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QUARTERLY PROGRESS REPORT NO 17

"Plasma Instabilities and Waves Excited
by an Anisotropic Energy Distribution
Produced by Electron Cyclotron Resonance"

U.S. Atomic Energy Commission
Contract No. AT(11-1) 1695

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Madison, Wisconsin 53706

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Plasma Instabilities and Waves Excited by an Anisotropic Energy
Distribution Produced by Electron Cyclotron Resonance

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U.S. Atomic Energy Commission

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INTRODUCTION

J. L. Shohet

This report covers the period between March 15, 1971 and June 14, 1971.

Basically the work performed covered three areas. They are

1. Polarization measurements of the free-free bremsstrahlung x-rays
2. Plasma kinetic theory
3. Plasma diagnostics

Both theory and experimental work have been done.

The bremsstrahlung measurements have been made to determine the polarization distribution of the free-free bremsstrahlung exciting radially along the mid-plane. The scattering method has been used with beryllium as the scattering material. Use of this material, together with improved pulse counting techniques has resulted in a 5-fold increase in the scattered x-ray flux, necessitating counting times of only 200-400 minutes. Both wall and free-free bremsstrahlung polarization measurements were made using this method. The results have been quite exciting and point to a valuable means for measuring the anisotropy of a hot electron plasma.

The theoretical work in this area is concentrating on a direct evaluation of the scattering matrix elements to obtain a definitive measure of the anisotropy of the free-free bremsstrahlung, a more definite statement on the nature of distribution functions that can be used to describe the plasma.

In the area of plasma diagnostics, a method of determining the amounts of cold and hot plasma present in the device is being developed. It will be extended to utilize the Boltzmann equation. The method is based on a different response to microwaves in a cavity when the plasma is considered warm than when it is cold. Further diagnostic work in the optical spectrum will commence shortly when a monochromator is installed on the system. Hopefully, optical bremsstrahlung will be measured. It is anticipated that a Lisitano type r.f. source will be installed in the low frequency system to improve coupling characteristics. Finally, further calibration of the scintillator probe used to measure the distribution of electrons in the plasma has been made.

THE POLARIZATION OF THE FREE-FREE BREMSSTRAHLUNG
IN AN ANISOTROPIC HOT-ELECTRON PLASMA

D.G.S. Greene, J.L. Shohet and P.A. Raimbault

The intensity of the free-free bremsstrahlung in a hot-electron plasma produced in a magnetic field has previously been found to be anisotropic.^{1,2} This measurement reflected the general anisotropy of the velocity distribution of the electrons in the plasma. However, the need for various geometrical and alignment corrections to the measurement of the intensity parallel and perpendicular to the field lines made the technique quite insensitive to significant variations in the plasma or machine parameters, such as impurity concentrations, mirror ratio, etc.

We present here a technique that circumvents many of the problems encountered with the previous method. The anisotropy of the electron velocity distribution can be effectively measured, at one location, by determining the degree of polarization of the free-free x-ray bremsstrahlung flux emitted in a direction perpendicular to the d.c. magnetic field lines.

Figure 1 shows a diagram of the experimental apparatus. The plasma is produced in helium gas at a pressure of 7×10^{-4} torr in a magnetic mirror by means of electron cyclotron resonance heating from a 100 kilowatt pulsed microwave source (360 2 microsecond pulses per second). The bremsstrahlung is observed through a collimated window at the end of a long vacuum path. This permits removal of the effects of the walls and significant absorption by the air. The x-rays are in the 5-50 keV regime.^{3,4} Hence, the most

efficient Compton scattering occurs in those materials that have a low Z .⁵ The scatterer used here is a block of pure beryllium metal. The NaI scintillator-photomultiplier detector can be oriented so as to measure the polarization of the bremsstrahlung parallel to or perpendicular to the field lines. The orientation shown in Figure 1 measures the flux polarized perpendicular to B . The scintillator-photomultiplier assembly is turned 90° about the axis of rotation shown in the Figure so that it can measure the flux polarized parallel to the field.

Figure 2 shows the results for 100 minute counts of the scattered flux measuring the polarization in both the parallel and perpendicular direction, respectively. Since the x-rays appeared only within the first 500 microseconds after each heating pulse, the system was gated on only during that time to improve its noise characteristics. The detection device was shielded from the magnetic field and produced identical spectra in the two orientations when an unpolarized radioactive source was used for calibration. From Figure 2 it can be seen that there is a greater flux of photons whose electric vectors are perpendicular to the magnetic field lines. The anisotropy in the fluxes is about 4 to 1, yielding a polarization of 0.6.

The degree of polarization is a function of impurity gas pressure. The normal base pressure of the system is about 5×10^{-7} torr. When the pumping rate was deliberately slowed down to produce a base pressure of 3×10^{-6} torr which increased the impurity concentration, the polarization at the same operating pressure was

Magnetic Field on
Axis of Vacuum Chamber

Figure 1

UPPER TRACE - Scattered flux polarized with electric field perpendicular to B

LOWER TRACE - Scattered flux polarized with electric field parallel to B

Figure 2

decreased to about 0.3 along with a 4 fold decrease in total flux because of the increased collision cross section. The shape of the profiles remained the same, however.

The degree of polarization was also found to be a function of the mirror ratio. The original mirror ratio was 1.19 to 1. When the ratio was decreased to 1.02 to 1 the degree of polarization increased and the absolute flux decreased as to be expected, since the confinement time is lowered with the smaller value of mirror ratio.

To examine the validity of these results, we consider the quantum-mechanical nonrelativistic theory of free-free bremsstrahlung polarization^{6,7} from a particle with momentum $\vec{p}_0$ deflected by a fixed scattering center. Figure 3 is a coordinate representation used to describe the problem. Its origin is at the scattering center. The magnetic field is in the z-direction and $\vec{k}$, the propagation vector of the photon in the direction of observation, is in the y-direction. $\vec{p}_0$ is the momentum vector of the electron before the collision, and $\vec{p}$ is its momentum after the collision. The polarization vector $\vec{J}$ (parallel to the photon's electric field) is always perpendicular to $\vec{k}$. Thus, for the directions shown, the polarization vector is in the x-z plane.

The degree of polarization, A, is given by equation (1).⁶

$$A = \frac{J_{\parallel} - J_{\perp}}{J_{\parallel} + J_{\perp}} = \frac{(1-3D^2) \log \frac{1+D}{1-D} + 6D}{(3-D^2) \log \frac{1+D}{1-D} + 2D}, \quad \text{where } \frac{|p|}{|p_0|} = D \quad (1)$$

The subscripts $\parallel$ and $\perp$ refer to directions parallel and perpendicular

Figure 3

to the plane of the incident momentum vector $\vec{p}_0$ and $\vec{k}$, (not to the direction of the magnetic field lines). A graph of this function is shown in Figure 4, as a function of ν/ν_0 , where ν is the photon frequency and ν_0 is the highest photon frequency obtainable from an electron of momentum p_0 . The conditions under which the x-rays are emitted in this system are at this high frequency limit. (That is, the photon energy is comparable to the electron energy.⁴) The degree of polarization should approach the value 1. Thus, in the y-direction more photons are emitted with polarization in the x-direction if the electron is incident in the x-direction.

For a plasma whose electron velocity distribution is anisotropic so that most of the electrons are in the x-y plane, we conclude that $\vec{J}$ tends to be in the direction perpendicular to the magnetic field lines. Electron cyclotron resonance heating does produce anisotropic velocity distributions of this type^{1,2} and the results are in excellent agreement with both theory and experiment. Integration over the velocity distribution will be performed, but it should not change the basic results.

