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TEMPERATURE DISTRIBUTION AND THERMAL STRESSES
IN A SLAB WITH EXPONENTIAL HEAT SOURCE

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TEMPERATURE DISTRIBUTION AND THERMAL STRESSES IN A SLAB WITH EXPONENTIAL HEAT SOURCE

INTRODUCTION

The knowledge of temperatures, temperature gradients and resulting thermal stresses in the reactor vessel and thermal shields is of considerable importance in order to check the soundness of the mechanical design of these components. The temperatures and temperature gradients are a result of heat generation within these components due to absorption of gamma radiation which falls off exponentially. A number of simple cases have been treated in literature. This report tries to find solutions for more complicated cases and to give a summary of available solutions for conditions which may arise in connection with the design of the reactor vessel and its components.

The basic stress formula for the case of an internal exponential heat source in slab geometry is derived, into which the appropriate temperature equation can be inserted.

HEAT CONDUCTION EQUATION

t_1 = surface temperature at $x = 0$

t_{f1} = fluid temperature at $x = 0$

h_1 = convection heat transfer coefficient at $x = 0$

t_2 = surface temperature at $x = L$

t_{f2} = fluid temperature at $x = L$

h_2 = convection heat transfer coefficient at $x = L$

q_R = radiation heat flux at $x = 0$

Q_0 = volumetric heat source at $x = 0$

μ = energy absorption coefficient

k = thermal conductivity

The basic equation for heat conduction in an infinite slab with exponential heat source is

$$-k \frac{d^2 t}{dx^2} = Q_0 e^{-\mu x} \quad (1)$$

of which the general solution is

$$t = - \frac{Q_o}{k\mu} e^{-\mu x} + C_1 x + C_2 \quad (2)$$

TEMPERATURE DISTRIBUTION IN SLAB

In order to find the particular solution to (2), it is necessary to apply the given boundary conditions for a specific case to compute C_1 and C_2 . In the following, a summary is given for the various boundary conditions and the corresponding computed C values for each case; in addition $x_{\max}$, the location of the highest temperature within the slab, is computed. The computed C_1 and C_2 values inserted into (2) will give the temperature distribution throughout the slab.

$$\text{For } x = 0 \quad t_1 = - \frac{Q_o}{k\mu} + C_2 \quad (3)$$

$$\text{For } x = L \quad t_2 = - \frac{Q_o}{k\mu} e^{-\mu L} + C_1 L + C_2 \quad (4)$$

In the following, it can be seen that there are many conditions which will yield similar solutions. It seems obvious to use one solution for several cases and modify it somewhat in order to satisfy the particular condition; however, in order to facilitate computation, it was considered to write each solution separately.

$$(1) \quad \begin{aligned} x = 0 & \quad t = t_1 \\ x = L & \quad t = t_2 \end{aligned}$$

$$C_1 = \frac{Q_o}{k\mu^2 L} (e^{-\mu L} - 1) + \frac{t_2 - t_1}{L}$$

$$C_2 = \frac{Q_o}{k\mu} + t_1$$

$$x_{\max} = - \frac{1}{\mu} \ln \left[(t_1 - t_2) \frac{k\mu}{Q_o L} + \frac{1 - e^{-\mu L}}{\mu L} \right]$$

$$(2) \quad t = t_1 = t_2$$

$$C_1 = \frac{Q_o}{k_\mu^2 L} (e^{-\mu L} - 1)$$

$$C_2 = \frac{Q_o}{k_\mu^2} + t$$

$$x_{\max} = -\frac{1}{\mu} \ln \frac{1 - e^{-\mu L}}{\mu L}$$

$$(3) \quad \begin{array}{ll} x = 0 & t = t \\ x = L & \frac{dt}{dx} = 0 \quad (\text{insulated}) \end{array}$$

$$C_1 = -\frac{Q_o}{k_\mu} e^{-\mu L}$$

$$C_2 = \frac{Q_o}{k_\mu^2} + t$$

$$(4) \quad \begin{array}{ll} x = 0 & \frac{dt}{dx}_0 = 0 \quad (\text{insulated}) \\ x = L & t = t \end{array}$$

