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INTERNAL RAREFIED GAS FLOWS WITH BACKSCATTERING
PART 3: KRAMER'S PROBLEM--THE EFFECT OF
BACKSCATTERING ON SLIP

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Printed in the United States of America. Available from
National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road, Springfield, Virginia 22151
Price: Printed Copy \$3.00; Microfiche \$0.95

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Date of Issue: June 16, 1971

Report Number: K-1798, Part 3

Subject Category: PHYSICS

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ABSTRACT

The dependence of the slip coefficient on the probability of backscattering is computed. The calculation uses the Bhatnagar, Gross, Krook model to solve the Kramer problem with modified boundary condition.

The slip coefficient is found to decrease monotonically with increasing backscattering. As an example, 6% backscattering reduces the slip coefficient by about 11% below that which would be computed for pure diffuse scattering at the solid surface.

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INTRODUCTION

In Parts 1 and 2 of this report the effects of the backscattering of molecules (from a wall collision) on free molecule flows were investigated. To provide a useful model for the interpretation of internal rarefied gas flows, the effects of the inclusion of backscattering in the boundary condition on internal gas flows other than free molecule flow must be studied and the results compared with experiment when possible.

To become familiar with the methods to be used in the examination of internal rarefied gas flows, we examine in this report the effect of backscattering on the slip coefficient. The problem selected for this purpose is the so-called Kramer's problem. The study leads to a prediction of the change in the slip coefficient due to the inclusion of backscattering in the boundary condition.

For the analysis, we replace the Boltzmann equation by the simplest model equation, namely, the Bhatnagar-Gross-Krook [1] model (BGK) with constant collision frequency. Cercignani and co-workers [2,3,4] have shown that this model qualitatively accounts for the behavior of internal rarefied gas flows.

KRAMER'S PROBLEM

Consider a gas in contact with an infinite flat plate. The distance from the plate to a point in the gas is y , and far from the plate ($y \rightarrow \infty$) the component of gas velocity q_z parallel to the plate approaches a linear function of y ,

$$q_z \sim k(y + \zeta) , \quad (1)$$

where ζ is the slip coefficient whose value we wish to estimate. This

problem has been solved by Cercignani and co-workers [5] for the case of pure diffuse ($\alpha = 1$), or partial specular reflection at the plate.

The molecular velocity distribution function $f(\vec{c}, \vec{r})$ depends, in this problem, only on the molecular velocity $\vec{c}$ and the distance from the plate y so that the BGK model equation is

$$c_y \frac{\partial f(\vec{c}, y)}{\partial y} + \nu f(\vec{c}, y) = \nu f_{eq} \quad (2)$$

where c_y is the y component of molecular velocity, ν is the collision frequency (independent of $\vec{c}$), and

$$f_{eq} = \frac{n}{(2\pi RT)^{3/2}} e^{-(\vec{c} - \vec{q})^2/2RT} \quad (3)$$

Here, n , $\vec{q}$, and T are the local gas density, velocity, and temperature, respectively, and R is the gas constant per gram.

From Part 1, the boundary condition at the plate surface is

$$f(\vec{c}, 0) = \frac{\alpha A}{(2\pi RT_0)^{3/2}} e^{-c^2/2RT_0} + (1 - \alpha)f(-\vec{c}, 0); c_y > 0 \quad (4)$$

where T_0 is the plate temperature and the superscripts on f have been deleted to simplify the notation.

It is convenient to introduce the dimensionless quantities

$$Y = \nu_0 y / \sqrt{2RT_0}; \quad \vec{\xi} = \vec{c} / \sqrt{2RT_0}; \quad w = q_z / \sqrt{2RT_0} \quad (5)$$

where ν_0 is the collision frequency in a gas in equilibrium at T_0 . Equation (2) becomes

$$\xi_y \frac{\partial f(\vec{\xi}, Y)}{\partial Y} + \frac{\nu}{\nu_0} f(\vec{\xi}, Y) = \frac{\nu f_{eq}}{\nu_0} \quad (6)$$

with the boundary condition

$$f(\vec{\xi}, 0) = \frac{\alpha A}{(2\pi RT_0)^{3/2}} e^{-\xi^2} + (1 - \alpha)f(-\vec{\xi}, 0); \xi_y > 0 \quad (7)$$

and with

$$f_{eq} = \frac{n}{(2\pi RT)^{3/2}} e^{-(\xi^2 + w^2 - 2w\xi_z)(T_0/T)} \quad (8)$$

where we have used the fact that the only nonvanishing component of gas velocity is the z component parallel to the plate.

