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DEVELOPMENT OF THIN SHELL EQUATIONS FOR REACTOR
SUBASSEMBLY DYNAMICS

by

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ABSTRACT

By means of Hamilton's principle the elastic equations of motion for thin cylindrical shells are developed. Explicitly included in these equations are the possibilities of loading due to interior pressure pulses and to thermal stresses. A test problem is considered for the case of a uniform pressure pulse traveling along the axis; an exact analytic solution is compared with the results found by a finite-difference method. The two solutions agree within 1%; which agreement is satisfactory for present purposes.

1. INTRODUCTION

We develop and study the equations of motion of thin deformed cylindrical shells. The deformations are assumed small and the general equations of motion of elasticity are considerably simplified when applied to thin shells.

The motivation for this study is that of modeling the structural dynamics of subassembly can walls for assumed core disruptive accidents of a liquid-metal fast breeder reactor (LMFBR) during extended fuel motion. The restraint of fluid motion by the cans confines flow mainly to the axial direction until failure and determines the timing and growth of a disrupted region within the core. The treatment of wall failure involves dynamic pressure loads, thermal stresses, melting (ablation), and multi-can interactions.

The formalism discussed here has certain advantages and limitations with respect to LMFBR Safety Research in its current form. The method treats true structural dynamics; not quasi-static dynamics. Rotary inertia is included and axial variations in properties are allowed. The elastic response of the can wall to dynamic thermal stresses and pressure loading can be computed. Features which are currently being added to the method include plasticity

and space/time-dependent shell thickness (ablation). The most significant limitation of the technique is the assumption of circular cylindrical geometry; future studies may have to attempt to account for the real hexagonal geometry involved.

We consider closed circular cylindrical thin shells of thickness h under axisymmetric loading. A shell is termed thin if h is small compared to other dimensions. A surface midway between the bounding surfaces, known as the shell middle surface, is chosen as a reference surface (see Fig. 1). The general deformation of the middle surface is a combination of pure stretching and pure bending, although it is possible in certain instances to have one without the other. Stretching is due to stresses which give rise to forces which act on an element of the shell in a direction tangential to the middle surface of the element. Bending is due to stresses which cause changes in the curvature of the middle surface. It is usually assumed in theories of bending that (i) lines normal to the middle surface remain normal to the middle surface after the deformation and (ii) stress components appearing in $\sigma_{ij}n_j$, where $\vec{n}$ is the normal vector to the cylindrical surfaces, can be neglected in comparison with the remaining components of the stress tensor. The latter is due to the fact

Fig. 1. Thin shell element.

that although stresses acting on the lateral surfaces cause the bending, they may in general be neglected compared to the internal stresses σ_{zz} and $\sigma_{\theta\theta}$. Thus on the inner and outer surfaces $\sigma_{rr} = \sigma_{r\theta} = 0$. Then for a thin cylinder these stresses can be neglected within the cylinder. With these approximations the number of independent spatial variables can be reduced from two to one.

By means of Hamilton's principle we formulate the equations of motion for a cylindrical shell for axisymmetric interior loading. The loading may be due, e.g., to pulses traveling along the axis of the can walls in the core of a reactor; or it may be due to thermal stresses generated by heat flow from the reactor core. Both of these types of loading are included explicitly in the equations of motion in the approximate treatment of thin shells presented here.

II. DERIVATION OF THE EQUATIONS OF MOTION

A. Hamilton's Principle

For our problem, in cylindrical coordinates with θ -symmetry, the components of displacement are u_r, u_z , with $u_\theta = 0$. The components of the strain tensor are defined by

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u_r}{r}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \epsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}. \quad (1)$$

The stress-strain relation (Hooke's law) is

$$\sigma_{ij} = \frac{E}{1+\nu} (\epsilon_{ij} + \frac{\nu}{1-2\nu} \epsilon_{kk} \delta_{ij}) + K(T - T_0) \delta_{ij}, \quad (2)$$

where E is Young's modulus, ν is Poisson's ratio and K is the bulk modulus

$$K = \frac{E}{3(1-2\nu)}.$$

The volume coefficient of thermal expansion is denoted by α , T is temperature, and T_0 is a reference temperature. Now since $\sigma_{rr} = 0$, Eq. (2) yields

$$\epsilon_{rr} = -\frac{\nu}{1-\nu} \left[\epsilon_{\theta\theta} + \epsilon_{zz} - \frac{1+\nu}{3\nu} \alpha(T - T_0) \right] \quad (3)$$

and thus ϵ_{rr} can be eliminated from Eqs. (2) leaving

$$\sigma_{zz} = \frac{E}{1-\nu^2} \left(\epsilon_{zz} + \nu \epsilon_{\theta\theta} \right) - \frac{1-\nu}{1+\nu} K \alpha (T - T_0) \quad (4)$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2} \left(\epsilon_{\theta\theta} + \nu \epsilon_{zz} \right) - \frac{1-\nu}{1+\nu} K \alpha (T - T_0).$$