$$C_1 = -\frac{Q_o}{k_\mu}$$

$$C_2 = \frac{Q_o}{k_\mu^2} e^{-\mu L} + \frac{Q_o L}{k_\mu} + t$$

$$(5) \quad \begin{array}{lll} x = 0 & h = h_1 & t_f = t_{f1} \\ x = L & h = h_2 & t_f = t_{f2} \end{array}$$

$$C_1 = - \frac{\frac{Q_o}{\mu} \left(\frac{1}{h_1} + \frac{1}{k_u} \right) + \frac{Q_o}{\mu} \left(\frac{1}{h_2} - \frac{1}{k_u} \right) e^{-\mu L} + (t_{f1} - t_{f2})}{\frac{k}{h_1} + \frac{k}{h_2} + L}$$

$$C_2 = \frac{Q_o}{\mu} \left(\frac{1}{h_1} + \frac{1}{k_u} \right) + t_{f1} + \frac{k}{h_1} C_1$$

$$x_{\max} = \frac{1}{\mu} \ln \frac{\frac{1}{h_1} + \frac{1}{h_2} + \frac{L}{k}}{\left(\frac{1}{h_1} + \frac{1}{k_u} \right) + \left(\frac{1}{h_2} - \frac{1}{k_u} \right) e^{-\mu L} + \frac{\mu}{Q_o} (t_{f1} - t_{f2})}$$

(6) $x = 0 \quad h = h_1 = h_2 \quad t_f = t_{f1}$
 $x = L \quad t_f = t_{f2}$

$$C_1 = - \frac{\frac{Q_o}{h_u} (1 + e^{-\mu L}) + \frac{Q_o}{k_u 2} (1 - e^{-\mu L}) + (t_{f1} - t_{f2})}{2 \frac{k}{h} + L}$$

$$C_2 = \frac{Q_o}{\mu} \left(\frac{1}{h} + \frac{1}{k_u} \right) + t_{f1} + \frac{k}{h} C_1$$

$$x_{\max} = \frac{1}{\mu} \ln \frac{\frac{2}{h} + \frac{L}{k}}{\frac{1}{h} (1 + e^{-\mu L}) + \frac{1}{k_u} (1 - e^{-\mu L}) + \frac{\mu}{Q_o} (t_{f1} - t_{f2})}$$

(7) $x = 0 \quad h = h_1 \quad t_f = t_{f1} = t_{f2}$
 $x = L \quad h = h_2$

$$C_1 = - \frac{\frac{Q_o}{\mu} \left(\frac{1}{h_1} + \frac{1}{k_u} \right) + \frac{Q_o}{\mu} \left(\frac{1}{h_2} - \frac{1}{k_u} \right) e^{-\mu L}}{\frac{k}{h_1} + \frac{k}{h_2} + L}$$

$$C_2 = \frac{Q_o}{\mu} \left(\frac{1}{h_1} + \frac{1}{k_u} \right) + t_f + \frac{k}{h_1} C_1$$

$$x_{\max} = \frac{1}{\mu} \ln \frac{\frac{1}{h_1} + \frac{1}{h_2} + \frac{L}{k}}{\left(\frac{1}{h_1} + \frac{1}{k_u} \right) + \left(\frac{1}{h_2} - \frac{1}{k_u} \right) e^{-\mu L}}$$

$$(8) \quad h = h_1 = h_2$$

$$t_f = t_{f1} = t_{f2}$$

$$C_1 = - \frac{\frac{Q_o}{h_u} \left(1 + e^{-\mu L} \right) + \frac{Q_o}{k_u^2} \left(1 - e^{-\mu L} \right)}{2 \frac{k}{h} + L}$$

$$C_2 = \frac{Q_o}{\mu} \left(\frac{1}{h} + \frac{1}{k_u} \right) + t_f + \frac{k}{h} C_1$$

$$x_{\max} = \frac{1}{\mu} \ln \frac{\frac{2}{h} + \frac{L}{k}}{\frac{1}{h} \left(1 + e^{-\mu L} \right) + \frac{1}{k_u} \left(1 - e^{-\mu L} \right)}$$

