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NONLINEAR VIBRATION

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MASTER

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ABSTRACT

This memorandum introduces nonlinear vibration at a fundamental level and compares it with linear vibration. The basic intentions of this memorandum are: (1) to acquaint test personnel with nonlinear vibration phenomena and suggest methods to be used in testing a nonlinear system, collecting and analyzing the data, and interpreting the results for presentation; and (2) to enable the project personnel requesting the tests to understand and utilize the data and analyses they receive in the report.

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LIST OF SYMBOLS

m = mass

F_R = restoring force

$|X|$ = absolute response amplitude

f = frequency

F_n = input force amplitude

C = damping constant

ω = circular frequency = $2\pi f$

$f(X)$ = function of displacement X

t = time

g = acceleration ratio = $\frac{a}{32.2}$

a = acceleration response

X = displacement

$\dot{X} = \frac{dx}{dt}$ = velocity

$\ddot{X} = \frac{d^2x}{dt^2}$ = acceleration

Definitions

Critical Frequency -- Any frequency at which vibration causes structural failure or malfunction of the test item. It may or may not be the same as the resonant frequency.

Distortion -- A frequency (or frequencies), other than the fundamental, that is present in a signal.

Bandpass Filter -- Attenuates frequencies above and below the specified bandwidth of the filter.

Forced Vibration -- Vibration under the influence of an external force.

Free Vibration -- Vibration not under the influence of an external force.

Fundamental Frequency -- The particular frequency of the external force.

Fundamental Response -- That portion of the response that has the frequency of the fundamental frequency.

Resonant Frequency -- That frequency at which an increase or decrease in the frequency of excitation causes a decrease in the fundamental response of the test item.

NONLINEAR VIBRATION

Introduction

Everyone involved in vibrations should be aware of the limitations in using linear vibration theory and should be capable of detecting, understanding, and analyzing nonlinear vibration when it occurs. As will be shown, linear vibration theory can be a hindrance rather than an aid if used when vibration response of a test item is predominately nonlinear.

Beyond a doubt, the foundation of present vibration testing is linear vibration theory. In almost every case a vibration test is specified, the response data are analyzed, and the results are interpreted with the assumption that the test item has linear response characteristics. In the majority of these cases, this assumption is justified since vibration tests have historically shown that linear theory is closely approximated. However, there will always be tests where nonlinear response characteristics are predominant. In these cases, linear theory is an insufficient tool since it neither predicts nor implies the nonlinear vibration phenomenon.

The basic intentions of this memorandum are:

1. To acquaint test personnel with nonlinear vibration phenomenon and suggest methods to be used in (a) testing a nonlinear system, (b) collecting and analyzing the data, and (c) interpreting the results for presentation.
2. To enable project personnel requesting the tests to understand and utilize the data and analyses they receive in the report. This memorandum does not offer solutions to design problems which may be encountered in systems design.

In the Discussion section, nonlinear vibrations are introduced by considering the restoring force characteristics of vibration systems. For those interested, a mathematical introduction is presented in Appendix I.

Discussion

Restoring Force Characteristics

In this presentation the exciting force will always be assumed sinusoidal or zero, and the damping will be assumed constant or negligible. Thus, the only nonlinear factor will be the spring force.

Consider the linear spring-mass system shown in Figure 1(a). If the amount of force required to displace the mass (m) from the position X_0 to any number of positions $X_1, X_2, X_3, \dots, X_n$ were measured, and a plot of "restoring force" (F_R) versus "displacement" (X) were constructed, the restoring force curve would be as shown in Figure 1(b).

Figure 1(a). Simple Spring Mass System

Figure 1(b). Restoring Force Curve for System in 1a

Figure 1(b) simply shows the force (F_R) which is acting on the mass (m) to return it to its static equilibrium position. For example, it requires F_1 to hold the mass at X_1 .

It is the characteristics of this restoring force curve and the displacement of the mass encountered during a vibration test that determine the linearity or nonlinearity of the vibration response. A linear response during vibration will result when the restoring force curve is straight (Figure 1(b)). A nonlinear vibration response will result when the restoring force curve is other than a straight line. For example, consider the dual-spring mass system shown in Figure 2(a), and its respective restoring force curve shown in Figure 2(b).

If the mass is pulled down to X_1 , only the spring 1 will produce a restoring force. If pulled from X_1 on to X_2 , the restoring force acting on m will increase to that produced by both springs 1 and 2. Figure 2(b) shows the resulting restoring force curve. Now, the vibration response would be nonlinear if the test were conducted over the range of X_2 to X_{-2} since the restoring force curve is not a straight line over this range. If the deflection during a test were not to exceed X_{-1} to X_0 to X_1 , the vibration response would be linear since this portion of the restoring force curve is a straight line.

