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ON THE CONNECTION BETWEEN THE DISTORTED WAVE AND DISPERSION

THEORIES OF DIRECT NUCLEAR REACTIONS

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In a recent series of papers, Shapiro and his collaborators, and Saperstein have outlined a program for treating direct nuclear reactions which is based on dispersion theory techniques. This program, which superficially appears to be much different from the widely used method of distorted waves, was motivated by an observation of Amado that the amplitude for a stripping reaction contains a pole in the momentum transfer variable quite close to the physical region.

It was originally hoped that the residue at the stripping pole, which is proportional to the reduced width of the captured particle state, could be determined by extrapolating experimental data. Attempts to determine reduced widths by extrapolation techniques, however, tend to be frustrated by distortions due to initial and final state interactions and the form factors for the deuteron and captured particle vertex functions.

The program of Shapiro et al and Saperstein is not so much concerned with the determination of reduced widths as it is with constructing the direct reaction amplitude starting with known (or assumed) vertex form factors and the (complex) phase shifts associated with initial and final state scattering. It is claimed that the dispersion theory method is superior to the method of distorted waves since it uses only the asymptotic behavior (phaseshifts) of the scattering wave functions.
The distorted wave method makes use of the entire scattering wave functions and thus would yield ambiguous results if several different optical potentials account for the same set of scattering phase shifts. The dispersion theory method, as formulated up to now, does not seem to suffer from such ambiguities.

The purpose of this note is to show that the distorted wave and dispersion methods are not as dissimilar as has been suggested and in fact probably give essentially the same results. The ambiguity mentioned above in connection with the distorted wave approach is also present in the dispersion theory method when the latter is properly formulated.

Let us consider the non-relativistic stripping process $A(d,p)B$ corresponding to zero angular momentum for the captured neutron state. For simplicity, we ignore Coulomb effects and the spin and statistics of the particles and assume that there are only two possible channels, $(A + d)$ and $(B + p)$. In the following discussion, these channels will be referred to as $\alpha$ and $\beta$ respectively.

The center of mass energy of the system is

$$E = \frac{n^2 k_d^2}{2\nu_\alpha} + \epsilon_\alpha = \frac{n^2 k_p^2}{2\nu_\beta} + \epsilon_\beta$$

where

$$\nu_\alpha = \frac{M_d M_A}{M_d + M_A}, \quad \nu_\beta = \frac{M_p M_B}{M_p + M_B}$$

and $M_d$, $M_A$, $M_p$ and $M_B$ are respectively the masses of the deuteron, nucleus $A$, the proton and nucleus $B$. $\epsilon_\alpha$ and $\epsilon_\beta$ are the internal energies.
The reduced two by two scattering matrix for center of mass angular momentum \( l \) satisfies the unitarity condition

\[
\begin{align*}
\text{Im}(t_\ell) &= (t_\ell) \left( e^{i\delta_\alpha} \sin \delta_\alpha \right) \xi(\xi - \epsilon_\alpha) \\
&\quad + (t_\ell) \left( e^{i\delta_\beta} \sin \delta_\beta \right) \xi(\xi - \epsilon_\beta)
\end{align*}
\]

since the right hand side of (5) is real, we see that the phase of \( t_\ell \beta \alpha \) is given by the phase of

\[
\begin{align*}
e^{i\delta_\alpha} \sin \delta_\alpha \xi(\xi - \epsilon_\alpha) - e^{i\delta_\beta} \sin \delta_\beta \xi(\xi - \epsilon_\beta)
\end{align*}
\]
The above result, which is a generalization of the Fermi-Watson final-state interaction theorem, was derived by Saperstein in reference 5. His result, however, contains an error in sign (see his equations (17) and (18) and the discussion following them.)