Figure 4

ON ENERGY DISTRIBUTIONS FOR ELECTRONS IN A PLASMA

D.G.S. Greene and J.L. Shoet

Introduction

A plasma is a collection of approximately equal numbers of positively and negatively charged particles, for example, electrons and ions. In the laboratory these charged particles are sometimes contained by magnetic fields. As an example, one type of confinement geometry, a pair of magnetic mirrors, is illustrated in Figure 5. The z axis is defined to be the axis of symmetry of the magnetic field. Electrons in such a field configuration tend to travel in helices about tubes of constant magnetic flux. The plasma contained in such a mirror field may conveniently be heated by an electron cyclotron resonance technique as follows. Microwaves enter the plasma region through the microwave horns labelled B in Figure 5. Since the wavelength of the microwaves is small compared to the dimensions of the cavity, the electric field of the microwaves tends to be distributed uniformly throughout the cavity. In most regions there is a component of the field oriented perpendicular to the d.c. magnetic field. The frequency of the microwaves is set equal to the frequency of rotation of the electrons in their helical orbits (the electron cyclotron frequency) at some point in the magnetic field. Thus, it is possible, in this resonant region of the cavity, that the rotational components of the electron velocities are greatly increased, sometimes reaching relativistic velocities. The electron velo-

Figure 5

- A magnet coils
- B microwave heating input
- C region of plasma production and confinement

cities usually only become large in the directions perpendicular to the z axis and tend to remain low in the direction parallel to the z axis. Hence, an anisotropic, relativistic electron energy distribution function is needed to describe such a plasma.

General Characteristics of Distribution Functions

A distribution function for one species of plasma particles will be developed. This single species of particles will be called a system. Let the number of plasma particles in the system be N_0 . First, a phase space for these N_0 particles will be defined to be a $2L$ coordinate space, consisting of L independent position coordinates for the total of these N_0 particles, plus the L conjugate momentum coordinates. Note, if each individual plasma particle has ℓ independent spatial coordinates, then

$$L = \ell \times N_0 \quad (2)$$

where: ℓ is the number of independent spatial coordinates per plasma particle,

N_0 is the number of plasma particles, and

L is the total number of degrees of freedom for the system.

For this plasma system various quantities may be conserved; for example: total energy, momentum, etc. These constants are used as restrictions on the way particles are to be arranged in phase space, together with any other constraints based on the nature of the particles themselves, and the effects of the surroundings,

if any. In addition, if the effects of special relativity need to be included, a further complication results. A state of the system is specified by a single point in its $2L$ dimensional phase space since such a point specifies the positions and momenta of all the particles in the plasma system. This single point is called a micro-state. The fundamental postulate of statistical mechanics, which is used in this development, is that any micro-state of the system subject to restrictions such as those above, is equally likely to occur. Finally, the distribution function which best describes the actual physical plasma system (a macro-state) will be that function which is consistent with the maximum number of micro-states. That is, many different points in phase space may lead to the same macroscopic state of the system. The distribution function with the greatest number of microstates is the proper distribution function, because it is the most likely to occur.

If over a "short" period of time (compared to the time of observation) the plasma distribution function does not change appreciably, the plasma is sometimes considered to be in a state of "quasi-equilibrium". That is, as long as the distribution function is essentially constant over the time of observation, the plasma will be considered to be unchanging and hence in quasi-equilibrium. It should be noted that a confined plasma cannot be in strict equilibrium, since the confining forces produce accelerations of the rapidly moving charged particles, which in turn produce radiative energy losses. Moreover, even if radiative

energy losses may be neglected, a plasma confined by electric and magnetic fields will still not be in equilibrium. The reason is as follows.

The Hamiltonian for a charged particle in a plasma confined by electric and magnetic fields is given by:

$$H = \frac{1}{2}mV^2 + q\phi \quad (3)$$

where: ϕ is the potential function proposed for confinement,

q is the charge,

m is the mass, and

V is the velocity of the particle.

Now, suppose the plasma is confined, In a given location, the confining force is:

$$F = -\nabla H = -q\nabla\phi \quad (4)$$

The force on the positively charged plasma particles will be oppositely directed to the force on the negatively charged particles due to the sign of q . At one point at least on the surface of the confined plasma (according to the "shaggy dog" theorem of topology), there must exist either

- a) a component of the force which is perpendicular to the surface of the confined plasma or
- b) a zero net confining force.

In both cases, the confinement is destroyed either by

- a) an actual expelling force for one or the other species of charged particle or
- b) a lack of a confining force for any particle.

Finally, still neglecting radiation losses which otherwise prevent confinement, confinement may be achieved if the confining force is independent of the sign of the electric charge of the particles. An example of such confinement is the gravitational forces which confine the plasma in a star.

Under the conditions of quasi-equilibrium an acceptable distribution function for plasma particles should not change with time. The distribution function for all the plasma particles will not change in time, if the distribution function for a typical single particle of the plasma does not change in time. A single particle distribution function that is time invariant satisfies the Vlasov equation which is:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H] = 0 \quad (5)$$

where: f is a single particle distribution function,

H is now the Hamiltonian of a single particle

$[f, H]$ is the Poisson bracket equal to

$$\sum_{i=1}^l \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right),$$

q_i is the i th generalized coordinate,

p_i is the conjugate momentum to q_i , and

l is the number of independent coordinates for the particle.

To prove that a given single particle distribution function satisfies equation (5), it is sufficient to express the given distribution function as a function of only the constants of the motion.

Since the constants of the motion by definition do not change with time along the trajectory of the particle, a function of the constants will likewise not change with time, which therefore satisfies equation (5) directly. Hence, a distribution function for all the particles in a plasma may be used to describe a quasi-equilibrium plasma, if the distribution function for a single particle may be expressed as a function of the constants of the motion for that single particle.

Historical Distribution Functions

a) The Maxwell-Boltzmann distribution

This review of various relevant distribution functions begins with the Maxwell-Boltzmann distribution. The usual derivation of the Maxwellian is done using the techniques of statistical mechanics previously indicated. The resulting N_0 -particle distribution function is given in equation (6):

$$n = N_0 (m/2\pi kT)^{3/2} \exp(-\frac{1}{2}mV^2/kT) \quad (6)$$

where: m is the mass of a particle in the distribution,

V is the velocity of the particle,

n is the density of particles with kinetic energy $\frac{1}{2}mV^2$,

N_0 is the total number of particles in the distribution,

k is the Boltzmann constant equal to $1.3804 \cdot 10^{-23} \frac{\text{joule}}{\text{°K}}$,
and

T is the characteristic temperature of the distribution.

This is a nonrelativistic, isotropic distribution function, since the energy expression is for nonrelativistic velocities and no part of the distribution has any angular dependency. It describes particles in a condition of thermal equilibrium. The constraints imposed on the particles which yielded the N_0 -particle Maxwellian distribution are (1) the total number of particles in the plasma does not change, (2) the total kinetic energy of the particles does not change, (3) all of phase space is accessible to a given particle, and (4) any occupation of phase space by all of the particles (subject to the other constraints) is equally likely to occur. The single particle distribution function is derived from the N_0 -particle distribution function by an integration of equation (6) over $(L-l)$ spatial and $(L-l)$ momentum coordinates; the coordinates not included in the integration are the l spatial and the l momentum coordinates of a given single particle. The result of this integration is given in equation (7):

$$f = (m/2\pi kT)^{1/2} \exp(-\frac{1}{2}mv^2/kT) \quad (7)$$

where: f is the probability of finding a single particle in a particular region of a phase space for a single particle (note that this single particle phase space has $2l$ dimensions).

