Algebra used by DWUCK

with applications to the normalization
of typical reactions

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A. General Description of DWUCK

The computer program DWUCK calculates the scattering differential cross section for a general form of the distorted wave Born approximation (DWBA). The incoming and outgoing waves may be in any combination of spin 0, spin 1/2, or spin 1 particles. The spin dependent parts of the optical potentials for the distorted waves must be diagonal in the total and orbital angular momentum. The calculations are performed in a zero range form between the coordinates of the incoming and outgoing waves. This specialization is not important for the usual inelastic scattering theories. The particle transfer reactions can be treated with this program with a local energy approximation for the finite range effects. However, reactions such as heavy particle stripping and knock-on reactions cannot be calculated with the program since the finite range effects are important and cannot be included in the present computer code.

The DWBA computer code DWUCK calculates a transition amplitude for the reaction $A(a,b)B$ of the form

$$T = J \int d\vec{r}_b \int d\vec{r}_a \chi_f^{-*}(\vec{k}_f, \vec{r}_b) \langle bB | V | aA \rangle \chi_i^+(\vec{k}_i, \vec{r}_a)$$

where χ_f^- and χ_i^+ are the distorted waves, $\vec{r}_a$ and $\vec{r}_b$ are the relative coordinates for the systems (a,A) and (b,B) respectively, and J is the Jacobian of the transformation to these coordinates. The quantity $\langle bB | V | aA \rangle$ is the form factor for the reaction and must contain a delta function for the coordinates $\vec{r}_a$ and $\vec{r}_b$. In the absence of spin the distorted wave χ_i^+ describes asymptotically a plane wave of momentum $\vec{k}$ plus an outgoing scattered wave which in the case of no Coulomb potential has the form

$$\chi_i^+(\vec{k}, \vec{r}) \xrightarrow{r \rightarrow \infty} e^{i\vec{k} \cdot \vec{r}} + f(\theta) \frac{e^{ikr}}{r}$$

The final state distorted wave which has an incoming scattered wave condition is related to the solution with outgoing waves by time reversal

$$\chi_{\underline{\hat{r}}}^{-*}(k, r) = \chi_{\underline{\hat{r}}}^{+}(-k, r).$$

When the particles a and b have spin the functions $\chi^{\pm}$ become matrices in spin space $\chi_{m'm}^{\pm}$. The relevant time reversal condition is

$$\chi_{m'm}^{-*}(k, r) = (-)^{m-m'} \chi_{-m', -m}^{+}(-k, r).$$

The factor $\langle bB | V | aA \rangle$ which contains the nuclear structure information is expressed as

$$\begin{aligned} \langle J_B M_B \Lambda_B m_B | V | J_A M_A \Lambda_A m_A \rangle &= \sum_{l s j} B_{l s j} \langle J_A \Lambda_A M_A m_A - M_B | J_B M_B \rangle \\ &\times \langle \Delta_B \Lambda_B m_B m_A - m_B | \Lambda_A M_A \rangle \langle l s m m_A - m_B | j M_B - M_A \rangle \\ &\times f_{l s j}(r_a) \delta(r_b - \frac{A}{B} r_a) i^{-l} Y_l^m(\hat{r}_a)^*. \end{aligned}$$

The quantity $B_{l s j}$ is a measure of the strength of the interaction and is equal to $\sqrt{\frac{2\Delta+1}{2\Delta_A+1}} A_{l s j}$ where $A_{l s j}$ is the spectroscopic coefficient used in JULIE.

