PARTICLE LOSSES DUE TO DIFFUSION PROCESSES IN PRESENCE OF AN APERTURE LIMITATION

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Particle Distribution Function

\[ f = f(x, x', t) \]

\[ N(t) \text{, number of particles} \]

\[ N(t) = \int \int f(x, x', t) \, dx \, dx' \]

Transform the variables from \( x, x' \) to \( \epsilon, \phi \)

\[ f = f(\epsilon, \phi, t) \]

\( \epsilon \) is the amplitude – excitation

What is the transport equation that \( f \) satisfies?
Assume there is no difference. The motion preserves energy. Dianville theorem applies.

Total time derivative: \[ \frac{df}{ds} = 0 \]

\[ \frac{df}{ds} = \frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + x'' \frac{\partial f}{\partial x'} = 0 \]

Or

\[ \frac{\partial f}{\partial s} + \phi' \frac{\partial f}{\partial \phi} + \varepsilon' \frac{\partial f}{\partial \varepsilon} = 0 \]

In a storage ring the time \( t \) is replaced by the longitudinal coordinate \( s \). \( \varepsilon \) is an invariant of motion, \( \varepsilon' = 0 \).

Dine,

\[ \frac{df}{ds} + \frac{\partial f}{\partial \phi} = 0 \]

where \( \phi' = \varepsilon \) is about the betatron time.
We can assume that

\[ \frac{\partial f}{\partial \phi} = 0 \]

and

\[ f = f(\varepsilon, t) \]

For a stationary distribution \( \frac{\partial f}{\partial t} = 0 \) and

\[ f = f(\varepsilon) \] of \( \varepsilon \) alone.

In conclusion

\[ \frac{\partial f}{\partial t} = 0 \]

If one introduce diffusion \( D \) the equation changes to

\[ \frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial \varepsilon} \left( \varepsilon D \frac{\partial f}{\partial \varepsilon} \right) \]

F. Kiss, Planck equation.
$D$ is the average true increase of the emission per unit of time.

$$D = \frac{d\langle e \rangle}{dt}$$

It is a strong function of the amplitude-emittance $\varepsilon$.

We will assume here

$$D = D(t)$$
Let us change variables

\[ x = \frac{1}{\varepsilon_a} \int_0^t D(t') \, dt' \]

\[ y = \frac{\varepsilon}{\varepsilon_a} \]

where \( \varepsilon_a \) is an aperture limit.

The final equation is

\[ \frac{\partial f}{\partial t} = -\frac{\partial}{\partial y} \left( y \frac{\partial f}{\partial y} \right) \]

This is a partial differential equation. To solve it, one requires knowledge of two boundary conditions.

1) \( f(y, x=0) = f_0(y) \) initial distribution.

   In particular, \( f_0(y=L) = 0 \)

2) \( f(y=L, x) = 0 \) that is, there are no particles at the well (sink) at any time \( t \).
Let us solve the equation

\[ f = \sum_{n} c_n f_n(y) e^{-\beta_n x} \]

By insertion one can find

\[ f_n(y) = \mathcal{J}_0(\lambda_n \sqrt{y}) \]

and

\[ \beta_n = \frac{\lambda_n^2}{4} \]

\( \lambda_n \) are the zeros of the Bessel function \( \mathcal{J}_0 \). Boundary condition 2) is automatically satisfied since

\[ \mathcal{J}_0(\lambda_m) = 0 \]

The coefficient \( c_n \) are to be calculated from the boundary condition 4).
\[ \sum \frac{c_n}{n} J_0 (\lambda_n \sqrt{y}) = f_0 (y) \]

or by inversion:

\[ c_n = \frac{1}{\sqrt{J_2 (\lambda_n)}} \int_0^1 f_0 (y) J_0 (\lambda_n \sqrt{y}) \, dy \]

We are interested in the number of particles \( N(\tau) \) surviving after time \( \tau \):

\[ N(\tau) = \int_0^1 f(y, \tau) \, dy \]

\[ = 2 \sum \frac{e^{-\lambda_n^2 \tau / 4}}{\sqrt{\lambda_n J_2 (\lambda_n)}} \int_0^1 f_0 (y) J_0 (\lambda_n \sqrt{y}) \, dy \]

Observe that at no time a gaussian distribution is possible.
The results for some interesting cases:

(1) \( f_0(y) = \delta(y) \)

\[
N(t) = 2 \sum_n e^{-\lambda_n t/4} / \lambda_n J_1(\lambda_n)
\]

(2) \( f_0(y) = 1 \quad 0 \leq y < 1 \)

\[
N(t) = 4 \sum_n e^{-\lambda_n t/4} / \lambda_n^2
\]

(3) \( f_0(y) = \lambda \cdot J_0(\lambda \sqrt{y}) / 2 J_1(\lambda) \)

\[
N(t) = e^{-\lambda^2 t/4}
\]
The good field aperture is

\[ +25.2 \, \text{mm} \]

We take

\[ \beta_{\text{max}} = 51.6 \, \text{m} \]
\[ \eta_{\text{max}} = 1.385 \, \text{m} \]

For \( \Delta p/p = 0 \), \[ \varepsilon_a = 12.3 \, \pi \, \text{mm-mrad} \]

For \( \Delta p/p = 0.5\% \), \[ \varepsilon_a = 6.5 \, \pi \, \text{mm-mrad} \]

The initial normalized emittance is

\[ 10 \, \pi \, \text{mm-mrad} \] for \((95\% \, \text{of beam})\)

Even at \( y = 10 \),

\[ \varepsilon_{\text{init}} = 4 \, \pi \, \text{mm-mrad} \] (""")

So,

\[ \varepsilon_{\text{init}} \ll \varepsilon_a \]

We can take for the initial distribution a delta function.
Observe that:

\[ \tau = \frac{1}{\varepsilon_a} \int_0^t D(t') \, dt' = \frac{1}{\varepsilon_a} \int_0^t \left< \Delta \varepsilon \right> \, dt \]

\[ = \frac{\Delta \varepsilon(t)}{\varepsilon_a} \]

where \( \Delta \varepsilon(t) \) is the mean emissance increase after time \( t \).