Only one independent variable, v^2 , appears in equation (7) so only one constant of the motion is needed to eliminate this variable. This constant is the kinetic energy of the particle. Expressing the single particle Maxwellian as a function of this constant of

the motion yields equation (8):

$$f = (m/2\pi kT)^{1/2} \exp(-\epsilon/kT) \quad (8)$$

where: ϵ is the kinetic energy of the single particle.

In this form it is clear that the total derivative of f with respect to time is zero thus satisfying the Vlasov equation, which means that f is an acceptable single particle distribution function for describing a quasi-equilibrium plasma. Therefore n in equation (6) is an acceptable N_0 -particle distribution function for describing a quasi-equilibrium plasma.

b) The Jüttner relativistic distribution

For relativistic energies E . Jüttner derived the following isotropic distribution function:

$$n = \frac{N_0 \exp(-E/kT)}{\frac{\pi}{4} u_0^2 kT [-iH_2^{(1)}(iu_0/kT)]} \quad (9)$$

where: u_0 = electron rest mass energy = $m_0 c^2$,

c = speed of light,

E = total electron energy (including rest mass energy),

n = density of particles with total energy E ,

$H_2^{(1)}$ is the Bessel function of the second kind with imaginary argument.

The denominator of equation (9), $\mathcal{D}$, results from the normalization requirement; that is, integration of the right hand side of equation (9) over energy values from u_0 to infinity must yield N_0 electrons. Equation (9) is an N_0 -particle distribution function.

It will be shown, on considering the properties of the plasma, to be time invariant and hence may be used to describe a quasi-equilibrium plasma system. The steps in the demonstration will be the same as those followed for the Maxwellian: 1) a single particle distribution will be written down; 2) the single particle distribution will be written in terms of only the constants of the motion; and 3) since the constants are time invariant, the single particle distribution function will be time invariant, and so also will the N_0 -particle distribution function. Integrating equation (9) over all the coordinates of the $2L$ -dimensional phase space except for the $2l$ coordinates of a particular single particle yields equation (10):

$$\hat{r} = \frac{1}{\vartheta} \exp(-E/kT) \quad (10)$$

where: ϑ is the denominator of equation (9).

Equation (10) is a single particle distribution function and is seen to already be expressed in terms of the constants of the motion. Again, only one constant is needed which is the total energy, E , (which now includes the rest mass energy). Thus, the Jüttner distribution is also an acceptable distribution function for describing a quasi-equilibrium plasma. The constraints imposed in deriving the N_0 -particles Jüttner distribution function are (1) the total number of particles in the plasma does not change, (2) the total energy (including rest mass energy) does not change, (3) only velocities less than C may be considered (not all of phase space is accessible to the plasma particles), and (4) any

occupation of phase space (subject to the first 3 constraints) is equally likely.

c) The Harris anisotropic distribution

To introduce anisotropy E.G. Harris assumed that the energy distribution could be represented by a nonrelativistic Maxwellian in any given direction, but that the electron energy used in the distribution is dependent on the chosen direction. The Harris distribution function, assuming an anisotropy in the energies parallel to and perpendicular to the z-axis, is given in equation (11):

$$n(v_{\parallel}, v_{\perp}) = \frac{N_0 m^{3/2}}{(2\pi k)^{3/2} T_{\parallel}^{1/2} T_{\perp}} \exp\left(\frac{-mv_{\parallel}^2}{2kT_{\parallel}} - \frac{mv_{\perp}^2}{2kT_{\perp}}\right) \quad (11)$$

where: $v_{\parallel}$ = electron velocity component in the z direction,

$$v_{\parallel} = v_z,$$

$v_{\perp}$ = electron velocity component perpendicular to

$$\text{the z direction, } v_{\perp}^2 = v_x^2 + v_y^2,$$

$n(v_{\parallel}, v_{\perp})$ = density of electrons with velocity components $v_{\parallel}$ and $v_{\perp}$,

$T_{\parallel}$ = electron temperature associated with the components of electron velocities in the z direction,

$T_{\perp}$ = electron temperature associated with the components of electron velocities perpendicular to the z direction, and

k = Boltzmann's constant.

Defining angle A as the polar angle relative to the z direction as shown in Fig. 6, $V_{\parallel}$ equals $V \cos A$ and $V_{\perp}$ equals $V \sin A$. Substituting these expressions into equation (11) yields:

$$n(V, A) = \frac{N_0 m^{3/2}}{(2\pi k)^{3/2} T_{\parallel}^{1/2} T_{\perp}} \exp\left(-\frac{1}{2} m V^2 (\cos^2 A / k T_{\parallel} + \sin^2 A / k T_{\perp})\right) \quad (12)$$

Equation (12) will be referred to as the H distribution.

In the development of the H distribution from a normal Maxwellian distribution, V^2 is set equal to $V_{\parallel}^2 + V_{\perp}^2$ and a characteristic temperature is assigned to each term. From equation (12) a single particle distribution function may be derived. The constants of the motion which can eliminate the coordinates in the single particle distribution are the kinetic energy, ϵ , and the magnetic moment, $\mu = \frac{1}{2} m V_{\perp}^2 / B$, where B is the magnetic field, assumed to be constant in both space and time. μ is not a true constant of the motion, but is rather an adiabatic invariant. This means that in a region where the magnetic field does not change too rapidly in space or time, μ may be considered to be a constant of the motion. Expressing the single particle H distribution as a function of these constants yields equation (13):

$$f = \frac{m^{3/2}}{(2\pi k)^{3/2} T_{\parallel}^{1/2} T_{\perp}} \exp(-\epsilon / k T_{\parallel} - \mu B (1 / k T_{\perp} - 1 / k T_{\parallel})) \quad (13)$$

Note again that B must be constant (not for example a function of z) in order that the H distribution may be used to describe a quasi-equilibrium plasma.

where: z is the direction of the d.c. magnetic field,

V is the particle velocity,

A is the angle between V and the z axis,

$V_{\perp}$ is the component of V perpendicular to the z direction, and

$V_{\parallel}$ is the component of V parallel to the z direction.

Figure 6

Note that the H distribution is limited to nonrelativistic particles because, in relativistic mechanics the energy appearing in the exponential expression is the total energy, E , not $\frac{1}{2}mv^2$. Since:

$$E = \sqrt{u_0^2 + P_x^2 + P_y^2 + P_z^2} \quad (14)$$

where: $P_i = mV_i C$ (P_i is the electron momentum component in the i th direction times C),

a different characteristic temperature cannot conveniently be assigned to parallel and perpendicular terms. Thus the Jüttner distribution function cannot be generalized to an anisotropic distribution using a development technique similar to that used to develop the H distribution.