For the middle surface denote the components of displacement by

$$u_r(R, z) = \xi(z), \quad u_z(R, z) = \eta(z), \quad (5)$$

i.e., ξ is the radial displacement of a point on the middle surface, η is its axial displacement and together they determine the stretching and bending of the middle surface. Actually the dependent variables will also be explicitly time dependent in the dynamical problem being considered; this explicit time dependence is suppressed temporarily. Since $\sigma_{rz} = 0$ we have from Eqs. (1) and (2)

$$2\epsilon_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} = 0. \quad (6)$$

In accordance with assumption (i) it is sufficient to take $u_r = \xi$ in Eq. (6). Then

$$\frac{\partial u_z}{\partial r} = -\xi'(z),$$

where the prime indicates differentiation with respect to z . The solution is

$$u_z(r, z) = \eta(z) - (r - R)\zeta'(z), \quad (7)$$

where the constant of integration has been chosen to make $u_z = \eta(z)$ at the middle surface $r = R$. The components of strain become

$$e_{\theta\theta} = \frac{z}{R}, \quad e_{zz} = \eta' - (r - R)\zeta'', \quad (8)$$

We could carry higher order corrections to $e_{\theta\theta}$. However, radial motion gives rise mainly to stretching, bending being of secondary importance; the former is calculated by z/R . In contrast, for the axial direction bending is often of primary importance and is included. In addition, it is not completely clear how to pick the higher order terms. Results obtained for such terms by the method described here differ from most textbook results obtained by different methods; also most textbook results differ from each other.^{1,2,3}

The free energy per unit volume given as an expansion about the reference state with no deformation and at temperature T_0 is

$$F(T) = F_0(T) - K\alpha(T - T_0)(e_{\theta\theta} + e_{zz}) + \frac{E}{2(1+\nu)} \left[e_{\theta\theta}^2 + \frac{\nu}{1-2\nu} e_{zz}^2 \right], \quad (9)$$

differentiation of this expression with respect to $e_{\theta\theta}$ gives Eq. (2) for $\sigma_{\theta\theta}$. Elimination of e_{zz} by means of Eq. (1) reduces (9) to

$$F(T) = F_0(T) - \frac{1-2\nu}{1-\nu} K\alpha(T - T_0)(e_{\theta\theta} + e_{zz}) + \frac{E}{2(1-\nu^2)} \left[e_{\theta\theta}^2 + e_{zz}^2 + 2\nu e_{\theta\theta}e_{zz} \right], \quad (10)$$

where

$$F_0(T) = F_\alpha(T) - \frac{(1+\nu)}{6(1-\nu)} K\alpha^2(T - T_0)^2.$$

Thus with Eq. (8) the free energy becomes

$$F(T) = F_0(T) - \frac{1-2\nu}{1-\nu} K\alpha(T - T_0) \left[\frac{z}{R} + \eta' - (r - R)\zeta'' \right] + \frac{E}{2(1-\nu^2)} \left[\frac{z^2}{R^2} + \eta'^2 + (r - R)^2 \zeta''^2 - 2(r - R)\eta'\zeta'' \right] + 2\nu \left(\frac{z}{R} \eta' - \frac{r - R}{R} \zeta \zeta'' \right). \quad (11)$$

In order to form the Lagrange density we also need the kinetic energy density

$$\tau = (1/2)\rho(\dot{u}_r^2 + \dot{u}_z^2) = (1/2)\rho \left[\dot{\zeta}^2 + (\dot{\eta} - (r - R)\dot{\zeta}')^2 \right], \quad (12)$$

where a dot indicates differentiation with respect to time t . The Lagrange density is then given by

$$\mathcal{L} = \tau - F, \quad (13)$$

with F and T given by Eqs. (11) and (12). The Lagrangian L is then the integral of $\mathcal{L}$ over the volume of the cylinder.

$$L = \int \mathcal{L} dV = 2\pi \int_0^L \int_{R-h/2}^{R+h/2} \mathcal{L} r dr dz = 2\pi R h \int_0^L dz \left\{ (1/2)\rho \left[\dot{\zeta}^2 + \dot{\eta}^2 + \frac{h^2}{12} \dot{\zeta}'^2 - \frac{h^2}{6R} \dot{\eta}' \dot{\zeta}' \right] - \frac{E}{2(1-\nu^2)} \left[\frac{z^2}{R^2} + \eta'^2 + 2\nu \frac{z}{R} \eta' + \frac{h^2}{12} \zeta''^2 - \frac{h^2}{6R} \eta' \zeta'' \right] + \frac{1-2\nu}{1-\nu} K\alpha(T - T_0) \left[\frac{z}{R} + \eta' - \frac{h^2}{12R} \frac{\zeta''}{R} \right] - F_0(T) \right\}, \quad (14)$$

where h^2/R^2 has been neglected compared to unity.