In summary, the vibration response will be linear or nonlinear, depending upon the nature of the restoring force curve and the deflection encountered by the mass during vibration. Linear and nonlinear systems will thus have a restoring force curve which is straight or bent, respectively.

Figure 2(a). Dual Spring Mass System

Figure 2(b). Restoring Force Curve for System in 2a

Hardening and Softening Systems

Nonlinear systems can be further broken down into hardening and softening systems. The restoring force curves of a linear system, hardening system, and softening system are shown in Figures 3(a), 3(b) and 3(c), respectively.

Figure 3(a). Linear

Figure 3(b). Hardening

Figure 3(c). Softening

Restoring Force Curves for Linear, Hardening, and Softening Systems
(Free Vibration)

Response Amplitude Versus Frequency

The vibration behavior of linear and nonlinear systems can be represented by curves of response amplitude versus frequency. Such curves simply show the response experienced by the mass at one or more frequencies of vibration. First, consider a free vibration. A good example of free vibration would be to pull the mass in Figure 1(a) to X_3 and then release it. The motion experienced after releasing the mass is termed free vibration. Experiments on linear and nonlinear systems would result in the response amplitude curves shown in Figures 4(a), 4(b) and 4(c) for the linear, softening, and hardening systems, respectively.

Figure 4(a). Linear System Figure 4(b). Softening System Figure 4(c). Hardening System

Now, consider the response curves for a forced input to the system. The force will be assumed to be a sinusoidal exciting force throughout this discussion. Figures 5(a), 5(b) and 5(c) show the respective curves for a linear softening and hardening system. Curves are shown for force input levels of F_1 , F_2 , F_3 , where the lower subscript represents the lower force amplitude.

Figure 5. Response Curves for Undamped Systems

Damping Effects

Damping, which will always be assumed to be linear, has the effect of lowering the response amplitude at the resonant frequency. Thus, the damping effects on the responses shown in Figure 5 would be as shown in Figure 6.

Figure 6. Response Curves for Linearly Damped Systems

Linear and Nonlinear Response Characteristics

A large majority of the vibration test results exhibit linear response characteristics, even though practically all physical systems have nonlinear response curves (either hardening or softening). Some of the reasons why the test results appear to be linear are: (1) restoring force curves are nearly linear; (2) the deflection of the mass covers only a small portion of the response curve, and this portion of the curve may be nearly linear; (3) damping is high; and (4) the force input is low. However, there will always be cases in vibration testing where the nonlinear response characteristics are predominant. Because of the scarcity of this type of results, the phenomenon may not be recognized, and/or the meaning of the results may be misinterpreted. In these cases, linear vibration theory is insufficient to explain the nonlinear phenomenon, since it is not predicted or implied in linear theory. The response characteristics of linear and nonlinear vibration are outlined below.

Linear Response Characteristics

1. Consider the free vibration of the linear system shown in Figure 4(a). As can be seen, free vibration of a linear system will always occur at the resonant frequency. This frequency is independent of the amplitude of response. Damping will cause the vibration motion to fade out, but, as long as motion exists, it will occur at the resonant frequency.
2. Consider forced vibration of a linear system shown in Figure 5(a). The resonant frequency of the system, when subjected to a sinusoidal force, is independent of the amplitude of response, and hence the magnitude (F_0) of the force input.

3. Consider a damped linear system subjected to a sinusoidal force input (Figure 6(a)). If a vibration test consisted of a frequency sweep from a to b and/or b to a, the results would show the resonant frequency to be at the same value regardless of the direction of vibration sweep.
4. In a linear system, either free or forced, the degree of linear damping influences the amplitude of response, but does not alter the resonant frequency of the system. The effect of different degrees of damping (C_n) on a linear system subjected to a harmonic force F_0 are shown in Figure 7.

Figure 7. The Response Amplitude as a Function of Linear Damping of Various Amounts

Nonlinear Response Characteristics

1. Consider the free-nonlinear systems vibration response shown in Figures 4(b) and 4(c). The free vibration frequency (resonant frequency) is dependent upon the response amplitude. Therefore, as the response dampens out, the frequency will change. This change will be increasing for softening systems, and decreasing for hardening systems. This is shown in Figure 8.