The upper limit of energy for which the H distribution is accurate is usually given in the literature as 14 keV which corresponds to a 10^8 °K electron (note that this is not the characteristic temperature of a distribution which contains a negligible number of electrons with energies above 14 keV, since a distribution with characteristic temperature of only 2.5×10^7 °K (i.e. 3.5 keV has about 5% of its population with energies greater than 14 keV). A 14 keV electron has a speed of 7×10^7 m/sec = .2 C which means a relativistic mass increase of 2%. Thus the H distribution at best is suitable only for low temperature plasmas.

d) The Noerdlinger anisotropic, relativistic distribution

To produce both an anisotropic and also relativistic distribution, P. D. Noerdlinger started with Jüttner's relativistic

Maxwellian and applied an anisotropic, adiabatic compression maintaining the magnetic moment and the action integral as invariants. The exponential argument was changed from the isotropic case:

$$-\xi \left(1 + \frac{P^2}{m_o^2 c^2} \right)^{1/2}$$

to the anisotropic case of:

$$-\xi \left(1 + \frac{P_{\perp}^2 A^2}{m_o^2 c^2} + \frac{P_{\parallel}^2 D^2}{m_o^2 c^2} \right)^{1/2}$$

where: $B_o/B = A^2$,

$L/L_o = D$, and

$$\xi = \frac{m_o c^2}{kT}.$$

This function describes the particle energies in a slab of plasma of length L_o in a magnetic field, B_o , which has been adiabatically compressed to a length L and containing a field B such that:

$P_{\perp}^2/B = P_{o\perp}^2/B_o$, conserving the magnetic moment ($\mu = P_{\perp}^2/2mB$),

and $P_{\parallel}L = P_{o\parallel}L_o$, conserving the action integral ($J = \oint P_{\parallel} ds$).

This distribution is limited to a situation reached by an adiabatic change from an equilibrium Maxwellian. To express this distribution in terms of the characteristic temperatures used in this paper, a Taylor expansion of the argument of the exponent of the following function is made:

$$F = \exp(-\sqrt{aP_{\parallel}^2 + P_{\perp}^2 + u_o^2} / kT) \quad (15)$$

where: a and T are constants to be evaluated as follows:

$$F = \exp((- \sqrt{P_{\perp}^2 + u_o^2} / kT) \sqrt{1 + aP_{\parallel}^2 / (P_{\perp}^2 + u_o^2)}), \quad (16)$$

$$F = \exp((- \sqrt{P_{\perp}^2 + u_o^2} / kT) (1 + aP_{\parallel}^2 / 2(P_{\perp}^2 + u_o^2) + \dots)). \quad (17)$$

For motion in the z direction, $P_{\perp} = 0$, so:

$$F = \exp((-u_o / kT) (1 + aP_{\parallel}^2 / 2u_o^2 + \dots)). \quad (18)$$

Assuming $aP_{\parallel}^2$ is small relative to u_o^2 , the terms represented by dots may be neglected relative to the first two terms in equation (18). Thus

$$F \approx C \exp(-aP_{\parallel}^2 / 2u_o kT), \quad (19)$$

where C is a constant equal to $\exp(-u_o / kT)$

Which corresponds to the classical distribution (where $P_{\parallel}$ is non-relativistic):

$$C_1 \exp(-P_{\parallel}^2 / 2u_o kT_{\parallel}).$$

Thus:

$$a = T / T_{\parallel}. \quad (20)$$

For motion perpendicular to the z direction, $P_{\parallel} = 0$, so:

$$F = \exp(-\sqrt{P_{\perp}^2 + u_o^2} / kT), \quad (21)$$

which corresponds to the relativistic distribution:

$$C_2^2 \exp(-\sqrt{P_{\perp}^2 + u_o^2} / kT_{\perp}).$$

So

$$T = T_{\perp}, \text{ and } a = T / T_{\parallel} = T_{\perp} / T_{\parallel}.$$

Substituting these values for a and T into equation (15) yields:

$$n = \frac{N_0}{\vartheta_R} \exp\left(-\sqrt{\left(\frac{T_{\perp}}{T_{\parallel}}\right) P_{\parallel}^2 + P_{\perp}^2 + u_0^2} / kT_{\perp}\right). \quad (22)$$

where: ϑ_R is the normalization constant.

Finally replacing $P_{\parallel}$ by $P \cos A$ and $P_{\perp}$ by $P \sin A$ yields:

$$N = \frac{N_0}{\vartheta_R} \exp\left(-\sqrt{\frac{T_{\perp}}{T_{\parallel}} p^2 \cos^2 A + P^2 \sin^2 A + u_0^2} / kT\right) \quad (23)$$

Equation (23) will be referred to as the R distribution. The R distribution may likewise be used to describe a quasi-equilibrium plasma, since the corresponding single particle distribution function may be written as a function of the constants of the motion E & μ :

$$f = \frac{1}{\vartheta_R} \exp\left(-\sqrt{\frac{T_{\perp}}{T_{\parallel}}(E^2 - u_0^2 - 2mBu) + 2mBu + u_0^2} / kT_{\perp}\right) \quad (24)$$

Again, note that B must be constant in order that this f satisfy the Vlasov equation.

e) The Loss Cone distribution

The loss cone distribution has frequently been used as an anisotropic distribution function to describe the particle energies found in a magnetic mirror machine. Equation (25) is a loss cone distribution for N_0 plasma particles with energies including relativistic energies:

$$\begin{aligned}
 n &= N_0 \exp(-E/kT)/\vartheta / \cos \theta_c & \text{for } A > \theta_c \\
 &= 0 & \text{for } A < \theta_c
 \end{aligned}
 \tag{25}$$

where: A is the angle between the z axis and the direction of motion of the plasma particle (A is illustrated in Figure 6),

ϑ is the normalization constant described in equation (7),

E is the total relativistic energy, and

θ_c is the critical angle defined by equation (26):

$$\sin \theta_c = B_0/B_{\max} \tag{26}$$

where: B_0 is the magnetic field on the axis at the center of the magnetic mirror machine and

$B_{\max}$ is the maximum magnetic field at the mirror throats.

The parameters for the loss cone distribution are illustrated in Figure 7. The N_0 -particle loss cone distribution function can be integrated to yield a single particle distribution function. The result is expressed as a function of a constant of the motion, the total energy of the single particle, in equation (27):

$$\begin{aligned}
 f &= \exp(-E/kT)/\vartheta / \cos \theta_c & \text{for } A > \theta_c \\
 &= 0 & \text{for } A < \theta_c
 \end{aligned}
 \tag{27}$$

Thus, since it is also only a function of the constants of the motion, the loss cone distribution may also be used to describe a quasi-equilibrium plasma.

Illustration of Loss Cone Distribution Function Parameters

Figure 6

Criticism

Background to understand a criticism of both the H and R distributions is presented in the following two examples:

(1) A Maxwellian distribution, N_1 , has a characteristic temperature T_1 . A second Maxwellian distribution, N_2 , has a characteristic temperature T_2 , where $T_2 = 2T_1$. Thus

$$\begin{aligned} N_1 &= N_{01} \left(\frac{m}{2\pi kT_1} \right)^{3/2} \exp(-\epsilon/kT_1), \text{ and} \\ N_2 &= N_{02} \left(\frac{m}{2\pi kT_2} \right)^{3/2} \exp(-\epsilon/kT_2), \end{aligned} \quad (28)$$

where: N_{0i} is the number of particles in the i th distribution.

To describe the change from distribution N_1 to distribution N_2 the characteristic temperature is said to be "doubled"; that is: the energy, ϵ , is not said to be "doubled" nor is it said to be "halved". Thus, for a given maximum N_1 , the parameters which can be varied to describe different physical situations are kT_1 and N_{01} . As a result of these variations the values of N_1 for various energies, ϵ , are determined.