Hamilton's principle states that the equations of motion are determined by setting the variation of the line integral of L between fixed times t_1 and t_2 equal to the variation of the same line integral of

the potential energy of the cylinder due to the external forces acting on its lateral surface. The virtual displacements on the varied paths are for constant z as well as constant t and are taken to be zero at the end points t_1 and t_2 . Thus

$$\delta \int_{t_1}^{t_2} L dt = - 2\pi R \int_{t_1}^{t_2} \int_0^L [P_r \delta \zeta + P_z \delta \eta] dz dt, \quad (15)$$

where $\vec{P}$ is the external force per unit lateral surface area. Combining Eqs. (14) and (15) and performing the variation we find

$$\begin{aligned} 2\pi R h \int_{t_1}^{t_2} \int_0^L dz \left\{ \rho \left[\dot{\zeta} \delta \zeta + \dot{\eta} \delta \eta + \frac{h^2}{12} \ddot{\zeta}' \delta \zeta' - \frac{h^2}{12R} \ddot{\eta} \delta \zeta' - \frac{h^2}{12R} \ddot{\zeta}' \delta \eta \right] - \frac{E}{(1-v^2)} \left[\frac{\zeta}{R^2} \delta \zeta + \eta' \delta \eta' + \frac{v \eta'}{R} \delta \zeta + \frac{v \zeta}{R} \delta \eta' \right. \right. \\ \left. \left. + \frac{h^2}{12} \zeta'' \delta \zeta'' - \frac{h^2}{12R} \eta' \delta \zeta'' - \frac{h^2}{12R} \zeta'' \delta \eta' \right] + \frac{1-2v}{1-v} K \alpha (T - T_0) \left(\frac{\delta \zeta}{R} + \delta \eta' - \frac{h^2}{12R} \delta \zeta'' \right) \right. \\ \left. + \left(\frac{P_r}{h} \delta \zeta + \frac{P_z}{h} \delta \eta \right) \right\} = 0. \end{aligned}$$

By using the fact that the variation operation commutes with ordinary differentiation and by integrating by parts, the above reduces to

$$\begin{aligned} \int_{t_1}^{t_2} dt \left\{ \left[-\rho \frac{h^2}{12} \ddot{\zeta}' + \rho \frac{h^2}{12R} \ddot{\eta} - \frac{E}{(1-v^2)} \left(-\frac{h^2}{12} \zeta'''' + \frac{h^2}{12R} \eta'' - (1+v) \frac{\alpha}{3} \frac{h^2}{12R} T' \right) \right] \delta \zeta - \frac{E}{(1-v^2)} \left[\frac{h^2}{12} \zeta'' \right. \right. \\ \left. \left. - \frac{h^2}{12R} \eta' + (1+v) \frac{\alpha}{3} \frac{h^2}{12R} (T - T_0) \right] \delta \zeta' - \frac{E}{(1-v^2)} \left[\eta' + v \frac{\zeta}{R} - \frac{h^2}{12R} \zeta'' - (1+v) \frac{\alpha}{3} (T - T_0) \right] \delta \eta \right\} \Big|_0^L \\ + \int_{t_1}^{t_2} \int_0^L dz \left\{ \left[-\rho \ddot{\zeta} + \rho \frac{h^2}{12} \ddot{\zeta}'' - \rho \frac{h^2}{12R} \ddot{\eta}' - \frac{E}{(1-v^2)} \left(\frac{\zeta}{R^2} + v \frac{\eta'}{R} + \frac{h^2}{12} \zeta'''' - \frac{h^2}{12R} \eta'' - (1+v) \frac{\alpha}{3} \frac{(T - T_0)}{R} \right) \right. \right. \\ \left. \left. + (1+v) \frac{\alpha}{3} \frac{h^2}{12R} T' \right) + \frac{P_r}{h} \right] \delta \zeta + \left[-\rho \ddot{\eta} + \rho \frac{h^2}{12R} \ddot{\zeta}' - \frac{E}{(1-v^2)} \left(-\eta'' - v \frac{\zeta'}{R} + \frac{h^2}{12R} \zeta'''' + (1+v) \frac{\alpha}{3} T' \right) \right. \right. \\ \left. \left. + \frac{P_z}{h} \right] \delta \eta \right\} = 0. \end{aligned}$$