Figure 8. Time History of Damped-Free Vibration Nonlinear Systems

2. Consider the nonlinear-forced vibration response curves for hardening and softening systems (Figure 9). The two curves are for different force inputs F_1 and F_2 . The resonant frequency is a function of the response amplitude, which in turn is a function of the force input amplitude. Resonant frequencies for the different forces are shown as points a and b. For a hardening system, it can be seen that the resonant frequency is a direct function of the force input magnitude, whereas the resonant frequency is inversely a function of a force input magnitude for a softening system.

Figure 9. Response Curves for Nonlinear-Forced Vibration Systems

3. The direction of vibration cycling has a direct influence on the indicated resonant frequency, and the corresponding response amplitudes. For example, consider the response curves of the hardening and softening system shown in Figure 10. For the hardening system shown in (a), an increasing frequency sweep from a to d would have the following results. Following the response curve, at point b the response amplitude has to jump to point c. This jump is required, since the curve from b to c is impossible to follow for an increasing fundamental frequency. From point c the curve is followed to d. The resulting data would show the resonant frequency and maximum amplitude to be at point b.

Now, consider a test sweep from d to a on the same response curve. The curve would be followed from d to e, and would then jump from e to f. After the jump, the curve would be followed from f to a. Data from this test would show the resonant frequency and maximum amplitude to be at point f. It can thus be seen that the direction of sweep results in unequal values for the resonant frequency and the respective response amplitudes are unequal. The same reasoning, applied to the softening system (Figure 10(b)) would result in resonance points at c and e for the increasing and decreasing frequency sweeps, respectively.

Test Specifications for Nonlinear Systems

Physically, a majority of the systems subjected to vibration testing are nonlinear to some degree. However, the degree of nonlinearity is usually minor and the results are linear, or nearly so, in character. An example of a nonlinear response characteristic found in most vibration tests is the shifting of the resonant

Figure 10. Response Curves for Nonlinear-Forced Vibration

frequency for different force input levels. However, this frequency shift is normally small and hence neglected. Therefore, test specifications and data analysis, based on a linear system, are normally justified. However, when a system is known to have, or found to possess, a high degree of nonlinearity, the test should be treated with caution. For example, it was pointed out previously that the resonant frequency and response are a function of the input force amplitude. To find the resonant frequency that would occur under "in use" conditions, the force input levels during a test should be quite similar to those found in the field. It is conceivable that a test item sees several types of vibration environment during use, so it may be necessary to test the unit at several input levels if it were desired to determine the response characteristics under the different environmental conditions. Also, frequency sweeps should be run from both directions in order to determine the two values of the resonant frequency. In summary, the force input and the direction of vibration sweep should be specified for a nonlinear-system test only after the above points have been considered. Analysis of the nonlinear data will be discussed later.

Fundamental, Subharmonic, and Superharmonic Responses

A linear, single degree of freedom system that is subjected to an undistorted input will have a response at the exciting frequency only. A response containing other than the fundamental response results when a distorted input contains frequencies other than the fundamentals that excite the system. Typical causes of input distortion are the power amplifier, rattling of the test item and/or vibration fixture (such as loose bolts), flexure resonance, and nonlinearity of the shaker system.

It can be shown, both theoretically and experimentally, that steady-state responses of a nonlinear system can occur at the fundamental frequencies and at multiples and submultiples of this frequency. This is the case for either a pure or distorted input signal.

At this point, the following definitions will be included for future reference. These definitions apply only when the input is undistorted.

Fundamental response--the response that occurs at the input frequency.

Subharmonic response--the response that occurs at $\frac{\omega}{n}$, where ω is the input frequency and n is some integer.

Superharmonic response--the response that occurs at $n\omega$, where ω is the input frequency and n is an integer.

Subharmonic resonance--the system's resonant frequency $\frac{\omega}{n}$ is excited by the fundamental frequency ω .

Superharmonic resonance--the system's resonant frequency $n\omega$ is excited by the fundamental frequency ω .

There is no simple-physical explanation for the phenomenon of subharmonic and/or superharmonic responses in nonlinear vibration systems. The only explanation found during a recent literature survey is as follows: "The nonlinear character of the restoring force provides a mechanism for the transfer of energy from the driving frequency to the higher harmonics of the fundamental of the motion." Subharmonic response was explained as follows: "If the system is being forced by a forcing function, the frequency of which is in the proper range, and a disturbance to the system allows a 'free' vibration to occur, the nonlinear character of the restoring force allows the forcing frequency to lock in with one of the higher harmonics of the 'free' vibration and thus to feed energy into the fundamental of the motion. Thus it is possible for a forcing function to sustain an oscillation, the frequency of which is a submultiple of its own."