(2) In analytic geometry the formula for a circle of radius r and centered at the point (a,b) is:

$$(x-a)^2 + (y-b)^2 = r^2 \quad (29)$$

The parameters which are varied to describe different circles are a , b , and r . As a result of these variations the values of y are determined for various x values in the different circles. In

terms of these two examples, in order to introduce an anisotropy (a dependence on the angle A) into the Maxwellian distribution, the variable ϵ should not be modified, but rather the parameter kT . This is done in the following section for a proposed anisotropic, relativistic, distribution function.

An Anisotropic, Relativistic Distribution Function

For an N_0 -particle distribution that is only a function of energy,

$$n = N_0 f(E) \quad (30)$$

The average value of a quantity F that depends on E , in terms of this distribution function is given by:

$$\bar{F} = \int F f(E) dE \quad (31)$$

By partial integration of equation (10) for the relativistic Maxwellian, kT will be found to be proportional to the average value of $v^2/\sqrt{1-\beta^2}$ where $\beta = v/c$ as follows: First, recalling equation (10):

$$n = N_0 \exp(-E/kT)/D \quad (10)$$

This n times a volume element will give the number of particles in that volume element, dN . A spherically symmetric volume element in momentum space is $4\pi p^2 dp$. Thus:

$$dN = n 4\pi p^2 dp \quad (32)$$

Since:

$$P^2 dP = PE dE = (E^2 - u_0^2)^{1/2} E dE, \quad (33)$$

the substitution of equation (33) into equation (32) yields:

$$dN = 4\pi n (E^2 - u_0^2)^{1/2} E dE \quad (34)$$

By partial integration of equation (34), equation (35) is obtained:

$$\begin{aligned} N_0 = \int dN &= 4\pi \int d[(E^2 - u_0^2)^{3/2} \exp(-E/kT)/\vartheta] \\ &+ \int \frac{4\pi}{kT} \exp(-E/kT) \frac{2}{3} (E^2 - u_0^2)^{3/2} dE/\vartheta \end{aligned} \quad (35)$$

The first term on the right hand side of equation (35) is zero because $E^2 - u_0^2$ is zero at the lower limit of integration (where E equals the rest mass energy) and $\exp(-E/kT)$ is zero at the upper limit of integration (where E goes to infinity). Solving for kT :

$$kT = \frac{2}{3} \int [(E^2 - u_0^2)/E] \frac{4\pi}{N_0} \exp(-E/kT) (E^2 - u_0^2)^{1/2} E dE/\vartheta \quad (36)$$

From the definition of average value kT in equation (36) can be found to be $\frac{2}{3} [(E^2 - u_0^2)/E]$.

Since:

$$\frac{E^2 - u_0^2}{E} = \frac{P^2}{E} = \frac{m^2 v^2 c^2}{m c^2} = m v^2 = u_0 \frac{v^2}{\sqrt{1 - \beta^2}} \quad (37)$$

Thus, the desired result may be written:

$$kT = D \quad v^2 / \sqrt{1 - \beta^2} \quad (38)$$

where: D is the proportionality constant equal to $\frac{2}{3} u_0$.

Previously, it was noted that the total relativistic energy could

not be separated into components. However, kT may be separated, since v^2 has components, as follows:

$$kT = D \left((v_{\parallel}^2 + v_{\perp}^2) / \sqrt{1-\beta^2} \right) \quad (39)$$

$$\begin{aligned} &= D \frac{v_{\parallel}^2}{\sqrt{1-\beta^2}} + D \frac{v_{\perp}^2}{\sqrt{1-\beta^2}} \\ &= D \frac{v^2}{\sqrt{1-\beta^2}} \cos^2 A + (D \frac{v^2}{\sqrt{1-\beta^2}}) \sin^2 A \end{aligned} \quad (40)$$

Anisotropy is now introduced by making the characteristic temperature a function of the angle A as follows:

$$kT(A) = kT_{\parallel} \cos^2 A + kT_{\perp} \sin^2 A \quad (41)$$

$$\text{where: } kT_{\parallel} = D \frac{v^2}{\sqrt{1-\beta^2}} \text{ for } A = 0 \text{ and}$$

$$kT_{\perp} = D \frac{v^2}{\sqrt{1-\beta^2}} \text{ for } A = 90^\circ.$$

Thus, this anisotropic relativistic distribution may be written as:

$$n = N_0 \exp(-E/(kT_{\parallel} \cos^2 A + kT_{\perp} \sin^2 A)) / \mathcal{D}_M \quad (42)$$

where: E is the total relativistic energy and

$\mathcal{D}_M$ is the normalization constant.

Equation (42) will be referred to as the M distribution. The M distribution may be integrated to yield a single particle distribution function, which in turn may be expressed as a function of the constants of the motion E & μ , as follows:

$$f = \exp(-EC^2(E^2 - u_0^2) / (kT_{\parallel}(C^2(E^2 - u_0^2) - E^2 \mu B) - kT_{\perp} E^2 \mu B)) / \mathcal{D}_M \quad (43)$$

Hence, the M distribution may also be used to describe a quasi-equilibrium plasma.

To contrast the M and H distributions a graph of characteristic temperature versus polar angle A is given in Figure 8, in which $T_{\perp}$ is taken to be ten times $T_{\parallel}$. Thus in Figure 8 for distribution M:

$$T(A)_M = T_{\parallel} \cos^2 A + T_{\perp} \sin^2 A = T_{\parallel} (1 + 9 \sin^2 A) \quad (44)$$

and for distribution H:

$$T(A)_H = 1/(\cos^2 A/T_{\parallel} - \sin^2 A/T_{\perp}) = T_{\parallel} (10/(10 - 9 \sin^2 A)) \quad (45)$$

Note that at angle A equals to 45° for the H distribution, the characteristic temperature is always less than twice $T_{\parallel}$ regardless of how large the ratio of $T_{\perp}$ to $T_{\parallel}$ is; whereas, for the M distribution at A equal to 45° , the characteristic temperature is equal to the average of $T_{\parallel}$ and $T_{\perp}$.

In these last few sections including the historical distribution functions, many distribution functions have been found which satisfy the Vlasov equation and hence may be used to describe a quasi-equilibrium plasma. Many more distribution functions may be found. Therefore additional criteria must be presented to aid in the selection of the proper distribution function to describe an actual physical plasma. The ultimate criterion, which will determine the proper selection, will be the experimental evidence.

Comparison of Characteristic Temperatures
 Expected at Various Polar Angles
 For the M and H Distribution Functions
 For an Anisotropy of $T_{\perp}/T_{\parallel} = 10$

Figure 8

A	H	M
0	1	1
30	1.291	3.25
45	1.818	5.50
60	3.077	7.75
90	10	10

HYBRID COMPUTER SIMULATION OF A SINGLE CHARGED PARTICLE

S.K. Nijhawan

The University of Wisconsin has a hybrid computer which is a combination of AD 256 analog computer and SDS 930 digital computer. An analog model was made for the simulation of the motion of a charged particle in a magnetic mirror machine and externally applied electric field in a 3 dimensional plane. The differential equations of the model were solved by numerical integration to see the validity of the model. The projection of the path of the charged particle was studied in the xy, xz, and yz planes. The motion was plotted by use of an x,y recorder.