The distorted waves $\chi_{m'm}^{\pm}(k, r)$ can be written as

$$\begin{aligned} \chi_{m'm}^{\pm}(k, r) &= \frac{\sqrt{4\pi}}{kr} \sum_{JL} i^L \langle L \Delta 0 m | J m \rangle \sqrt{2L+1} \chi_{JL}(k, r) \\ &\times \langle L \Lambda m' - m' m' | J m' \rangle Y_L^{m' - m'}(\hat{r}) D_{m m'}^J(\hat{k})^* \end{aligned}$$

where D_{mm}^J are the rotation functions.¹⁾ The radial part of the distorted waves satisfies the equation

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{L(L+1)}{r^2} - \frac{\partial \mu}{\hbar^2} (U + U_c + U_L^J) \right] \chi_{JL}(k, r) = 0$$

1) A. R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton University Press, Princeton, N.J. (1957).

where $\chi_{JL}(k,0) = 0$ and for large r (where $U + U_L^J$ may be neglected) and also satisfies the boundary condition

$$\chi_{JL}(k,r) \rightarrow \frac{[H_L^*(kr) - \eta_L^J H_L(kr)] e^{i\sigma_L}}{2i}$$

Here $H_L(kr) = G_L + iF_L$ is the outgoing wave Coulomb function, η_L^J is the reflection coefficient and σ_L is the Coulomb phase shift. The quantity $\frac{\eta_L^J - 1}{2i}$ is the usual partial wave scattering amplitude. The functions χ_{JL} are computed by numerically integrating the differential equations to a value of r where either the potentials $U + U_L^J$ are negligible or where the form factor $f_{\alpha\beta}(\vec{r})$ makes a negligible contribution to the reaction. The χ_{JL} are matched to the Coulomb functions G_L and F_L at the last two integration points to determine η_L^J and the overall normalization constants.

With the definitions of $\langle bB|V|aA \rangle$ and the distorted waves $\chi_{m'm}(kr)$ we can write down the transition amplitude as

$$\begin{aligned} T = & \frac{4\pi}{k_a k_b} \sum_{\alpha\alpha'} \sqrt{2\alpha+1} B_{\alpha\alpha'} \langle J_A J_A M_A M_B - M_A | J_B M_B \rangle \\ & \times \sum_{\substack{J_A L_A \\ J_B L_B}} \langle L_a \Delta_a 0 m_a | J_a m_a \rangle \langle L_b \Delta_b 0 m_b | J_b m_b \rangle \\ & \times \langle J_b J m_a - m m | J_a m_a \rangle (2L_b+1) \langle L_b 2 0 0 | L_a 0 \rangle \\ & \times \sqrt{(2\Delta_a+1)(2J+1)(2J_b+1)(2L_a+1)} \left\{ \begin{matrix} L_b \Delta_b J_b \\ J \Delta J' \\ L_a \Delta_a J_a \end{matrix} \right\} d_{m_b m_a - m}^{J_b}(\hat{k}_b) \frac{B}{A} I_{J_a L_a J_b L_b}^{\alpha\alpha'}. \end{aligned}$$

In the above expression the usual nine-j symbol appears and

$$d_{m'm}^J(\theta) = D_{m'm}^J(0,0,0).$$

The radial integrals are defined as

$$I_{J_a L_a J_b L_b}^{l \lambda j} = \frac{A}{B} \int dr_a \chi_{J_b L_b}(k_b, \frac{A}{B} r_a) f_{l \lambda j}(r_a) \chi_{J_a L_a}(k_a r_a).$$

If we define the new quantities

$$\begin{aligned} \beta_{l \lambda j}^{m m_a m_b} &= \sum_{J_a L_a L_b} i^{L_a - L_b - l} \langle L_a \Delta_a 0 m_a | J_a m_a \rangle \langle L_b \Delta_b 0 m_b | J_b m_b \rangle \\ &\times \langle J_b j m_a - m m | J_a m_a \rangle (2L_b + 1) \langle L_b l 0 0 | L_a 0 \rangle \\ &\times \sqrt{(2\Delta_a + 1)(2j + 1)(2J_b + 1)(2L_a + 1)} \left\{ \begin{matrix} L_b \Delta_b J_b \\ l \quad \Delta \quad j \\ L_a \Delta_a J_a \end{matrix} \right\} I_{J_a L_a J_b L_b}^{l \lambda j} \end{aligned}$$

and

$$S_{l \lambda j}^{m m_a m_b} = \sum_{J_b} \beta_{l \lambda j}^{m m_a m_b} d_{m_b m_a - m}^{J_b}(\hat{k}_b)$$

then we may write the transition amplitude in the form

$$T = \frac{4\pi}{k_a k_b} \frac{B}{A} \sum_{l \lambda j} \sqrt{2l+1} B_{l \lambda j} \langle J_a j m_a m_b - m_a | J_a m_a \rangle S_{l \lambda j}^{m m_a m_b}$$

