi^0(\rho)} = \frac{2}{R B_\phi^0(\rho)} \frac{dp^0}{d\rho} \cos \phi, \end{aligned} \quad (\text{B-20})$$

the following is obtained:

$$\delta J_\alpha(\rho, \phi) = \left[\frac{\rho}{R} \frac{2}{B_\phi^0(\rho)} \frac{dp^0(\rho)}{d\rho} - J_\theta^0(\rho) \right] \cos \phi. \quad (\text{B-21})$$

From Equations B-17 and B-21:

$$B'_\theta(\rho, \phi) = \left[1 - B_\theta^0(\rho) \frac{\rho}{R} + \delta(\rho) \frac{dB_\theta^0(\rho)}{d\rho} \right] \cos \phi, \quad (\text{B-22})$$

and:

$$J'_\theta(\rho, \phi) = \left\{ \left[\frac{2}{B_\theta^0(\rho)} \frac{dp^0(\rho)}{d\rho} - J_\theta^0(\rho) \right] \frac{\rho}{R} + \delta(\rho) \frac{dJ_\theta^0(\rho)}{d\rho} \cos \phi \right\}. \quad (\text{B-23})$$

Thus, all toroidal corrections are expressed directly in terms of this displacement, $\Delta(\rho)$. In turn, this displacement can be expressed as an integral over the plasma cross section of $\Delta(\rho)$ which is called the "asymmetry coefficient" of the poloidal field. To understand this relationship, $\vec{B}$ must be expressed in terms of:

$$B_\phi = \frac{1}{2\pi r} \frac{\partial \Psi}{\partial \rho} = \frac{1}{2\pi r} \left(\frac{\partial \Psi^0}{\partial \rho} + \frac{\partial \Psi^1}{\partial \rho} \right) = \frac{1}{2\pi r} \left[2\pi R B_\phi^0(\rho) + \frac{d}{d\rho} \left(2\pi R B_\phi^0(\rho) \delta(\rho) \cos\phi \right) \right],$$

and, expanding $\frac{1}{r} \approx 1 - \frac{\rho}{R} \cos\phi$:

$$B_\phi^1(\rho, \phi) = -B_\phi^0(\rho) \frac{\rho}{R} \cos\phi + \frac{d}{d\rho} B_\phi^0(\rho) \delta(\rho) \cos\phi. \quad (B-24)$$

Comparing Equations B-4 and B-24:

$$B_\phi^0(\rho) \frac{\rho}{R} \Lambda(\rho) + \delta(\rho) \frac{dB_\phi^0}{d\rho} = -B_\phi^0(\rho) \frac{\rho}{R} + \delta(\rho) \frac{dB_\phi^0}{d\rho} + B_\phi^0(\rho) \frac{d\delta}{d\rho}.$$

Therefore, for $\Lambda(\rho) = \frac{\bar{p}(\rho) - p^0(\rho)}{\frac{B_\phi^0(\rho)^2}{2\mu_0}} + \frac{l_i(\rho)}{2} - 1$:

$$\frac{d\delta}{d\rho} = \frac{\rho}{R} [\Lambda(\rho) + 1] d\rho^1,$$

and:

$$\delta(\rho) = \int_0^\rho \frac{\rho^1}{R} [\Lambda(\rho^1) + 1] d\rho^1. \quad (B-25)$$

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ACKNOWLEDGEMENTS

The author wishes to express his appreciation to J. M. Bailey, Electrical Engineering Department, The University of Tennessee, for his continual advice and guidance during the course of this work. Special thanks are due R. J. Colchin, Oak Ridge National Laboratory, for his unhesitating assistance in the magnetic field and plasma physics areas of the work. The author also appreciates all the assistance and cooperation that were provided by the ORMAK experimental and theory groups of the Thermonuclear Division, Oak Ridge National Laboratory, and R. E. Textor of the Computer Science Division, Union Carbide Corporation-Nuclear Division.

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