The equation of motion of a charged particle is given by

$$\vec{F} = q (\vec{E} + \vec{V} \times \vec{B}) \quad (46)$$

$$\vec{F} = m \frac{d^2 \vec{V}}{dt^2} \quad (47)$$

If we assume $\frac{q}{m} = -1.76 \times 10^{11}$ for an electron and resolve the equation into component form we have

$$\frac{dV_x}{dt} = -1.76 \times 10^{11} [E_x + V_y \cdot B_z - V_z \cdot B_y] \quad (48)$$

$$\frac{dV_y}{dt} = -1.76 \times 10^{11} [E_y + V_z \cdot B_x - V_x \cdot B_z] \quad (49)$$

$$\frac{dV_z}{dt} = -1.76 \times 10^{11} [E_z + V_x \cdot B_y - V_y \cdot B_x] \quad (50)$$

We shall be taking the z axis as the axis of symmetry.

The z component of the magnetic field was taken as $B_0(1+\alpha|z|)$. $B_0(1+\alpha z^2)$ is sometimes a better approximation just to make sure that the particle is reflected somewhere under all circumstances because its gradient is continuous. The x , y , z components of the magnetic field that we considered are as follows:

$$B(z) = B_0(1+\alpha|z|) \quad (51)$$

$$B(x) = - \frac{B_0 \cdot \alpha \cdot x}{2} \frac{|z|}{z} \quad (52)$$

$$B(y) = - \frac{B_0 \cdot \alpha \cdot y}{2} \frac{|z|}{z} \quad (53)$$

The desire to examine the validity of the programs and to observe differences in the motion of the particle from one orbit to another required us to consider a very low magnetic field and a high α (B_0 as low as 3 gauss and $\alpha=3$).

A rough estimate for calculating B_0 is as follows. We would like that the particle is reflected in 10 cyclotron orbits within a distance of 20 cm. If the particle travels 2 cm. in the z -direction in one orbit, then the average velocity in z -direction is 10^6 m/sec. The particle should therefore complete one orbit in $2 \times 10^{-2} / 10^6$ sec. Thus, ω_c should be of the order of $\frac{10^8}{2}$, and the magnetic field should be of the order of 3 gauss.

Since a charged particle is reflected in a magnetic mirror machine along a gradient in the magnetic field, it is considered essential to solve the exact equation of motion of the particle.

The nonvalidity of the guiding center theory due to rapidly changing magnetic fields does not affect our results since we are solving the exact equations.

The problem was scaled to simulate the large velocities, high frequency, and small radius of gyration that appear in reality.

The values used for scaling are the following

$$\begin{aligned}
 x &= 2x_c \\
 y &= 2y_c \\
 z &= .2z_c \\
 V_x &= 10^8 V_{cx} \\
 V_y &= 10^8 V_{cy} \\
 V_z &= 10^7 V_{cz} \\
 t &= .2 \times 10^7 \tau
 \end{aligned}$$

where the symbol c represents the variable in the computer and V_z, V_y, V_x, x, y, z represent the velocities and position in x, y, z direction. τ and t are computer and real time,

The values of the magnetic field after scaling and assuming $B_0 = 3 \text{ gauss} = 3 \times 10^{-4} \text{ webers}$ becomes the following.

$$B_x = - \frac{3 \times 10^{-4}}{2} x_c \cdot \alpha \frac{|z_c|}{z_c} \quad (54)$$

$$B_y = -3 \times 10^{-4} y_c \cdot \alpha \frac{|z_c|}{z_c} \quad (55)$$

$$B_z = 3 \times 10^{-4} \left(1 + \alpha \cdot \frac{|z_c|}{5} \right) \quad (56)$$

The scaling also changed the differential equations of motion as shown, assuming $E_y = E_z = 0$.

$$\frac{dx_c}{d\tau} = V_{cx} \quad (57)$$

$$\frac{dy_c}{d\tau} = V_{cy} \quad (58)$$

$$\frac{dz_c}{d\tau} = V_{cz} \quad (59)$$

$$\frac{dV_x}{d\tau} = -3.5 \left[\frac{E_x}{10^5} + .30 V_{cy} \left(1 + \frac{|z_c|}{5} \right) + .03 V_{cz} x_{yc} \frac{|z_c|}{2} \right] \quad (60)$$

$$\frac{dV_y}{d\tau} = 1.05 \left[\frac{V_{cz} x_c}{10} + V_c \left(1 + \frac{|z_c|}{5} \right) \right] \quad (61)$$

$$\frac{dV_z}{d\tau} = -10.50 [V_{cy} \cdot x_c - V_{cx} \cdot y_c] \frac{|z_c|}{z_c} \quad (62)$$

The new scaled flow diagram is shown on the next page.

The following results were obtained when the differential equations were solved by numerical integration method on the digital computer.

Case A No electric field.

- 1) The total energy remained constant.
- 2) As the particle goes to the higher magnetic fields its radius of gyration keeps on decreasing.
- 3) The particle velocity in the z direction is a sine function of time.

Fig. 11 shows the particle motion projected on the x,y plane and

Fig. 13 shows a graph of V_z vs. time.

Case B Electric field $E_x = 100$ V/m.

Fig. 12 shows the projection on xy planes of the motion.

FIG 9

Fig 10

PROJECTION OF PARTICLE MOTION in
x-y plane.

$$E_0 = 0.$$

Fig 11

PROJECTION OF PARTICLE MOTION
in x-y plane.^{43.}

$$E_D = 100$$

Fig 12

NO ELECTRIC FIELD.

FIG 13

Time →

CALCULATION OF EIGENVALUES OF A PLASMA IN A MICROWAVE CAVITY CONSIDERING THE BOLTZMANN EQUATION

R. T. Pedley and J. L. Shohet

The eigenvalues of a microwave cavity and plasma system will be computed. The geometry of the problem is assumed as follows: the plasma is cylindrical and suspended coaxially in the cylindrical cavity as shown.

It will first be assumed that no surface currents exist and that the plasma dielectric constant has the following form

$$\frac{\epsilon}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2 [1 + \frac{i\nu}{\omega}]}$$

where ω_p is the electron plasma frequency and ν is the electron collision frequency. This result defines the relationship between frequency and wave numbers, which can be used in an eigenvalue equation for the cavity modes.

The following boundary conditions must be satisfied.

1) The tangential components of $\vec{E}$ and $\vec{H}$ must be continuous across the plasma-vacuum interface.

2) The tangential component of $\vec{E}$ must vanish at the cavity boundary.

Condition (2) is equivalent to letting $\frac{\partial H_z}{\partial r} = 0$ at the walls.

The fields in the system have the following form for particular values of the mode indices, ℓ , m , n , corresponding to the 3 degrees of freedom of the system.