The differential cross section for the A(a,b)B reaction is defined in terms of T by

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\mu_a \mu_b}{(2\pi \hbar^2)^2} \frac{k_b}{k_a} \frac{1}{2J_a + 1} \frac{1}{2\Delta_a + 1} \sum_{m_a m_b m_a m_b} |T|^2 \\ &= \frac{2J_b + 1}{2J_a + 1} \frac{4\pi}{E_a E_b} \frac{k_a}{k_b} \left(\frac{B}{A}\right)^2 \frac{1}{2\Delta_a + 1} \sum_{j m} \frac{1}{2j + 1} \left| \sum_{l \lambda} \sqrt{2l+1} B_{l \lambda j} S_{l \lambda}^{m m_a m_b} \right|^2 \end{aligned}$$

where E_a and E_b are the center of mass energies for the entrance and exit channels respectively.

The program DWUCK computes the quantity

$$\sigma_{DW}^{2qj} = \frac{4\pi}{E_a E_b} \frac{k_a}{k_b} \left| \frac{B}{A} \right|^2 \frac{1.0 \times 10^4}{2\lambda_a + 1} (2\ell + 1) \sum_{m m_a m_b} \left| S_{2qj}^{m m_a m_b} \right|^2 \quad a \neq b$$

$$\sigma_{DW}^{2qj} = \frac{4\pi}{E_a E_b} \frac{k_a}{k_b} \left| \frac{B}{A} \right|^2 \frac{1}{2\lambda_a + 1} \sum_{m m_a m_b} \left| S_{2qj}^{m m_a m_b} \right|^2 \quad a = b$$

in units of F^2/s^2 $[1F^2 = 10 \text{ mb} = 10^{-26} \text{ cm}^2]$.

Therefore the cross section is

$$\frac{d\sigma^{2qj}}{d\Omega} = \frac{2J_B + 1}{2J_A + 1} \frac{|B_{2qj}|^2}{1.0 \times 10^4} \frac{\sigma_{DW}^{2qj}}{2j + 1} \quad a \neq b$$

$$\frac{d\sigma^{2qj}}{d\Omega} = \frac{2J_B + 1}{2J_A + 1} \frac{2\ell + 1}{2j + 1} |B_{2qj}|^2 \sigma_{DW}^{2qj} \quad a = b.$$

[Note : $a=b$ also means $p=n$ and $H_e^3 = T, eZ_c$]

B. Application to Reactions

a) Collective nuclear excitations by inelastic scattering

To first order in the deformations we can write the interaction term V

as

$$V = -\beta_2 \frac{R_0}{a} V_0 \frac{df}{dx} Y_2^0(\theta)$$

where $x = (r - R_0)/a$ and θ is the angle between the scattering particle and the nuclear symmetry axis. If we take the strong coupling form of the initial and final state wave function

$$\psi_{JM_K}^j = \sqrt{\frac{2J+1}{16\pi^2}} \left[\phi_K^j D_{KM}^J + (-)^{J-j} \phi_{-K}^j D_{-KM}^J \right]$$

we find that

$$\langle J_B M_B \lambda_B m_B | V | J_A M_A \lambda_A m_A \rangle = -\beta_2 \frac{R_0}{a} V_0 \frac{df}{dx} \sqrt{\frac{2J_A+1}{2J_B+1}}$$

$$\times \langle J_A \ell K 0 | J_B K \rangle \langle J_A \ell M_A M_B - M_A | J_B M_B \rangle Y_\ell^m(\hat{n}_0)^*$$

The form factor computed by DWUCK is

$$- \frac{R_0}{a} V_0 \frac{df}{dx}$$

and since $s = 0$ we have

$$B_{\text{pol}} = \beta_l \langle J_A l K 0 | J_B K \rangle \sqrt{\frac{2J_A+1}{2J_B+1}}$$

