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Interaction of Relativistic Electron Beams With High Z Plasmas

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ABSTRACT

A set of relativistic multigroup diffusion equations has been derived for the study of electron beam-target interactions. Included are transport, Coulomb collisions, electric and magnetic fields, bremsstrahlung, and hydrodynamic motion of the background plasma. LASNEX, the Laser-Fusion code, is being modified to include these equations and will be used for modeling electron beam fusion.

INTRODUCTION

Recent interest in the possibility of controlled electron beam fusion has created a need for computer codes which model these processes. The laser fusion code LASNEX has been modified so that it is now suitable for studying Electron beam-target interaction. In Section 1 the basic equations are derived. In Section 2 the physical implication of the various terms is discussed. In Section 3 the finite difference equations used to numerically solve the basic equations are discussed.

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I. Basic Equations

The relativistic Boltzmann equation with the collision term derived by D. Mosher\textsuperscript{1} will be taken as a starting point:

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p} \cdot \mathbf{E}}{\epsilon} \cdot \frac{\partial f}{\partial \mathbf{x}} - \epsilon [\mathbf{E} + \frac{\mathbf{p}}{\epsilon} \times \mathbf{B}] \cdot \frac{\partial f}{\partial \mathbf{p}} = - \frac{\partial \mathbf{P}}{\partial \mathbf{p}},$$  \hspace{1cm} (1)

where, number of electrons = \( \int d^3\mathbf{x} d^3\mathbf{p} \ f(\mathbf{x},\mathbf{p}) \),  \hspace{1cm} (2)

$$\epsilon = \sqrt{(mc^2)^2 + (pc)^2}, \quad \gamma = \epsilon/(mc^2),$$

$$\Gamma = \frac{\text{C}}{\text{p}^2} \sum_{k=1}^{n} \text{C}_k \ln \Lambda_k \left[ \sum_{j} (\delta_{ij} \mathbf{p}^2 - \mathbf{p}_i \mathbf{p}_j) \frac{\partial f}{\partial \mathbf{p}_j} + 2\gamma (\frac{\mathbf{m}}{m_k}) \mathbf{p}_i f \right],$$

\( k = 1 \) is electron scattering and \( k = 2, \ldots, n \) is scattering off the various ion species,

$$\text{C}_k \ln \Lambda_k = 2\pi n_k \frac{Z_k^2}{\mathbf{r}_o^2} (m_e c^2)^2 \ln (\frac{b_k^\text{MAX}}{b_k^\text{MIN}}).$$

Since high \( Z \) plasmas are assumed, it is reasonable to make the diffusion approximation, but since the background plasma is in motion it is important when making the diffusion approximation to assume near isotropy in the local rest frame of the moving matter rather than in the laboratory rest frame. Thus let \( \mathbf{v}(\mathbf{x},t) \) be the hydrodynamic velocity of the background plasma \( (v/c \ll 1) \) and since it follows from Eq. (2) that \( f \) is a Lorentz scalar, one finds that

$$\mathbf{p}' = \mathbf{p} - \epsilon \mathbf{v}/c^2\quad (3a)$$

$$\epsilon' = \epsilon - \mathbf{p}' \cdot \mathbf{v}\quad (3b)$$
\[ f'(p') = f(p) \quad (3c) \]
\[ \varepsilon \frac{\partial f'}{\partial p'} = \frac{\partial f}{\partial p} \quad (3d) \]

where the primed quantities are all matter rest frame quantities.