$$(9) \quad x = 0 \quad \frac{dt}{dx} = 0 \quad (\text{insulated})$$

$$x = L \quad h = h \quad t_f = t_f$$

$$C_1 = - \frac{Q_o}{k_u}$$

$$C_2 = \frac{Q_o}{h_u} \left(1 - e^{-\mu L} \right) + \frac{Q_o}{k_u^2} e^{-\mu L} + \frac{Q_o L}{k_u} + t_f$$

$$(10) \quad x = 0 \quad h = h \quad t_f = t_f$$

$$x = L \quad \frac{dt}{dx} = 0 \quad (\text{insulated})$$

$$C_1 = - \frac{Q_o}{k\mu} e^{-\mu L}$$

$$C_2 = \frac{Q_o}{h\mu} \left(1 - e^{-\mu L} \right) + \frac{Q_o}{k\mu^2} + t_f$$

$$(11) \quad \begin{array}{llll} x = 0 & h = h_1 & t_f = t_{f1} & q_R \text{ (radiation heat flux)} \\ x = L & h = h_2 & t_f = t_{f2} & \end{array}$$

$$C_1 = - \frac{\left(\frac{q_1}{h_1} - \frac{q_2}{h_2} \right) + \frac{Q_o}{\mu} \left(\frac{1}{h_1} + \frac{1}{k\mu} \right) + \frac{Q_o}{\mu} \left(\frac{1}{h_2} - \frac{1}{k\mu} \right) e^{-\mu L} + \left(t_{f1} - t_{f2} \right)}{\frac{k}{h_1} + \frac{k}{h_2} + L}$$

$$C_2 = \frac{q_1}{h_1} + \frac{Q_o}{\mu} \left(\frac{1}{h_1} + \frac{1}{k\mu} \right) + t_{f1} + \frac{k}{h_1} C_1$$

$$x_{\max} = \frac{1}{\mu} \ln \frac{\frac{1}{h_1} + \frac{1}{h_2} + \frac{L}{k}}{\left(\frac{1}{h_1} + \frac{1}{k\mu} \right) + \left(\frac{1}{h_2} - \frac{1}{k\mu} \right) e^{-\mu L} + \frac{\mu}{Q} \left[\left(\frac{q_1}{h_1} - \frac{q_2}{h_2} \right) + \left(t_{f1} - t_{f2} \right) \right]}$$

where

$$q_1 = \frac{1}{1 + \frac{kh_2}{Lh_1h_2 + kh_1}} \quad q_R = \frac{t_{f1} - t_{f2}}{\frac{1}{h_1} + \frac{Lh_2 + k}{kh_2}}$$

$$q_2 = \frac{1}{1 + \frac{Lh_1h_2 + kh_1}{kh_2}} \quad q_R = \frac{t_{f1} - t_{f2}}{\frac{1}{h_1} + \frac{Lh_2 + k}{kh_2}}$$

$$(12) \quad \begin{array}{llll} x = 0 & h = h_1 = h_2 & t_f = t_{f1} & q_R \text{ (radiation heat flux)} \\ x = L & & t_f = t_{f2} & \end{array}$$

$$C_1 = \frac{\frac{1}{h} (q_1 - q_2) + \frac{Q_o}{h\mu} (1 + e^{-\mu L}) + \frac{Q_o}{k\mu^2} (1 - e^{-\mu L}) + (t_{f1} - t_{f2})}{2 \frac{k}{h} + L}$$

$$C_2 = \frac{q_1}{h} + \frac{Q_o}{\mu} \left(\frac{1}{h} + \frac{1}{k_u} \right) + t_{f1} + \frac{k}{h} C_1$$

$$x_{\max} = \frac{1}{\mu} \ln \frac{\frac{2}{h} + \frac{L}{k}}{\frac{1}{h} (1 + e^{-\mu L}) + \frac{1}{k_u} (1 - e^{-\mu L}) + \frac{\mu}{Q_o} \left(\frac{1}{h} (q_1 - q_2) + (t_{f1} - t_{f2}) \right)}$$

where

$$q_1 = \frac{1}{1 + \frac{k}{Lh + k}} q_R - \frac{t_{f1} - t_{f2}}{\frac{2}{h} + \frac{L}{k}}$$

$$q_2 = \frac{1}{1 + \frac{Lh + k}{k}} q_R + \frac{t_{f1} - t_{f2}}{\frac{2}{h} + \frac{L}{k}}$$