Breakdown of Nonlinear Responses

Mathematically, the solution of the nonlinear form of Equation 1A (Appendix A) may be of the general form

$$\text{Total Response} = f(X) = \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t) \quad (1)$$

where A_n = constants, some of which may be zero
 B_n = constants, some of which may be zero
 $n\omega$ = subharmonic, fundamental, and superharmonic frequencies.

Equation 1 can be interpreted as follows. The total response $f(X)$ which is periodic is nothing more than the sum of the individual responses occurring at one or more of the submultiples and/or multiples of the fundamental input frequency. This sum in Equation 1 is called a Fourier series. If, during a nonlinear vibration test, the total vibration response of the system is observed on an oscilloscope, the wave shape will be highly distorted and periodic. This wave shape supports the fact that some form of the Fourier series represents the total response. A function which can be represented by a Fourier series does not necessarily contain all the sine and cosine terms, since some of the constants can be zero.

Equation 1 introduces the fact that subharmonics and superharmonics are present in the total response. Subharmonics may occur, but the damping has to be very low. Damping can completely eliminate subharmonics; therefore, not all nonlinear systems experience them. On the other hand, superharmonics are always present, and damping merely reduces the amplitude of the superharmonics. It has been found that subharmonics and superharmonics will not die out with time when a nonlinear system is vibrated at some specific frequency.

*Burgess, John C., Nonlinear Phenomena in Environmental Engineering, Inst. of Environmental Sciences, 1963 Proceedings, p. 115.

Examples of Nonlinear Responses Found in Vibration Tests

Two highly nonlinear systems were recently vibration-tested at Sandia. The results, which illustrate the response characteristics outlined previously, are discussed below.

A weight, setting on rubber, was vibrated while loose on the shaker table. The nonlinear characteristics were due to the weight and rubber bouncing on the table.

Both subharmonic and superharmonic responses were experienced during this test. The subharmonic response was found to be a subharmonic resonance. This means that the resonant frequency ω was being excited when the input frequency was 2ω . The amplitudes of the subharmonic, fundamental, and superharmonic portions of the total response were distinguished and compared. The subharmonic response, since it was at the resonant frequency, had an amplitude nearly twice as large as the fundamental response. The superharmonics had an amplitude which was less than the fundamental response amplitude. These superharmonics were not investigated in detail.

The approximate restoring force curve for this nonlinear system was as shown in Figure 11.

Figure 11. Nonlinear Vibration System

Figure 11(b) shows the restoring force curve while the system bounced on the table. At high frequencies, the system ceased to bounce and the restoring force curve was as shown in Figure 11(c). The displacement of the test item in this case was not great enough to cause the nonlinear-type response. Thus, after bouncing, only a response at the fundamental frequency was seen (undistorted response).

A weight was vibrated while setting loosely on the table. The nonlinear characteristics were due to the weight bouncing on the table.

There were no subharmonic responses during this test. Therefore, considerable time was spent analyzing the superharmonic characteristics. At a 15-cps fundamental, the weight was resonating. The resonance was not a superharmonic resonance. An analysis of the distorted response at resonance was made; the results are shown below. All values represent zero-to-peak measurements.

Total Response	= 15.0g	
15 cps Response	= 4.8g	→ resonant frequency
30 cps Response	= 2.8g	→ 1st superharmonic
45 cps Response	= 2.5g	→ 2nd superharmonic
60 cps Response	= 1.9g	→ 3rd superharmonic
75 cps Response	= 1.4g	→ 4th superharmonic
90, 105, 120...	< 1.0g	→ 5th, 6th, ... nth superharmonics

Analyses of the total response were also made at 10, 20, and 25 cps. All these frequencies had a total response which consisted of the fundamental and superharmonics similar to that outlined above. This type of breakdown serves well to illustrate the validity of the Fourier series shown in Equation 1, where the A_n^s and B_n^s are the acceleration amplitudes, and the ωn^s are the corresponding frequencies of the fundamental and the various harmonics.

Two points of interest are brought out by the above analysis: (1) Odd and even superharmonics can occur simultaneously in nonlinear responses. (2) Superharmonics may occur at any fundamental frequency in the nonlinear range.

The restoring force curve for this nonlinear system is approximated below (Figure 12).