Now \( t_2(E) \) has branch cuts on the physical sheet of the energy plane which start below the threshold for physical scattering \(^6,7\). The cut nearest the physical region is determined by partial wave contributions of the Butler stripping diagram (Figure 1.), i.e. the ordinary plane wave Born approximation. If we assume constant form factors (neglect the internal structure of the deuteron and nuclei A and B). The Butler pole contribution to \( t_2(E) \) is, aside from constant factors, \(^8,9,10\)

\[
\frac{1}{(k_p k_d)^2} \int_0^\infty r^2 dr e^{-k_p r} J_\lambda(k_d r) J_\lambda \left( k_p \frac{M_A}{M_A + M_n} \right)
\]

\[
= \frac{1}{2(k_p k_d)^{2+1}} Q_\lambda \left( \frac{k_n^2 + k_d^2 + \left( \frac{M_A}{M_A + M_n} \right)^2 k_p^2}{\left( \frac{M_A}{M_A + M_n} \right)} \right)
\]

\[
k_n = \sqrt{\frac{2 M_A M_n}{M_n + M_A}} \frac{|\epsilon_d - \epsilon_a + \epsilon_d|}{n^2}
\]

where \( \epsilon_d \) is the internal energy of the deuteron, \( M_n \) is the neutron mass, \( J_\lambda \) is the spherical Bessel function \(^13\) and \( Q_\lambda \) is the Legendre function of the second kind. \(^14\)

The branch points of (7) on the physical energy sheet are determined by the equations

\[
\left( k_p \pm \frac{M_A}{M_A + M_n} k_p \right)^2 = -k_n^2
\]
\[
\text{Im } k_p, \quad \text{Im } k_d > 0
\]

(7) is real on the real energy axis to the right of \( E = E_o \) where \( E_o \) is the largest value of \( E \) satisfying (9) and (10).

The form factors may be calculated on the basis of models for the deuteron and neutron - A bound state wave functions \( 2, 3, 5, 8, 9 \) or alternatively in terms of the effective (optical) potentials for \( n - p \) and \( n - A \) scattering \( 2,3 \). Such form factors lead to additional terms in (7) which have branch cuts starting to the left of \( E_o \). The positions of these cuts have been determined by Shapiro \( 2,3 \) on the basis of assumed optical potentials and by Schnitzer using the Feynman diagram techniques of Landau. Since the dispersion theory calculation of form factors must rely, at least for the present, on the wave function-potential picture of the distorted wave theory, the consideration of form factors sheds no light on the difference between the two methods.

The dispersion theory calculation \( 1,3,5 \) of the stripping amplitude is based on the Omnes-Muskhelishvili \( 6,7,15,16 \) representation of the scattering amplitude in terms of its phase for energies above the physical scattering threshold [i.e. the phase of the expression (6)] and discontinuities across various branch cuts in the unphysical energy region. Now the only branch cuts in the unphysical region considered in the references cited above are those associated with diagrams of the Butler type with possible non-constant form factors. However, other branch cuts further removed from the physical region may also be important. In fact we shall see below that the branch cuts associated with interactions in the initial and final states, when properly accounted for, bring the dispersion theory into essential congruence with the method of distorted waves.
According to the method of distorted waves \(8,9\), the stripping amplitude \(t_2\) is, aside from constant factors,

\[
\frac{1}{(k_p k_d)^2} \int_0^\infty r^2 dr e^{-k_n^r} \mu (-)^m (k_p, r) \mu (+) (k_d, r)
\]

\[
\left(-\frac{\hbar^2}{2\mu_a} \frac{d^2}{dr^2} + V_a(r)\right) \left\{r \mu (+) (k_d, r)\right\} = \frac{\hbar^2 k_d^2}{2\mu_a} \left\{r \mu (+) (k_d, r)\right\}
\]

\[
\left(-\frac{\hbar^2}{2\mu_b} \frac{d^2}{dr^2} + V_b(r)\right) \left\{r \mu (-) (k_p, r)\right\} = \frac{\hbar^2 k_p^2}{2\mu_b} \left\{r \mu (-) (k_p, r)\right\}
\]

\[
\mu (+) (k_d, r) + e^{i\delta_a(k_d)} \sin (k_d r - \frac{k_d}{2} + \delta_a (k_d) / k_d r
\]

\[
\mu (-) (k_p, r) + e^{-i\delta_b(k_p)} \sin (k_p r - \frac{k_p}{2} + \delta_b (k_p) / k_p r)
\]

where \(V_a\) and \(V_b\) are respectively the optical potentials for \(d - A\) and \(p - B\) scattering. We have employed the zero range approximation \(8,9\) and neglected internal structure effects in writing (11).* We have also assumed that \(\frac{M_A}{M_A + M_n} \approx 1\) which is usually the case in practice and helps simplify the discussion.