The diffusion approximation is

\[ f'(p', \hat{x}, t) = \left( \frac{1}{4\pi} \right) n(p', \hat{x}, t) + \left( \frac{3}{4\pi} \right) \frac{c^2 p'}{2} \cdot \Phi(p', \hat{x}, t), \quad (4a) \]

where

\[ n(p', \hat{x}, t) = \int d\Omega^- f'(p', \hat{x}, t), \quad (4b) \]
\[ \Phi(p', \hat{x}, t) = \int d\Omega^- \vec{u'} f'(p', \hat{x}, t), \quad (4c) \]
\[ \vec{u'} = \frac{c^2 p'}{\varepsilon} . \]

Multiplying Eq. (1) by \( \varepsilon \) and rewriting it in terms of the primed quantities, one obtains,

\[ (\varepsilon^- + \vec{v} \cdot \vec{p}^-) \frac{\partial f'}{\partial t} - (\varepsilon^- \frac{c^2}{\varepsilon} \vec{v} \cdot \frac{\partial f'}{\partial p'}) \]
\[ + c^2 (\vec{p} + \frac{\varepsilon^- \vec{v}}{c^2}) \cdot \frac{\partial f'}{\partial \vec{x}} - \varepsilon^- \sum_{i,j} \frac{\partial f'}{\partial x_i} \frac{\partial f'}{\partial p_j} \]
\[ - \varepsilon \vec{E} \cdot \vec{B} \cdot \frac{\partial f'}{\partial \vec{p}} = -\varepsilon \frac{\partial f}{\partial p}. \quad (5) \]

Taking the $\int d\Omega^*$ (zeroth moment) of Eq. (5) and neglecting terms which are smaller by factors of $v/c$, $v/u + m_e/m_i \ll 1$, one obtains,

$$\frac{3n}{\delta t} + (\vec{\nabla} \cdot \vec{\nabla}) n = -\frac{\dot{\vec{v}}}{p} + 1/3 \ p \frac{3n}{\partial p} \ (\vec{v} \cdot \vec{v})$$

$$+ \frac{e}{(pc)^2} \frac{3}{\partial p} (\epsilon p \vec{E} \cdot \vec{F}) + \frac{2m_e}{p^2} \frac{3}{\partial p} (\gamma^2 n c_e \epsilon n A_e) \ , \ (6)$$

where all primes have been dropped and from now on all unprimed quantities (except for $\vec{x}$, $t$ and $\vec{v}$) are assumed to be local matter rest frame quantities.

Similarly, taking the $\int d\Omega^* \vec{u}^*$ (first moment) of Eq. (5), neglecting terms smaller by factors of $v/c + 1/Z^2$, and assuming that on time scales of interest the flux equation (first moment equation) is always relaxed so that $\partial \vec{F}/\partial t$ may be neglected, one obtains,

$$\frac{-2m_e\gamma}{p^2} \left[ \sum_{k} \epsilon_{n_k} \right] \vec{F} - \frac{e}{m \gamma c} \ (\vec{p} \times \vec{B})$$

$$= \frac{1}{3} \ u^2 \ \dot{\vec{v}} \ n - \frac{e \vec{p}}{3m \gamma} \ p \frac{\partial n}{\partial p} \ . \ (7)$$

Equations (6) and (7) are our basic equations for modeling the evolution of the relativistic electrons in electron beam interactions with a high Z plasma target.
II. Physical Significance of the Various Terms.

It will prove convenient to use a different normalization. Define

\[ np^2 dp = n \, d\varepsilon \, , \]
\[ \mathcal{F} p^2 dp = \mathcal{F} \, d\varepsilon \, , \]

or,

\[ n = n/(pm\gamma) \, , \]
\[ \mathcal{F} = \mathcal{F}/(pm\gamma) \, , \]

so that number of electrons is \( \int d^3 \mathbf{x} \, d\varepsilon \, n(\varepsilon, \mathbf{x}, t) \).

Then Eqs. (6) and (7) become after some rearrangement,

\[
\begin{align*}
\frac{\partial n}{\partial t} + \mathbf{v} \cdot (\mathbf{v} n) &= - \mathbf{v} \cdot \mathbf{\Pi} + \frac{1}{3} (\mathbf{v} \cdot \mathbf{v}) \frac{\partial}{\partial \varepsilon} (pmn) \\
+ \frac{3}{\varepsilon} (eE \cdot \mathbf{\Pi}) + \frac{2}{m_e} \frac{\partial}{\partial \varepsilon} \left[ \left( \frac{c_e \gamma n_e}{u} \right) n \right] ,
\end{align*}
\]