Figure 12. Nonlinear Vibration System

Figure 12(b) represents the response curve when the weight bounced on the table. At high frequencies (>100 cps), bouncing ceased, and the displacement decreased so that the response curve was linear enough (Figure 13(c)) to result in linear response characteristics.

Subharmonic resonance at one-third the fundamental frequency was observed during a vibration test.

During a component-vibration test conducted at Sandia (T-22809), the component was found to have nonlinear response characteristics. The resonant frequency at 2.0g input was found to be 145 cps. At this frequency, both the fundamental and superharmonic responses were experienced. The fundamental response was 1.5 times larger than the highest superharmonic response.

At 483 cps fundamental frequency (5.0g input), resonance occurred as a one-third subharmonic of 160 cps. The 160 cps response amplitude was about 1.5 times the 483 cps fundamental response amplitude. Note that the resonant frequency differs by 15 cps in these two cases. This is due to different force input levels at 145 and 483 cps, which suggests that it is a hardening system.

At the 483 cps fundamental, there were no superharmonics detected on the Panoramic Analyzer. This was probably due to a low, damped response amplitude of the superharmonics.

Analysis of Nonlinear Vibration Response Data

There are many facets of nonlinear response characteristics which may be of interest when analyzing and interpreting the test data. The methods of gaining this information are outlined below.

Magnetic Tape Machines

Magnetic tape machines are used to record the output signals of the accelerometers which are located at points of interest on the test item. A diagram of the test setup is shown in Appendix B. Through the use of tape machines, permanent electrical records of the test run are possible. In order to use electrical instruments to study the nonlinear phenomenon occurring during a test, it is essential that the test data be recorded.

Oscilloscope

An oscilloscope can be used to visually monitor the vibration responses during a test. In some cases, nonlinearity can be detected during a test if the oscilloscope is used. For example, it may be possible to observe frequency shifts or amplitude changes in the response. These are nonlinear characteristics previously discussed. The instrumentation diagram for the oscilloscope is shown in Appendix B.

Filtered X-Y Plots of g Versus Frequency

Filtered (narrow band pass) plots of acceleration response (g) versus frequency can be used to separate the fundamental frequency response from the total response. The bandwidth of the filter should be narrow enough so that the subharmonic and superharmonic responses are attenuated. This is especially true at the lower frequencies. If part or all of a subharmonic and/or a superharmonic response lies within the filter bandwidth, the true fundamental response will not be shown. At higher fundamental frequencies, where the Δf between responses is large, a narrow band pass filter would show the fundamental response only. The instrumentation diagram for filtered XY plots is shown in Appendix B.

Unfiltered XY Plots of g Versus Frequency

The unfiltered XY plots will show the total acceleration response of the system at any fundamental frequency. This plot can be compared with the filtered plots to determine the percent of the total response caused by subharmonic and/or superharmonic responses. The instrumentation diagram for obtaining unfiltered plots is shown in Appendix B.

Oscillograph Records

These records provide a good, but old-fashioned, means of studying the wave shape and amplitude of the responses in detail. Unfiltered and filtered oscillograph records can be obtained for the purpose of studying total and fundamental responses, respectively.

Oscillograph records were recently used to study in detail the following response characteristics:

1. Total response waveshape and amplitude.
2. Subharmonic resonance which was first discovered from the oscillograph record.
3. Instability of the response when the jump phenomenon occurred (Figure 10). The response was seen to fluctuate between two amplitudes.

These oscillograph records also furnish pictures of the response to be used in reports. The instrumentation for this setup is shown in Appendix B.

A more convenient and faster way of accomplishing this would be to use a Panoramic Analyzer to obtain a complete breakdown of the total response. Pictures of the waveshape can be taken from the oscilloscope by a camera.

Tape Loop

A complete frequency and amplitude (Fourier) breakdown of the total response can be obtained in the following way. A tape loop is cut from the tape recording at any fundamental frequency of interest. This tape loop is then played back on the tape machine. By sweeping through the frequency range with a narrow band pass filter, the amplitudes and frequencies of the fundamental, subharmonic, and superharmonic responses can be plotted as frequency versus maximum g. The instrumentation is shown in Appendix B.

Oscilloscope, High- and Low-Pass Filter, and Voltmeter

To gain a quick insight into the characteristics of the response, the taped record can be played back, and the waveshape of the response can be observed on the oscilloscope. Amplitudes and frequencies of any portion of the total response can be studied with the oscilloscope, high- and low-pass filters, and voltmeters.