The variations $\delta \zeta$ and $\delta \eta$ are arbitrary and independent. The variational problem is solved by setting the coefficients of $\delta \zeta$ and $\delta \eta$ in the integrand of the second integral equal to zero and by setting the corresponding boundary terms in the first integral equal to zero. Thus the equations of motion are

$$\rho \xi - \rho \frac{h^2}{12} \xi'' + \rho \frac{h^2}{12R} \bar{\eta}' + \frac{E}{(1-\nu^2)} \left[\frac{\xi}{R^2} + \nu \frac{\eta'}{R} \right. \\ \left. + \frac{h^2}{12} \xi'''' - \frac{h^2}{12R} \eta'''' - (1+\nu) \frac{\alpha}{3} \frac{(T-T_0)}{R} \right. \\ \left. + (1+\nu) \frac{h^2}{12R} \frac{\alpha}{3} T'' \right] = \frac{P_r}{h}, \quad (16)$$

$$\rho \bar{\eta} - \rho \frac{h^2}{12R} \xi' - \frac{E}{(1-\nu^2)} \left[\eta'' + \nu \frac{\xi'}{R} - \frac{h^2}{12R} \xi'''' \right. \\ \left. - (1+\nu) \frac{\alpha}{3} T' \right] = \frac{P_z}{h} \quad (17)$$

and the end conditions are

$$\left[\rho \xi' - \rho \frac{\bar{\eta}}{R} - \frac{E}{(1-\nu^2)} \left(\xi'''' - \frac{\eta''}{R} \right. \right. \\ \left. \left. + (1+\nu) \frac{\alpha}{3} \frac{T'}{R} \right) \right] \delta \xi + \frac{E}{(1-\nu^2)} \left[\xi'' - \frac{\eta'}{R} \right. \\ \left. + (1+\nu) \frac{\alpha}{3} \frac{(T-T_0)}{R} \right] \delta \xi' \Big|_0^l = 0 \quad (18)$$

$$\left[\eta' + \nu \frac{\xi}{R} - \frac{h^2}{12R} \xi'' \right. \\ \left. - (1+\nu) \frac{\alpha}{3} (T-T_0) \right] \delta \eta \Big|_0^l = 0. \quad (19)$$

Equations (16) and (17) represent a simplification of the problem in that there are two dependent variables and only one independent spatial variable z . The end condition (18) is mainly related to rotary inertia and to changes of curvature at the ends in planes containing the axis. The end condition (19) with the aid of Eqs. (4) and (8) can be written as

$$\sigma_{zz} \delta \eta \Big|_0^l = 0, \quad (20)$$

where σ_{zz} is the average stress over a cross section of the cylinder.

B. Discussion of End Conditions

We can distinguish various types of end conditions: Example 1. Suppose the cylinder is clamped at both ends. Then the boundary conditions to be imposed are

$$\begin{aligned} \eta(0) &= 0 & \xi(0) &= 0 & \xi'(0) &= 0 \\ \eta(l) &= 0 & \xi(l) &= 0 & \xi'(l) &= 0. \end{aligned}$$

Since the variations of specified conditions vanish, the end conditions (18) and (20) are automatically satisfied with these boundary conditions. Example 2. The ends are fixed but not clamped. Then

$$\begin{aligned} \eta(0) &= 0 & \xi(0) &= 0 \\ \eta(l) &= 0 & \xi(l) &= 0. \end{aligned}$$

Here the variations $\delta \xi'(0)$ and $\delta \xi'(l)$ are arbitrary and in order to satisfy the end conditions (18) and (20) two additional conditions are required

$$\xi''(z) - \frac{\eta'(z)}{R} + (1+\nu) \frac{\alpha}{3} \frac{(T(z) - T_0)}{R} = 0$$

at $z = 0$ and $z = l$. Example 3. The end at $z = 0$ is clamped; the other end is free. Then

$$\eta(0) = 0, \quad \xi(0) = 0, \quad \xi'(0) = 0.$$

Here the additional conditions required are

$$\sigma_{zz}(l) = 0$$

$$\rho \xi'(l) - \rho \frac{\bar{\eta}(l)}{R} - \frac{E}{(1-\nu^2)} \left[\xi''''(l) - \frac{\eta''(l)}{R} \right. \\ \left. + (1+\nu) \frac{\alpha}{3} \frac{T'(l)}{R} \right] = 0$$

$$\xi''(l) - \frac{\eta'(l)}{R} + (1+\nu) \frac{\alpha}{3} \frac{(T(l) - T_0)}{R} = 0.$$