Now if the stripping amplitude is negligible in comparison with that of \(d - A\) and \(n - B\) elastic scattering, the potentials \(V_a\) and \(V_b\)

* This procedure is practically equivalent to assuming constant form factors.
are Hermitian and the phase of (11) is just \( \delta_a + \delta_b \). Thus (5) is exactly satisfied by (11) and the phase of the distorted wave stripping amplitude in the physical region is the same as that used in the dispersion theory calculation.

Furthermore, it can be shown, using the method of A. Martin and de Alfaro and Rossetti that the nearest cut of (11) to the left of the physical scattering threshold is correctly described by the Butler pole contribution (7).

If \( V_a \) and \( V_b \) can be written as superpositions of Yukawa wells, i.e.

\[
V_{a,b} = \sum_{\alpha,\beta} C_{\alpha,\beta} (r) e^{-\alpha r} d\alpha
\]

(14)

the next nearest cuts will occur for

\[
(k_p + k_d)^2 < -(k_n + \alpha_{a,\beta})^2
\]

(15)

\[
\text{Im } k_p, \text{ Im } k_d > 0
\]

(16)

and correspond to the Feynman-Landau diagrams in Figures 2a and 2b. There will also be poles associated with bound states of d-A and n-B systems.

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Nucleon and deuteron optical potentials are usually described by the Saxon-Woods well which, as is well known, cannot be represented by (14). If, however, the potentials can be represented by (14) at large distances and have the general Saxon-Woods shape at small distances, there will still be branch points at the locations given by (15) and (16). In addition, there will be an essential singularity at infinity which can easily be accounted for in a dispersion theory calculation.

(21) and (22), for the case when only initial or only final state scattering is important may be immediately deduced from equations (12), (13), (33), (34) and (35) of reference 16. Note that the discontinuity across the Butler branch cut is not modified by initial or final state scattering except where the cuts described by (15) and (16) overlap with it.
If these additional cuts and poles are to be accounted for in a dispersion theory calculation, they will at present have to be treated via the potential-wave function picture and it should be now fairly obvious that both the distorted wave and dispersion methods in the present case are practically equivalent. It is also clear that the consideration of branch cuts due to initial and final state interactions adds to the dispersion method the ambiguity, mentioned in connection with the distorted wave method at the beginning of this note. Potentials which yield practically the same scattering results over a limited energy region may, in general, lead to different singularities in the unphysical region. If the position of singularities described by (15) and (16) are sufficiently far from the physical region, for the class of potentials fitting the initial and final state scattering data, then both methods should yield fairly unambiguous results.

The above discussion has been based on the assumption of negligible inelastic scattering in the initial and final states. A small amount of inelastic scattering should not change the general conclusions drastically. When there is a large amount of inelastic scattering the relationship between the two approaches is not particularly clear. The importance of short range interactions means that singularities considerably removed from the physical region must be carefully considered in the dispersion approach. Also one must probably take all important channels into account explicitly. All of the points mentioned in this note will be dealt with at greater length in a forthcoming paper to be published elsewhere.

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The Butler stripping pole diagram

Fig 1

Fig 2

Lowest order diagrams representing interactions in the initial and final states. The dotted lines represent the exchange of a particle of mass

\[ \frac{\hbar A_\alpha}{c} \quad \text{and} \quad \frac{\hbar A_\beta}{c} \]

respectively [see (14)]. Note that the exchanged particle cannot be a pion in (a) since the deuteron has an isotopic spin of zero.