In a recent test, high- and low-pass filters were used to separate the subharmonic and fundamental responses from the total response. Once these were separated, as could be determined on the oscilloscope, the amplitudes of these responses were determined on a voltmeter. The instrumentation for this setup is shown in Appendix B.

As a note of caution, the voltmeter should not be used when trying to measure accurately the amplitude of a distorted response, since it is not 0.707 times the peak amplitude in this case. For true peak measurements of a total (distorted) nonlinear response, a peak meter should be used.

Limitations of the Log Converter in g Versus Frequency Plotting

With the foregoing procedures, one particular note of caution should be mentioned. The log converter which is used in obtaining XY plots of g versus frequency is a half-wave peak rectifier. Nonlinear responses can be such that high acceleration peaks occur on one side of the zero line only. Should this happen, half-wave rectification may miss the high peaks, thus resulting in an incomplete analysis of the true response. Therefore, an oscilloscope or oscillograph record should be studied to predetermine the response that the log converter should detect. As an example, a recent nonlinear system was vibration-tested. The total response, as seen on the oscilloscope and oscillograph record, had the waveshape shown in Figure 13(a). As it turned out, the log converter rectified the high half of the response curve. However, had the other side been sensed by the log converter, the error in measuring the maximum response would have been large. The potential error is best exemplified by the results recently obtained by Dr. Kuehl.* In this study, the waveshape shown in Figure 13(b) was introduced. The signal was then inverted as shown in Figure 13(c). Due to the half-wave rectification, there was a 15-decibel difference between the two amplitudes shown on the XY plot.

Figure 13. Sensing Characteristics of Log Converter

Test Specifications and Data Interpretation

Input to the Test Item

Since the resonant frequency of nonlinear systems is a function of the input force magnitude, the input level should be considered during data interpretation. Also, depending upon the degree of nonlinearity, it may be advisable to test over some frequency range at more than one input level. For example, if one field environment shows that the test should be at 2.0g between 10 and 500 cps, and another environmental condition calls for a 5.0g input between 10 and 500 cps, the higher input level would normally be specified for a vibration test. However, in nonlinear systems, the resonant frequency may be quite different for the two input levels. It may therefore be advisable to test at both inputs.

*Memorandum from D. R. Kuehl, 7324-2, to C. T. Schafer, 7324-2, dated June 12, 1960.

When the resonant frequency and/or the critical frequency are a function of the input magnitude, the different responses and frequencies may be valuable in determining (a) the operational and/or structural integrity of a system and/or (b) the environment of internal components for component test specifications.

Direction of Vibration Sweep

The resonant frequency and the response amplitude at resonance have been shown to be a function of the direction of vibration sweep. Therefore, if the system being tested is thought to be nonlinear, a sweep should be made in both directions. This would not only locate the two resonant points, but it would also subject the unit to both the increasing and decreasing amplitude jumps which occur in the region of instability. This jump phenomenon, discussed previously, is shown in Figure 10.

Input Accelerometer Control Signal

The test specimen should be vibrated at the specified acceleration level by controlling the amplitude of the fundamental frequency. Based on the previous discussion, it is conceivable that an unfiltered control signal would be highly distorted, because of the vibration exciter, the test fixture, and the test item. Therefore, a filter located between the input control accelerometer and the control sensing device is essential. A variable low-pass or band-pass filter is recommended.

Data Interpretation

Filtered and Unfiltered Plots -- Data analysis of nonlinear system tests should include the total (unfiltered) response and a Fourier breakdown of the total response. The total response shows the acceleration being experienced at any time, but it tells nothing of the frequencies involved. The Fourier breakdown is necessary to determine the subharmonic, fundamental, and superharmonic frequencies at which the system is vibrating simultaneously.

For example, consider the test results on the nonlinear system previously discussed. At resonance, the total response of 15. g consisted of the following Fourier terms. All are zero-to-peak measurements.

Total Response	= 15.0g
15 cps	= 4.8g (resonant and input frequency)
30 cps	= 2.8g
45 cps	= 2.5g
60 cps	= 1.9g
75 cps	= 1.4g
90, 105, 120 ...	< 1.0g

Assume that the system test was run to determine the vibration experienced by an internal component, from which a component vibration test was to be specified. This test specification should not call for 15.0g at 15 cps fundamental. The input at 15 cps should be based on the fundamental response of 4.8g. The only valid way to determine the effect of the 15.0g total response during the component test would be to introduce a complex input, consisting of the Fourier terms, to the shaker table.