Inside the plasma.

$$E_r^i = \left[\frac{i\hbar}{\lambda_1} J_\ell'(\lambda_1 r) a_\ell^i - \frac{\mu\omega\ell}{\lambda_1^2 r} J_\ell(\lambda_1 r) b_\ell^i \right] F_\ell$$

$$E_\theta^i = - \left[\frac{\ell\hbar}{\lambda_1^2 r} J_\ell(\lambda_1 r) a_\ell^i + \frac{i\mu\omega}{\lambda_1} J_\ell'(\lambda_1 r) b_\ell^i \right] F_\ell$$

$$E_z^i = [J_\ell(\lambda_1 r) a_\ell^i] F_\ell$$

$$H_r^i = \left[\frac{\ell k_1^2}{\mu\omega\lambda_1^2 r} J_\ell(\lambda_1 r) a_\ell^i + \frac{i\hbar}{\lambda_1} J_\ell'(\lambda_1 r) b_\ell^i \right] F_\ell$$

$$H_\theta^i = \left[\frac{ik_1^2}{\mu\omega\lambda_1} J_\ell'(\lambda_1 r) a_\ell^i - \frac{\ell\hbar}{\lambda_1^2 r} J_\ell(\lambda_1 r) b_\ell^i \right] F_\ell$$

$$H_z^i = [J_\ell(\lambda_1 r) b_\ell^i] F_\ell$$

Outside the plasma.

$$E_r^e = \left[\frac{i\hbar}{\lambda_2} (J_\ell'(\lambda_2 r) a_\ell^e + Y_\ell'(\lambda_2 r) c_\ell^e) - \frac{\mu\omega\ell}{\lambda_2^2 r} (J_\ell(\lambda_2 r) b_\ell^e + Y_\ell(\lambda_2 r) d_\ell^e) \right] F_\ell$$

$$E_{\theta}^e = -\left[\frac{\ell h}{\lambda_2^2 r} (J_{\ell}(\lambda_2 r) a_{\ell}^e + Y_{\ell}(\lambda_2 r) c_{\ell}^e) + \frac{1}{\lambda_2} (J'_{\ell}(\lambda_2 r) b_{\ell}^e + Y'_{\ell}(\lambda_2 r) d_{\ell}^e)\right] F_{\ell}$$

$$E_z^e = [J_{\ell}(\lambda_2 r) a_{\ell}^e + Y_{\ell}(\lambda_2 r) c_{\ell}^e] F_{\ell}$$

$$H_r^e = \left[\frac{\ell k_2^2}{\mu \omega \lambda_2^2 r} (J_{\ell}(\lambda_2 r) a_{\ell}^e + Y_{\ell}(\lambda_2 r) c_{\ell}^e) + \frac{i h}{\lambda_2} (J'_{\ell}(\lambda_2 r) b_{\ell}^e + Y'_{\ell}(\lambda_2 r) d_{\ell}^e)\right] F_{\ell}$$

$$H_{\theta}^e = \left[\frac{i k_2^2}{\mu \omega \lambda_2} (J'_{\ell}(\lambda_2 r) a_{\ell}^e + Y'_{\ell}(\lambda_2 r) c_{\ell}^e) - \frac{\ell h}{\lambda_2^2 r} (J_{\ell}(\lambda_2 r) b_{\ell}^e + Y_{\ell}(\lambda_2 r) d_{\ell}^e)\right] F_{\ell}$$

$$H_z^e = [J_{\ell}(\lambda_2 r) b_{\ell}^e + Y_{\ell}(\lambda_2 r) d_{\ell}^e] F_{\ell}$$

In these relations

$$\lambda_1^2 = k_1^2 - \left(\frac{n\pi}{L}\right)^2$$

$$\lambda_2^2 = k_2^2 - \left(\frac{n\pi}{L}\right)^2$$

$$F_{\ell} = \exp(i\ell\theta + i h z - i \omega t)$$

$$h = \frac{n\pi}{L}$$

$$k_2 = \omega(\epsilon_0 \mu_0)^{1/2} \quad (\text{the free space wave number})$$

k_1 , the wavenumber in the plasma, can be obtained from the dispersion relation

$$k_1 = \frac{\omega}{c} \left(1 - \frac{\omega_p^2}{\omega^2(1+i\nu/\omega)} \right)$$

$a_\ell^i, b_\ell^i, a_\ell^e, b_\ell^e, c_\ell^e, d_\ell^e$, are the unknowns. The index m will appear because the eigenvalue equation has multiple roots. The cylindrical functions for inside the plasma are Bessel functions of the first kind. The cylindrical functions for the vacuum region are Bessel functions of the first and second kind. A prime above a function denotes differentiation with respect to its argument.

The problem now to be considered is that of finding the relation between k_1 and ω utilizing the Vlasov equation. This permits a more general development and allows consideration of various distribution functions. A dispersion relation for the plasma region will be derived based on the above method. It is desired to obtain a relation for the transverse waves of the electromagnetic mode. The collisionless Boltzmann Equation will be used and the motion of the ions will be considered small. The following equations will be used.

The Vlasov Equation may be written

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f - \frac{e\vec{E}}{m} \cdot \frac{\partial f}{\partial \vec{v}} = 0 \quad (63)$$

Assuming that we can neglect the effects of the wave magnetic field. Maxwell's Equations become

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} [n_0 - \iiint f(\vec{r}, \vec{v}, t) d^3v] \quad (64)$$

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}_1}{\partial t} \quad (65)$$

$$\nabla \cdot \vec{E} = \mu_0 \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (66)$$

$$\nabla \cdot \vec{B} = 0 \quad (67)$$

where B is the wave magnetic field, and E is the wave electric field. A distribution of the form

$$f = n_0 f_0(\vec{v}) + f_1(\vec{r}, \vec{v}, t) \quad (68)$$

will be used.

Next, we substitute Equation (68) into Equation (63) and eliminate the nonlinear terms, yielding

$$\frac{\partial f_1(\vec{r}, \vec{v}, t)}{\partial t} + \vec{v} \cdot \nabla f_1 - \frac{en_0 \vec{E}}{m} \frac{\partial f_0(\vec{v})}{\partial \vec{v}} = 0 \quad (69)$$

We now define

$$\rho(\vec{r}, t) = -e \int f_1(\vec{r}, \vec{v}, t) d\vec{v} \quad (70)$$

$$\vec{J}(\vec{r}, t) = -e \int \vec{v} f_1(\vec{r}, \vec{v}, t) d\vec{v} \quad (71)$$

Assuming wave type solutions, we get

$$f_1(\vec{v}) = f(\vec{v}) e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{E} = \vec{E} e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{B} = \vec{B}_1 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

and Fourier analyzing equations (64), (65), (66), (67) and (68) we get

$$1(\vec{k} \cdot \vec{v} - \omega) f_1(\vec{k}, \vec{v}) - \frac{en_0 \vec{E}(\vec{k}) \cdot \frac{\partial f_0(\vec{v})}{\partial \vec{v}}}{m} = 0 \quad (72)$$

$$\vec{k} \cdot \vec{B} = 0 \quad (73)$$

$$\vec{k} \cdot \vec{E} = \frac{-1}{\epsilon_0} \rho \quad (74)$$

$$\vec{k} \times \vec{E} = \omega \vec{B}_1 \quad (75)$$

$$\vec{k} \times \vec{B} = i\mu_0 \vec{j} - \omega \vec{E} \quad (76)$$

We can solve Equation (72) for $f_1(\vec{k}, \vec{v})$

$$f_1(\vec{k}, \vec{v}) = \frac{1}{1(\vec{k} \cdot \vec{v} - \omega)} \left[\frac{en_0}{m} \vec{E}(\vec{k}) \cdot \frac{\partial f_0(\vec{v})}{\partial \vec{v}} \right] \quad (77)$$

Then Maxwell's Equations are solved for $\vec{j}(\vec{k})$ in terms of $\vec{E}$

$$\vec{j}(\vec{k}) = \frac{i}{\mu_0 \omega} [\vec{E}(\omega^2 - k^2) + \vec{k} \vec{k} \cdot \vec{E}] \quad (78)$$

Using Equation (71), we combine it with Equation (77) to obtain

$\vec{j}(\vec{k})$ in terms of $\vec{E}$, namely

$$\vec{j}(\vec{k}) = \frac{ie^2 n_0}{m} \int \frac{\vec{v} d\vec{v}}{(\vec{k} \cdot \vec{v} - \omega)} \frac{\partial f_0(\vec{v})}{\partial \vec{v}} \cdot \vec{E}(\vec{k}) \quad (79)$$