The cross section is

$$\frac{d\sigma}{d\Omega} = \beta_l^2 |\langle J_A l K 0 | J_B K \rangle|^2 \sigma_{\text{DW}}^l(\theta)$$

b) Stripping reactions (d,p)

The matrix element for a (d,p) reaction may be written as

$$\begin{aligned} \langle J_B M_B \lambda_B m_B | V | J_A M_A \lambda_A m_A \rangle &= \sum_{j\ell} \sqrt{S_{j\ell}} R_{j\ell}(r_{xA}) (-i)^\ell Y_{\ell m}(\hat{r}_{xA}) \\ &\times \langle \ell s m \mu - m | j \mu \rangle \langle \lambda_B \lambda m_B m_B - m_A | \lambda_A m_A \rangle \langle J_A J M_A M_B - M_A | J_B M_B \rangle \\ &\times D(\hat{r}_{xb}) \end{aligned}$$

where $S_{j\ell}$ is the spectroscopic factor, $R_{j\ell}(r)$ is a normalized radial function, and $D(\hat{r}_{xb})$ is the product of the neutron-proton potential times the deuteron internal wave function. By making use of the zero range approximation which gives

$$D(\hat{r}_{xb}) = D_0 \delta(\hat{r}_x - \hat{r}_b)$$

we find that

$$B_{\text{pol}} = \sqrt{S_{\text{pol}}} D_0$$

where the form factor is

$$f_{lsj}(r) = R_{jl}(r).$$

The cross section then is written as

$$\frac{d\sigma}{d\Omega} = \frac{2J_B+1}{2J_A+1} S_{ej} \frac{D_0^2}{1.0 \times 10^4} \frac{\sigma_{DW}}{2j+1}$$

A value of D_0 when the deuteron wave function is described by a Hulthén function

$$\phi_0(r_{np}) = N \frac{(e^{-\alpha r_{np}} - e^{-\beta r_{np}})}{r}$$

where

$$N = \sqrt{\frac{\alpha\beta(\alpha+\beta)}{2\pi(\alpha-\beta)^2}}$$

is

$$D_0 = - \left(1 + \frac{\alpha}{\beta}\right)^{3/2} \frac{\hbar^2}{m} \sqrt{8\pi\alpha}$$

If we take $\beta = 1.48 \text{ F}^{-1}$ then we find

$$D_0 = -1.235 \times 10^2 \text{ MeV} - F^{3/2}$$

$$\text{or } (D_0/10^2)^2 = 1.53 \text{ MeV}^2 - F^3.$$

c) Pick-up reaction (pd)

From time reversal invariance of the d,p reaction described above we

may write the cross section for the p,d reaction in terms of the Dp cross section,

$$\begin{aligned} \frac{d\sigma_{po}}{d\Omega} &= \frac{2J_A+1}{2J_B+1} \frac{2D_0+1}{2D_0+1} \frac{k_a^2}{k_b^2} \frac{d\sigma_{Dp}}{d\Omega} \\ &= \frac{3}{2} S_{ej} \frac{D_0^2}{1.0 \times 10^4} \frac{\sigma_{DW}}{2j+1} \end{aligned}$$

where σ_{DW} is calculated with protons in the incident channel and deuterons in the exit channel.

Microscopic Interaction Model for Inelastic Scattering

a) Central interactions

The potential between the projectile and one of the target nucleons may be expanded in a Legendre polynomial series

$$V(r_0 - r) = V_0 \sum_l (2l+1) v_l(r_0, r) P_l(\hat{r}_0 \cdot \hat{r})$$

$$= 4\pi V_0 \sum_l v_l(r_0, r) \sum_m Y_l^m(\hat{r}_0)^* Y_l^m(\hat{r})$$

Using this form we find for the matrix element of V

$$\langle J_B M_B \Delta_B m_B | V | J_A M_A \Delta_A m_A \rangle =$$

$$4\pi V_0 \sum_l \langle J_A \Delta_A M_A M_B - M_A | J_B M_B \rangle \langle J_B || v_l(r_0, r) i^2 Y_l || J_A \rangle$$

$$\times \langle \Delta_B \Delta m_B m_B - m_B | \Delta_A m_A \rangle (-i)^2 Y_l^m(\hat{r}_0)^*$$