Maximum Response at a Particular Frequency -- Because of the phenomenon of subharmonic and superharmonic responses, a vibration response will occur at frequency ω_1 when ω_1 is the fundamental frequency and also when ω_1 is a subharmonic or superharmonic of fundamental frequencies ω_n . During a vibration test where the input level changes over the frequency test range, the ω_1 response at the fundamental frequency ω_1

may be more or less than the ω_1 response at some ω_n fundamentals. For example, assume that the input level changes from 5.0g to 2.0g at 100 cps. Now, at ω_1 fundamental of 80 cps (5.0g input) the ω_1 response may be 10.0g. A superharmonic of ω_1 may occur at $\omega_2 = 160$ cps, and be 5.0g in magnitude. Now, at ω_2 fundamental of 160 cps (2.0g input) the fundamental response may be 3.0g. Thus, unless a complete analysis of the response at the ω_1 and ω_2 fundamental frequencies is made, the maximum ω_2 response would be missed. As pointed out above, this misinterpretation could result in an untrue component test.

Responses Outside the Frequency Test Range -- Because of subharmonic and superharmonic phenomena, it is possible that vibration response of a nonlinear system may contain responses above and/or below the specified test range. For example, a nonlinear system was subjected to a vibration test from 10 to 500 cps, the object being to determine the response experienced by an internal component. The system had a resonant frequency at 5 cps, which occurred at 10 cps fundamental in the form of a subharmonic. Therefore, the resulting component test was specified to include vibration at 5 cps.

Also, superharmonic responses of a high-frequency fundamental may exist outside the specified test range. Thus, it may be necessary to specify that a component test be designed to include these frequencies since they may be critical.

Locating the Source of Nonlinearity

In some instances, it may be desirable to locate the source of the nonlinear vibration response. If the source is not obvious by inspection, the alternative would be to locate accelerometers throughout the system and observe their individual response characteristics. Provided that feedback during vibration is not too complex, it may be possible to pinpoint the nonlinear source.

Complex-Wave Vibration Tests on Nonlinear Systems

A complex-wave vibration test is one where the test item is vibrated at several frequencies simultaneously. The rms acceleration input level within each bandwidth is preset with a peak-notch equalizer before the test is conducted. Because of the nature of nonlinear responses, it is conceivable that the test setup will be very difficult, since the acceleration level in any bandwidth may be due to fundamental, subharmonic, or superharmonic responses or a combination of these. Thus, to preset the acceleration level in a particular bandwidth, it may be necessary to adjust the input energy in this bandwidth as well as in others.

Analysis of the response data will also be difficult, since more than one exciting frequency is always present simultaneously. The response amplitude will be known, but it will be almost impossible to determine whether the response is a fundamental, subharmonic, superharmonic, or a combination.

If an automatic equalizer is used, it may not always be possible to control the input within each bandwidth. This is because the automatic equalizer provides no way of controlling the subharmonic and/or superharmonic responses of a fundamental frequency.

In summary, if a test item is predominately nonlinear in its response characteristics, it is recommended that a sinusoidal vibration test be specified.

APPENDIX A LINEAR AND NONLINEAR VIBRATION

The equation of motion for a single-degree-of-freedom system can be represented by the following differential equation:

$$D(\ddot{X}) + E(\dot{X}) + F(X) = G(t) \quad (1A)$$

where

$$\begin{aligned} D(\ddot{X}) &= \text{inertial force} \\ E(\dot{X}) &= \text{damping force} \\ F(X) &= \text{restoring force (spring force)} \\ G(t) &= \text{external exciting force} \\ X, \dot{X}, \ddot{X} &= \text{displacement, velocity, and acceleration of the mass } m \\ t &= \text{time.} \end{aligned}$$

Several forms of Equation 1A will be considered during this discussion. However, the equation can first be simplified by assuming that the inertial force is linear [$D(\ddot{X}) = m\ddot{X}$], and that the external force is sinusoidal. Thus, Equation 1A reduces to

$$m\ddot{X} + E(\dot{X}) + F(X) = F_0 \sin \omega t \quad (2A)$$

where

$$\begin{aligned} m &= \text{constant of proportionality (mass)} \\ F_0 &= \text{maximum amplitude of external force} \\ \omega &= \text{angular frequency of the external force } (\omega = 2\pi f) \end{aligned}$$

There are now two terms in Equation 2A which can be either linear or nonlinear. If c and k are respectively the damping and spring constants of proportionality, the linearized form of Equation 2A becomes

$$m\ddot{X} + c\dot{X} + kX = F_0 \sin \omega t \quad (3A)$$

An example of linear damping is viscous damping at low mass velocities, where the damping force is defined by $E(\dot{X}) = c\dot{X}$. At higher velocities, the damping force is more accurately expressed by $E(\dot{X}) = c(\dot{X})^2$, which is nonlinear. Another way to explain linear and nonlinear damping is to consider a plot of damping force versus velocity. Such a plot is shown in Figure A-1, where the curves for $E(\dot{X}) = c\dot{X}$ and $E(\dot{X}) = c(\dot{X})^2$

are shown. Note that the linear damping is represented by a straight line. Any linear damping term has a constant slope, whereas all nonlinear terms have variable slopes.