For our present purpose, we seek a solution of (6) and (7) for the case $w = q_z/\sqrt{2RT_0} \ll 1$. This suggests linearizing the problem by taking

$$f(\vec{\xi}, Y) = \frac{n_0}{(2\pi RT_0)^{3/2}} e^{-\xi^2} [1 + h(\vec{\xi}, Y)] \equiv f_0(1 + h) \quad (9)$$

where $h(\vec{\xi}, Y)$ is the perturbation of the global equilibrium distribution function f_0 which results from the presence of a velocity gradient in the gas. Similarly, one linearizes the expression for f_{eq} [equation (8)] by setting

$$n = n_0(1 + \eta) \quad (10)$$

$$T = T_0(1 + t) \quad (11)$$

where η and t are the small density and temperature perturbations, respectively, and by ignoring all products of the small quantities η , t , and w . One obtains

$$f_{eq} = f_0[1 + \eta + 2w\xi_z + (\xi^2 - \frac{3}{2})t] \quad (12)$$

The collision frequency ν in equation (6) may be taken proportional to $n\sqrt{T}$ or to $n_0\sqrt{T_0}(1 + \frac{t}{2})(1 + \eta)$ so that the ratio ν/ν_0 to first order is

$$\frac{\nu}{\nu_0} = 1 + \frac{t}{2} + \eta \quad (13)$$

Substitution of (9), (12), and (13) into (6) and retention of first order terms gives the linearized BGK equation

$$\xi_y \frac{\partial h(\vec{\xi}, Y)}{\partial Y} + h(\vec{\xi}, Y) = \eta + 2w\xi_z + (\xi^2 - \frac{3}{2})t \quad (14)$$

The boundary condition (7) is also linearized to obtain

$$h(\vec{\xi}, 0) = (1 - \alpha)h(-\vec{\xi}, 0) + \alpha\left(\frac{A}{n_0} - 1\right) ; \xi_y > 0 . \quad (15)$$

In terms of dimensionless variables and in linearized form, the perturbations in the density, velocity, and temperature of the gas are given by

$$\eta = \frac{1}{\pi^{3/2}} \int_{\vec{\xi}} h e^{-\xi^2} d\vec{\xi} \quad (16)$$

$$w = \frac{1}{\pi^{3/2}} \int_{\vec{\xi}} \xi_z h e^{-\xi^2} d\vec{\xi} \quad (17)$$

$$t = \frac{2}{3\pi^{3/2}} \int_{\vec{\xi}} \xi^2 h e^{-\xi^2} d\vec{\xi} - \eta . \quad (18)$$

These equations are compatible with a solution in which the density and temperature perturbations η and t are zero while the velocity w is non-vanishing. We seek such a solution, and in equation (15) assign $A = n_0$ which is compatible with constant density and temperature everywhere in the flow field.

Our linearized, constant density, constant temperature problem now becomes

$$\xi_y \frac{\partial h(\vec{\xi}, Y)}{\partial Y} + h(\vec{\xi}, Y) = 2w\xi_z \quad (19)$$

with the boundary condition

$$h(\vec{\xi}, 0) = (1 - \alpha)h(-\vec{\xi}, 0) ; \xi_y > 0 . \quad (20)$$

Equation (19) may be formally integrated along lines of constant $\vec{\xi}$ to obtain

$$h(\vec{\xi}, Y) = h(\vec{\xi}, 0) e^{-Y/\xi_y} + 2 \int_0^Y \frac{\xi_z}{\xi_y} w(Y') e^{-(Y - Y')/\xi_y} dY' ; \xi_y > 0 \quad (21)$$

$$h(\vec{\xi}, Y) = -2 \int_Y^\infty \frac{\xi_z}{\xi_y} w(Y') e^{-(Y - Y')/\xi_y} dY' ; \xi_y < 0 \quad (22)$$

where we have assumed that the perturbation h is sufficiently well behaved as Y approaches infinity. From (22)

$$h(\vec{\xi}, 0) = -2 \int_0^{\infty} \frac{\xi_z}{\xi_y} w(Y') e^{-Y'/\xi_y} dY'$$