(6a)

and,

\[
-\frac{2(m_e)^2}{p^3} \left[ \sum \xi_k \xi_k \right] \mathbf{\Pi} - \frac{e}{c} (\mathbf{\Phi} \times \mathbf{B})
= \frac{1}{3} pu \mathbf{v} \cdot (p \mathbf{v}) - \frac{eB}{3} p^3 \frac{\partial}{\partial \varepsilon} \left( \frac{n}{pm\gamma} \right) .
\]

(7a)

The first two terms of Eq. (6a) are

\[
\frac{\partial n}{\partial t} + \mathbf{v} \cdot (\mathbf{v} n) = \frac{1}{V} \frac{d(Vn)}{dt} ,
\]

(8)

where \( V = (\text{matter density})^{-1} = \text{specific volume} \), and
\[ \frac{d}{dt} = \left( \frac{\partial}{\partial t} + \hat{\mathbf{v}} \cdot \hat{\mathbf{v}} \right). \] So the left-hand side of (6a) is just the total Lagrangian time derivative. In a Lagrangian code (e.g. LASNEX) the left-hand side of Eq. (6a) is just the change in the number of electrons in a given zone divided by the volume of the zone.

The transport term in Eq. (6a), \(- \hat{\mathbf{v}} \cdot \hat{\mathbf{I}}\), makes a multi group diffusion treatment vital. D. Mosher\(^2\) has shown that in a spatially homogeneous plasma a monoenergetic beam stays monoenergetic as the average energy decreases due to collisional energy loss. However, in a spatially inhomogeneous plasma such as an electron beam target, electrons will transport into the target, reach a certain density and then reflect out again. The electrons just coming in for the first time will have a much higher energy than those at the same location that are coming out from the target and have undergone considerable energy loss. Thus, at a given point in space, the electron distribution will be far from monoenergetic, as is born out by Monte Carlo computer simulations\(^3\), and \(n\) will in general be an arbitrary function of \(e\).

The \((\hat{\mathbf{v}} \cdot \hat{\mathbf{v}})/3 \frac{\partial}{\partial e} (\text{pue})\) term is just a Doppler shift of the electrons to compensate the \(P_e dV\) work done by the beam electrons on the matter. To see this note that from Eqs. (5a) and (8),
\[
\frac{d}{dt} \int d\varepsilon \left( n\varepsilon V \right) = \frac{V (\hat{\nabla} \cdot \hat{\mathbf{J}})}{3} \int \varepsilon \frac{d}{d\varepsilon} \left( \frac{E_n}{3} \right) \frac{d\varepsilon}{\varepsilon} + \text{other terms}
\]

\[
= -\frac{dV}{dt} \int d\varepsilon \left( \frac{E_n}{3} \right) \frac{d\varepsilon}{\varepsilon} + \text{other terms},
\]

but \( \int d\varepsilon \left( \frac{E_n}{3} \right) \frac{d\varepsilon}{\varepsilon} = P_e \) is just the beam electron contribution to the matter pressure.

The \( \partial \varepsilon / \partial \varepsilon \left( \varepsilon \hat{E} \cdot \hat{\mathbf{E}} \right) \) term is the energy loss of the beam electrons due to ohmic heating of the background plasma. From Eqs. (6a) and (8) one obtains

\[
\frac{d}{dt} \int d\varepsilon \left( n\varepsilon V \right) = V \int \varepsilon \frac{d}{d\varepsilon} \left( \frac{\varepsilon \hat{E} \cdot \hat{\mathbf{E}}}{3} \right) d\varepsilon + \text{other terms}
\]

\[
= -V \left( \hat{\mathbf{E}} \cdot \hat{\mathbf{J}} \right) + \text{other terms},
\]

where \( \hat{\mathbf{J}} = e \int d\varepsilon \hat{\mathbf{V}} \) is the total beam current density.

The \( \frac{2}{m_e} \frac{\partial}{\partial \varepsilon} \left[ \left( \frac{C_{v} \ln n_e}{u} \right)n \right] \) term is the beam energy loss due to collisions with thermal electrons.