We can now set Equation (78) equal to Equation (79)

$$\frac{i}{\omega \mu_0} [\vec{E}(\omega^2 - k^2) + \vec{k} \vec{k} \cdot \vec{E}] = \frac{ie^2 n_0}{m} \int \frac{\vec{v} d\vec{v}}{(\vec{k} \cdot \vec{v} - \omega)} \frac{\partial f_0(\vec{v})}{\partial \vec{v}} \cdot \vec{E}(\vec{k})$$

or

$$(\omega^2 - k^2) \vec{E} + \vec{k} \vec{k} \cdot \vec{E} = \frac{\omega e^2 n_0 \mu_0}{m} \int \frac{\vec{v} d\vec{v}}{(\vec{k} \cdot \vec{v} - \omega)} \frac{\partial f_0(\vec{v})}{\partial \vec{v}} \cdot \vec{E}(\vec{k}) \quad (80)$$

We shall now assume $\vec{E}$ has longitudinal and transverse components and may be written as

$$\vec{E} = \vec{E}_t + \frac{\vec{k}\vec{k}\cdot\vec{E}}{k^2} \quad \text{with } \vec{k}\cdot\vec{E}_t = 0$$

Equation (80), after substituting in the above becomes

$$\begin{aligned} (\omega^2 - k^2)\vec{E} + \vec{k}\vec{k}\cdot\vec{E} - \omega\omega_p^2 \mu_o \epsilon_o \frac{\vec{k}\cdot\vec{E}}{k^2} \int \frac{\vec{v} d\vec{v}}{(\vec{k}\cdot\vec{v} - \omega)} \vec{k} \cdot \frac{\partial f_o}{\partial \vec{v}}(\vec{v}) \\ - \omega\omega_p^2 \mu_o \epsilon_o \int \frac{\vec{v} d\vec{v}}{(\vec{k}\cdot\vec{v} - \omega)} \vec{E}_t \cdot \frac{\partial f_o}{\partial \vec{v}}(\vec{v}) = 0 \end{aligned} \quad (81)$$

The transverse modes can be obtained by operating with $\vec{k} \times$ on Equation (81), viz.,

$$\begin{aligned} \vec{k} \times \vec{E} (\omega^2 - k^2) + \cancel{\vec{k} \times \vec{k} \cdot \vec{E}}^0 - \omega\omega_p^2 \mu_o \epsilon_o \frac{\vec{k} \times \vec{E}}{k^2} \int \frac{d\vec{v} (\vec{k} \times \vec{v})}{(\vec{k}\cdot\vec{v} - \omega)} \vec{k} \cdot \frac{\partial f_o}{\partial \vec{v}}(\vec{v}) \\ - \omega\omega_p^2 \mu_o \epsilon_o \int \frac{d\vec{v} (\vec{k} \times \vec{v})}{(\vec{k}\cdot\vec{v} - \omega)} \vec{E}_t \cdot \frac{\partial f_o}{\partial \vec{v}}(\vec{v}) = 0 \end{aligned} \quad (82)$$

$f_o(\vec{v})$ will initially be assumed a Maxwellian.

The fourth term must be evaluated by integration by parts.

$$\begin{aligned} \int_{-\infty}^{+\infty} \frac{d\vec{v}}{(\vec{k}\cdot\vec{v} - \omega)} (\vec{k} \times \vec{v}) \vec{E}_t \cdot \frac{\partial f_o}{\partial \vec{v}}(\vec{v}) \\ = \frac{(\vec{k} \times \vec{v}) \vec{E}_t f_o}{\vec{k}\cdot\vec{v} - \omega} \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f_o \vec{E}_t \cdot \frac{\partial}{\partial \vec{v}} \left(\frac{\vec{k} \times \vec{v}}{\vec{k}\cdot\vec{v} - \omega} \right) d\vec{v} \\ = f_o \frac{\partial}{\partial \vec{v}} \left(\frac{\vec{k} \times \vec{v}}{\vec{k}\cdot\vec{v} - \omega} \right) \Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} \vec{E}_t \cdot \frac{\vec{k} \times \vec{v}}{(\vec{k}\cdot\vec{v} - \omega)} \frac{\partial f_o(\vec{v})}{\partial \vec{v}} d\vec{v} \\ = \frac{\vec{v}}{(\vec{k}\cdot\vec{v} - \omega)} f_o(\vec{v}) \Big|_{-\infty}^{+\infty} - \vec{E}_t \times \vec{k} \int \frac{f_o(\vec{v})}{\vec{k}\cdot\vec{v} - \omega} d\vec{v} \end{aligned}$$

The third term in Equation (82) vanishes. This can be seen by noting that it is of exactly the same form as the 4th term except that k is written in place of E_t . Thus, by inspecting the last expression above, it can be seen that the third term is zero. Thus, Equation (82) becomes

$$\vec{k} \times \vec{E}(\omega^2 - k^2) + \omega \omega_p^2 \mu_o \epsilon_o \vec{k} \times \vec{E}_t \int \frac{f_o(\vec{v})}{\vec{k} \cdot \vec{v} - \omega} d\vec{v} = 0 \quad (83)$$

Since $\vec{k} \times \vec{E} \equiv \vec{k} \times \vec{E}_t$, we may obtain a dispersion relation from (83). It is

$$k^2 = \omega^2 + \omega \omega_p^2 \mu_o \epsilon_o \int_{-\infty}^{\infty} \frac{f_o(\vec{v})}{\vec{k} \cdot \vec{v} - \omega} d\vec{v} \quad (84)$$

It is now necessary to evaluate the integral in Equation (84). We assume f_o is a function of the complex variable v

$$P(v) = \int_{-\infty}^{\infty} \frac{f_o(v)}{v - \frac{\omega}{k}} dv \quad (85)$$

which has a singularity at $v = \frac{\omega}{k}$. If f_o is assumed to be a Maxwellian ($f_o = v^2 e^{-v^2/\alpha^2}$), $k > 0$, and $I_m \omega > 0$, then by use of the Cauchy Residue Theorem, Equation (85) becomes

$$P\left(\frac{\omega}{k}\right)_{\text{contour 1}} = P\left(\frac{\omega}{k}\right)_{\text{contour 2}} + 2\pi i f\left(\frac{\omega}{k}\right)$$

Assuming F has no other singularities, contour 2 may be pushed to infinity. Thus

$$F\left(\frac{\omega}{k}\right) = 2\pi i \frac{\omega^2}{k^2} e^{-\omega^2/k^2/\alpha^2}$$

Equation (84) now becomes

$$k^2 = \omega^2 - \omega \omega_p^2 \mu_0 \epsilon_0 \left(2\pi i \frac{\omega^2}{k^2} e^{-\omega^2/k^2/\alpha^2} \right)$$

which is a transcendental equation that can be solved for k for various values of ω by iterative techniques on the computer. This program is being developed at the present time.

For particular values of l and n , a set of homogeneous linear equations in terms of a_l^i , b_l^i , a_l^e , b_l^e , c_l^e , d_l^e can be obtained from Equations (63) - (68) and (69) - (74) by application of the boundary conditions. The resulting system of equations has a nontrivial solution only when the system 6×6 determinant is set equal to zero. The program for this is also being worked on at this time. This eventually will be extended to anisotropic energy distributions.

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