Note that our problem was formulated in such a way that the end conditions are such that the corresponding forces do no work on the cylinder. If we

By setting the determinant of the coefficients of ζ_0 and η_0 equal to zero we find the dispersion relations

$$\begin{vmatrix} -\omega^2 + c_1^2 k^2 & ik\omega^2 \frac{h^2}{12R} - c_1^2 ik \left(\frac{v}{R} + k^2 \frac{h^2}{12R} \right) \\ -ik\omega^2 \frac{h^2}{12R} + c_1^2 ik \left(\frac{v}{R} + k^2 \frac{h^2}{12R} \right) & -\left(\omega^2 + c_1^2 k^2 \frac{h^2}{12} \right) + c_1^2 \left(\frac{1}{R^2} + k^4 \frac{h^2}{12} \right) \end{vmatrix} = 0,$$

had wished to include end forces which do work and cause displacements then we would have had to have included the associated variation in potential energy in Hamilton's principle just as we did for the applied lateral forces. These applied forces would have contributed to the end conditions (18) and (20), and analyses similar to those in the examples above would show what boundary conditions are necessary to satisfy given problems. For example, suppose we wanted to impose a time-dependent force per unit cross-sectional area of the cylinder $f_z(t)$ at $z = 0$. Then end condition (20) would be modified to

$$\sigma_{zz}(L) \delta \eta(L) - [\sigma_{zz}(0) - f_z] \delta \eta(0) = 0.$$

C. Dispersion Relations

In order to investigate the nature of the equations of motion (16) and (17) we look at the dispersion relations. Set $T = T_0$ and $\vec{P} = 0$ and take

$$\zeta = \zeta_0 e^{i(kz - \omega t)}, \quad \eta = \eta_0 e^{i(kz - \omega t)}, \quad (21)$$

and substitute these into Eqs. (16) and (17). We obtain

$$\begin{aligned} & -\left(\omega^2 + \omega^2 k^2 \frac{h^2}{12} \right) \zeta_0 - ik\omega^2 \frac{h^2}{12R} \eta_0 \\ & + c_1^2 \left[\frac{\zeta_0}{R^2} + ikv \frac{\eta_0}{R} + k^4 \frac{h^2}{12} \zeta_0 + ik^3 \frac{h^2}{12R} \eta_0 \right] = 0 \\ & -\omega^2 \eta_0 + ik\omega^2 \frac{h^2}{12R} \zeta_0 \\ & - c_1^2 \left[-k^2 \eta_0 + ikv \frac{\zeta_0}{R} + ik^3 \frac{h^2}{12R} \zeta_0 \right] = 0. \end{aligned} \quad (22)$$

where

$$c_1^2 = \frac{E}{\rho(1 - \nu^2)}.$$

There are two roots for ω^2 and to order k^2 they are

$$\omega^2 = \begin{cases} \frac{c_1^2}{R^2} (1 + \nu^2 R^2 k^2) \\ c_1^2 (1 - \nu^2) k^2. \end{cases} \quad (23)$$

The first root corresponds to mainly radial oscillations and the second root corresponds to longitudinal waves. To see this, substitute the first root into either of Eqs. (22). To lowest order in wave number one finds

$$\eta_0 = -ikvR\zeta_0$$

or from Eq. (21)

$$\eta = \eta_0 e^{i(kz - \omega t)} = -ikvR\zeta_0 e^{i(kz - \omega t)} = -\nu R\zeta'.$$

Substituting this result into Eq. (16) yields

$$\begin{aligned} \rho \zeta'' - \rho \frac{h^2}{12} (1 + \nu) \zeta'''' + \frac{E}{(1 - \nu^2)} \left[\frac{\zeta}{R^2} - \nu^2 \zeta'' \right. \\ \left. + \frac{h^2}{12} (1 + \nu) \zeta'''' - (1 + \nu) \frac{\alpha}{3} \frac{(T - T_0)}{R} \right. \\ \left. + (1 + \nu) \frac{h^2}{12R} \frac{\alpha}{3} T'' \right] = \frac{P}{h} \zeta, \end{aligned} \quad (24)$$

which predicts mainly radial oscillations.