The worst case is at \( y = 12 \) inch (6 in.)

\[ \Delta \varepsilon(t=2 \text{ hours}) = \frac{34.5 - 10}{6 \times 12} = 0.34 \text{ ft mm mm} \]

So it is very small anyway. There are very negligible losses (see also next Table) certainly less than 0.1%.
\[ x = \frac{\Delta \varepsilon(n)}{\varepsilon_0} \]

<table>
<thead>
<tr>
<th>( y_0 = 0 )</th>
<th>( y_0 = 0.2 )</th>
<th>( y_0 = 0.4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.99987</td>
<td>0.99958</td>
</tr>
<tr>
<td>0.04</td>
<td>0.99972</td>
<td>0.99945</td>
</tr>
<tr>
<td>0.06</td>
<td>0.99959</td>
<td>0.99932</td>
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<tr>
<td>0.08</td>
<td>0.99945</td>
<td>0.99906</td>
</tr>
<tr>
<td>0.10</td>
<td>0.99924</td>
<td>0.99829</td>
</tr>
</tbody>
</table>
Diffusion in ke Momentum Plane

Let \( f = f(p, t) \) be the distribution function.

The F.P. equation now is

\[
\frac{\partial f}{\partial t} = \frac{\partial}{\partial p} \left( D_p \frac{\partial f}{\partial p} \right)
\]

where now

\[
D_p = \frac{d \langle p^2 \rangle}{dt}
\]

we can assume again

\[
D_p = D_p(t)
\]

Introduce \( p_o \) as outgoing limit and

\[
z = \frac{p}{p_o}, \quad \tau = \frac{t}{p_o} \int D_p(t') dt'
\]

\[
= \left( \frac{D_p(t)}{p_o} \right)^2
\]

where \( D_p(t) \) is the rms increase in momentum during after time \( t \).
Then we have

\[ f = f(x, z) \]

and

\[ \frac{\partial f}{\partial z} = \frac{\partial^2 f}{\partial z^2} \]

which has solution

\[ f = \sum_{n=0}^{\infty} c_n (\cos \beta_n z) e^{-\beta_n^2 z} \]

having assumed that \( f(-z) = f(z) \)

There are again two boundary conditions:

1) \( f(\pm z, 0) = 0 \) \quad ("sink")

This is satisfied by letting

\[ \beta_n = \frac{\pi}{2} (2n + 1) \]

2) \( f(z, 0) = f_0(z) \) : initial distribution
This condition requires

$$\sum_{n=0}^{\infty} c_n \cos(\beta_n \xi) = f_0(\xi)$$

that is, by inversion

$$c_n = 2 \int_0^1 f_0(\xi) \cos(\beta_n \xi) \, d\xi$$

For $f_0(\xi) = \delta(\xi) \Rightarrow c_n = \frac{2}{\beta_n}$

For the number of particles surviving at the time $\tau$

$$N(\tau) = \sum_{n=0}^{\infty} c_n \cos(\beta_n \xi) \int_0^1 \cos(\beta_n \xi) \, d\xi$$

$$= \sum_{n=0}^{\infty} c_n (-1)^n \frac{e^{-\beta_n^2 \tau}}{\beta_n}$$

For $f_0(\xi) = \delta(\xi)$

$$N(\tau) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta_n} e^{-\beta_n^2 \tau}$$
Results for $\tilde{f}_0(t) = \tilde{S}(t)$

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>$N(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02</td>
<td>0.999893</td>
</tr>
<tr>
<td>0.04</td>
<td>0.999979</td>
</tr>
<tr>
<td>0.06</td>
<td>0.992102</td>
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<tr>
<td>0.08</td>
<td>0.975039</td>
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<td>0.10</td>
<td>0.949173</td>
</tr>
<tr>
<td>0.12</td>
<td>0.917406</td>
</tr>
<tr>
<td>0.14</td>
<td>0.882289</td>
</tr>
<tr>
<td>0.16</td>
<td>0.845546</td>
</tr>
<tr>
<td>0.18</td>
<td>0.808682</td>
</tr>
<tr>
<td>0.20</td>
<td>0.772152</td>
</tr>
</tbody>
</table>
From & Parameter Calculations \( (V=1MV) \)
\[ t=2 \text{ hours} \]

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>12</th>
<th>20</th>
<th>30</th>
<th>50</th>
<th>75</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_a )</td>
<td>3.49</td>
<td>6.13</td>
<td>9.08</td>
<td>3.93</td>
<td>2.91</td>
<td>2.45</td>
</tr>
<tr>
<td>( \Delta P )</td>
<td>1.22</td>
<td>1.37</td>
<td>1.57</td>
<td>1.14</td>
<td>.921</td>
<td>.789</td>
</tr>
<tr>
<td>( \tau )</td>
<td>0.122</td>
<td>0.050</td>
<td>0.030</td>
<td>0.084</td>
<td>0.100</td>
<td>0.104</td>
</tr>
<tr>
<td>( N(t) )</td>
<td>0.91</td>
<td>0.99676</td>
<td>0.998</td>
<td>0.949</td>
<td>0.94</td>
<td></td>
</tr>
</tbody>
</table>

\[ \tau = \left( \frac{\Delta P}{\rho_a} \right)^2 = \left( \frac{1}{2.5} \right)^2 = 0.16 \]

\[ N(t) = 0.84565 \]