(13)

$$x = 0 \quad h = h_1 \quad q_R \text{ (radiation heat flux)}$$

$$x = L \quad h = h_2 \quad t_f = t_{f1} = t_{f2}$$

$$C_1 = - \frac{\left(\frac{q_1}{h_1} - \frac{q_2}{h_2} \right) + \frac{Q_o}{\mu} \left(\frac{1}{h_1} + \frac{1}{k_u} \right) + \frac{Q_o}{\mu} \left(\frac{1}{h_2} - \frac{1}{k_u} \right) e^{-\mu L}}{\frac{k}{h_1} + \frac{k}{h_2} + L}$$

$$C_2 = \frac{q_1}{h_1} + \frac{Q_o}{\mu} \left(\frac{1}{h_1} + \frac{1}{k_u} \right) + t_f + \frac{k}{h_1} C_1$$

$$x_{\max} = \frac{1}{\mu} \ln \frac{\frac{1}{h_1} + \frac{1}{h_2} + \frac{L}{k}}{\left(\frac{1}{h_1} + \frac{1}{k_u} \right) + \left(\frac{1}{h_2} - \frac{1}{k_u} \right) e^{-\mu L} + \frac{\mu}{Q_o} \left(\frac{q_1}{h_1} - \frac{q_2}{h_2} \right)}$$

where

$$q_1 = \frac{1}{1 + \frac{kh_2}{Lh_1 h_2 + kh_1}} q_R$$

$$q_2 = \frac{1}{1 + \frac{Lh_1 h_2 + kh_1}{kh_2}} q_R$$

$$(14) \quad x = 0 \quad q_R \quad (\text{radiation heat flux})$$

$$h = h_1 = h_2$$

$$t_f = t_{f1} = t_{f2}$$

$$C_1 = - \frac{\frac{1}{h} (q_1 - q_2) + \frac{Q_o}{h\mu} (1 + e^{-\mu L}) + \frac{Q_o}{k\mu^2} (1 - e^{-\mu L})}{2 \frac{k}{h} + L}$$

$$C_2 = \frac{q_1}{h} + \frac{Q_o}{\mu} \left(\frac{1}{h} + \frac{1}{k\mu} \right) + t_f + \frac{k}{h} C_1$$

$$x_{\max} = \frac{1}{\mu} \ln \frac{\frac{2}{h} + \frac{L}{k}}{\frac{1}{h} (1 + e^{-\mu L}) + \frac{1}{k\mu} (1 - e^{-\mu L}) + \frac{\mu}{Q_o} \left(\frac{1}{h} (q_1 - q_2) \right)}$$

where

$$q_1 = \frac{1}{1 + \frac{k}{Lh + k}} q_R$$

$$q_2 = \frac{1}{1 + \frac{Lh + k}{k}} q_R$$

$$(15) \quad x = 0 \quad h = h \quad t_f = t_f \quad q_R \quad (\text{radiation heat flux})$$

$$x = L \quad \frac{dt}{dx} = 0 \quad (\text{insulated})$$

$$C_1 = - \frac{Q_o}{k\mu} e^{-\mu L}$$

$$C_2 = \frac{Q_o}{k\mu^2} + \frac{Q_o}{h\mu} \left(1 - e^{-\mu L} \right) + \frac{q_R}{h} + t_f$$

THERMAL STRESS IN SLAB

Thermal stresses in a plate are treated in article 132 of "Theory of Elasticity," by S. Timoshenko and J. M. Goodier, and the appropriate equation is

$$\sigma = \frac{\alpha E}{1-\nu} \left(-t + \frac{1}{2c} \int_{-c}^c t dx + \frac{3x}{2c^3} \int_{-c}^c t x dx \right) \quad (5)$$

where $C = \frac{L}{2}$ = half thickness of the plate
 α = temperature coefficient of expansion
 E = Young's modulus of elasticity
 ν = Poisson's ration