Similarly the second root of (23) when substituted into Eqs. (22) yields

$$\zeta_0 = -ikRv\eta_0$$

or from (21)

$$\zeta = \zeta_0 e^{i(kz - \omega t)} = -ikRv\eta_0 e^{i(kz - \omega t)} = -Rv\eta'.$$

Substitution of this result into Eq. (17) gives

$$\rho \ddot{\eta} + \rho \frac{h^2}{12} \ddot{\eta}'' - \frac{E}{(1 - \nu^2)} \left[(1 - \nu^2) \eta'' + \frac{h^2}{12} \eta'''' - (1 + \nu) \frac{\alpha}{3} T' \right] = \frac{P_z}{h}, \quad (25)$$

which predicts longitudinal waves traveling along the cylinder with essentially the "bar" velocity $\sqrt{E/\rho}$ modified by some dispersion.

The type of problem we wish to consider will be that of a pressure or a thermal pulse traveling along the cylinder axis. Thus we will be concerned mainly with the type of motion predicted by Eq. (24). Furthermore, the motion will mostly be radial involving stretching of the middle surface and we will ignore those terms proportional to h^2 which relate to pure bending and rotary inertia. Thus we have

$$\zeta + c_1^2 \left[\frac{\zeta}{R^2} - \nu^2 \zeta'' - (1 + \nu) \frac{\alpha}{3} \frac{(T - T_0)}{R} \right] = \frac{P_r}{\rho h}. \quad (26)$$

Consistent with this equation we have

$$u_r = \zeta$$

$$u_z = -Rv\zeta'$$

$$e_{\theta\theta} = \frac{\zeta}{R}$$

$$e_{zz} = -Rv\zeta''$$

$$e_{rr} = -\frac{\nu}{1 - \nu} \left[\frac{\zeta}{R} - Rv\zeta'' - \frac{1 + \nu}{\nu} \frac{\alpha}{3} (T - T_0) \right].$$

It should be noted that Eq. (26) gives the same solution for the static case of a long cylinder with constant uniform internal pressure as does the thin shell limit of the solution of the exact equations of elasticity. The usual static equilibrium equation derived in textbooks³ differs from the static part of Eq. (26) and is derived on the assumption that the average force corresponding to σ_{zz} is zero. This would be correct for static problems of cylinders with finite length and no applied longitudinal forces at the ends. However, for dynamic problems with pulse traveling along the cylinder we would expect Eq. (26) to pertain as a pulse does not know of the finite length until reflection occurs.

In the next section we investigate the properties of Eq. (26) to see if it is adequate, once we allow for plastic deformation, for problems related to reactor safety.

III. A TEST PROBLEM

A. Analytic Solution

We next consider a test problem for Eq. (26) wherein an exact analytic solution is compared with a computer finite-difference solution. First it is convenient to introduce dimensionless variables defined by

$$\begin{aligned} \hat{t} &= \frac{c_1}{R} t, \quad \hat{z} = \frac{1}{Rv} z, \quad \hat{\zeta} = \frac{1}{R} \zeta, \quad \hat{T} = \frac{1}{R} T \\ &= (1 + \nu) \frac{\alpha}{3} (T - T_0), \quad \hat{P}_r = \frac{R}{\rho c_1^2 h} P_r. \end{aligned} \quad (27)$$

Then Eq. (26) becomes, dropping the hats,

$$\zeta + \zeta - \zeta'' - (T - T_0) = P_r. \quad (28)$$

For P_r we choose

$$P_r = P_0 S \left(t - \frac{z}{V} \right), \quad (29)$$

where P_0 is a constant and $S(t)$ is the unit step function

$$S(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0. \end{cases} \quad (30)$$

Thus (29) represents a step-function pressure pulse traveling along the axis of a cylindrical tube with dimensionless velocity V . V may be subsonic, sonic, or supersonic according to $0 < V < 1$, $V = 1$, $V > 1$, respectively. This problem was chosen because it models reactor subassembly can wall elastic radial displacement for axial fuel motion. The temperature is here taken to be uniform.

The Laplace transform of ζ is defined by

$$\bar{\zeta}(z, p) = \int_0^{\infty} dt e^{-pt} \zeta(z, t),$$

and the transforms of Eqs. (28) and (29) yield

$$p^2 \bar{\zeta} + \bar{\zeta} - \bar{\zeta}'' = \frac{P_0}{p} e^{-pz/V}, \quad (31)$$

where for convergence of the transform p must have a positive real part ϵ . A solution of (31) is

$$\bar{\zeta} = A e^{-\sqrt{p^2+1} z} + B e^{\sqrt{p^2+1} z} - \frac{P_0}{p} \frac{e^{-pz/V}}{p^2 \left(\frac{1-V^2}{V^2} \right) - 1}.$$