Eq. (7a) is a multigroup version of Ohm's law. The coefficient of the flux, \( \frac{-2(m_e)^2}{p^3} \sum_k [C_{v} n_k] = R(\varepsilon, x) \), is the resistance. The forces driving the flux are

\[
\frac{-e}{c} (\hat{\mathbf{V}} \times \hat{\mathbf{B}}), \text{ the } \frac{1}{c} (\hat{\mathbf{J}} \times \hat{\mathbf{B}}) \text{ force, where } \hat{\mathbf{J}} = -e\hat{\mathbf{V}} = \text{current density per unit energy, } 1/3 \text{ pu}^2, \text{ the } -\nabla p_e(\varepsilon) \text{ force, where } p_e(\varepsilon) = 1/3 \text{ pu}^2 \text{ electron pressure per unit energy, and }
\]

\[
\frac{e\hat{\mathbf{E}}}{3} \frac{d}{d\varepsilon} \left( \frac{n}{p^3 \varepsilon} \right) \text{ which is just the multigroup form of the electric}
\]
field force as can be seen from

\[
\frac{\partial F}{\partial t} = \int \rho^3 \frac{\partial}{\partial \varepsilon} (n \varepsilon m_y) d\varepsilon = -e\varepsilon \int \eta d\varepsilon = -e\varepsilon \eta_e.
\]

Finally, the effects of bremsstrahlung are included by adding to Equation (6a) an additional term of the form.

\[
n_i \left[ \int_{0}^{\infty} dk \eta (\varepsilon + k) u_k \frac{d\sigma}{dk} - \eta(\varepsilon) u \int_{0}^{\varepsilon} dk \frac{d\sigma}{dk} \right],
\]

where \( n_i \) is the ion density, \( k \) is the photon energy, \( u_k = c \sqrt{(\varepsilon + k)^2 - (mc^2)^2} / (\varepsilon + k) \), and \( \frac{d\sigma}{dk} \) is the bremsstrahlung cross section.

For a typical high Z plasma, e.g., Pb, above 15-20 MeV more energy is lost through bremsstrahlung than through scattering of plasma electrons. Even at 1 MeV in Pb, bremsstrahlung losses are 10% of the total energy loss. Therefore bremsstrahlung is an important loss mechanism. This term is also important for diagnostic purposes and for the study of preheat problems.

Eqs. (6a) and (7a) can be cast into a more elegant form in the special case that the electric field has no transverse part. Then \( \vec{E} = -\nabla \phi \) and defining

\[
\varepsilon_T = \varepsilon - e\phi (x,t),
\]

\[
\eta_T (\varepsilon_T, x, t) = \eta(\varepsilon, x, t),
\]

\[
\Phi_T (\varepsilon_T, x, t) = \Phi (\varepsilon, x, t),
\]

(10a) (10b) (10c)
Equations (6a) and (7a) become

\[
\frac{\partial \eta_T}{\partial t} + \nabla \cdot \left( \eta_T \nabla T \right) = -\nabla \cdot \left( \eta_T \nabla \phi \right) + \frac{2}{3} \frac{e}{\varepsilon_T} \frac{\partial \eta_T}{\partial \varepsilon_T} (pu \eta_T)
\]

\[
+ \frac{2}{m_e} \frac{\partial}{\partial \varepsilon_T} \left[ \frac{C e \ln \Lambda}{u} \eta_T \right] - e \frac{\partial \phi}{\partial t} \frac{\partial \eta_T}{\partial \varepsilon_T}, \quad (6b)
\]

and

\[
-\frac{2(m_e)^2}{p^3} \left[ \Sigma C_k \eta_k \right] \eta_T = \frac{e}{c} \left( \dot{\eta}_T \times \dot{B} \right)
\]

\[
+ \frac{p^3}{3} \frac{\eta_T}{(pu)^2}. \quad (7b)
\]

This form has the advantage that the electric field no longer appears explicitly in the diffusion equations.