Inserting equation (2) into (5) and transforming the reference axis of (2) in order to correspond to the integral limits $-c$ to $+c$, (5) takes the form

$$\begin{aligned} \sigma = \frac{\alpha E}{1-\nu} & \left[\frac{Q_o}{k_u^2} e^{-\mu x} - C_1 x - C_2 + \frac{1}{2c} \int_{-c}^c \left(\frac{Q_o}{k_u^2} e^{-\mu(x+c)} + C_1(x+c) + C_2 \right) dx \right. \\ & \left. + \frac{3x}{2c^3} \int_{-c}^c \left(\frac{Q_{ox}}{k_u^2} e^{-\mu(x+c)} + C_1(x+c)x + C_2 x \right) dx \right] \end{aligned} \quad (6)$$

which after integration becomes

$$\sigma = \frac{\alpha E}{1-\nu} \left[\frac{Q_o}{k_u^2} \left(e^{-\mu x} + \frac{1}{\mu L} (e^{-\mu L} - 1) + \frac{6}{\mu L} \frac{x}{L} (e^{-\mu L} + 1) + \frac{12}{(\mu L)^2} \frac{x}{L} (e^{-\mu L} - 1) \right) + \frac{L}{2} C_1 \right] \quad (7)$$

For $x = 0$

$$\sigma = \frac{\alpha E}{1-\nu} \left[\frac{Q_o}{k_u^2} \left(1 + \frac{1}{\mu L} (e^{-\mu L} - 1) \right) + \frac{L}{2} C_1 \right] \quad (8)$$

For $x = L$

$$\sigma = \frac{\alpha E}{1-\nu} \left[\frac{Q_o}{k_u^2} \left(e^{-\mu L} + \frac{1}{\mu L} (e^{-\mu L} - 1) + \frac{6}{\mu L} (e^{\mu L} + 1) + \frac{12}{(\mu L)^2} (e^{-\mu L} - 1) \right) + \frac{L}{2} C_1 \right] \quad (9)$$

Assuming that the plate is restricted in bending only, the last term can be dropped. This assumption is applicable in case of reactor vessel wall and thermal shields. Reference is made to C. F. Bonilla's "Nuclear Engineering" and B. F. Langer's "Design Basis for Reactor Vessels," Westinghouse WAPD-CE-43. This assumption is conservative and furthermore simplifies (5) to

$$\sigma = \frac{\alpha E}{1-\nu} \left[\frac{Q_o}{k_u^2} e^{-\mu x} - C_1 x - C_2 + \frac{1}{2c} \int_{-c}^c \left(\frac{Q_o}{k_u^2} e^{-\mu(x+c)} + C_1(x+c) + C_2 \right) dx \right] \quad (10)$$

Performing integration gives the final result as

$$\sigma = \frac{\alpha E}{1-\nu} \left[\frac{Q_o}{k_\mu^2} \left(e^{-\mu x} + \frac{1}{\mu L} (e^{-\mu L} - 1) \right) + \left(\frac{L}{2} - x \right) C_1 \right] \quad (11)$$

which for

$$x = 0 \quad \sigma = \frac{\alpha E}{1-\nu} \left[\frac{Q_o}{k_\mu^2} \left(1 + \frac{1}{\mu L} (e^{-\mu L} - 1) \right) + \frac{L}{2} C_1 \right] \quad (12)$$

$$x = L \quad \sigma = \frac{\alpha E}{1-\nu} \left[\frac{Q_o}{k_\mu^2} \left(e^{-\mu L} + \frac{1}{\mu L} (e^{-\mu L} - 1) \right) - \frac{L}{2} C_1 \right] \quad (13)$$

It should be mentioned that the maximum thermal stress will always occur at the colder face of the slab.

CONCLUSION

The derived C and $x_{\max}$ values which have been summarized in section 1 to 15 should cover almost any possible boundary condition.

If the temperature distribution is of concern only, compute C_1 , C_2 , and $x_{\max}$ and insert into (3), (4), and (2) in order to obtain t_1 , t_2 , and $t_{\max}$, respectively.

If the maximum stress is required, compute the appropriate C_1 value and insert into equation (12) and (13).

Although the equations derived deal with slab geometry, it can be easily proved that this is a good approximation for thin wall cylinders like reactor vessel walls, thermal shields, etc. For diameter ratio $\frac{d_o}{d_i} \leq 1.4$ the maximum error will be $\leq 1\%$ for the temperature distribution.