The constants A and B are determined by the boundary conditions which we take to be

$$\zeta(\infty, t) = 0$$

$$\zeta'(0, t) = 0.$$

Thus

$$\bar{\zeta} = \frac{VP_0}{1-V^2} \frac{e^{-\sqrt{p^2+1} z}}{\sqrt{p^2+1} \left[p^2 - \frac{V^2}{1-V^2} \right]} - \frac{V^2}{1-V^2} \frac{P_0}{p} \frac{e^{-pz/V}}{\left[p^2 - \frac{V^2}{1-V^2} \right]}. \quad (32)$$

The inverse Laplace transform is given by

$$\zeta(z, t) = \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp e^{pt} \bar{\zeta}(z, p).$$

Consider first the inverse transform I_1 of the second term in Eq. (32); it can be evaluated by residue theory. There are three cases:

(i) $V < 1$. There are three poles which lie on the real axis at $\frac{-V}{\sqrt{1-V^2}}$, 0 , $\frac{V}{\sqrt{1-V^2}}$. For $t > z/V$

the contour must be closed in the left half-plane and the poles are enclosed. For $t < z/V$ the contour must be closed in the right half-plane and no poles are enclosed. Thus

$$I_1 = \begin{cases} P_0 \left[1 - \cosh \left[\frac{V}{\sqrt{1-V^2}} (t - z/V) \right] \right], & t > z/V \\ 0, & t < z/V. \end{cases} \quad (33)$$

(ii) $V > 1$. The three poles now lie on the imaginary axis; the contour must be closed to the left for $z < Vt$ and all three poles are enclosed. Thus

$$I_1 = \begin{cases} P_0 \left[1 - \cos \left[\frac{V}{\sqrt{1-V^2}} (t - z/V) \right] \right], & t > z/V \\ 0, & t < z/V. \end{cases} \quad (34)$$

(iii) $V = 1$. There is only one pole, at $p = 0$, and

$$I_1 = P_0 S(t - z). \quad (35)$$

Next we evaluate the inverse transform of the first term in Eq. (32),

$$I_2 = \frac{VP_0}{1-V^2} \frac{1}{2\pi i} \int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp e^{pt} \frac{e^{-\sqrt{p^2+1} z}}{\sqrt{p^2+1} \left[p^2 - \frac{V^2}{1-V^2} \right]}. \quad (36)$$

This can be written as a convolution since

$$f^{-1} \left\{ \frac{e^{-\sqrt{p^2 + 1} z}}{\sqrt{p^2 + 1}} \right\} = \begin{cases} 0, & 0 < t < z \\ J_0 \left(\sqrt{t^2 - z^2} \right), & t > z. \end{cases} \quad (37)$$

Thus we have for the three cases:

(i) $V < 1$.

$$I_2 = \frac{-P_0}{2\sqrt{1-V^2}} \int_z^t d\tau J_0 \left(\sqrt{\tau^2 - z^2} \right) \sinh \left[\frac{V}{\sqrt{1-V^2}} (t - \tau) \right] \quad (38)$$

(ii) $V > 1$.

$$I_2 = \frac{-P_0}{\sqrt{V^2 - 1}} \int_z^t d\tau J_0 \left(\sqrt{\tau^2 - z^2} \right) \sin \left[\frac{V}{\sqrt{V^2 - 1}} (t - \tau) \right] \quad (39)$$

(iii) $V = 1$.

$$I_2 = -J_0 \left(\sqrt{t^2 - z^2} \right) S(t - z). \quad (40)$$

By combining the above results, $\zeta = I_1 + I_2$, we have:

(i) $V < 1$.

$$\zeta = \begin{cases} P_0 \left[1 - \cosh \left[\frac{V}{\sqrt{1-V^2}} (t - z/V) \right] \right], & z < Vt, \\ 0, & z > Vt, \end{cases} + \begin{cases} -\frac{P_0}{2\sqrt{1-V^2}} \int_z^t d\tau J_0 \left(\sqrt{\tau^2 - z^2} \right) \sinh \left[\frac{V}{\sqrt{1-V^2}} (t - \tau) \right], & z < t, \\ 0, & z > t. \end{cases} \quad (41)$$

(ii) $V > 1$.

$$\zeta = \begin{cases} P_0 \left[1 - \cos \left[\frac{V}{\sqrt{1-V^2}} (t - z/V) \right] \right], & z < Vt \\ 0, & z > Vt, \end{cases} + \begin{cases} -\frac{P_0}{\sqrt{V^2 - 1}} \int_z^t d\tau J_0 \left(\sqrt{\tau^2 - z^2} \right) \sin \left[\frac{V}{\sqrt{V^2 - 1}} (t - \tau) \right], & z < t, \\ 0, & z > t. \end{cases} \quad (42)$$

(iii) $V = 1$.

$$\zeta = P_0 \left[1 - J_0 \left(\sqrt{t^2 - z^2} \right) \right] S(t - z). \quad (43)$$

These solutions are next compared to solutions of the same problem as found from a finite-difference scheme on a computer.