so that the boundary condition (20) yields

$$h(\vec{\xi}, 0) = -2(1 - \alpha) \int_0^{\infty} \frac{\xi_z}{\xi_y} w(Y') e^{-Y'/\xi_y} dY'$$

which permits us to rewrite the integrated BGK equation (21) as

$$\begin{aligned} h(\vec{\xi}, Y) = & -2(1 - \alpha) \int_0^{\infty} \frac{\xi_z}{\xi_y} w(Y') e^{-(Y + Y')/\xi_y} dY' \\ & + 2 \int_0^Y \frac{\xi_z}{\xi_y} w(Y') e^{-(Y - Y')/\xi_y} dY' ; \xi_y > 0 . \end{aligned} \quad (23)$$

The component of gas velocity parallel to the plate is given by (17).

With the half range expressions for h provided by (22) and (23) and with integration over ξ_z and ξ_x , the following integral equation for w is obtained.

$$\begin{aligned} \pi^{1/2} w(Y) = & \int_0^{\infty} w(Y') J_{-1}(|Y - Y'|) dY' \\ & - (1 - \alpha) \int_0^{\infty} w(Y') J_{-1}(Y + Y') dY' \end{aligned} \quad (24)$$

where the Abramowitz [5] functions J_n are defined by

$$J_n(x) = \int_0^{\infty} t^n \exp[-t^2] J_n(x/t) dt . \quad (25)$$

To continue the development we could seek the asymptotic behavior of $w(Y)$, the solution to (24). Comparison with equation (1) would then furnish the slip coefficient ζ . However, it turns out to be somewhat simpler to obtain a bound on the slip coefficient by the use of the variational principle discussed in Part 2.

We begin by writing equation (1) in dimensionless form

$$w(Y) \sim K(Y + \beta) \quad (26)$$

where

$$\beta = (\sqrt{\pi}/2\lambda)\zeta ; \quad (27)$$

the mean free path λ is defined in terms of the gas viscosity $\mu = (\rho\lambda/2) \sqrt{8RT_0/\pi}$, and where we have used $v_0 = p_0/\mu$ for the collision frequency and the gas law $p_0 = \rho_0 RT_0$.

It is convenient to define a new velocity variable $\phi(Y)$ by the equation

$$w(Y) = K[Y + \phi(Y)]$$

so that $\phi(Y) \sim \beta$ for large Y . The integral equation for $\phi(Y)$ is

$$\pi^{1/2} \phi(Y) = \alpha J_1(Y) + \int_0^\infty \phi(Y') [J_{-1}(|Y - Y'|) + (1 - \alpha)J_{-1}(Y + Y')] dY' . \quad (28)$$

To obtain (28) some of the properties of the Abramowitz functions have been used. These properties are given in the Appendix to this report.

An integral equation can be derived from (28) which, together with the variational method, will furnish a bound to the slip coefficient. However, two integrations of (28) with respect to Y from Y to infinity are required. To insure convergence of such integrals we define

$$\psi(Y) = \phi(Y) - \beta$$

so that $\psi(Y) \sim 0$ for large Y . The integral equation for $\psi(Y)$ is

$$\begin{aligned} \pi^{1/2} \psi(Y) = & \alpha J_1(Y) - (2 - \alpha)\beta J_0(Y) \\ & + \int_0^\infty \psi(Y') [J_{-1}(|Y - Y'|) - (1 - \alpha)J_{-1}(Y + Y')] dY' . \end{aligned} \quad (29)$$

One integration* of (29) from Y to infinity gives

*Details of this integration may be found in the Appendix.

$$\begin{aligned}
(2 - \alpha)\beta J_1(Y) - \alpha J_2(Y) &= \int_0^\infty \psi(Y') J_0(|Y - Y'|) dY' \\
&- (1 - \alpha) \int_0^\infty \psi(Y') J_0(Y + Y') dY' .
\end{aligned} \tag{30}$$

A second integration leads to

$$(2 - \alpha)\beta J_2(Y) - \alpha J_3(Y) = \int_0^\infty \psi(Y') [J_1(|Y - Y'|) - (1 - \alpha)J_1(Y + Y')] dY' . \tag{31}$$

Setting $Y = 0$ in (31) and using $\phi(Y) = \psi(Y) + \beta$, one obtains

$$\int_0^\infty \phi(Y) J_1(Y) dY = (\beta \sqrt{\pi}/2\alpha) - (1/2) . \tag{32}$$