The program DWUCK calculates for the form factor the expression

$$4\pi V_0 \langle j_1' j_2 J_B || Y_l(\hat{r}) || j_1 j_2 J_A \rangle \int R_{l,j_1'}(r_1) v_l(r_0, r_1) R_{l,j_1}(r_1) r_1^2 dr_1$$

where the $R_{lj}(r_1)$ are the normalized radial functions for the initial and final nucleon states, and the reduced matrix element $\langle j_1' j_2 J_B || Y_l || j_1 j_2 J_A \rangle$ is defined by Edmonds^v. The quantity j_2 is the spin for the coupling of the A-1 nucleons. The reduced matrix element $\sqrt{4\pi} \langle j_1' j_2 J_B || Y_l || j_1 j_2 J_A \rangle$ is printed out as RME in the output.

Since $B_{\text{bol}} = (2J_B + 1)^{-\frac{1}{2}}$ we have

$$\frac{d\sigma^{\text{el}}}{d\Omega} = \frac{1}{2J_A + 1} \sigma_{\text{bw}}^{\text{el}}$$

b) $\sigma_i \cdot \sigma_j$ interaction

A central interaction $V(r_0 - r_f)$ with a $\sigma_0 \cdot \sigma_f$ dependence can be written in the following form

$$V(\underline{r}_0 - \underline{r}_1) \underline{\sigma}_0 \cdot \underline{\sigma}_1 = 4\pi V_0 \sum_{\ell j \mu} V_\ell(r_0, r_1) (-i)^{\ell-j+\mu} Y_{\ell j -\mu}(0) Y_{\ell j \mu}(1)$$

where

$$Y_{\ell j \mu} = \sum_m \langle \ell s \mu \mu-m | j \mu \rangle i^\ell Y_\ell^m(\theta) \sigma^{\mu-m}$$

The matrix element for V in the transition amplitude is

$$\begin{aligned} & \langle J_B M_B \Delta_b m_b | V | J_A M_A \Delta_a m_a \rangle = \\ & 4\pi V_0 (-i)^{\ell-j} \langle \ell s m \mu-m | j \mu \rangle (-i)^\ell Y_\ell^m(\hat{r}_0)^* \\ & \times \langle \Delta_b \Delta m_b m_a - m_b | \Delta_a m_a \rangle 2\sqrt{\Delta_b(\Delta_b+1)} (-i)^{2j} \langle J_B || V_\ell(r_0, r_1) i^\ell Y_\ell(\hat{r}_1) || J_A \rangle \\ & \times \langle J_A J M_A M_B - M_A | J_B M_B \rangle \frac{1}{\sqrt{2J_B+1}} \end{aligned}$$

where $s=1$ and $s_a=s_b$.

The form factor computed by DWUCK is

$$4\pi V_0 \sqrt{3} \langle j_1' j_2 J_B || V_\ell(r_0, r_1) i^\ell Y_{\ell j'}(1) || j_1 j_2 J_A \rangle$$

and the quantity

$$B_{\ell j'} = (-i)^{2j} \frac{2}{\sqrt{2J_B+1}} \sqrt{\frac{\Delta_b(\Delta_b+1)}{3}}$$

We obtain the cross section

$$\frac{d\sigma}{d\Omega} = \frac{4}{2J_A+1} \frac{\Delta_b(\Delta_b+1)}{3} \sigma_{0w}^{\ell j'}$$

Note: It should be pointed out that the cross section for the microscopic interaction is done in terms of a single particle transition $j_1 \rightarrow j_1'$. If there is more than one equivalent particle then the matrix element ratio

$$\sum_i \langle J_B || Y_j(i) || J_A \rangle / \langle j_1' j_2 J_B || Y_j(1) || j_1 j_2 J_A \rangle$$

will have to multiply the $B_{\ell s j}$ factors which are outlined here.