Figure A-1. Graphical Representation of Linear and Nonlinear Damping

A type of nonlinear damping which is usually apparent in vibration tests at Sandia is structural damping. Structural damping increases with increasing stress. The test results show that the response ratios (response/input) are inversely proportional to the input level.

Restoring force is linear when $F(X) = kX$, where k is the spring constant of proportionality. An example of a nonlinear restoring force, the form of which is used in a majority of the theoretical work, is $F(X) = k(X + \alpha X^3)$. These expressions are shown graphically in Figure A-2. Again note that the linear restoring force is represented by a straight line (constant slope). A constant slope indicates that $\frac{dF(X)}{dX} = \text{constant}$.

Figure A-2. Graphical Representation of Linear and Nonlinear Restoring Forces

Some classical examples of linear and nonlinear spring-mass systems are shown in Figure A-3. Their respective restoring force curves are shown in Figure A-4.

Figure A-3. Linear and Nonlinear Spring-Mass Systems

Figure A-4. Respective Restoring Force Curves for Figure A-3 Spring-Mass Systems

The restoring force characteristics of any vibration system can be represented by a restoring force curve similar to those shown in Figure A-4. Depending upon the slope characteristics of these curves, a vibration system can be classified as linear, hardening, or softening. For a linear system the slope is constant, while a positive and negative slope are defined as a hardening and softening system, respectively. Examples of each type of system are shown in Figure A-5.

Figure A-5. Examples of Restoring Force Characteristics for Linear, Hardening and Softening Systems

To further simplify Equation 3A, damping will be assumed negligible for the present. Equation 3A thus reduces to

$$m\ddot{X} + F(X) = F_0 \sin \omega t \quad (4A)$$

Equation 4A can then be written in the forms

$$m\ddot{X} + kX = F_0 \sin \omega t \quad \text{linear restoring force} \quad (5A)$$

$$m\ddot{X} + F(X) = F_0 \sin \omega t \quad \text{nonlinear restoring force} \quad (6A)$$

With no external force (free vibration system), Equations 5A and 6A become

$$m\ddot{X} + kX = 0 \quad \text{free vibration with linear restoring force} \quad (7A)$$

$$m\ddot{X} + F(X) = 0 \quad \text{free vibration with nonlinear restoring force} \quad (8A)$$

The vibration behavior of systems represented by Equations 5A through 8A can best be shown by curves of response amplitude, $|X|$, versus fundamental exciting frequency ω . This type of curve can be found both experimentally and theoretically. The general forms of the linear response curves are shown in Figure A-6, while the curves for the nonlinear systems are shown in Figure A7.

Figure A-6. Response Curves for Linear Vibration Systems Represented in Equations (6) and (8)

Figure A-7. Response Curves for Nonlinear Vibration Systems Represented in Equations (6) and (8)

Any vibration system possesses damping to some degree. The principal effect of damping is to limit the response amplitude at resonance. For example, damping would alter the theoretical response curves shown in Figures A-6(b) and (c) to those shown in Figure A-8(a) and (b), respectively.

Figure A-8. Response Curves for Damped Linear and Nonlinear Vibration Systems
 (a) Linear Forced (b) Nonlinear Forced ($F_1 < F_2 < \dots < F_n$)

(See the section, Linear and Nonlinear Response Characteristics, for further discussion.)

APPENDIX B

INSTRUMENTATION FOR DATA ANALYSIS

TEST SETUP FOR MAGNETIC TAPE AND OSCILLOSCOPE

INSTRUMENTATION FOR PLAYBACK OF TAPE RECORDING

INSTRUMENTATION FOR PLAYBACK ON OSCILLOGRAPH

INSTRUMENTATION FOR PLAYBACK OF TAPE LOOP

INSTRUMENTATION FOR TAPE PLAYBACK INTO OSCILLOSCOPE, FILTER, AND TRMS METER

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