The variation principle provides the bound

$$\max F\{g(x)\} \leq \frac{\alpha}{\sqrt{\pi}} \int_0^\infty J_1(Y) \phi(Y) dY \tag{33}$$

which, together with (32) and (27), gives for the slip coefficient

$$\zeta \geq (2\alpha\lambda/\pi) + (4\lambda/\sqrt{\pi}) \max F\{g(x)\} . \tag{34}$$

We choose the simple trial function $g(Y) \equiv c$, a constant, and find

$$F\{c\} = (1/2)[\alpha c - (2 - \alpha)c^2/\sqrt{\pi}] .$$

The value of c which leads to an extremal of $F\{c\}$ is

$$c = [\alpha\sqrt{\pi}/2(2 - \alpha)]$$

and the maximum value of $F\{c\}$ is

$$\max F\{c\} = \alpha^2 \sqrt{\pi}/8(2 - \alpha) .$$

The bound on the slip coefficient provided by (34) becomes

$$\zeta \geq \alpha\lambda[(2/\pi) + (\alpha/2)/(2 - \alpha)] . \quad (35)$$

Equation (35) indicates that the slip coefficient decreases monotonically with increasing backscattering (decreasing α) and vanishes for pure backscattering ($\alpha = 0$). The slip constant (ζ/λ) is shown in Figure 1 as a function of the fraction of backscattering ($1 - \alpha$). For pure diffuse scattering ($\alpha = 1$), equation (35) gives $\zeta = 1.137\lambda$. This result is identical to that obtained by Cercignani and Pagani [6] for the case $\alpha = 1$. For $\alpha = 0.94$, the value of α found in Part 2 for argon in nickel capillaries, the slip coefficient is $\zeta = 1.015\lambda$.

SUMMARY AND CONCLUSIONS

An estimate of the dependence of the slip coefficient on the fraction of backscattering has been obtained by a solution of the Kramer problem using the Bhatnagar, Gross, Krook equation and a variational method. It is found that the slip coefficient decreases monotonically with increasing backscattering. The effect is appreciable; 6% backscattering lowers the slip coefficient by about 11%.

In Part 4 of this report we will examine the effects of backscattering on the flow between two infinite, parallel flat plates (plane Poiseuille flow).

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Figure 1
EFFECT OF BACKSCATTERING ON SLIP

APPENDIX A

The Abramowitz Functions

The Abramowitz functions J_n are defined by

$$J_n(x) = \int_0^{\infty} t^n \exp[-t^2 - (x/t)] dt. \quad (A-1)$$

A short table of values may be found in Abramowitz and Stegun, Handbook of Mathematical Functions, Dover, New York, 1965, page 1003. This reference also provides power series and asymptotic representations of the functions.

Some properties of the J_n 's which prove useful are the following:

$$x \frac{d^3 J_n(x)}{dx^3} - (n-1) \frac{d^2 J_n(x)}{dx^2} + 2J_n(x) = 0 \quad (A-2)$$

$$\frac{dJ_n(x)}{dx} = -J_{n-1}(x) \quad (A-3)$$

$$2J_n(x) = (n-1)J_{n-2}(x) + xJ_{n-3}(x); \quad n \geq 3 \quad (A-4)$$

$$J_n(0) = \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right). \quad (A-5)$$

The asymptotic representation of the functions is

$$J_n(x) \sim \frac{\sqrt{\pi}}{3} 3^{-n/2} v^{n/2} e^{-v} \left(a_0 + \frac{a_1}{v} + \frac{a_2}{v^2} + \dots + \frac{a_n}{v^n} + \dots \right)$$

with

$$v = 3\left(\frac{x}{2}\right)^{2/3}$$

and

$$a_0 = 1$$

$$a_1 = (1/12)(3n^2 + 3n - 1)$$

$$\begin{aligned}
(k+2)a_{k+2} &= -(12k^2 + 36k - 3n^2 - 3n + 25)a_{k+1} \\
&+ (1/2)(n-2k)(2k+3-n)(2k+3+2n)a_k; \quad k = 0, 1, 2, \dots
\end{aligned}
\tag{A-6}$$