III. Numerical Questions

The computer code LASNEX\(^5\) already incorporates a complete description of the thermal background plasma so only the numerical coding of the superthermal beam electron equations need be discussed.

The basic point is that one requires fully implicit differencing (i.e., all quantities on the right-hand side of Eq. (6a) are evaluated at the end of the time step) so that when the time step gets much larger than the relaxation time, the code will calculate the steady state solution (with the given
electromagnetic fields, background plasma conditions and boundary conditions). This is very difficult to do for Eqs. (6a) and (7a) because in addition to the usual diffusion in space the electric field terms give rise to a diffusion term in energy of the form

\[ \frac{(eE)^2}{3} \frac{3}{\partial \epsilon} \left( \frac{p^3}{R(\epsilon,x)} \right) \nabla^2 \eta, \]

and a term of the form \[ \frac{1}{3} \sum \epsilon E \frac{3}{\partial \epsilon} \left( \frac{p^2}{R(\epsilon,x)} \right) (\nabla \epsilon). \]

These terms couple all energy bins and spatial zones in such a complicated way that it becomes impossible to solve the set of linear equations generated by implicit differencing with available computing power.

It is for this reason that equations (6b) and (7b), where the electric fields no longer appear, are far easier to solve with finite difference methods. Equations (6b) and (7b) were derived under the assumption that \[ \hat{\mathbf{E}} = -\nabla \psi, \]

but this is always true in the steady state anyway, so in the general case we can include only the longitudinal part of \( \mathbf{E} \) in the implicit calculation and then add on explicit terms in the transverse part of \( \hat{\mathbf{E}} \) to recover our full equation. In solving Eq. (7b) for \( \hat{\mathbf{E}} \), a flux limiter is included so that the basic inequality, \[ |\psi| \leq \psi_{\text{un}}, \]

will not be violated. There are three terms in Eq. (6b) which still couple different energy bins. The first one,

\[ \left( \nabla \cdot \nabla \right) \frac{3}{\partial \epsilon} \left( p \epsilon \nabla \epsilon \right), \]
is done explicitly since the hydro time step controller requires that $(\vec{\nabla} \cdot \vec{v})(\Delta t) < (1/10)$ so the energy shift in one time step is always small. The thermal loss term,

$$\frac{3}{m_e} \frac{\partial}{\partial \varepsilon} \left( \frac{C_e \ln \Lambda}{n} \right) n_T,$$

is implicitly differenced as

$$\frac{n(\varepsilon_1, t+\Delta t) - n(\varepsilon_1, t)}{(\Delta t)} = \alpha(\varepsilon_{i+1}) \frac{n(\varepsilon_{i+1}, t+\Delta t) - \alpha(\varepsilon_i) n(\varepsilon_i, t+\Delta t)}{\Delta \varepsilon_i}$$

+ other implicit terms,

where $\alpha(\varepsilon_i) = \frac{2}{m_e} \frac{C_e \ln \Lambda}{u}$. This form of differencing guarantees that $(\eta(\varepsilon_i, t) > 0$ implies $(\eta(\varepsilon_i, t+\Delta t) > 0)$, and also enables one to solve the ordinary spatial diffusion equation for each energy bin separately and then include $\alpha(\varepsilon_{i+1}) \eta(\varepsilon_{i+1}, t+\Delta t)$ as a source term in the ordinary diffusion equation for the next lower (the i'th) energy bin.

The third term, $- \frac{2\phi(x, t)}{\Delta t} \frac{\partial n_T}{\partial \varepsilon_m}$, is automatically taken care of when one shifts the energy bins in each zone at the beginning of each time step. This shift takes care of the shift in $\varepsilon_T$ in each zone due to the change in $\phi(x, t)$ during the last time step (see Eq. (10a)).

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REFERENCES

2. Ibid, Appendix A.
3. These simulations were done with a relative electron beam entering a static high Z plasma using the computer code SANDYL. H.M. Colbert, SANDYL, Sandia Lab. Pub. SLL-74-0012, May 1974.