B. Finite Difference Solution

For purposes of differencing, Eq. (28) was rewritten as

$$\zeta = \zeta'' - \zeta + P_r. \quad (44)$$

At cycle n ζ'' was computed from,

$$(\zeta'')_j^n \approx \left[\left(\frac{\Delta \zeta}{\Delta z} \right)_{j+1/2}^n - \left(\frac{\Delta \zeta}{\Delta z} \right)_{j-1/2}^n \right] / (z_{j+1/2} - z_{j-1/2}), \quad (45)$$

where

$$\begin{aligned} \left(\frac{\Delta \zeta}{\Delta z} \right)_{j+1/2}^n &= (\zeta_{j+1} - \zeta_j) / (z_{j+1} - z_j); \left(\frac{\Delta \zeta}{\Delta z} \right)_{j-1/2}^n \\ &= (\zeta_j - \zeta_{j-1}) / (z_j - z_{j-1}). \end{aligned} \quad (46)$$

Thus ζ is centered at the j^{th} grid point. With a suitable prescription for P_r the right-hand side of (44), denoted as $(\text{RHS})_j^n$, is then evaluated for the n^{th} cycle. ζ is then advanced in time by the following equations,

$$\left(\frac{\Delta \zeta}{\Delta t} \right)_j^{n+1/2} - \left(\frac{\Delta \zeta}{\Delta t} \right)_j^{n-1/2} = (\text{RHS})_j^n (t^{n+1/2} - t^{n-1/2}) \quad (47)$$

$$(\zeta)_j^{n+1} - (\zeta)_j^n = \left(\frac{\Delta \zeta}{\Delta t} \right)_j^{n+1/2} (t^{n+1} - t^n). \quad (48)$$

Use was made of the WONDY⁴ Lagrangian code as a vehicle for input and output and to provide a grid of 400 cells with $\Delta z = 0.1$. The Lagrangian dynamics of WONDY was nullified and three problems were run with $V = 1/2, 1, 2$. In each case a value of $P_r = 1$ was loaded into those cells where

$$z \leq Vt. \quad (49)$$

A constant time step of $\Delta t = 0.08$ was chosen to satisfy the Courant condition, because the nondimensional sound speed is 1 unit.

The results of the calculations are shown in Figs. 2, 3, 4 for $V = 1/2, 1, 2$, respectively. In each figure the continuous line is the difference solution and the asterisks denote the analytic solution. Computer profiles of ζ vs z at time

intervals of four units were taken so that we could watch the progress of the wave. Figures 2, 3, 4 show these profiles at times when the wave front is near the end of the grid, i.e., $t = 36, 36, \text{ and } 16$ respectively. In the $V = 1$ case some difficulty was encountered in resolving the high frequency oscillations immediately behind the wave front; for this case it was necessary to double the mesh to 800 cells with $\Delta z = 0.05$ and a corresponding change in Δt in order to produce agreement with the analytic solution. Apart from this difficulty the agreement between the difference solutions and the analytic solutions was at worst about 1% and often within 0.1%. For our purposes this agreement is satisfactory.

IV. DISCUSSION

A theory for the dynamics of cylindrical elastic thin shells has been developed from Hamilton's principle. The resulting coupled equations of motion are further approximated, from an analysis of the dispersion relations, by a single equation of motion involving only the radial displacement as dependent variable. Analytic solutions for the problem of an interior step pressure pulse traveling along the axis compare exceptionally well with finite difference solutions. This agreement validates the finite difference scheme. Future studies will compare results from this approximate theory with those from the exact equations of elasticity to ascertain the class of problems for which the approximate theory is suitable. In addition the theory will be extended to include plastic deformation, ablation, and failure in order to treat LMFBR structural dynamics.

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Fig. 2. Radial displacement, ζ , vs z for subsonic case, $V=1/2$. The solid line shows the finite difference solution and asterisks depict the analytic solution for $t = 36$.

Fig. 3. Radial displacement, ζ , vs z for sonic case, $V=1$. The solid line shows the finite difference solution and asterisks depict the analytic solution for $t = 36$.

Fig. 4. Radial displacement, ζ , vs z for supersonic case, $V=2$. The solid line shows the finite difference solution and asterisks depict the analytic solution for $t = 16$.