Integration Details

We wish to integrate equation (29) from Y to infinity. This equation, with the variable Y replaced by a dummy integration variable, say z , is

$$\begin{aligned}
\pi^{1/2} \psi(z) &= \alpha J_1(z) - (2-\alpha)\beta J_0(z) + \int_0^z \psi(Y') J_{-1}(z-Y') dY' \\
&+ \int_z^\infty \psi(Y') J_{-1}(Y'-z) dY' - (1-\alpha) \int_0^\infty \psi(Y') J_{-1}(z+Y') dY'.
\end{aligned}
\tag{A-7}$$

We wish to evaluate

$$\begin{aligned}
\pi^{1/2} \int_Y^\infty \psi(z) dz &= \alpha \int_Y^\infty J_1(z) dz - (2-\alpha)\beta \int_Y^\infty J_0(z) dz + \int_Y^\infty \int_0^z \psi(Y') J_{-1}(z-Y') dY' dz \\
&+ \int_Y^\infty \int_z^\infty \psi(Y') J_{-1}(Y'-z) dY' dz - (1-\alpha) \int_Y^\infty \int_0^\infty \psi(Y') J_{-1}(z+Y') dY' dz.
\end{aligned}
\tag{A-8}$$

The first two integrals and the last integral on the right-hand side are easily evaluated with the aid of (A-3). Thus,

$$\alpha \int_Y^\infty J_1(z) dz = \alpha \int_Y^\infty -J_2'(z) dz = -\alpha J_2(z) \Big|_Y^\infty = \alpha J_2(Y)$$

since $J_2(z) \rightarrow 0$ as $z \rightarrow \infty$ as may be seen from the expansion (A-6). Similarly,

$$-\beta(2-\alpha) \int_Y^\infty J_0(z) dz = \beta(2-\alpha) J_1(z) \Big|_Y^\infty = -\beta(2-\alpha) J_1(Y).$$

Also,

$$\begin{aligned}
-(1 - \alpha) \int_Y^\infty \int_0^\infty \psi(Y') J_{-1}(z + Y') dY' dz &= -(1 - \alpha) \int_0^\infty \psi(Y') dY' \int_Y^\infty J_{-1}(z + Y') dz \\
&= -(1 - \alpha) \int_0^\infty \psi(Y') J_0(Y + Y') dY' .
\end{aligned}$$

The evaluation of the remaining double integrals in (A-8) proceeds readily if one notes that

$$\begin{aligned}
\frac{d}{dz} \int_0^z \psi(Y') J_0(z - Y') dY' &= \int_0^z \psi(Y') \frac{dJ_0(z - Y')}{dz} dY' \\
+ \psi(z) J_0(0) &= - \int_0^z \psi(Y') J_{-1}(z - Y') dY' + \psi(z) J_0(0) ,
\end{aligned}$$

with which we obtain for one of the double integrals

$$\begin{aligned}
&\int_Y^\infty \int_0^z \psi(Y') J_{-1}(z - Y') dY' dz \\
&= \int_Y^\infty \left\{ \psi(z) J_0(0) - \frac{d}{dz} \int_0^z \psi(Y') J_0(z - Y') dY' \right\} dz \\
&= J_0(0) \int_Y^\infty \psi(z) dz - \left[\int_0^z \psi(Y') J_0(z - Y') dY' \right]_{z=Y}^{z=\infty} \\
&= J_0(0) \int_Y^\infty \psi(z) dz + \int_0^Y \psi(Y') J_0(Y - Y') dY' .
\end{aligned}$$

Similarly, one obtains for the remaining double integral

$$\begin{aligned}
\int_Y^\infty \int_z^\infty \psi(Y') J_{-1}(Y' - z) dY' dz &= \int_Y^\infty \left\{ \frac{d}{dz} \int_z^\infty \psi(Y') J_0(Y' - z) dY' + \psi(z) J_0(0) \right\} dz \\
&= - \int_Y^\infty \psi(Y') J_0(Y' - Y) dY' + J_0(0) \int_Y^\infty \psi(z) dz .
\end{aligned}$$

Using the fact that $J_0(0) = \sqrt{\pi}/2$, equation (A-8) can be put in the form

$$0 = \alpha J_2(Y) - \beta(2 - \alpha)J_1(Y) - (1 - \alpha) \int_0^{\infty} \psi(Y') J_0(Y + Y') dY' \\ + \int_0^Y \psi(Y') J_0(Y - Y') dY' - \int_Y^{\infty} \psi(Y') J_0(Y' - Y) dY'$$

which is equivalent to equation (30).

The second integration with respect to Y proceeds in an analogous fashion to give equation (31).